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Counting Tilings of the $n \times m$ Grid, Cylinder, and Torus

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Abstract

We count tilings of the rectangular grid, cylinder, and torus with arbitrary tile designs up to arbitrary symmetries of the square and rectangle, along with cyclic shifting of rows and columns, generalizing and classifying a a tiling problem first enumerated by M. C. Escher in May 1942. This provides a unifying framework for understanding a family of counting problems, expanding on the work by Ethier and Lee counting tilings of the torus by tiles of two colors.

In 1704 the Dominican priest, mathematician, and typographer Sébastien Truchet, wrote a manuscript *Mémoire sur les combinaisons* [29], which illustrates designs that can be made from many copies and rotations of the "Truchet tile" ◩, one of which is reproduced in Figure 1. In 1722, Douat published a book containing futher analysis and illustrations of these tilings [9]. Truchet's and Douat's work resurfaced in Cyril Stanley Smith and Pauline Boucher's translation [27], which also introduced another tile design which is also, somewhat ambiguously, called a Truchet tile: ▧.

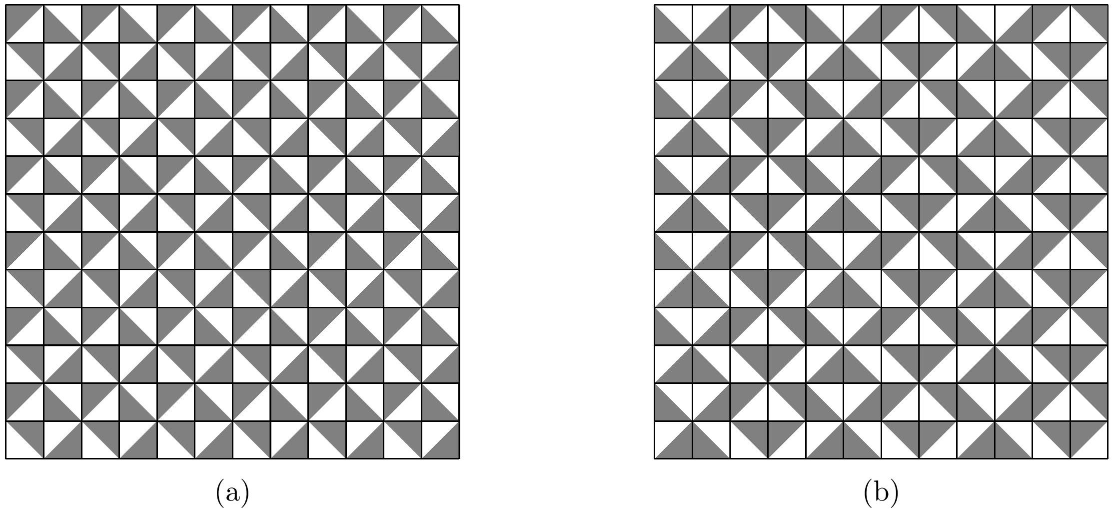


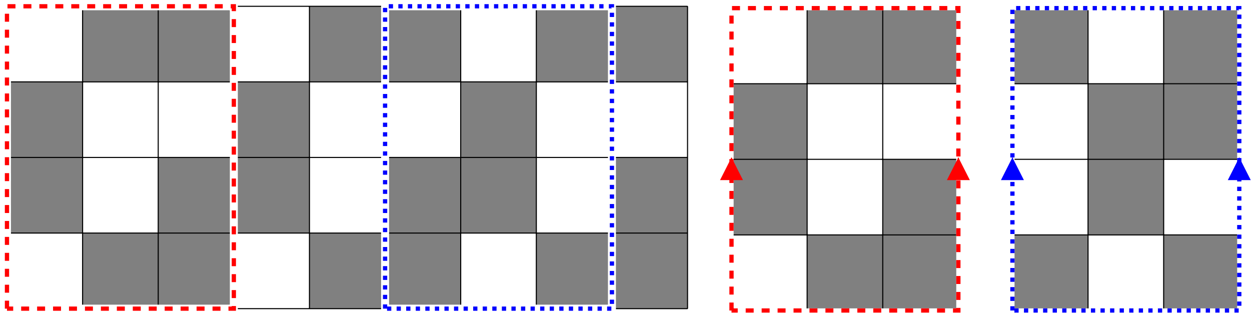
Figure 1: Subfigure (a) is a reproduction of tiling D from Truchet's Plate 1. It is a representation of the 17 ways of tiling the 2×2 torus with Truchet tiles up to dihedral action of the square. Subfigure (b) is a reproduction of a pattern 1 from Douat. It is a representation of one of the 2196 ways of tiling the 4×2 torus by Truchet tiles up to reflection of rows and columns.

The earliest record of attempting to count these configurations was perhaps the artist M. C. Escher, who in May 1942 explicitly enumerated all of the 23 configurations of what we call the 2×2 grid on a torus by rotationally asymmetric tiles up to 90° rotation, as verified by Schattschneider [25, 26]. (An illustration of this can be found in the appendix in Figure 81.)

Given some set of arbitrary tile designs, we are interested in counting ways of tiling the $n \times m$ square grid, of tiling the infinite strip in a periodic way, and of tiling the Euclidean plane in a way that is periodic both left-to-right and top-to-bottom, up to various symmetries. Both Truchet's and Douat's work were, at least in part, meant to be useful as a reference for artists, architects, and designers. In this way, the counting problems are of physical interest as they count the essentially different ways of tiling a square table, knitting a scarf with a repeating motif, or tiling the floor of a large room with a repeating design.

Ultimately, we will construct a framework for counting the number of ways of tiling the grid up to various symmetries. This provides a unifying theory for a family of problems that appear to have only been analyzed in an *ad hoc* manner. This will give a unifying framework for over a dozen OEIS sequences including but not limited to A047937, A054247, A086675, A179043, A184271, A184277, A184284, A200564, A222187, A222188, A225910, A255015, A255016, A295223, A295229, A302484, A343095, and A343096. Applying this framework has resulted in the addition of 49 new sequences to the On-Line Encyclopedia of Integer Sequences (OEIS) [24].

When we consider at the grid up to symmetries of the square or rectangle, we call this



(a) (b) (c)

Figure 2: Part (a) shows a 3×4 cylinder repeated three times horizontally with two 3×4 regions selected. Part (b) is one of the grid representations of this cylinder. Part (c) is an equivalent grid representation if 180° rotation is allowed.

the $n \times m$ grid, an example of which is illustrated in Figure 1. When we additionally allow cyclic shifting of columns, we call this the $n \times m$ cylinder, which is illustrated in Figure 2. When we allow cyclic shifting of the rows in addition to the above symmetries, we call this the $n \times m$ torus, which is illustrated in Figure 3.

The number of such tilings depends on the size of the grid, the symmetries of the grid under consideration, the symmetries of the tile designs, and the number of tile designs with a given symmetry. We formalize each of these four notions below, and use them to give a formula that counts the number of corresponding tile designs.

1 Background

Many mathematicians, architects, artists, and others have studied and generalized Truchet tiles, starting with Douat [9], who illustrated examples of rosettes, which are tilings of the grid with dihedral symmetry, and which were studied mathematically by Hall, Almeida, and Teixeira [14]. M. C. Escher, Schattschneider [26], and [8] were perhaps the earliest to count these tilings on a torus, and Schattschneider is perhaps the earliest to ask about higher-dimensional analogs in the form of cubic tiles.

Many others have considered generalizations of these tiles, in terms of specific tile designs, other polygons, and higher dimensional analogs. Lord and Ranganathan [19] also looked at generalizations on rhombuses and on cubes. Browne considered different scales [6] and generalizations to hexagons [6]. Krawczyk generalized to other square tile designs, including tiling the faces of a cube [18]. Ahmed gave a catalog of square and hexagonal tile designs, including tile designs for octagons and the truncated square tiling [1]. Borlenghi produced perhaps the richest catalog of examples of square tile designs and also has a related US Patent [3]. Beveridge looked at generalizations of Truchet tiles to all $2n$-gons and the faces of a cube [2]. Carlson considered putting together tiles of different scales in a compatible manner [7]. Mitchell gave further examples of triangular, square, hexagonal, and octagonal

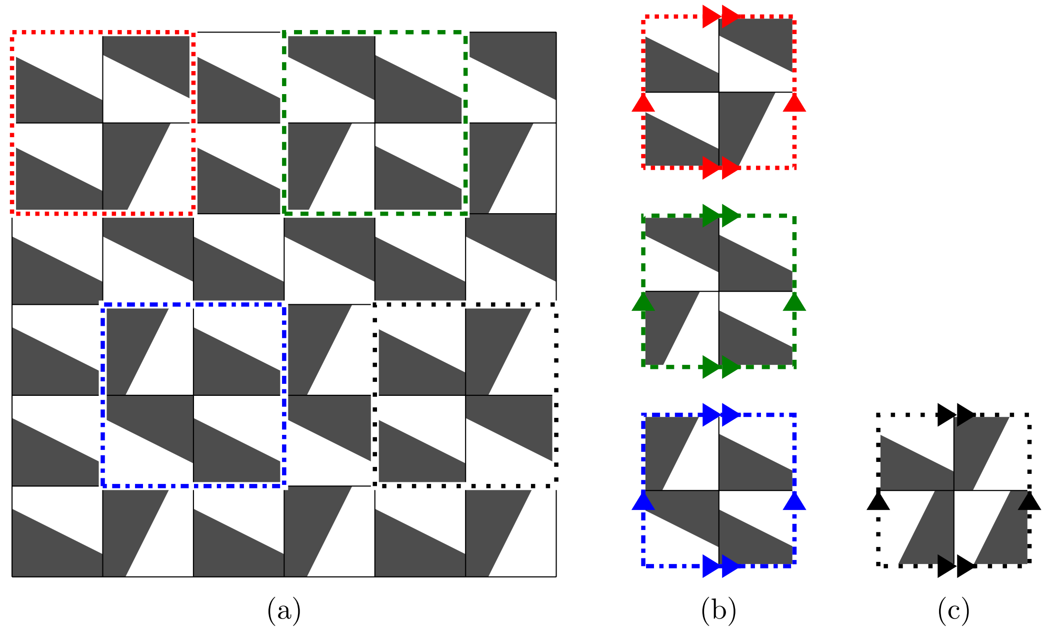


Figure 3: Part (a) shows a 2×2 torus repeated three times horizontally and three times vertically, with three 2×2 regions selected. Part (b) shows three tilings of the 2×2 grid that are equivalent under the toroidal action $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Part (c) shows a 2×2 torus that is equivalent to the other tori under the dihedral action r^3.

tile designs and corresponding Archimedean tilings of the Euclidean plane [21]. Walter, Ligler, and Gürsoy utilized "shape rules" to generate tile designs on the equilateral triangle, square, regular hexagon, and other convex polygons [31]. Knoll, McLellan, and Cox studied the 2×2 grid of Truchet tiles including some novel group actions related to flipping woven versions along with some constructions related to de Bruijn sequences [17].

In the 1980s the following one-line Commodore 64 computer program was a popular way to create an interesting output:

```
10 PRINT CHR$(205.5+RND(1));: GOTO 10
```

The program printed `\` and `/` to the display as an endless loop, which created maze-like figures that are closely related to those shown in Figure 72. This computer program is also the title of a book by Montfort along with nine other authors which discusses the program along with its cultural importance [22].

In addition to the connections to M. C. Escher, art, programming, and recreational mathematics, related concepts also come up in the context of statistical mechanics (especially with Smith's version of the Truchet tile) with the Completely Packed Loops model or $O(n)$ Loop Model. See, for example, Fonseca and Zinn-Justin's analysis of the $O(\tau)$-loop model on a cylider [13], Hooper's analysis of probabilities related to closed curves [15], and Nahum, Serna, Somoza, and Ortuño's loop models with crossings [23].

2 Notation and preliminaries

In this section, we will formalize the notation of a grid and its size, the symmetries of the grid that we count up to, the symmetries of the tile designs, and the number of tile designs with a given symmetry.

2.1 Tilings and the grid

In order to talk about tilings of the $n \times m$ grid, cylinder, and torus, it is important to first formalize what these are. All of these ideas start with the fundamental idea of the $n \times m$ grid, which we define as follows.

Definition 1. The $n \times m$ **grid** is the set $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$, and the elements of this set are called **cells**.

When illustrating grids, we use the convention that the $n \times m$ grid has n columns and m rows, which are described using 0-indexed Cartesian coordinates, where $(0, 0)$ is the cell in the lower left corner and $(n-1, m-1)$ is the cell in the upper right corner.

2.2 Symmetries of the grid

We will count grids up to various symmetries, some of which may be specified by subgroups of the dihedral group of the square, $R \leq D_8 = \langle r, f \mid r^4 = f^2 = (rf)^2 = \mathrm{id}\rangle$. Because this is a group of rotations and reflections of the grid, we call this subgroup R. When considering the $n \times m$ grid for $n \neq m$, we will further specify that $R \leq D_4$, where $D_4 = \langle r^2, f \mid (r^2)^2 = f^2 = \mathrm{id}\rangle$ is the dihedral group of the rectangle.[1]

In all cases, we will use the convention that our symmetry groups act on the grid via *right actions*, illustrated in Figure 4. We will also use the conventions that r acts on the grid by $+90°$ rotations and f acts on the grid by horizontal reflection (i.e., over the vertical line). Because the group acts on the right, rf reflects the square grid over the line $y = x$ and $r^3 f$ corresponds to matrix transposition.

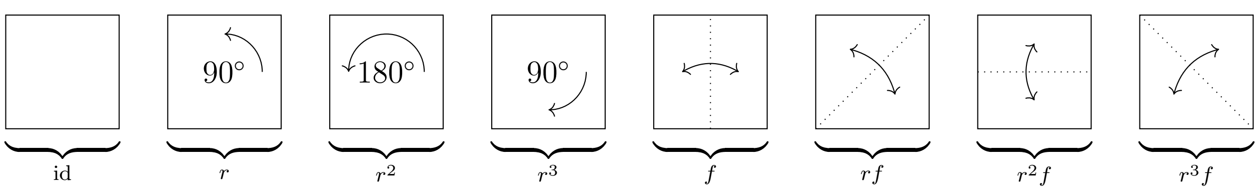


Figure 4: Illustrations of the eight group actions of the dihedral group of the square D_8, which we call "identity", "90° rotation", "180° rotation", "−90° rotation", "horizontal reflection", "diagonal reflection", "vertical reflection", and "antidiagonal reflection" respectively.

Formally, we describe the action on the cell by specifying how the generators of D_8 act.

[1]We will later see that we use D_4 in the case of the $n \times n$ cylinder as well.

Definition 2. The (right) action of an element of D_4 on a cell $(x, y) \in \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ is given by

$$\begin{aligned}(x, y) \cdot f &= (n-1-x, y), \qquad \text{and} \\ (x, y) \cdot r^2 &= (n-1-x, m-1-y).\end{aligned}$$

In the case of an $n \times n$ grid, the action of $r \in D_8$ is given by

$$(x, y) \cdot r = (n-1-y, x). \tag{1}$$

These actions can be extended to all of D_4 and D_8 via the binary operation of the group, since the group action is specified for the generators.

2.3 Symmetries of the cylinder and torus

Now that we know how the dihedral group acts on the $n \times n$ and $n \times m$ grids, we can also look at symmetries of the grid by cyclic shifting of rows and/or columns. When we shift just the columns, we call this a *cylindrical action*, which we describe with the group $\mathbb{Z}/n\mathbb{Z}$; when we shift the rows and columns, we call this a *toroidal*, which we describe with the group $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$. Both of these are named in reference to the corresponding topological identification of the square.

Definition 3. The **cylindrical action** of $a \in \mathbb{Z}/n\mathbb{Z}$ on a cell $(x, y) \in \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ corresponds to a (rightward) cyclic shift of columns:

$$(x, y) \cdot a = (x + a, y).$$

Definition 4. The **toroidal** of $(a, b) \in \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ on a cell $(x, y) \in \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ corresponds to a (rightward) cyclic shift of columns and an (upward) cyclic shift of rows:

$$(x, y) \cdot (a, b) = (x + a, y + b).$$

Definition 4 is illustrated in Figure 5.

2.4 Compatibility of grid symmetries

Notice that we can act on the grid with both the dihedral actions and the cylindrical/toroidal actions. In order to make the group actions of the dihedral group compatible with the cylindrical action ($\mathbb{Z}/n\mathbb{Z}$) or the toroidal action ($\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$), we define their (outer)

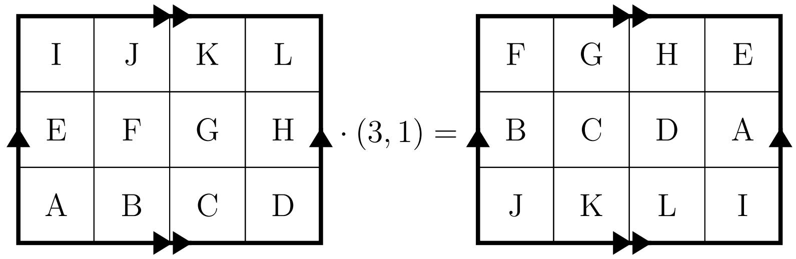


Figure 5: The $(3, 1) \in \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ acts on the 4×3 grid identified as a torus by cyclically shifting columns to the right by 3 and cyclically shifting rows by 1.

semidirect product, $\mathbb{Z}/n\mathbb{Z} \rtimes R$ or $(\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}) \rtimes R$ respectively. As in the examples above, we define this as a right action, thinking of this as first cyclically shifting the rows and columns and then rotating or reflecting according to the element R.

The outer semidirect products are defined with respect to the homomorphisms $\psi\colon D_4 \to \mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$ and $\phi\colon D_8 \to \mathrm{Aut}(\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z})$ respectively. (We use D_4 rather than D_8 in the case of the cylinder because a 90° rotation is not an isometry of the infinite strip, which is the universal cover of the cylinder.)

Definition 5. Let $\psi\colon D_4 \to \mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$ be given by

$$\begin{aligned} \psi_f(x) = \psi_{r^2}(x) &= -x \text{ and} \\ \psi_{\mathrm{id}}(x) = \psi_{r^2 f}(x) &= x. \end{aligned}$$

Then the product of two elements in $\mathbb{Z}/n\mathbb{Z} \rtimes D_4$ is given by

$$(a_1, g_1)\,(a_2, g_2) = \big(a_1 + \psi_{g_1}(a_1), g_1 g_2\big)\,.$$

In the case of the torus, the definition of the semidirect product $(\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}) \rtimes R$, where $R \leq D_8$, is essentially similar.

Definition 6. Let $\phi\colon D_8 \to \mathrm{Aut}(\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z})$ be defined on the generators r and f by

$$\begin{aligned} \phi_f((x, y)) &= (-x, y) \text{ and} \\ \phi_r((x, y)) &= (y, -x), \end{aligned}$$

and extended to the other elements of D_8. Then the binary operation of the semidirect product of $(\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}) \rtimes D_8$ is given by

$$\big((a_1, b_1), g_1\big)\,\big((a_2, b_2), g_2\big) = \big((a_1, a_2) + \phi_{g_1}(a_1, a_2), g_1 g_2\big)\,.$$

Using the facts that r and f generate D_8 and ϕ is a homomorphism, together with function composition yields:

$$\begin{aligned}
\phi_{\mathrm{id}}((x,y)) &= (x,y), & \phi_r((x,y)) &= (y,-x),\\
\phi_{r^2}((x,y)) &= (-x,-y), & \phi_{r^3}((x,y)) &= (-y,x),\\
\phi_f((x,y)) &= (-x,y), & \phi_{rf}((x,y)) &= (y,x),\\
\phi_{r^2f}((x,y)) &= (x,-y), & \phi_{r^3f}((x,y)) &= (-y,-x).
\end{aligned}$$

Example 7. We check our work on an individual cell. For every choice of tile and pair of symmetries, we should have

$$\Big((x,y)\cdot\big((a_1,b_1),g_1\big)\Big)\cdot\big((a_2,b_2),g_2\big) = (x,y)\cdot\Big(\big((a_1,b_1),g_1\big)\,\big((a_2,b_2),g_2\big)\Big).$$

In particular, we check in the case of the 4×4 torus with $(x,y)=(1,0)$, $\big((a_1,b_1),g_1\big) = \big((1,1),f\big)$ and $\big((a_2,b_2),g_2\big) = \big((2,0),r\big)$.

$$\begin{aligned}
\Big((1,0)\cdot\big((1,1),f\big)\Big)\cdot\big((2,0),r\big) &= \big((2,1)\cdot f\big)\cdot\big((2,0),r\big)\\
&= (1,1)\cdot\big((2,0),r\big)\\
&= (3,1)\cdot r\\
&= (2,3).
\end{aligned}$$

Now using the semidirect product,

$$\begin{aligned}
(1,0)\cdot\Big(\big((1,1),f\big)\,\big((2,0),r\big)\Big) &= (1,0)\cdot\big((1,1)+\phi_f(2,0), fr\big)\\
&= (1,0)\cdot\Big((1,1)+(-2,0), r^3f\Big)\\
&= (1,0)\cdot\Big((3,1), r^3f\Big)\\
&= (0,1)\cdot r^3f\\
&= (2,3).
\end{aligned}$$

This suggests that this semidirect product is the appropriate way to make the dihedral actions compatible with the toroidal action.

2.5 Symmetries of tile designs

We are now ready to start filling in our grid with tiles. Before defining what tiles are, we introduce the following definition for convenience.

Definition 8. If X is a set and G has a group action on X, then we call X a G**-set**.

Definition 9. Given $R \leq D_8$, a **set of tile designs** is simply an R-set. A **tile design** is any element of such a set.

We will always illustrate our tile designs with squares that have designs in them, but any abstract R-set will work in place of these illustrations. When we specify one of these tile designs together with a cell, we get a tile.

Definition 10. A **tile** in the $n \times m$ grid with the set of tile designs T is an element of $(\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}) \times T$.

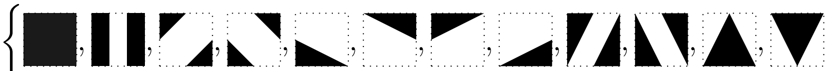
Now we are ready to define a tiling of the grid, which is a specification of a tile design for each cell.

Definition 11. A **tiling** of the $n \times m$ grid with the set of tile designs T is a map $f\colon (\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}) \to T$. The tiles associated with the tiling are elements of the graph of the map

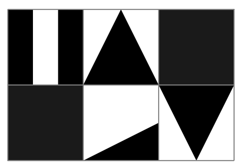
$$\{\big((x,y), f(x,y)\big) \mid (x,y) \in \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}\}.$$

We next give an example to illustrate all of these definitions.

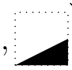
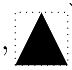
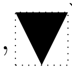
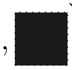
Example 12. Suppose that $R = D_4 = \langle r^2, f\rangle$, the dihedral group of the rectangle. Then

$T = \Big\{$  $\Big\}$.

is a set of tile designs, because it is an R-set. The 3×2 tiling



consists of the six tiles

$\Big\{\Big((0,0),$  $\Big)\ \Big((0,1),$  $\Big)\ \Big((1,0),$  $\Big)\ \Big((1,1),$  $\Big)\ \Big((2,0),$  $\Big)\ \Big((2,1),$  $\Big)\Big\}$.

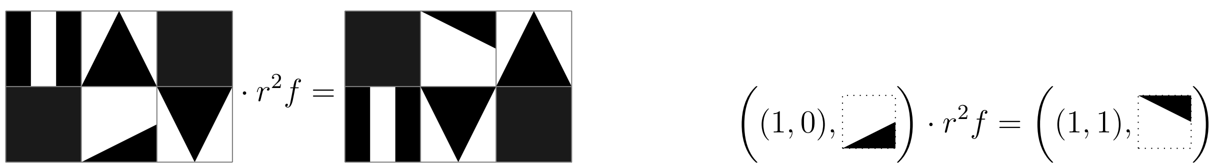
Since we have now defined tiles, we are ready to talk about how the symmetries of our grid, cylinder, or torus act on tiles: dihedral actions can act on the tile design nontrivially, but cylindrical/toroidal actions always act on a tile design by the identity. We extend Definitions 2, 3, and 4 to this setting in a direct way, and illustrate this in Figure 6.

Definition 13. If $\big((x,y),d\big)$ is a tile in an $n \times m$ grid, then $g \in D_4$ acts on $\big((x,y),d\big)$ by

$$\begin{aligned}
\big((x,y),d\big) \cdot \mathrm{id} \;&= \big((x,y),d\big), \\
\big((x,y),d\big) \cdot r^2 \;&= \Big((n-1-x, m-1-y), d \cdot r^2\Big), \\
\big((x,y),d\big) \cdot f \;&= \big((n-1-x, y), d \cdot f\big)\,, \text{ and} \\
\big((x,y),d\big) \cdot r^2 f &= \Big((x, m-1-y), d \cdot r^2 f\Big)\,.
\end{aligned}$$

Furthermore, if $\big((x,y),d\big)$ is a tile in an $n \times n$ grid, then $g \in D_8$ acts on $\big((x,y),d\big)$ by the above actions together with

$$\begin{aligned}
\big((x,y),d\big)\cdot r \quad &= \big((n-1-y,x), d\cdot r\big)\,,\\
\big((x,y),d\big)\cdot r^3 \quad &= \Big((y,n-1-x), d\cdot r^3\Big)\,,\\
\big((x,y),d\big)\cdot rf \quad &= \big((y,x), d\cdot rf\big)\,,\ \text{and}\\
\big((x,y),d\big)\cdot r^3 f &= \Big((n-1-y,n-1-x), d\cdot r^3 f\Big)\,.
\end{aligned}$$



(a)

(b)

Figure 6: An example of the action of $r^2 f$ (a vertical reflection) on (a) a tiling of the 3×2 grid and on (b) a specific tile.

In order to understand what makes two tiles *essentially different* with respect to counting tilings, it is useful to define the notion of a stabilizer subgroup.

Definition 14. Let X be a G-set. Then the **stabilizer subgroup** of an element $x \in X$ is the subgroup

$$G_x = \{g \in G \mid x \cdot g = x\} \leq G.$$

Because our set of tile designs T is an R-set, the relevant difference between different tile designs for the purpose of counting tilings is their stabilizer subgroups.

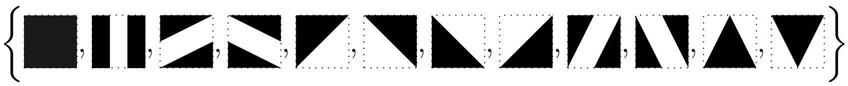
2.6 Classifying sets of tile designs

In order to describe the essential features of a set of tile designs, we partition it into orbits with respect to R. By counting up the number of orbits and classifying each orbit by the (conjugacy class of the) stabilizer subgroup of one of its representatives we can understand the combinatorics of the set of tile designs completely.

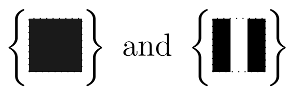
Definition 15. Let $R \subseteq D_8$ and let T be a set of tile designs (with respect to R). Then for each (conjugacy class of) a subgroup $S \leq R$, let $\mathcal{O}_S^R$ denote the number of orbits that contain a tile whose stabilizer subgroup is conjugate to S.

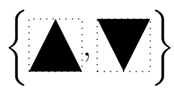
Notice that we classify up to conjugacy class because if d is stable under S, then $d \cdot g$ is stable under $g^{-1}Sg$ since $(d \cdot g) \cdot g^{-1}Sg = d \cdot Sg = d \cdot g$.

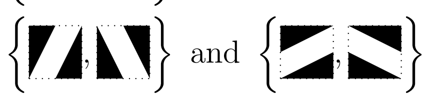
Example 16. Suppose we are counting tilings of the grid, cylinder, or torus up to horizontal and vertical reflection with a set of tile designs given by

$T =$  .

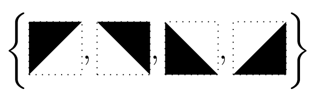
Since the symmetry group $D_4 = \langle r^2, f \mid (r^2)^2 = f^2 = \mathrm{id}\rangle$ has 5 conjugacy classes of subgroups, there are five types of orbits:

$\mathcal{O}_{\langle r^2,f\rangle}^{\langle r^2,f\rangle} = 2$ via 

$\mathcal{O}_{\langle f\rangle}^{\langle r^2,f\rangle} = 1$ via 

$\mathcal{O}_{\langle r^2\rangle}^{\langle r^2,f\rangle} = 2$ via 

$\mathcal{O}_{\langle r^2 f\rangle}^{\langle r^2,f\rangle} = 0$

$\mathcal{O}_{\mathbb{1}}^{\langle r^2,f\rangle} = 1$ via  .

Lemma 17. *The number of tilings for a given R-tiling T only depends on the tuple*

$$\bigoplus_{S\in\mathrm{conj}(R)} \mathcal{O}_S^R,$$

where $\mathrm{conj}(R)$ *is the set of equivalence classes of subgroups of R up to conjugacy.*

Proof. Suppose that we have two sets of tile designs T and T' with the same number of orbits for each stabilizer conjugacy class. There exists a bijection $f\colon T \to T'$ such that $f(d) = d'$ whenever d and d' have the same stabilizer subgroup in R, that is, $R_d = R_{d'}$. Then the induced map of f to the tilings is also bijection of tilings. $\square$

In Appendix A.5, we explicitly enumerate all of the R-sets of tile designs that consist of a single orbit for each subgroup $R \leq D_4$ or $R \leq D_8$.

2.7 Counting strategy

In order to count how many tilings exist up to various symmetries, we will use Burnside's lemma.

Theorem 18 (Burnside's lemma)**.** *Let X be a G-set. Then the size of X up to the action of G is*

$$|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|,$$

where $|X^g|$ is the number of elements of X that are fixed under the action of $g \in G$.

We want to understand how many tilings are fixed under various symmetries. To do this it is necessary to categorize the symmetries of various tiles.

Definition 19. Let T be a set of tile designs, where $R \leq D_8$. For each $g \in R$, the set of tiles that are fixed by g is denoted

$$T^g = \{d \in T \mid d \cdot g = d\},$$

and the size of this set is denoted

$$t_g = |T^g|.$$

Note that $T^{\mathrm{id}} = T$, so t_{id} is the total number of tile designs.

The following theorem gives us a strategy for counting the number of tilings that are fixed under a given symmetry, which is illustrated in Figure 7. For more thorough treatment in the case of Truchet tiles in particular, see Hall, Almeida, and Teixeira [14].

Theorem 20. *Suppose that $s = g \in R$, $s = (a, g) \in \mathbb{Z}/n\mathbb{Z} \rtimes R$, or $s = \big((a, b), g\big) \in (\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}) \rtimes R$.*

Since the set of cells $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ form an $\langle s\rangle$-set, we partition the cells into orbits with respect to the cyclic subgroup $\langle s\rangle$, which we call Θ_s, so that $\bigsqcup_{\vartheta\in\Theta_s}\vartheta = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$.

Then if X^s is the set of tilings of the $n \times m$ grid that are stable under s,

$$|X^s| = \prod_{\vartheta\in\Theta_s} t_{g^{|\vartheta|}}.$$

Proof. Because Θ_s partitions the cells into orbits with respect to the cyclic subgroup $\langle s\rangle$, the number of tilings that are fixed under s is equal to the product of the number of tilings of each orbit of cells under $\langle s\rangle$.

The tiling of an orbit of cells can be specified by a single tile d, which then determines the rest of the orbit by $\big((x, y), d\big) \cdot s^k$ for $0 \leq k < |\vartheta|$. The only requirement for a valid tiling of a orbit is that

$$\big((x, y), d\big) = \big((x, y), d\big) \cdot s^{|\vartheta|} = \Big((x, y), d \cdot g^{|\vartheta|}\Big),$$

thus d must be fixed by $g^{|\vartheta|}$, and so $d \in T^{g^{|\vartheta|}}$. Therefore there are $t_{g^{|\vartheta|}}$ choices for d and thus for the orbit of cells containing (x, y). □

Thus, this reduces the problem to a matter of counting the orbits of cells under each symmetry s along with counting the sizes of each of these orbits.

We proceed with Sections 3, 4, and 5, which all implement the above strategy. Each section consists broadly of fixed point counting theorems, which count tilings of the grid that are fixed under the actions of D_4 or D_8 for arbitrary sets of tile designs.

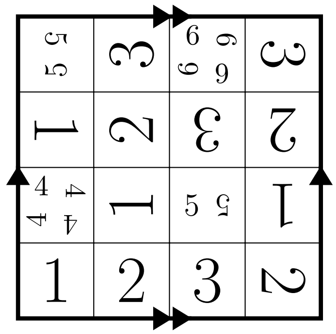


Figure 7: An illustration showing a tiling of the 4×4 torus that is fixed under $\big((1,2), r\big)$, and the six orbits of its cells with respect to the subgroup generated by this symmetry. There are three orbits of size 4 (whose tiles are stable under $r^4 = \mathrm{id}$), one orbit of size 2 (whose tiles are stable under r^2), and two orbits of size 1 (whose tiles are stable under r).

3 Grid

For counting tilings of the $n \times m$ rectangular grid or $n \times n$ square grid under subgroups $R \leq D_4$ and $R \leq D_8$ respectively, we count the number of tilings that are fixed under each element of R.

In the following two subsections, we denote the rectangular grid by RG and the square grid SG.

3.1 The $n \times m$ grid

We begin by specifying the number of tilings that are fixed under each symmetry.

Definition 21. For a given set of tile designs T, and an element $g \in R \leq D_4$, the number of tilings of the $n \times m$ grid by tile designs in T that are fixed by g is denoted $\mathrm{fxpt}_g^{\mathrm{RG}}(n,m)$.

Theorem 22. *For a given set of tile designs T and an element $g \in R \leq D_4$, the number of tilings of the $n \times m$ grid by tile designs in T that are fixed by g is*

$$\mathrm{fxpt}_{\mathrm{id}}^{RG}(n,m) = t_{\mathrm{id}}^{nm}. \tag{2}$$

$$\mathrm{fxpt}_{r^2}^{RG}(n,m) = \begin{cases} t_{\mathrm{id}}^{nm/2}, & \text{if } nm \text{ is even;} \\ t_{\mathrm{id}}^{(nm-1)/2} t_{r^2}, & \text{if } nm \text{ is odd.} \end{cases} \tag{3}$$

$$\mathrm{fxpt}_{f}^{RG}(n,m) = \begin{cases} t_{\mathrm{id}}^{nm/2}, & \text{if } n \text{ is even;} \\ t_{\mathrm{id}}^{(m(n-1))/2} t_f^m, & \text{if } n \text{ is odd.} \end{cases} \tag{4}$$

$$\mathrm{fxpt}_{r^2 f}^{RG}(n,m) = \begin{cases} t_{\mathrm{id}}^{nm/2}, & \text{if } m \text{ is even;} \\ t_{\mathrm{id}}^{(n(m-1))/2} t_{r^2 f}^n, & \text{if } m \text{ is odd.} \end{cases} \tag{5}$$

Then the number of distinct tilings of the $n \times m$ grid up to action of R is given by

$$\frac{1}{|R|} \sum_{g \in R} \operatorname{fxpt}_g^{RG}(n, m). \tag{6}$$

Proof. The proof will consist of three cases.

Equation (2). This follows from the fact that t_{id} is the number of distinct tiles, and every tiling is fixed under $\mathrm{id} \in D_4$.

Equation (3). This follows from the fact that the (right) action of r^2 on the cell (x, y) is

$$(x, y) \cdot r^2 = (n - x - 1, m - y - 1).$$

Since r^2 has order 2, each cell is in an orbit of size 1 or 2. The cell (x, y) is fixed under the action of r^2 if and only if n and m are both odd and $(x, y) = (\frac{n-1}{2}, \frac{m-1}{2})$.

Therefore when nm is even, the grid is partitioned into $nm/2$ orbits of size 2, so any fixed tiling can be specified by choosing any tile design in T for each orbit. When nm is odd, the grid is partitioned into one orbit of size 1 together with $(nm - 1)/2$ orbits of size 2, so any fixed tiling can be specified by choosing a tile design in T^{r^2} for the fixed point and any tile design in T for each orbit.

Equations (4) and (5). These two equations are essentially the same, so without loss of generality, we will prove the case of Equation (4). The right action of f on $\big((x, y), d\big)$ is

$$\big((x, y), d\big) \cdot f = ((n - x - 1, m), d \cdot f).$$

Because f is order 2, we can conclude that (x, y) is either a fixed point or a 2-cycle with respect to f. It follows that (x, y) is a fixed point if and only if n is odd and $x = (n - 1)/2$, therefore when n is odd the tiling has m fixed cells. The fixed cells can be specified by any tile design in T^f, and the cells in orbits of size 2 can be specified by any tile design in T.

Equation (6) Finally, this is a direct application of Burnside's lemma.

$\square$

3.2 The $n \times n$ grid

There are more symmetries and some specializations in the case of the $n \times n$ grid, which we denote SG for "square grid".

Definition 23. For a given set of tile designs T, and an element $g \in R \leq D_8$, the number of tilings of the $n \times n$ grid by tile designs in T that are fixed by g is denoted $\operatorname{fxpt}_g^{\mathrm{SG}}(n)$.

Theorem 24. *For a given set of tile designs T and an element $g \in R \leq D_8$, the number of tilings of the $n \times m$ grid by tile designs in T that are fixed by g is*

$$\mathrm{fxpt}^{SG}_{\mathrm{id}}(n) = \mathrm{fxpt}^{RG}_{\mathrm{id}}(n,n) = t_{\mathrm{id}}^{n^2} \tag{7}$$

$$\mathrm{fxpt}^{SG}_{r^2}(n) = \mathrm{fxpt}^{RG}_{r^2}(n,n) = \begin{cases} t_{\mathrm{id}}^{n^2/2}, & \textit{if } n \textit{ is even;} \\ t_{\mathrm{id}}^{(n^2-1)/2} t_{r^2}, & \textit{if } n \textit{ is odd.} \end{cases} \tag{8}$$

$$\mathrm{fxpt}^{SG}_{f}(n) = \mathrm{fxpt}^{RG}_{f}(n,n) = \begin{cases} t_{\mathrm{id}}^{n^2/2}, & \textit{if } n \textit{ is even;} \\ t_{\mathrm{id}}^{(n^2-n)/2} t_{f}^{n}, & \textit{if } n \textit{ is odd.} \end{cases} \tag{9}$$

$$\mathrm{fxpt}^{SG}_{r^2f}(n) = \mathrm{fxpt}^{RG}_{r^2f}(n,n) = \begin{cases} t_{\mathrm{id}}^{n^2/2}, & \textit{if } n \textit{ is even;} \\ t_{\mathrm{id}}^{(n^2-n)/2} t_{r^2f}^{n}, & \textit{if } n \textit{ is odd.} \end{cases} \tag{10}$$

$$\mathrm{fxpt}^{SG}_{r}(n) = \mathrm{fxpt}^{SG}_{r^3}(n) = \begin{cases} t_{\mathrm{id}}^{n^2/4}, & \textit{if } n \textit{ is even;} \\ t_{\mathrm{id}}^{(n^2-1)/4} t_{r}, & \textit{if } n \textit{ is odd.} \end{cases} \tag{11}$$

$$\mathrm{fxpt}^{SG}_{rf}(n) = t_{\mathrm{id}}^{(n^2-n)/2} t_{rf}^{n}. \tag{12}$$

$$\mathrm{fxpt}^{SG}_{r^3f}(n) = t_{\mathrm{id}}^{(n^2-n)/2} t_{r^3f}^{n}. \tag{13}$$

Then the number of distinct tilings of the $n \times n$ grid up to the dihedral action of the square is given by

$$\frac{1}{|R|} \sum_{g \in R} \mathrm{fxpt}^{SG}_{g}(n). \tag{14}$$

Proof. This proof proceeds with four cases.

Equations (7), (8), (9), and (10). These follow directly from Theorem 22, by specifying $m = n$.

Equation (11). Firstly, notice that the tilings that are fixed under r are identically those that are fixed under $r^{-1} = r^3$. The (right) action of $r \in D_8$ on a cell (x, y) is

$$(x, y) \cdot r = (n - y - 1, x)$$

therefore (a, b) is a fixed point if and only if it satisfies the system of equations

$$a = n - b - 1 \tag{15}$$

$$b = a, \tag{16}$$

which has an integer solution only when n is odd and when $(a, b) = (\frac{n-1}{2}, \frac{n-1}{2})$. Also, there are no cells that occur in 2-cycles. This can be seen by noticing that cells that occur in 2-cycles are also fixed points under r^2, and by the proof of Theorem 22, we know that this occurs under the same conditions as the fixed points under f. Therefore all other cells occur in 4-cycles.

Therefore when n is even, the grid is partitioned into $n^2/4$ orbits of size 4, each of which can be specified by any tile design in T; when n is odd, the grid has one fixed point, which must be tiled with a tile design in T^r, and the remaining cells can be partitioned into $(n^2-1)/4$ orbits of size 4, each of which can be specified by any tile design in T.

Equations (12) and (13). Because rf and r^3f are conjugate, these are essentially similar, so without loss of generality, we will prove Equation (12).

The (right) action of $rf \in D_8$ on a cell (x, y) is

$$(x, y) \cdot rf = (y, x).$$

Thus, (x, y) is a fixed point if and only if $x = y$, otherwise it is a part of a 2-cycle. Therefore there are n fixed points, which can be specified by a tile design in T^{rf} and $(n^2-n)/2$ 2-cycles, which can be specified with any tile design in T.

Equation (14). The final equation follows by a direct application of Burnside's lemma.

□

4 Cylinder

Here we use the convention that the $n \times m$ cylinder is identified along its left and right sides, as illustrated in Figure 2.

Even in the case of $n \times n$ grids, we only consider tilings of cylinders up to subgroups of the dihedral group of the rectangle, because other symmetries of the square would result in swapping the pair of identified sides (the right and left side) of the grid with the pair of non-identified sides (the top and bottom).

In both the case of the cylinder and the torus, we will repeatedly use the following observation.

Lemma 25. *For fixed values of n and a, the equation*

$$x \equiv -1 - x - a \pmod{n} \tag{17}$$

has solutions that depend on the parity of n and a.
When n is odd, there is one solution:

$$x \equiv \frac{n+1}{2}(-1-a) \pmod{n}. \tag{18}$$

When n is even and a is odd, there are two solutions:

$$x \equiv \frac{-1-a}{2} \pmod{n} \tag{19}$$

$$x \equiv \frac{n-1-a}{2} \pmod{n}. \tag{20}$$

When n and a are both even, there are no solutions.

Proof. In both cases, we write equation (17) as

$$2x \equiv -1-a \pmod{n}.$$

Odd n. When n is odd, 2 has a multiplicative inverse of $(n+1)/2$, so multiplying gives the unique solution described in equation (18).

Even n and odd a. When n is even and a is odd, $-1-a$ is even. Dividing by 2 gives the solution given in equation (19), and adding n and dividing by 2 gives the solution in equation (20).

Even n and a. When both n and a are even, $2x$ is even and $-1-a$ is odd, so there are no solutions.

□

Similarly, we will repeatedly use the following lemma when counting fixed points for both the cylinder and the torus.

Lemma 26 ([20])**.** *Given some $a \in \mathbb{Z}/n\mathbb{Z}$, if d is a minimal solution to the equation*

$$da \equiv 0 \pmod{n}$$

then $d \mid n$. Moreover, when d is a divisor of n, there are $\varphi(d)$ choices for a that result in d being a minimal solution, where φ is Euler's totient function.

Proof. First, we can see that the least value of da will occur when $da = \operatorname{lcm}(a, n)$, and thus

$$d = \operatorname{lcm}(a, n)/a = n/\gcd(a, n).$$

Therefore d must be a divisor of n. The $\varphi(d)$ choices for a such that d is a minimal solution to $da \equiv 0 \pmod{n}$ are $a \in \left\{kn/d \mid 1 \leq k \leq d \text{ and } \gcd(k, d) = 1\right\}$. □

4.1 The $n \times m$ cylinder

We denote the $n \times m$ cylinder by the superscript C.

Definition 27. For a given set of tile designs T, and an element $g \in R \leq D_4$ the sum over all cyclic shifts of the number of tilings of the $n \times m$ cylinder by tile designs in T that are fixed by $(a, g) \in \mathbb{Z}/n\mathbb{Z} \rtimes R$ is denoted

$$\operatorname{fxpt}_g^{\mathrm{C}}(n, m) = \sum_{a \in \mathbb{Z}/n\mathbb{Z}} X^{(a,g)},$$

where $X^{(a,g)}$ is the number of tilings fixed by (a, g).

Theorem 28. *The sum over all cyclic shifts of the number of tilings of the $n \times m$ cylinder by tile designs in T that are fixed by $(a, \mathrm{id}) \in \mathbb{Z}/n\mathbb{Z} \rtimes R$ is given by*

$$\operatorname{fxpt}_{\mathrm{id}}^{C}(n, m) = \sum_{d|n} \varphi(d) t_{\mathrm{id}}^{nm/d}. \tag{21}$$

Proof. For each element $(a, \mathrm{id}) \in \mathbb{Z}/n\mathbb{Z} \rtimes R$, the size of the orbits is the least d such that $da \equiv 0 \pmod{n}$. By Lemma 26, when $d \mid n$ there are $\varphi(d)$ choices for a that result in orbits of size d, and each choice partitions the $n \times m$ grid into nm/d orbits. □

The next theorem concerns the action of $(a, r^2) \in \mathbb{Z}/n\mathbb{Z} \rtimes R$, which is illustrated in Figure 8.

Theorem 29. *The sum over all cyclic shifts of the number of tilings of the $n \times m$ cylinder by tile designs in T that are fixed by $(a, r^2) \in \mathbb{Z}/n\mathbb{Z} \rtimes R$ is given by*

$$\operatorname{fxpt}_{r^2}^{C}(n, m) = \begin{cases} n t_{\mathrm{id}}^{nm/2}, & \textit{if } m \textit{ is even;} \quad (22\mathrm{a}) \\ n\left(\dfrac{1}{2} t_{\mathrm{id}}^{nm/2} + \dfrac{1}{2} t_{\mathrm{id}}^{(nm-2)/2} t_{r^2}^2\right), & \textit{if } m \textit{ is odd and } n \textit{ is even;} \quad (22\mathrm{b}) \\ n t_{\mathrm{id}}^{(nm-1)/2} t_{r^2}, & \textit{if } m \textit{ and } n \textit{ are odd.} \quad (22\mathrm{c}) \end{cases}$$

Proof. The right action of (a, r^2) on $\big((x, y), d\big)$ is

$$\big((x, y), d\big) \cdot (a, r^2) = ((n - 1 - x - a, m - 1 - y), d \cdot r^2). \tag{23}$$

By applying this map twice, we see that $\big((x, y), d\big)^2 = \mathrm{id}$, so each orbit is either size 1 or size 2. Orbits are size 1 precisely when

$$x \equiv -1 - x - a \pmod{n} \tag{24}$$
$$y = m - y - 1. \tag{25}$$

Equation (22a)**.** When m is even, $2y \neq m-1$, so there are no solutions to this system of equations. Thus all orbits have size 2 totaling $nm/2$ orbits. We can choose any tile design in T to start this orbit. Then we sum this over all n choices of $a \in \mathbb{Z}/n\mathbb{Z}$.

Equation (22b)**.** When m is odd, the second equation has the unique solution of $y = (m-1)/2$, which represents the middle row. The solutions for the second equation follow directly from Lemma 25, which states that when n is even, the equation has two solutions when x is odd and none otherwise; when n is odd the second equation has a single solution.

Therefore when m is odd and n is even, half of the choices of $a \in \mathbb{Z}/n\mathbb{Z}$ result in no orbits of size 1, and the other half of choices of x result in two orbits of size 1. In the former case, the grid decomposes into $nm/2$ orbits of size 2, each of which can be filled with any tile design. In the latter case, there are two orbits of size 1, which must be filled with a tile that is fixed under r^2, and the rest of the grid decomposes into $(nm-2)/2$ orbits all of size 2, which can be filled with any tile design.

Equation (22c)**.** Lastly, when both m and n are odd, there is a single orbit of size 1 that must be filled with a tile that is fixed under r^2, the remaining $nm-1$ cells are partitioned into $(nm-1)/2$ orbits of size 2 that can be filled with any tile design.

□

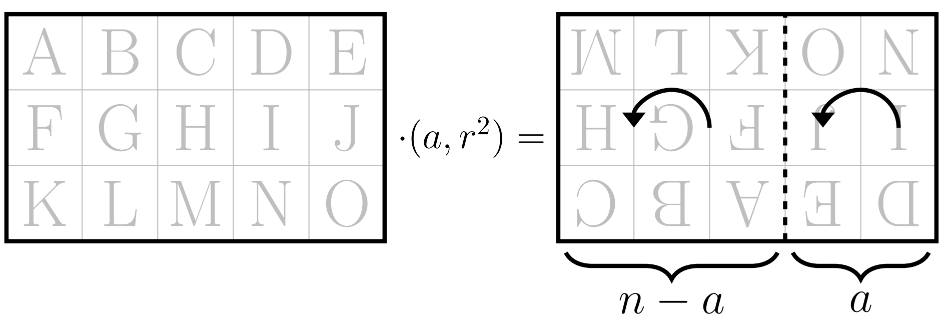


Figure 8: The action of $(a, r^2) \in \mathbb{Z}/n\mathbb{Z} \rtimes R$ on a tiling is equivalent to a 180° rotation of both the leftmost $n-a \times m$ sub-grid and the rightmost $a \times m$ sub-grid.

The next theorem concerns the action of $(a, f) \in \mathbb{Z}/n\mathbb{Z} \rtimes R$, which is illustrated in Figure 9.

Theorem 30. *The sum over all cyclic shifts of the number of tilings of the $n \times m$ cylinder by tile designs in T that are fixed by $(a, f) \in \mathbb{Z}/n\mathbb{Z} \rtimes R$ is given by*

$$\operatorname{fxpt}_f^C(n,m) = \begin{cases} n\left(\frac{1}{2}t_{\mathrm{id}}^{nm/2} + \frac{1}{2}t_{\mathrm{id}}^{(nm-2m)/2}t_f^{2m}\right), & \text{if } n \text{ is even;} \quad (26a) \\ nt_{\mathrm{id}}^{(nm-m)/2}t_f^{m}, & \text{if } n \text{ is odd.} \quad (26b) \end{cases}$$

Proof. Since $(a, f)^2 = \mathrm{id}$, every tile is either a fixed point or appears in a 2-cycle under (a, f). The right action of (a, f) on $\big((x, y), d\big)$ is

$$\big((x, y), d\big) \cdot (a, f) = \big((-x - a - 1, y), d \cdot f\big)$$

so the tiles that appear as fixed points are those that satisfy

$$x = -1 - x - a \pmod{n}.$$

Equation (26a)**.** When n and a are both even, there are no fixed points. When n is even and a is odd, there are two fixed points in each row:

$$x \equiv (n - a - 1)/2 \pmod{n} \qquad \text{and} \tag{27}$$
$$x \equiv (2n - a - 1)/2 \pmod{n}. \tag{28}$$

Equation (26b)**.** When n is odd, there is one fixed cell in each row, which occurs when $x \equiv (-a - 1)(n + 1)/2 \pmod{n}$.

$\square$

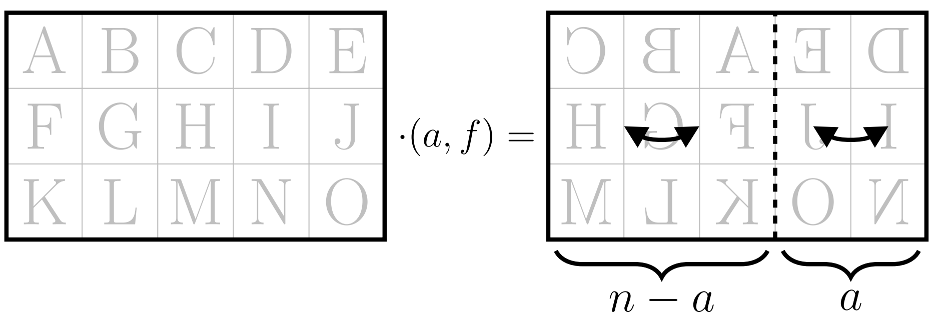


Figure 9: The action of $(a, f) \in \mathbb{Z}/n\mathbb{Z} \rtimes R$ on a tiling is equivalent to a horizontal flip of both the leftmost $n - a \times m$ sub-grid and the rightmost $a \times m$ sub-grid.

Theorem 31. *The sum over all cyclic shifts of the number of tilings of the $n \times m$ cylinder by tile designs in T that are fixed by $(a, r^2 f) \in \mathbb{Z}/n\mathbb{Z} \rtimes R$ is given by*

$$\mathrm{fxpt}^C_{r^2 f}(n, m) = \begin{cases} \displaystyle\sum_{d \mid n} \varphi(d) t_{\mathrm{id}}^{nm/\operatorname{lcm}(d,2)}, & \text{if } m \text{ is even;} \quad (29a) \\ \displaystyle\sum_{d \mid n} \varphi(d) t_{\mathrm{id}}^{(nm-n)/\operatorname{lcm}(d,2)} t_{(r^2 f)^d}^{n/d}, & \text{if } m \text{ is odd.} \quad (29b) \end{cases}$$

Proof. We can see that $(a, r^2 f)$ acts on the coordinates of (x, y) separately, that is,

$$(x,y)\cdot(a,r^2f)^k = \begin{cases} (x+ka, m-1-y), & \text{if } k \text{ is odd;} \\ (x+ka, y), & \text{if } k \text{ is even.} \end{cases}$$

Notice that the orbits of the y-coordinates have size 2, so it is enough to determine the size of the orbits of the x-coordinates. By Lemma 26, for each divisor $d \mid n$, there are $\varphi(d)$ choices for a such that the size of the orbits of the x-coordinate is d.

Equation (29a). When m is even, we see that $y \neq m - y - 1$ has no solutions, the orbit of every y-coordinate has size 2. For each x-orbit size $d \mid n$, each cell must be in an orbit of size $\operatorname{lcm}(d, 2)$, and there must $nm/\operatorname{lcm}(d, 2)$ of them. Each can be specified by any tile design, since all tile designs are stable under $(r^2 f)^{\operatorname{lcm}(d,2)} = \mathrm{id}$.

Equation (29b). When m is odd, we see that $y = m - y - 1$ has a solution precisely when $y = (m-1)/2$. For each x-orbit size $d \mid n$, if $y = (m-1)/2$, then the orbit has size d, otherwise it has size $\operatorname{lcm}(d, 2)$, as in the case above. Therefore there are n cells that are in orbits of size d resulting in n/d orbits that can be specified by any tile design that is stable under $(r^2 f)^{\operatorname{lcm}(d,2)}$. The remaining $n^2 - n$ cells are partitioned into $(n^2 - n)/\operatorname{lcm}(d, 2)$ orbits of size $\operatorname{lcm}(d, 2)$ that can be specified by any tile design.

□

Theorem 32. *For a given set of tile designs T, a symmetry group $R \leq D_4$, and an element $g \in R$, the number of distinct tilings of the $n \times m$ cylinder is*

$$\frac{1}{n|R|}\sum_{g\in R}\operatorname{fxpt}_g^C(n,m). \tag{30}$$

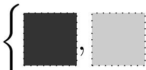
Proof. We will use the convention that when we index over g, implicitly $g \in R$; when we index over a, implicitly $a \in \mathbb{Z}/n\mathbb{Z}$; and when we index over (a, g), implicitly $(a, g) \in \mathbb{Z}/n\mathbb{Z} \rtimes R$.

Since $\operatorname{fxpt}_g^{\mathrm{C}}(n,m) = \sum_a X^{(a,g)}$, we can see that

$$\begin{aligned} \frac{1}{n|R|}\sum_g \operatorname{fxpt}_g^{\mathrm{C}}(n,m) &= \frac{1}{n|R|}\sum_g\sum_a X^{(a,g)} \\ &= \frac{1}{|\mathbb{Z}/n\mathbb{Z} \rtimes R|}\sum_{(a,g)} X^{(a,g)}, \end{aligned}$$

which counts the number of distinct tilings by a direct application of Burnside's lemma. □

5 Torus

This section builds on the work of Ethier [11] and Ethier and Lee [12]. By specializing to the set of tile designs $T = \{$  $\}$, we recover their work. Irvine [16] generalized this in the specific context of a set of tile designs of size n in the specific case that no rotation or reflection is allowed (only cyclic shifting of rows and columns), which we recover in Theorem 34.

We distinguish between the rectangular torus, which we denoted by RT, and the square torus, which we denote by ST.

5.1 The $n \times m$ torus

Definition 33. For a given set of tile designs T, and an element $g \in R \leq D_4$ the sum over all cyclic shifts of the number of tilings of the $n \times m$ cylinder by tile designs in T that are fixed by $\big((a,b), g\big) \in (\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}) \rtimes R$ is denoted

$$\mathrm{fxpt}_g^{\mathrm{RT}}(n,m) = \sum_{(a,b)\in\mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/m\mathbb{Z}} X^{\big((a,b),g\big)},$$

where $X^{\big((a,b),g\big)}$ is the number of tilings fixed by $\big((a,b), g\big)$.

Theorem 34. *The sum over all cyclic shifts of the number of tilings of the $n \times m$ torus by tile designs in T that are fixed by $(a, \mathrm{id}) \in (\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}) \rtimes R$ is given by*

$$\mathrm{fxpt}_{\mathrm{id}}^{RT}(n,m) = \sum_{c\mid m}\sum_{d\mid n} \varphi(c)\varphi(d) t_{\mathrm{id}}^{mn/\operatorname{lcm}(c,d)}. \tag{31}$$

Proof. The size of an orbit of a tile under $\big((a,b), g\big)^k$ is the set of solutions to

$$\begin{aligned} ka &\equiv 0 \pmod{n} \\ kb &\equiv 0 \pmod{m}. \end{aligned}$$

For each individual equation, the minimal choice for k must be a divisor of n. For a given divisor $d \mid n$, there are $\varphi(d)$ choices for $a \in \mathbb{Z}/n\mathbb{Z}$ so that $da \equiv 0 \pmod{n}$. Namely if i is coprime to d, then $a = i\,\big(n/d\big)$ will be a minimal solution. An analogous argument holds for the second equation.

Therefore if $d \mid n$ and $c \mid m$, there are $\varphi(d)\varphi(c)$ pairs $(a,b) \in \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ where a has order d and b has order c, and thus (a,b) has order $\operatorname{lcm}(d,c)$. Therefore, each orbit of cells has size $\operatorname{lcm}(d,c)$, and so the number of orbits of cells is $nm/\operatorname{lcm}(d,c)$.

Thus, the sum of the number of orbits over each pair $(a,b) \in \mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/m\mathbb{Z}$ gives equation (31), as desired. $\square$

The next theorem concerns the action of $\big((a,b),r^2\big) \in (\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}) \rtimes R$, which is illustrated in Figure 10.

Theorem 35. *The sum over all cyclic shifts of the number of tilings of the $n \times m$ torus by tile designs in T that are fixed by $(a, r^2) \in (\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}) \rtimes R$ is given by*

$$\mathrm{fxpt}^{RT}_{r^2}(n,m) = \begin{cases} nm\left(\frac{3}{4}t_{\mathrm{id}}^{nm/2} + \frac{1}{4}t_{\mathrm{id}}^{nm/2-2}t_{r^2}^4\right), & \text{if } n \text{ and } m \text{ are even;} \quad (32a) \\ nmt_{\mathrm{id}}^{(nm-1)/2}t_{r^2}, & \text{if } n \text{ and } m \text{ are odd;} \quad (32b) \\ nm\left(\frac{1}{2}t_{\mathrm{id}}^{nm/2} + \frac{1}{2}t_{\mathrm{id}}^{nm/2-1}t_{r^2}^2\right), & \text{otherwise.} \quad (32c) \end{cases}$$

Proof of Theorem 35. The orbits of cells under the group generated by $\big((a,b),r^2\big)$ have size either 1 or 2, because of how we defined the semidirect product, $\big((a,b),r^2\big)$ has order 2:

$$\big((a,b),r^2\big)^2 = \big((a,b),r^2\big)\big((a,b),r^2\big) = \big((a,b)+(-a,-b),r^2r^2\big) = \big((0,0),\mathrm{id}\big). \tag{33}$$

Therefore, it is enough to count how many cells are stable under $\big((a,b),r^2\big)$, which depends on the parity of n, m, a, and b.

The element $\big((a,b),r^2\big)$ fixes a cell (x,y) when

$$(x,y)\cdot\big((a,b),r^2\big) = (n-1-(x+a), n-1-(y+b)).$$

This corresponds to the system of equations

$$x \equiv -1-x-a \pmod{n} \tag{34}$$
$$y \equiv -1-y-b \pmod{m}, \tag{35}$$

whose solutions are given by Lemma 25.

Therefore we proceed by each case

Equation (32a). When n and m are even, there are fixed cells only when both a and b are odd, by Lemma 25; in this case, there are exactly 4 fixed cells, because each equation in the system of equations has two solutions. When this occurs, it partitions the cells into 4 orbits of size 1, which can be filled with tile designs that are fixed under r^2, and $(nm-4)/2$ orbits of size 2, which can be filled with any tile design.

When either a or b is even, there are no fixed cells, which partitions the cells into $nm/2$ orbits of size 2, each of which can be filled with any choice of tile design.

Since the 4 fixed cells occur for exactly one quarter of the pairs $(a,b) \in \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$, this results in the desired equation.

Equation (32b)**.** When n and m are both odd, Lemma 25 states that equations (34) and (35) have one solution.

When this occurs, it partitions the cells into 1 orbit of size 1, which can be filled only with a tile design that is fixed by r^2, and $(nm-1)/2$ orbits of size 2, which can be specified with any tile design.

Equation (32c)**.** Without loss of generality, we can assume that n is even and m is odd, because the proof is essentially similar in the opposite case. Lemma 25 states that equation (34) has no solutions when a is even and 2 solutions when a is odd; it also states that (35) has one solution.

Thus for half of the pairs (a, b), there are no fixed cells, and so there are $nm/2$ orbits, each of which can be specified by any tile design.

For the other half of the pairs, there are 2 fixed cells, which can be specified by any tile design that is fixed under r^2 and $(nm-2)/2$ orbits of size 2 that can be specified by any tile design.

□

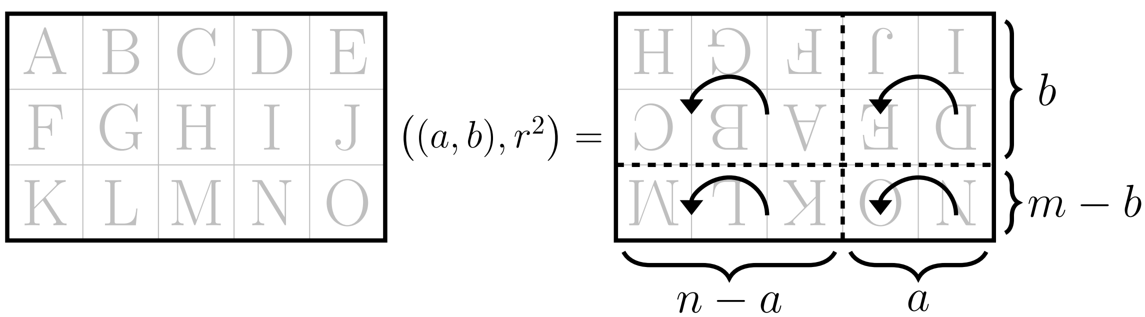


Figure 10: The action of $((a, b), r^2) \in (\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}) \rtimes R$ on a tiling is equivalent to a 180° rotations of the lower left $(n-a) \times (m-b)$ sub-grid, the lower right $a \times (m-b)$ sub-grid, the upper left $(n-a) \times b$ sub-grid, and the upper right $a \times b$ sub-grid.

Theorem 36. *The sum over all cyclic shifts of the number of tilings of the $n \times m$ torus by tile designs in T that are fixed by $(a, f) \in (\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}) \rtimes R$ is given by*

$$\operatorname{fxpt}_f^{RT}(n,m) = \begin{cases} n \displaystyle\sum_{c \mid m} \varphi(c) \left(\frac{1}{2} t_{\mathrm{id}}^{nm/\operatorname{lcm}(2,c)} + \frac{1}{2} t_{\mathrm{id}}^{(n-2)m/\operatorname{lcm}(2,c)} t_{f^c}^{2m/c} \right), & \textit{if } n \textit{ is even;} \quad (36a) \\ n \displaystyle\sum_{c \mid m} \varphi(c) t_{\mathrm{id}}^{(n-1)m/\operatorname{lcm}(2,c)} t_{f^c}^{m/c}, & \textit{if } n \textit{ is odd.} \quad (36b) \end{cases}$$

and

$$\mathrm{fxpt}^{RT}_{r^2f}(n,m) = \begin{cases} m\displaystyle\sum_{d\mid n}\varphi(d)\left(\frac{1}{2}t_{\mathrm{id}}^{nm/\operatorname{lcm}(d,2)} + \frac{1}{2}t_{\mathrm{id}}^{n(m-2)/\operatorname{lcm}(d,2)}t_{(r^2f)^d}^{2n/d}\right), & \textit{if } m \textit{ is even;} \\ m\displaystyle\sum_{d\mid n}\varphi(d)t_{\mathrm{id}}^{n(m-1)/\operatorname{lcm}(d,2)}t_{(r^2f)^d}^{n/d}, & \textit{if } m \textit{ is odd.} \end{cases}$$

Proof. Since f and r^2f are conjugate as elements of D_8, these fixed point formulas have essentially the same proof, so we will prove equations (36a) and (36b) specifically.

Since $(x,y)\cdot\big((a,b),f\big) = (n-1-(x+a), y+b)$, we can view $\big((a,b),f\big)$ as acting on each coordinate separately. Since $\big((a,b),f\big)^2 = ((0,2b),\mathrm{id})$, we can see that the orbits *of the first coordinate* have either size 1 or 2. Moreover, by Lemma 25, there are no fixed cells when n is even and a is even, there are 2 fixed cells when n is even and a is odd, and there is 1 fixed cell when n is odd.

Since $\big((a,b),f\big)$ acts on the second coordinate by shifting by b, we see that Lemma 26 applies. Thus for each divisor $c \mid m$, there are $\varphi(c)$ choices of b that produce orbits *of the second coordinate* with size c.

Equation (36a). When n is even, then half of the values of $a \in \mathbb{Z}/n\mathbb{Z}$ are even, and each orbit has size $\operatorname{lcm}(2,c)$ and these can be specified by any tile design.

The other half of values of a are odd, which results in 2 fixed points for the first coordinate, each of which results in an orbit of size c that can be specified by any tile design that is fixed by f^c. The remaining $(n-2)m$ cells then are partitioned into orbits of size $\operatorname{lcm}(2,c)$, which can be specified by any tile design.

Equation (36b). When n is odd, then there is one fixed point for the first coordinate, resulting in m cells where the first coordinate is fixed under the action of $\big((a,b),f\big)$. For each divisor $c \mid m$, there is a partition of these m cells into orbits of size c. The resulting m/c orbits can be specified by any tile design that fixes f^c. The remaining $n(m-1)$ cells are partitioned into orbits of size $\operatorname{lcm}(2,c)$, resulting in $n(m-1)/\operatorname{lcm}(2,c)$ orbits which can be specified by any tile design.

□

Theorem 37. *Then the number of distinct tilings of the $n\times m$ torus up to $R \subseteq D_4$ is given by*

$$\frac{1}{nm|R|}\sum_{g\in R}\mathrm{fxpt}^{RT}_g(n,m). \tag{37}$$

Proof. We will use the convention that when we index over g, implicitly $g \in R$; when we index over (a,b), implicitly $(a,b) \in \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$; and when we index over $\big((a,b),g\big)$, implicitly $\big((a,b),g\big) \in (\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}) \rtimes R$.

Since $\operatorname{fxpt}_g^{\mathrm{RT}}(n,m) = \sum_{(a,b)} X^{\left((a,b),g\right)}$, we can see that

$$\begin{aligned}\frac{1}{nm|R|}\sum_g \operatorname{fxpt}_g^{\mathrm{RT}}(n,m) &= \frac{1}{nm|R|}\sum_g \sum_{(a,b)} X^{\left((a,b),g\right)} \\ &= \frac{1}{|(\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}) \rtimes R|} \sum_{\left((a,b),g\right)} X^{\left((a,b),g\right)},\end{aligned}$$

which counts the number of distinct tilings by a direct application of Burnside's lemma. □

5.2 The $n \times n$ torus

Theorem 38. *The sum over all cyclic shifts of the number of tilings of the $n \times n$ square torus (denoted ST) by tile designs in T that are fixed by $\big((a,b),g\big) \in (\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}) \rtimes R$ is given by $\operatorname{fxpt}_g^{ST}(n)$ where*

$$\operatorname{fxpt}_{\mathrm{id}}^{ST}(n) = \sum_{d_1|n}\sum_{d_2|n} \varphi(d_1)\varphi(d_2) t_{\mathrm{id}}^{n^2/\operatorname{lcm}(d_1,d_2)}, \tag{38}$$

$$\operatorname{fxpt}_{r^2}^{ST}(n) = \begin{cases} n^2 t_{\mathrm{id}}^{(n^2-1)/2} t_{r^2}, & \text{if } n \text{ is odd;} \\ n^2\left(\dfrac{3}{4}t_{\mathrm{id}}^{n^2/2} + \dfrac{1}{4}t_{\mathrm{id}}^{n^2/2-2}t_{r^2}^4\right), & \text{if } n \text{ is even.}\end{cases} \tag{39}$$

$$\operatorname{fxpt}_{f}^{ST}(n) = \begin{cases} n\displaystyle\sum_{d|n}\varphi(d)\left(\frac{1}{2}t_{\mathrm{id}}^{n^2/\operatorname{lcm}(2,d)} + \frac{1}{2}t_{\mathrm{id}}^{(n^2-2n)/\operatorname{lcm}(2,d)}t_{f^d}^{2n/d}\right), & \text{if } n \text{ is even;} \\ n\displaystyle\sum_{d|n}\varphi(d) t_{\mathrm{id}}^{(n^2-n)/\operatorname{lcm}(2,d)} t_{f^d}^{n/d}, & \text{if } n \text{ is odd.}\end{cases} \tag{40}$$

$$\operatorname{fxpt}_{r^2 f}^{ST}(n) = \begin{cases} n\displaystyle\sum_{d|n}\varphi(d)\left(\frac{1}{2}t_{\mathrm{id}}^{n^2/\operatorname{lcm}(2,d)} + \frac{1}{2}t_{\mathrm{id}}^{(n^2-2n)/\operatorname{lcm}(2,d)}t_{(r^2f)^d}^{2n/d}\right), & \text{if } n \text{ is even;} \\ n\displaystyle\sum_{d|n}\varphi(d) t_{\mathrm{id}}^{(n^2-n)/\operatorname{lcm}(2,d)} t_{(r^2f)^d}^{n/d} & \text{if } n \text{ is odd.}\end{cases} \tag{41}$$

Proof. These equations follow directly from Theorems 34, 35 and 36 by specifying $m = n$. □

Theorem 39. *The sum over all cyclic shifts of the number of tilings of the $n \times n$ torus by* *tile designs in T that are fixed by $(a,r) \in (\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}) \rtimes R$ is given by*

$$\operatorname{fxpt}_r^{ST}(n) = \operatorname{fxpt}_{r^3}^{ST}(n) = \begin{cases} n^2 t_{\mathrm{id}}^{(n^2-1)/4} t_r, & \textit{if } n \textit{ is odd;} \quad (42a) \\ n^2 \left(\dfrac{1}{2} t_{\mathrm{id}}^{n^2/4} + \dfrac{1}{2} t_{\mathrm{id}}^{(n^2-4)/4} t_r^2 t_{r^2} \right), & \textit{if } n \textit{ is even.} \quad (42b) \end{cases}$$

Proof. First, note that the first equality comes from the fact that tilings that are stable under g are stable under g^{-1}.

Next, note that $\big((a,b),r\big)$ is an element of order 4, which follows from observing that $\big((a,b),r\big) = ((a',b'),r^2)$ together with equation (33). Therefore cells appear in orbits of size 1, 2, or 4 under $\big((a,b),r\big)$, and we will count how many cells appear in each.

We begin by counting cells (x,y) that are fixed by $\big((a,b),r\big)$, that is they satisfy the system of equations

$$x \equiv -y - b - 1 \pmod{n} \tag{43}$$
$$y \equiv x + a \pmod{n}, \tag{44}$$

where we can substitute y with $x+a$ in the first equation to get

$$x \equiv -x - a - b - 1 \pmod{n}. \tag{45}$$

Next we count cells (x,y) that are fixed by $\big((a,b),r\big)^2$, but are not solutions to the above system of equations. These cells satisfy the system of equations

$$x \equiv -1 - x - a - b \pmod{n} \tag{46}$$
$$y \equiv -1 - y - b + a \pmod{n}. \tag{47}$$

Equation (42a)**.** When n is odd, we can solve equation (45) using Lemma 25. We can see that this has one solution when n is odd, so in this case there is one fixed cell, which can be specified by any tile design that is stable under r.

We can add equations (46) and (47) and use Lemma 25 to see that this system has a single solution when n is odd. However, this is identically the solution that specifies the fixed point, so this does not describe an orbit of size 2.

Thus there are $n^2 - 1$ cells that are partitioned into $(n^2-1)/4$ orbits of size 4, which can be specified by any tile design.

Equation (42b) When n is even, we can see again by Lemma 25, that there are two fixed cells when $a+b$ is odd and none when $a+b$ is even.

When we check the number of orbits of size 2, Lemma 25 shows that we have 4 solutions when $a+b$ is odd and none when $a+b$ is even, 2 of which were the fixed cells, resulting in a single orbit of size 2.

Therefore, we can specify a fixed tiling by specifying tile designs that are fixed under r for each of the two fixed cells, specifying a tile design that is fixed under r^2 for the orbit of size 2, and specifying any tile designs for each of the $(n^2-4)/4$ orbits of length 4.

$\square$

Theorem 40.

$$\operatorname{fxpt}_{rf}^{ST}(n) = n \sum_{d|n} \begin{cases} \varphi(d) t_{\mathrm{id}}^{(n^2-n)/(2d)} t_{rf}^{n/d}, & \textit{if } d \textit{ is odd;} \quad (48a) \\ \varphi(d) t_{\mathrm{id}}^{n^2/(2d)}, & \textit{if } d \textit{ is even.} \quad (48b) \end{cases}$$

and

$$\operatorname{fxpt}_{r^3f}^{ST}(n) = n \sum_{d|n} \begin{cases} \varphi(d) t_{\mathrm{id}}^{(n^2-n)/(2d)} t_{r^3f}^{n/d}, & \textit{if } d \textit{ is odd;} \\ \varphi(d) t_{\mathrm{id}}^{n^2/(2d)}, & \textit{if } d \textit{ is even.} \end{cases} \tag{49}$$

Proof. These situations are essentially the same because rf and r^3f are conjugate in D_8, so we prove the case for $\operatorname{fxpt}_{rf}^{\mathrm{ST}}(n)$.

We can see that

$$(x,y) \cdot \big((a,b), rf\big)^{2k} = (x + k(a+b), y + k(a+b))$$
$$(x,y) \cdot \big((a,b), rf\big)^{2k+1} = (y + k(a+b) + b, x + k(a+b) + a)$$

and we can ask: what is the least k such that either

$$x \equiv x + k(a+b) \pmod{n} \tag{50}$$
$$y \equiv y + k(a+b) \pmod{n} \tag{51}$$

or

$$x \equiv y + k(a+b) + b \pmod{n} \tag{52}$$
$$y \equiv x + k(a+b) + a \pmod{n}. \tag{53}$$

In the first case, we want to know when $k(a+b) \equiv 0 \pmod{n}$, which occurs first when $k = n/\gcd(a+b,n)$. We call this d and note that $d \mid n$. In the second case, we can add equations (52) and (53) to get

$$(2k+1)(a+b) = 0 \pmod{n}, \tag{54}$$

which occurs when $2k+1 = n/\gcd(a+b,n)$. Again, this is a divisor of n, so we say $2k+1 = d$ and note that this solution occurs only when d is odd.

Equation (48a) Thus, when d is odd, we have the system of equations

$$x \equiv y + \frac{d-1}{2}(a+b) + b \pmod{n}$$
$$y \equiv x + \frac{d-1}{2}(a+b) + a \pmod{n},$$

which has n solutions: for each choice of x, there is a unique choice of y that satisfies both equations. Each of these solutions correspond to a cell in one of the n/d orbits of length d, each of which can be specified by any tile design that is stabilized by $(rf)^d = rf$, since d is odd.

The other $n^2 - n$ cells occur in one of the $(n^2 - n)/(2d)$ orbits of length $2d$ that are solutions to the first system of equations. Each of these orbits can be specified by any tile design at all.

The $\varphi(d)$ comes from the fact that for any choice of a there are precisely $\varphi(d)$ choices for b such that $n/\gcd(a+b, n) = d$.

Equation (48b) When d is even, there are no choices of (x, y) that simultaneously satisfy equations (52) and (53), so all n^2 of the tiles (x, y) occur in orbits of size $2d$. Each of these $n^2/(2d)$ orbits can be specified by any tile design.

□

Theorem 41. *The number of distinct tilings of the $n \times n$ torus up to $R \leq D_8$ is given by*

$$\frac{1}{n^2|R|} \sum_{g \in R} \mathrm{fxpt}_g^{ST}(n). \tag{55}$$

Proof. The proof of this theorem is essentially identical to the proof of Theorem 37, which follows by definition together with Burnside's lemma. □

Thus for any arbitrary $R \subseteq D_8$ and set of tile designs, we have a formula to count the number of tilings of the $n \times n$ torus up to R. A formula for each choice of R together with each R-set generated by a single tile design can be found in Appendix A.5; the corresponding illustrations can be found in Appendix B.5.

6 Next steps

In this section, we propose several different settings for studying similar kinds of problems. Many of these may be subtle research problems, many may be good undergraduate research problems, and many may be good homework problems for a combinatorics class. Many of them would make for interesting additions to the On-Line Encyclopedia of Integer Sequences.

6.1 Rectangular tori under 90° rotation

We have used the $n \times m$ torus as a model for a repeating tiling of the plane. However, in the case that $n \neq m$, we have only analyzed the case where we count tilings up to D_4, the dihedral group of the rectangle. However, for a given tile set, it is possible that a tiling of a $n \times m$ and a tiling of a $m \times n$ torus describe the same tiling of the plane; an example of this is given in Figure 11.

Conjecture 42. If a plane tiling described by an $n \times m$ torus is fixed under $\big((a,b), r\big)$, $\big((a,b), r^3\big)$, $\big((a,b), rf\big)$, or $\big((a,b), r^3 f\big)$, then it is equivalent to the tiling of a $\gcd(n,m) \times \gcd(n,m)$ torus.

Similarly, it might be interesting to count *irreducible* plane tilings: tilings of the plane corresponding to a tiling of the $n \times m$ torus that do not correspond to a smaller torus.

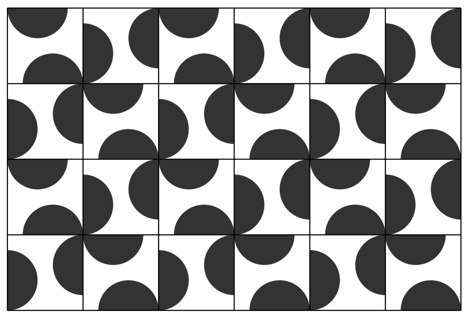


Figure 11: A periodic tiling of the plane arising from a 6×4 torus tiling that is fixed under 90° rotation. Notice that this tiling of the plane can also come from a 2×2 torus.

6.2 Other regions of the square grid.

We also are interested in counting the number of ways of tiling shapes like Aztec diamonds or centered square numbers, as shown in Figure 12.

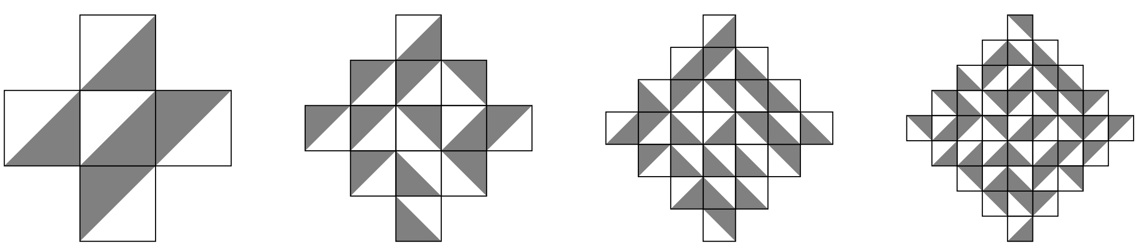


Figure 12: Order 1, 2, 3, and 4 centered square figures in the square tiling of the plane.

6.3 The Möbius strip and Klein bottle

Since we have looked at the orientable identifications of the grid, we are also interested in the non-orientable identifications. The Möbius strip has a universal cover that is $[0, 1] \times \mathbb{R}$, and the Klein bottle has a universal cover of $\mathbb{R} \times \mathbb{R}$, so we can visualize them analogously to how we visualized the cylinder and torus respectively. An illustration of a tiling of the Möbius strip in Figure 13. An illustration of a tiling of the Klein bottle in Figure 14.

We are also interested in counting tilings of the real projective plane, but because the universal cover is not the Euclidean plane, it cannot be illustrated in the same manner as the Klein bottle and the torus.

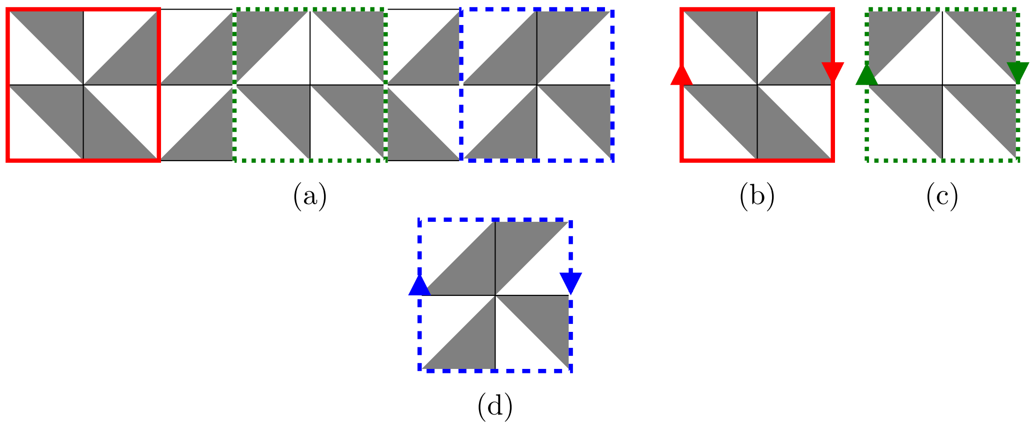


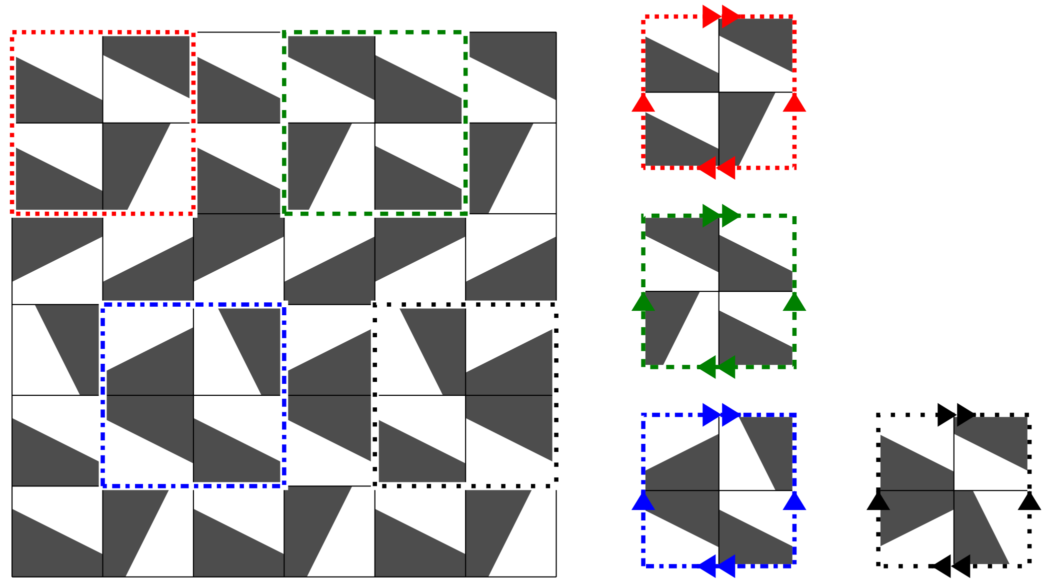
Figure 13: (a) A 2×2 Möbius strip repeated four times horizontally. Parts (b), (c), and (d) show equivalent tilings under this symmetry.

6.4 Tilings of the triangular and hexagonal grids

While this paper explored tilings of the square grid and related settings, it is equally natural to ask about tilings of the triangular grid and hexagonal grid. In particular, it would be interesting to explore the number of tilings of (1) triangular regions of the triangular grid, (2) hexagonal regions of the triangular grid, (3) triangular regions of the hexagonal grid, or (4) hexagonal regions of the hexagonal grid, all of which are illustrated in Figure 15.

Hexagonal tile designs have appeared in several tile-based edge-matching games such as Palago, Tantrix, Psyche-Paths, and Kaliko, which Van Ness has coined as "serpentiles" [30]. Triangular, hexagonal, and other polygonal tiles have been described by authors such as Ahmed [1], Mitchell [21], Beveridge [2], Walter [31], Bosch [4], Browne [5], and Lord and Ranganathan [19].

In the cases of tiling hexagons and triangles in the triangular grid, each can be extended to a tiling of the plane, as described in Figure 16.



(a) (b) (c)

Figure 14: Part (a) shows a 2×2 Klein bottle repeated three times horizontally and three times vertically, with three 2×2 regions selected. Part (b) shows three tilings of the 2×2 grid that are equivalent under the torus action $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Part (c) shows a 2×2 Klein that is equivalent to the other Klein bottles under 180° rotation.

6.5 Other tilings of the Euclidean plane

Extending this idea even further, we might be interested in ways of placing multiple shapes of tiles on various tilings of the Euclidean plane by convex polygons, such as the truncated trihexagonal tiling, the snub square tiling, or the triakis triangular tiling.

Figure 17 shows an example of such a setup on a region of the truncated square grid. Ahmed [1] and Mitchell [21] gave examples of tilings on the truncated square grid and other Archimedean tilings of the Euclidean plane. as well as

6.6 Polyhedra

We are also interested in settings related to polyhedra. For instance, one could use various tile designs to count the number of distinct tilings of a $2 \times 2 \times 2$ Rubik's cube-like object, as illustrated in Figure 18.

Similarly, one could do this analysis on other polyhedra. In 1997, Colour of Strategy released a puzzle called "Tantrix Rock," which featured square and hexagonal tiles placed on the vertices of a truncated octahedron [28]. Similar counting problems could be done on the other Platonic solids and Archimedean solids in addition to Johnson solids, prisms, antiprisms, and polyhedra whose faces are not regular polygons, such as the rhombic dodecahedron.

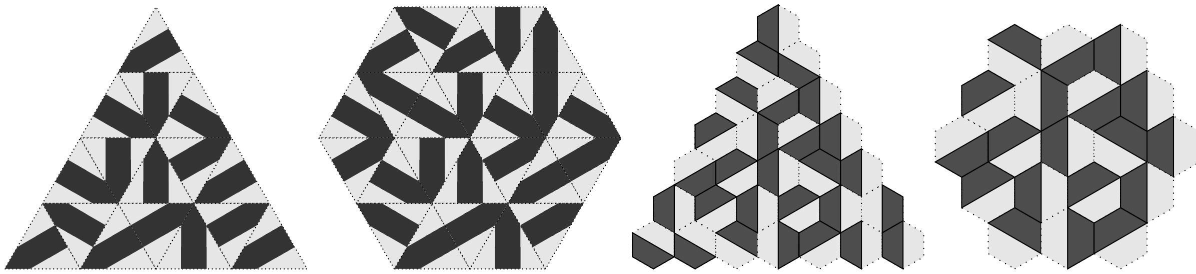
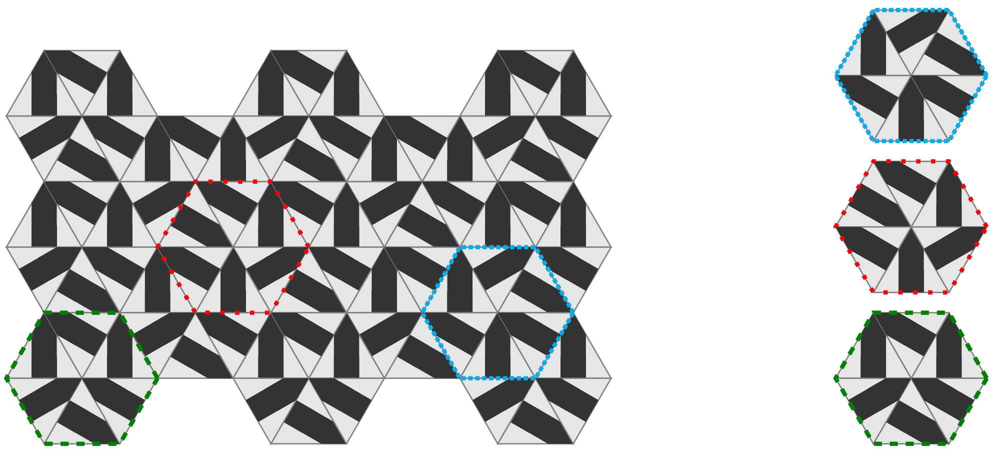


Figure 15: An illustration of a triangle in a triangular grid, a hexagon in a triangular grid, a triangle in a hexagonal grid, and a hexagon in a hexagonal grid.



(a) (b)

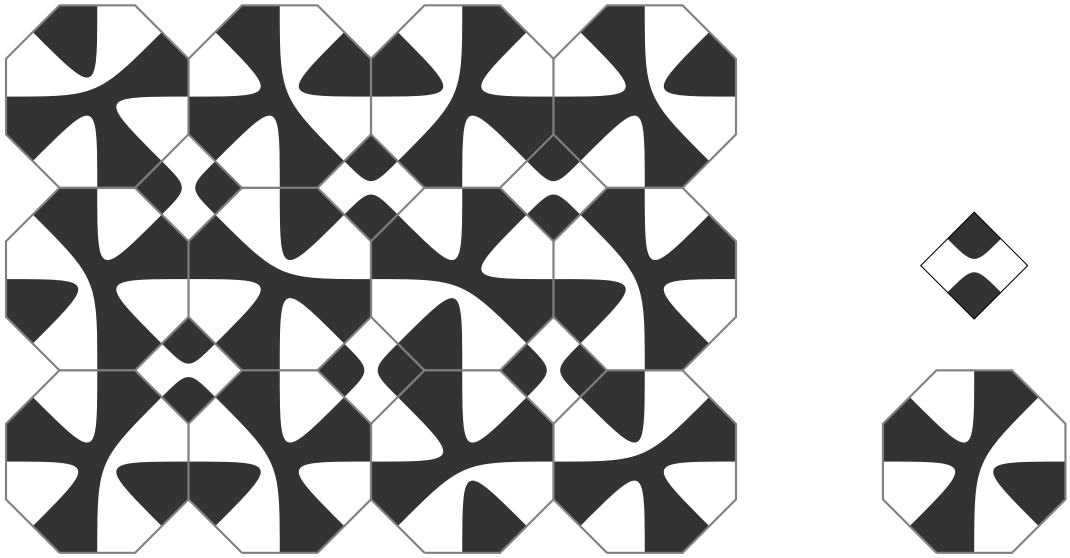
Figure 16: (a) A triangular tiling of the plane tiled with repeating patterns of size 1 hexagons. (b) Three equivalent tilings of the triangular hexagon under this symmetry.

6.7 Hyperbolic plane

In addition to the settings with no curvature (the plane) and positive curvature (polyhedra) it is also interesting to look at this in the negative curvature setting of the hyperbolic plane, as described by Dunham [10]. An example of this is illustrated in Figure 19.

6.8 Higher dimensional objects

We are also interested in computing higher-dimensional analogs, such as where the tile designs are space-filling polyhedra. In the context of cubes, these have been considered by Schattschneider [26], Lord and Ranganathan [19], Browne [6]



(a) (b)

Figure 17: (a) A 4×3 section of the truncated square tiling, and (b) the square and octagonal tile designs.

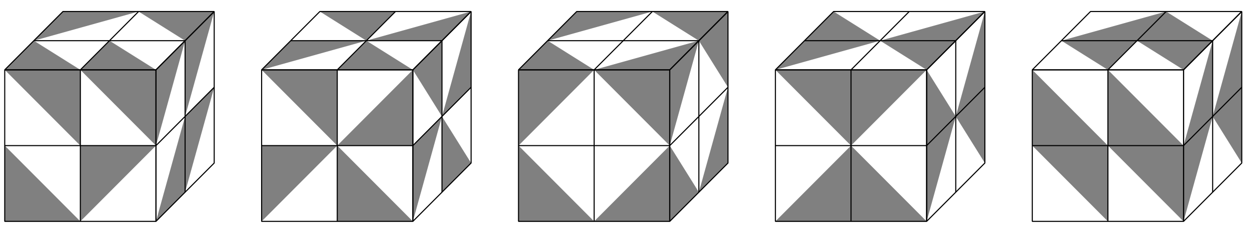


Figure 18: Five illustrations of $2 \times 2 \times 2$ cubes tiled with Truchet tiles.

6.9 Permuting tile colors

In some illustrations, one may observe that two tilings are equivalent up to swapping the colors of the tiles, as illustrated in Figure 20. In Appendix B.5, you may notice that for each tiling in Figures 72 and 74, swapping the colors of the tiling is equivalent to a 180° rotation, a property that we would like to understand better.

7 Acknowledgments

The authors would like to especially thank Ben Jeffers, Lani Canadilla, Christopher Havens, and all of the other volunteers at the Prison Mathematics Project for facilitating our conversation. Thank you also to Scott Reskey for your close reading.

Thank you to Shane Delmore; if you hadn't shown me the Commodore 64 program, `10 PRINT CHR\$(205.5+RND(1));: GOTO 10`, this paper likely would not exist.

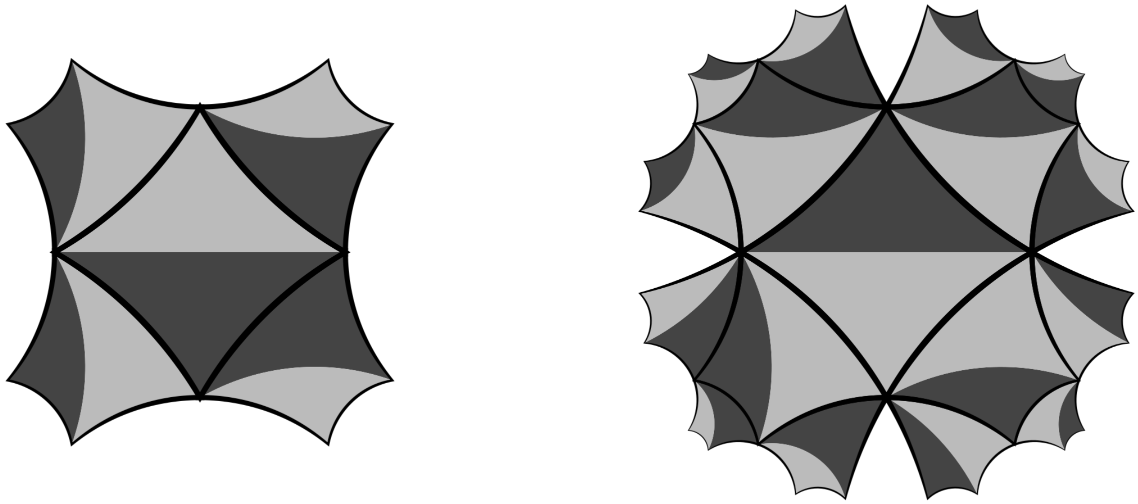


Figure 19: An illustration showing tilings of the size 2 and size 3 iterations of the order-5 square tiling of the hyperbolic plane.

Lastly, thank you to the reviewers. Your suggestions strengthened the paper tremendously, and without them we would have missed the connection to M. C. Escher's work, which makes this story even more compelling to us.

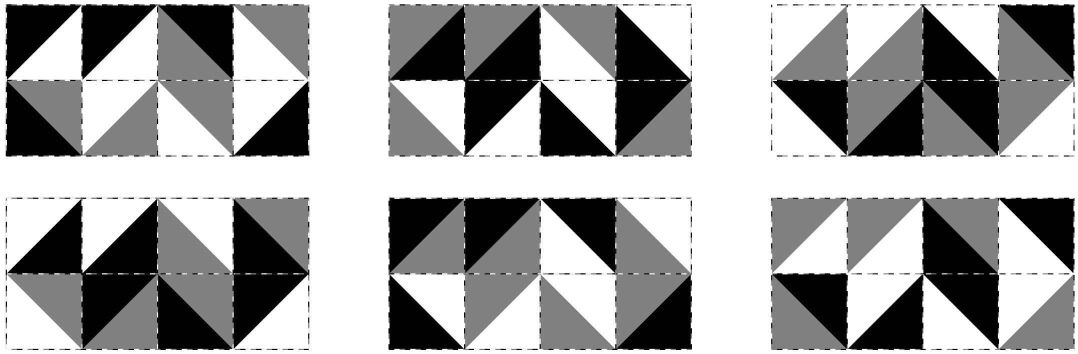


Figure 20: The following six tilings would be considered equivalent under permuting colors.

A Sequences

This section of the appendix gives examples of all of the different sequences and tables of integers that count tilings of the $n \times m$ grid, cylinder, and torus for all valid choices of $R \leq D_8$ and all sets of tile designs consisting of a single orbit.

A.1 The $n \times m$ grid

This section gives examples of every choice of symmetry of the $n \times m$ grid together with every essentially different set of tile designs that consists of a single orbit (or two orbits, in the case of a fully symmetric tile). Each sequence is annotated with its corresponding entry in the On-Line Encyclopedia of Integer Sequences. A table of all such sequences is given in Table 1.

	$\langle r^2, f\rangle$	$\langle f\rangle \cong C_2$	$\langle r^2\rangle \cong C_2$
$\mathcal{O}_{\langle r^2, f\rangle}$	Table 43 A225910	—	—
$\mathcal{O}_{\langle f\rangle}$	Table 44 A368218	Table 47 A368221	—
$\mathcal{O}_{\langle r^2\rangle}$	Table 45 A368219	—	Table 49 A368223
$\mathcal{O}_{\mathbb{1}}$	Table 46 A368220	Table 48 A368222	Table 50 A368224

Table 1: An index of tables that describe the number of tilings of the $n \times m$ grid.

A.1.1 Under horizontal and vertical reflection

When counting tilings of the grid up to $\langle r^2, f\rangle$, we have that

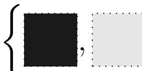
$$t_{\mathrm{id}} = \mathcal{O}_{\langle r^2,f\rangle}^{\langle r^2,f\rangle} + 2\mathcal{O}_{\langle f\rangle}^{\langle r^2,f\rangle} + 2\mathcal{O}_{\langle r^2 f\rangle}^{\langle r^2,f\rangle} + 2\mathcal{O}_{\langle r^2\rangle}^{\langle r^2,f\rangle} + 4\mathcal{O}_{\mathbb{1}}^{\langle r^2,f\rangle} \tag{56}$$

$$t_f = \mathcal{O}_{\langle r^2,f\rangle}^{\langle r^2,f\rangle} + 2\mathcal{O}_{\langle f\rangle}^{\langle r^2,f\rangle} \tag{57}$$

$$t_{r^2 f} = \mathcal{O}_{\langle r^2,f\rangle}^{\langle r^2,f\rangle} + 2\mathcal{O}_{\langle r^2 f\rangle}^{\langle r^2,f\rangle} \tag{58}$$

$$t_{r^2} = \mathcal{O}_{\langle r^2,f\rangle}^{\langle r^2,f\rangle} + 2\mathcal{O}_{\langle r^2\rangle}^{\langle r^2,f\rangle} \tag{59}$$

Proposition 43. *When $\mathcal{O}_{\langle r^2,f\rangle}^{\langle r^2,f\rangle} = 2$, such as when*

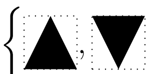
$T = \Big\{$  $\Big\}$,

the number of tilings of the $n \times m$ grid up to horizontal/vertical reflection by tile designs that are fixed horizontal/vertical reflection is given by the following table:

$n = 1$	2	3	6	10	20	36
$n = 2$	3	7	24	76	288	1072
$n = 3$	6	24	168	1120	8640	66816
$n = 4$	10	76	1120	16576	263680	4197376
$n = 5$	20	288	8640	263680	8407040	268517376
$n = 6$	36	1072	66816	4197376	268517376	17180065792

This is OEIS sequence A225910.

Proposition 44. *When $\mathcal{O}_{\langle f\rangle}^{\langle r^2,f\rangle} = 1$, such as when*

$T = \Big\{$  $\Big\}$,

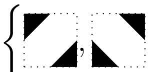
the number of tilings of the $n \times m$ grid up to horizontal and vertical reflection by tiles that are fixed under horizontal reflection but not vertical reflection is given by the following table:

$n = 1$	1	3	4	10	16	36
$n = 2$	2	7	20	76	272	1072
$n = 3$	3	24	144	1120	8448	66816
$n = 4$	6	76	1056	16576	262656	4197376
$n = 5$	10	288	8320	263680	8396800	268517376
$n = 6$	20	1072	65792	4197376	268451840	17180065792

The transpose of this table is the number of tilings by tiles fixed under vertical reflection but not horizontal reflection.

This has been added to the OEIS as sequence A368218.

Proposition 45. *When* $\mathcal{O}_{\langle r^2 \rangle}^{\langle r^2, f \rangle} = 1$*, such as when*

$T = \{$  $\}$,

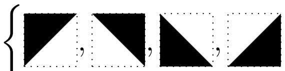
the number of tilings of the $n \times m$ *grid up to horizontal and vertical reflection by tiles that are fixed under* $180°$ *rotation, but not horizontal or vertical reflection is given by the following table:*

$n = 1$	1	2	3	6	10	20
$n = 2$	2	7	20	76	272	1072
$n = 3$	3	20	136	1056	8256	65792
$n = 4$	6	76	1056	16576	262656	4197376
$n = 5$	10	272	8256	262656	8390656	268451840
$n = 6$	20	1072	65792	4197376	268451840	17180065792

This table is symmetric across its main diagonal.

This has been added to the OEIS as sequence A368219.

Proposition 46. *When* $\mathcal{O}_{\mathbb{1}}^{\langle r^2, f \rangle} = 1$*, such as when*

$T = \{$  $\}$,

the number of tilings of the $n \times m$ *grid up to horizontal and vertical reflection by tiles that are fixed only under* $\mathrm{id} \in D_4$ *is given by the following table:*

$n = 1$	1	6	16	72	256
$n = 2$	6	76	1056	16576	262656
$n = 3$	16	1056	65536	4196352	268435456
$n = 4$	72	16576	4196352	1073790976	274878431232
$n = 5$	256	262656	268435456	274878431232	281474976710656
$n = 6$	1056	4197376	17180000256	70368756760576	288230376688582656

This has been added to the OEIS as sequence A368220.

A.1.2 Under horizontal (equivalently vertical) reflection

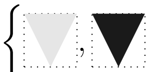
When counting tilings of the grid up to $\langle f \rangle$ (equivalently $\langle r^2 f \rangle$), we have that

$$t_{\mathrm{id}} = \mathcal{O}_{\langle f \rangle}^{\langle f \rangle} + 2\mathcal{O}_{\mathbb{1}}^{\langle f \rangle} \tag{60}$$

$$t_f = \mathcal{O}_{\langle f \rangle}^{\langle f \rangle} \tag{61}$$

Proposition 47. *When $\mathcal{O}_{\langle f\rangle}^{\langle f\rangle} = 2$, such as when*

$$T = \{\ \ \},$$



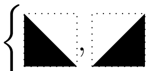
the number of tilings of the $n \times m$ grid up to horizontal reflection by two tiles that are fixed under horizontal reflection is given by the following table:

$n = 1$	2	3	6	10	20	36
$n = 2$	4	10	40	136	544	2080
$n = 3$	8	36	288	2080	16640	131328
$n = 4$	16	136	2176	32896	526336	8390656
$n = 5$	32	528	16896	524800	16793600	536887296
$n = 6$	64	2080	133120	8390656	537001984	34359869440

This has been added to the OEIS as sequence A368221.

Proposition 48. *When $\mathcal{O}_{\mathbb{1}}^{\langle f\rangle} = 1$, such as when*

$$T = \{\ \ \},$$



the number of tilings of the $n \times m$ grid up to horizontal reflection by tiles that are fixed only under id $\in \langle f\rangle$ *is given by the following table:*

$n = 1$	1	2	4	8	16	32
$n = 2$	3	10	36	136	528	2080
$n = 3$	4	32	256	2048	16384	131072
$n = 4$	10	136	2080	32896	524800	8390656
$n = 5$	16	512	16384	524288	16777216	536870912
$n = 6$	36	2080	131328	8390656	536887296	34359869440

This has been added to the OEIS as sequence A368222.

A.1.3 Under $180°$ rotation

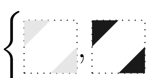
When counting tilings of the grid up to $180°$ rotation ($S = \langle r^2\rangle$),

$$t_{\text{id}} = \mathcal{O}_{\langle r^2\rangle}^{\langle r^2\rangle} + 2\mathcal{O}_{\mathbb{1}}^{\langle r^2\rangle} \tag{62}$$

$$t_{r^2} = \mathcal{O}_{\langle r^2\rangle}^{\langle r^2\rangle} \tag{63}$$

Proposition 49. *When $\mathcal{O}_{\langle r^2\rangle}^{\langle r^2\rangle} = 2$ such as when*

$$T = \{\ \ \},$$



the number of tilings of the $n \times m$ grid up to $180°$ rotation by tiles that are fixed under $180°$ rotation is given by the following table:

$n = 1$	2	3	6	10	20	36
$n = 2$	3	10	36	136	528	2080
$n = 3$	6	36	272	2080	16512	131328
$n = 4$	10	136	2080	32896	524800	8390656
$n = 5$	20	528	16512	524800	16781312	536887296
$n = 6$	36	2080	131328	8390656	536887296	34359869440

This has been added to the OEIS as sequence A368223.

Proposition 50. *When $\mathcal{O}_{\mathbb{1}}^{\langle r^2 \rangle} = 1$, such as when*

$T = \Bigg\{$  $\Bigg\}$,

the number of tilings of the $n \times m$ grid up to $180°$ rotation by tiles that are fixed only under $\mathrm{id} \in \langle r^2 \rangle$.

$n = 1$	1	3	4	10	16	36
$n = 2$	3	10	36	136	528	2080
$n = 3$	4	36	256	2080	16384	131328
$n = 4$	10	136	2080	32896	524800	8390656
$n = 5$	16	528	16384	524800	16777216	536887296
$n = 6$	36	2080	131328	8390656	536887296	34359869440

This has been added to the OEIS as sequence A368224.

A.2 The $n \times n$ grid

This section gives examples of every choice of symmetry of the $n \times n$ grid together with every essentially different set of tile designs that consists of a single orbit (or two orbits, in the case of a fully symmetric tile). Each sequence is annotated with its corresponding entry in the On-Line Encyclopedia of Integer Sequences. A table of all such sequences is given in Table 2.

A.2.1 Under symmetries of the square

When counting tilings of the grid up to $\langle r, f \rangle$, we have that

$$t_{\mathrm{id}} = \mathcal{O}_{\langle r,f \rangle}^{\langle r,f \rangle} + 2\mathcal{O}_{\langle r^2,f \rangle}^{\langle r,f \rangle} + 2\mathcal{O}_{\langle r^2,rf \rangle}^{\langle r,f \rangle} + 2\mathcal{O}_{\langle r \rangle}^{\langle r,f \rangle} + 4\mathcal{O}_{\langle f \rangle}^{\langle r,f \rangle} + 4\mathcal{O}_{\langle rf \rangle}^{\langle r,f \rangle} + 4\mathcal{O}_{\langle r^2 \rangle}^{\langle r,f \rangle} + 8\mathcal{O}_{\mathbb{1}}^{\langle r,f \rangle} \tag{64}$$

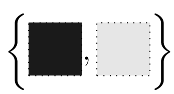
$$t_f = t_{r^2 f} = \mathcal{O}_{\langle r,f \rangle}^{\langle r,f \rangle} + 2\mathcal{O}_{\langle r^2,f \rangle}^{\langle r,f \rangle} + 4\mathcal{O}_{\langle f \rangle}^{\langle r,f \rangle} \tag{65}$$

$$t_{r^2} = \mathcal{O}_{\langle r,f \rangle}^{\langle r,f \rangle} + 2\mathcal{O}_{\langle r^2,f \rangle}^{\langle r,f \rangle} + 2\mathcal{O}_{\langle r^2,rf \rangle}^{\langle r,f \rangle} + 2\mathcal{O}_{\langle r \rangle}^{\langle r,f \rangle} + 4\mathcal{O}_{\langle r^2 \rangle}^{\langle r,f \rangle}. \tag{66}$$

	$\langle r, f\rangle$	$\langle r^2, rf\rangle$	$\langle r\rangle$	$\langle rf\rangle$
$\mathcal{O}_{\langle r,f\rangle}$	Sequence 51 A054247	—	—	—
$\mathcal{O}_{\langle r^2,f\rangle}$	Sequence 52 A367522	—	—	—
$\mathcal{O}_{\langle r^2,rf\rangle}$	Sequence 53 A295229	Sequence 59 A367526	—	—
$\mathcal{O}_{\langle r\rangle}$	Sequence 54 A367523	—	Sequence 63 A047937	—
$\mathcal{O}_{\langle f\rangle}$	Sequence 55 A367524	—	—	—
$\mathcal{O}_{\langle rf\rangle}$	Sequence 56 A302484	Sequence 60 A367527	—	Sequence 66 A200564
$\mathcal{O}_{\langle r^2\rangle}$	Sequence 57 A367524	Sequence 61 A367528	Sequence 64 A367531	—
$\mathcal{O}_{\mathbb{1}}$	Sequence 58 A367525	Sequence 62 A367529	Sequence 65 A367532	Sequence 67 A103488

Table 2: An index of tables that describe the number of tilings of the $n \times n$ grid.

Proposition 51. *When $\mathcal{O}^{\langle r,f\rangle}_{\langle r,f\rangle} = 2$, such as when*

$T =$  ,

the number of tilings of the $n \times n$ grid up to D_8 action by two distinct tile designs which are fixed under all elements of D_8 is given by

$$2, 6, 102, 8548, 4211744, 8590557312, 70368882591744, 2305843028004192256, \ldots$$

This is OEIS sequence A054247.

Proposition 52. *When $\mathcal{O}^{\langle r,f\rangle}_{\langle r^2,f\rangle} = 1$, such as when*

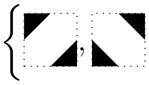
$T =$  ,

the number of tilings of the $n \times n$ grid up to D_8 action by tiles that are stable under horizontal and vertical reflections is given by

$$1, 4, 84, 8292, 4203520, 8590033024, 70368815480832, 2305843010824323072, \ldots$$

This has been added to the OEIS as sequence A367522.

Proposition 53. *When* $\mathcal{O}^{\langle r,f\rangle}_{\langle r^2,rf\rangle} = 1$*, such as when*

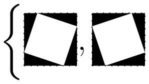
$T = \Bigg\{$  $\Bigg\},$

the number of tilings of the $n \times n$ *grid up to* D_8 *action by tiles that are stable under diagonal and antidiagonal reflections is given by*

$$1, 6, 84, 8548, 4203520, 8590557312, 70368815480832, 2305843028004192256, \ldots$$

This is OEIS sequence A295229.

Proposition 54. *When* $\mathcal{O}^{\langle r,f\rangle}_{\langle r\rangle} = 1$*, such as when*

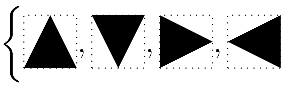
$T = \Bigg\{$  $\Bigg\},$

the number of tilings of the $n \times n$ *grid up to* D_8 *action by tiles that are stable under* $90°$ *rotations is given by*

$$1, 4, 70, 8292, 4195360, 8590033024, 70368748374016, 2305843010824323072, \ldots$$

This has been added to the OEIS as sequence A367523.

Proposition 55. *When* $\mathcal{O}^{\langle r,f\rangle}_{\langle f\rangle} = 1$*, (resp,* $\mathcal{O}^{\langle r,f\rangle}_{\langle r^2 f\rangle} = 1$*) such as when*

$T = \Bigg\{$  $\Bigg\},$

the number of tilings of the $n \times n$ *grid up to* D_8 *action by tiles that are stable under horizontal (resp. vertical) reflections is given by*

$$1, 39, 32896, 536895552, 140737496743936, 590295810384475521024, \ldots$$

This has been added to the OEIS as sequence A367524.

Proposition 56. *When* $\mathcal{O}^{\langle r,f\rangle}_{\langle rf\rangle} = 1$*, (resp.* $\mathcal{O}^{\langle r,f\rangle}_{\langle r^3 f\rangle} = 1$*) such as when*

$T = \Bigg\{$  $\Bigg\},$

the number of tilings of the $n \times n$ *grid up to* D_8 *action by tiles that are stable under antidiagonal (resp. diagonal) reflections is given by*

$$1, 43, 32896, 536911936, 140737496743936, 590295810401655390208, \ldots$$

This is OEIS Sequence A302484.

Proposition 57. *When* $\mathcal{O}_{\langle r^2\rangle}^{\langle r,f\rangle} = 1$*, such as when*

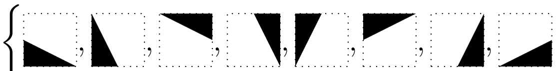
$T = \Bigg\{$  $\Bigg\}$,

the number of tilings of the $n \times n$ *grid up to* D_8 *action by tiles that are stable under* 180° *rotation is given by*

$$1, 39, 32896, 536895552, 140737496743936, 590295810384475521024, \ldots$$

Note that the above sequence agrees with sequence in Proposition 55, which has been added to the OEIS as sequence A367524.

Proposition 58. *When* $\mathcal{O}_{\mathbb{1}}^{\langle r,f\rangle} = 1$*, such as when*

$T = \Bigg\{$  $\Bigg\}$,

the number of tilings of the $n \times n$ *grid up to* D_8 *action by tiles that are stable under* 180° *rotation is given by*

$$1, 538, 16777216, 35184378381312, 4722366482869645213696, \ldots$$

This has been added to the OEIS as sequence A367525.

A.2.2 Under diagonal and antidiagonal reflection

When counting tilings of the grid up to $\langle r^2, f\rangle$, we have that

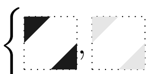
$$t_{\mathrm{id}} = \mathcal{O}_{\langle r^2,f\rangle}^{\langle r^2,f\rangle} + 2\mathcal{O}_{\langle f\rangle}^{\langle r^2,f\rangle} + 2\mathcal{O}_{\langle r^2 f\rangle}^{\langle r^2,f\rangle} + 2\mathcal{O}_{\langle r^2\rangle}^{\langle r^2,f\rangle} \tag{67}$$

$$t_f = \mathcal{O}_{\langle r^2,f\rangle}^{\langle r^2,f\rangle} + 2\mathcal{O}_{\langle f\rangle}^{\langle r^2,f\rangle} \tag{68}$$

$$t_{r^2 f} = \mathcal{O}_{\langle r^2,f\rangle}^{\langle r^2,f\rangle} + 2\mathcal{O}_{\langle r^2 f\rangle}^{\langle r^2,f\rangle} \tag{69}$$

$$t_{r^2} = \mathcal{O}_{\langle r^2,f\rangle}^{\langle r^2,f\rangle} + 2\mathcal{O}_{\langle r^2\rangle}^{\langle r^2,f\rangle} \tag{70}$$

Proposition 59. *When* $\mathcal{O}_{\langle r^2,rf\rangle}^{\langle r^2,rf\rangle} = 2$*, such as when*

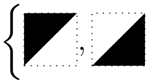
$T = \Bigg\{$  $\Bigg\}$,

the number of tilings of the $n \times n$ *grid up to diagonal and antidiagonal flipping by two colors of tiles that are stable under this symmetry is given by*

$$2, 9, 168, 16960, 8407040, 17180983296, 140737630961664, 4611686053860868096, \ldots$$

This has been added to the OEIS as sequence A367526.

Proposition 60. *When* $\mathcal{O}_{\langle rf \rangle}^{\langle r^2, rf \rangle} = 1$, *(resp* $\mathcal{O}_{\langle r^3 f \rangle}^{\langle r^2, rf \rangle} = 1$*) such as when*

$T = \{$  $\}$,

the number of tilings of the $n \times n$ *grid up to diagonal and antidiagonal flipping by the orbit of a tile that is stable under antidiagonal (resp. diagonal) flipping is given by*

$$1, 7, 144, 16704, 8396800, 17180459008, 140737555464192, 4611686036680998912, \ldots$$

This has been added to the OEIS as sequence A367527.

Proposition 61. *When* $\mathcal{O}_{\langle r^2 \rangle}^{\langle r^2, rf \rangle} = 1$, *such as when*

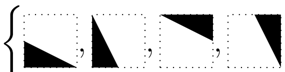
$T = \{$  $\}$,

the number of tilings of the $n \times n$ *grid up to diagonal and antidiagonal flipping by the orbit of a tile that is stable under* 180° *rotation is given by*

$$1, 5, 136, 16448, 8390656, 17179934720, 140737496743936, 4611686019501129728, \ldots$$

This has been added to the OEIS as sequence A367528.

Proposition 62. *When* $\mathcal{O}_{\mathbb{1}}^{\langle r^2, rf \rangle} = 1$, *such as when*

$T = \{$  $\}$,

the number of tilings of the $n \times n$ *grid up to diagonal and antidiagonal flipping by the orbit of a tile that is not stable under any of these symmetries is given by*

$$1, 68, 65536, 1073758208, 281474976710656, 1180591620734591172608, \ldots$$

This has been added to the OEIS as sequence A367529.

A.2.3 Under 90° rotation

Proposition 63. *When* $\mathcal{O}_{\langle r \rangle}^{\langle r \rangle} = 2$, *such as when*

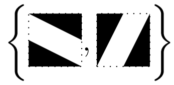
$T = \{$  $\}$,

the number of tilings of the $n \times n$ *grid up to* 90° *rotation by two colors of tiles that are fixed under this symmetry are*

$$2, 6, 140, 16456, 8390720, 17179934976, 140737496748032, 4611686019501162496, \ldots$$

This is in the OEIS as A047937*, which is column* 2 *of* A343095*.*

Proposition 64. *When* $\mathcal{O}_{\langle r^2\rangle}^{\langle r\rangle} = 1$, *such as when*

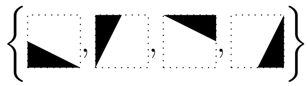
$T =$  ,

the number of tilings of the $n \times n$ *grid up to* $90°$ *rotation by tiles that are fixed under* $180°$ *rotations is given by*

$$1, 6, 136, 16456, 8390656, 17179934976, 140737496743936, 4611686019501162496, \ldots$$

This has been added to the OEIS as sequence A367531.

Proposition 65. *When* $\mathcal{O}_{\mathbb{1}}^{\langle r\rangle} = 1$, *such as when*

$T =$  ,

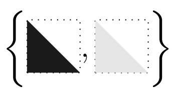
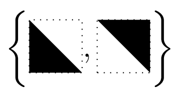
the number of tilings of the $n \times n$ *grid up to* $90°$ *rotation by an asymmetric tile*

$$1, 70, 65536, 1073758336, 281474976710656, 1180591620734591303680, \ldots$$

This has been added to the OEIS as sequence A367532.

A.2.4 Under diagonal (equivalently antidiagonal) reflection

Proposition 66. *When* $\mathcal{O}_{\langle rf\rangle}^{\langle rf\rangle} = 2$, *such as when*

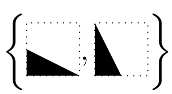
$T =$  *or* $T =$ 

the number of tilings of the $n \times n$ *grid up to flipping over the antidiagonal by tiles that are fixed under that symmetry is given by*

$$2, 12, 288, 33280, 16793600, 34360786944, 281475110928384, 9223372071214514176, \ldots$$

This is OEIS sequence A200564.

Proposition 67. *When* $\mathcal{O}_{\mathbb{1}}^{\langle rf\rangle} = 1$, *such as when*

$T =$  ,

the number of tilings of the $n \times n$ *grid up to flipping over the antidiagonal by asymmetric tiles is given by*

$$1, 8, 256, 32768, 16777216, 34359738368, 281474976710656, 9223372036854775808, \ldots$$

This is OEIS sequence A103488.

A.3 The $n \times m$ cylinder

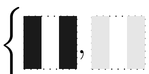
This section gives examples of every choice of symmetry of the $n \times m$ cylinder together with every essentially different set of tile designs that consists of a single orbit (or two orbits, in the case of a fully symmetric tile). Each sequence is annotated with its corresponding entry in the On-Line Encyclopedia of Integer Sequences. A table of all such sequences is given in Table 3.

	$\langle r^2, f\rangle$	$\langle f\rangle$	$\langle r^2 f\rangle$	$\langle r^2\rangle$	$\mathbb{1}$
$\mathcal{O}_V$	Table 68 A368253	—	—	—	—
$\mathcal{O}_{\langle f\rangle}$	Table 69 A368254	Table 73 A368258	—	—	—
$\mathcal{O}_{\langle r^2 f\rangle}$	Table 70 A368255	—	Table 75 A368260	—	—
$\mathcal{O}_{\langle r^2\rangle}$	Table 71 A368256	—	—	Table 77 A368262	—
$\mathcal{O}_{\mathbb{1}}$	Table 72 A368257	Table 74 A368259	Table 76 A368261	Table 78 A368263	Table 79 A368264

Table 3: An index of tables that describe the number of tilings of the $n \times m$ cylinder.

A.3.1 Under horizontal and vertical reflection

Proposition 68. *When $\mathcal{O}^{\langle r^2, f\rangle}_{\langle r^2, f\rangle} = 2$, such as when*

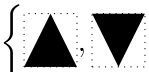
$T = \{$  $\}$ *or* $T = \{$  $\}$

the number of tilings of the $n \times m$ cylinder up to horizontal and vertical reflection by tiles that are fixed under those actions is given by

$n = 1$	2	3	6	10	20	36	72
$n = 2$	3	7	24	76	288	1072	4224
$n = 3$	4	13	74	430	3100	23052	179736
$n = 4$	6	34	378	4756	70536	1083664	17053728
$n = 5$	8	78	1884	53764	1689608	53762472	1718629200
$n = 6$	13	237	11912	709316	44900448	2865540112	183287416192
$n = 7$	18	687	77022	9608050	1227536100	157077883188	20105440563816

This has been added to the OEIS as sequence A368253.

Proposition 69. *When* $\mathcal{O}_{\langle f\rangle}^{\langle r^2,f\rangle} = 1$*, such as when*

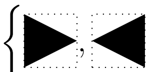
$$T = \left\{ \right.$$  $$\left. \right\}$$

the number of tilings of the $n \times m$ *cylinder up to horizontal and vertical reflection by tiles that are fixed under horizontal reflection is given by*

$n=1$	1	3	4	10	16	36	64
$n=2$	2	7	20	76	272	1072	4160
$n=3$	2	13	60	430	2992	23052	178880
$n=4$	4	34	346	4756	70024	1083664	17045536
$n=5$	4	78	1768	53764	1685920	53762472	1718511232
$n=6$	8	237	11612	709316	44881328	2865540112	183286192832
$n=7$	9	687	75924	9608050	1227395664	157077883188	20105422588224

This has been added to the OEIS as sequence A368254.

Proposition 70. *When* $\mathcal{O}_{\langle r^2 f\rangle}^{\langle r^2,f\rangle} = 1$*, such as when*

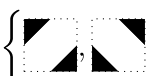
$$T = \left\{ \right.$$  $$\left. \right\}$$

the number of tilings of the $n \times m$ *cylinder up to horizontal and vertical reflection by tiles that are fixed under vertical reflection is given by*

$n=1$	1	2	3	6	10	20	36
$n=2$	2	5	14	44	152	560	2144
$n=3$	2	9	50	366	2780	22028	175128
$n=4$	4	26	298	4244	66184	1050896	16787488
$n=5$	4	62	1692	52740	1679368	53696936	1718039376
$n=6$	9	205	11272	701124	44761184	2863442960	183253337472
$n=7$	10	623	75486	9591666	1227208420	157073688884	20105365066344

This has been added to the OEIS as sequence A368255.

Proposition 71. *When* $\mathcal{O}_{\langle r^2\rangle}^{\langle r^2,f\rangle} = 1$*, such as when*

$$T = \left\{ \right.$$  $$\left. \right\}$$

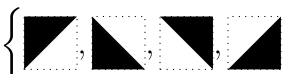
the number of tilings of the $n \times m$ cylinder up to horizontal and vertical reflection by tiles that are fixed under 180° rotation is given by

$n = 1$	1	2	3	6	10	20	36
$n = 2$	2	5	14	44	152	560	2144
$n = 3$	2	9	52	366	2800	22028	175296
$n = 4$	4	26	298	4244	66184	1050896	16787488
$n = 5$	4	62	1704	52740	1679776	53696936	1718052480
$n = 6$	8	205	11228	701124	44758448	2863442960	183253162688
$n = 7$	9	623	75412	9591666	1227199056	157073688884	20105363867968

This has been added to the OEIS as sequence A368256.

Proposition 72. *When $\mathcal{O}_{\mathbb{1}}^{\langle r^2, f\rangle} = 1$, such as when*

$$T = \Big\{ \quad \Big\}$$



the number of tilings of the $n \times m$ cylinder up to horizontal and vertical reflection by asymmetric tiles is given by

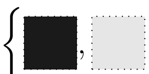
$n = 1$	1	6	16	72	256
$n = 2$	4	44	544	8384	131584
$n = 3$	6	366	21856	1399512	89478656
$n = 4$	23	4244	1050128	268472384	68719870208
$n = 5$	52	52740	53687104	54975896016	56294995342336
$n = 6$	194	701124	2863399264	11728132423744	48038396383286784

This has been added to the OEIS as sequence A368257.

A.3.2 Under horizontal reflection

Proposition 73. *When $\mathcal{O}_{\langle f\rangle}^{\langle f\rangle} = 2$, such as when*

$$T = \Big\{ \quad \Big\} \qquad \textit{or} \qquad T = \Big\{ \quad \Big\}$$

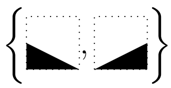
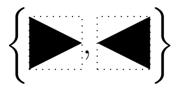


the number of tilings of the $n \times m$ cylinder up to horizontal reflection two distinct tiles that are stable under horizontal reflection is given by

$n = 1$	2	4	8	16	32	64	128
$n = 2$	3	10	36	136	528	2080	8256
$n = 3$	4	20	120	816	5984	45760	357760
$n = 4$	6	55	666	9316	139656	2164240	34084896
$n = 5$	8	136	3536	106912	3371840	107505280	3437022464
$n = 6$	13	430	23052	1415896	89751728	5730905440	366571686592
$n = 7$	18	1300	151848	19206736	2454791328	314154568000	40210845176448

This has been added to the OEIS as sequence A368258.

Proposition 74. *When $\mathcal{O}_{\mathbb{1}}^{\langle f\rangle} = 1$, such as when*

$T =$  *or* $T =$ 

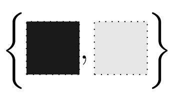
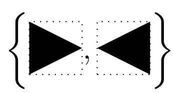
the number of tilings of the $n \times m$ cylinder up to horizontal reflection by a tile that is not stable under horizontal reflection is given by

$n = 1$	1	2	4	8	16	32	64
$n = 2$	2	6	20	72	272	1056	4160
$n = 3$	2	12	88	688	5472	43712	349568
$n = 4$	4	39	538	8292	131464	2098704	33560608
$n = 5$	4	104	3280	104864	3355456	107374208	3435973888
$n = 6$	9	366	22028	1399512	89489584	5726711136	366504577728
$n = 7$	10	1172	149800	19173968	2454267040	314146179392	40210710958720

This has been added to the OEIS as sequence A368259.

A.3.3 Under vertical reflection

Proposition 75. *When $\mathcal{O}_{\langle r^2 f\rangle}^{\langle r^2 f\rangle} = 2$, such as when*

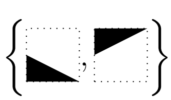
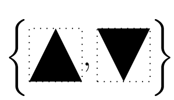
$T =$  *or* $T =$ 

the number of tilings of the $n \times m$ cylinder up to vertical reflection by two distinct tiles that are stable under vertical reflection is given by

$n = 1$	2	3	6	10	20	36	72
$n = 2$	3	7	24	76	288	1072	4224
$n = 3$	4	14	100	700	5560	43800	350256
$n = 4$	6	40	564	8296	131856	2098720	33566784
$n = 5$	8	108	3384	104968	3358736	107377488	3436078752
$n = 6$	14	362	22288	1399176	89505984	5726689312	366505626368
$n = 7$	20	1182	150972	19175140	2454416840	314146329192	40210730132688

This has been added to the OEIS as sequence A368260.

Proposition 76. *When $\mathcal{O}_{\mathbb{1}}^{\langle r^2 f\rangle} = 1$, such as when*

$T =$  *or* $T =$ 

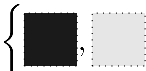
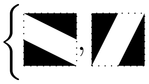
the number of tilings of the $n \times m$ cylinder up to vertical reflection by a tile that is not stable under vertical reflection is given by

$n = 1$	1	3	4	10	16	36	64
$n = 2$	2	7	20	76	272	1072	4160
$n = 3$	2	14	88	700	5472	43800	349568
$n = 4$	4	40	532	8296	131344	2098720	33558592
$n = 5$	4	108	3280	104968	3355456	107377488	3435973888
$n = 6$	8	362	21944	1399176	89484128	5726689312	366504228224
$n = 7$	10	1182	149800	19175140	2454267040	314146329192	40210710958720

This has been added to the OEIS as sequence A368261.

A.3.4 Under 180° rotation

Proposition 77. *When $\mathcal{O}_{\langle r^2 \rangle}^{\langle r^2 \rangle} = 2$, such as when*

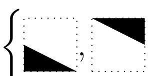
$T = \{$  $\}$ *or* $T = \{$  $\}$

the number of tilings of the $n \times m$ cylinder up to 180° rotation by two distinct tiles that are stable under 180° rotation is given by

$n = 1$	2	3	6	10	20	36	72
$n = 2$	3	7	24	76	288	1072	4224
$n = 3$	4	16	104	720	5600	43968	350592
$n = 4$	6	43	570	8356	131976	2099728	33568800
$n = 5$	8	120	3408	105376	3359552	107390592	3436104960
$n = 6$	13	382	22284	1400536	89505968	5726776672	366505626304
$n = 7$	18	1236	150824	19182160	2454398112	314147227968	40210727735936

This has been added to the OEIS as sequence A368262.

Proposition 78. *When $\mathcal{O}_{\mathbb{1}}^{\langle r^2 \rangle} = 1$, such as when*

$T = \{$  $\}$ *or* $T = \{$  $\}$

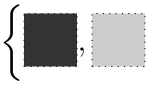
the number of tilings of the $n \times m$ cylinder up to 180° rotation by a tiles that is not stable under 180° rotation is given by

$n = 1$	1	3	4	10	16	36	64
$n = 2$	2	7	20	76	272	1072	4160
$n = 3$	2	16	88	720	5472	43968	349568
$n = 4$	4	43	538	8356	131464	2099728	33560608
$n = 5$	4	120	3280	105376	3355456	107390592	3435973888
$n = 6$	9	382	22028	1400536	89489584	5726776672	366504577728
$n = 7$	10	1236	149800	19182160	2454267040	314147227968	40210710958720

This has been added to the OEIS as sequence A368263.

A.3.5 Under cylindrical action only

Proposition 79. *When $\mathcal{O}_{\mathbb{1}}^{\mathbb{1}} = 2$, such as when*

$T = \Big\{$  $\Big\}$

the number of tilings of the $n \times m$ cylinder by two distinct tiles is given by

$n = 1$	2	4	8	16	32	64	128
$n = 2$	3	10	36	136	528	2080	8256
$n = 3$	4	24	176	1376	10944	87424	699136
$n = 4$	6	70	1044	16456	262416	4195360	67113024
$n = 5$	8	208	6560	209728	6710912	214748416	6871947776
$n = 6$	14	700	43800	2796976	178962784	11453291200	733008106880
$n = 7$	20	2344	299600	38347936	4908534080	628292358784	80421421917440

This has been added to the OEIS as sequence A368264.

A.4 The $n \times m$ torus

This section gives examples of every choice of symmetry of the $n \times m$ torus together with every essentially different set of tile designs that consists of a single orbit (or two orbits, in the case of a fully symmetric tile). Each sequence is annotated with its corresponding entry in the On-Line Encyclopedia of Integer Sequences. A table of all such sequences is given in Table 4.

	$\langle r^2, f\rangle$	$\langle f\rangle$	$\langle r^2\rangle$	$\mathbb{1}$
$\mathcal{O}_V$	Table 80 A222188	—	—	—
$\mathcal{O}_{\langle f\rangle}$	Table 81 A368302	Table 84 A368305	—	—
$\mathcal{O}_{\langle r^2\rangle}$	Table 82 A368303	—	Table 86 A368307	—
$\mathcal{O}_{\mathbb{1}}$	Table 83 A368304	Table 85 A368306	Table 87 A368308	Table 88 A184271

Table 4: An index of tables that describe the number of tilings of the $n \times m$ torus.

A.4.1 Under horizontal and vertical reflection

Proposition 80. *When* $\mathcal{O}^{\langle r^2,f\rangle}_{\langle r^2,f\rangle} = 2$, *such as when*

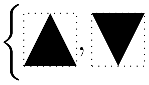
$$T = \Big\{$$  $$\Big\}$$

the number of tilings of the $n \times m$ *torus up to horizontal and vertical reflection by two distinct tiles with both horizontal and vertical reflectional symmetry is given by the following table:*

$n = 1$	2	3	4	6	8	13	18
$n = 2$	3	7	13	34	78	237	687
$n = 3$	4	13	36	158	708	4236	26412
$n = 4$	6	34	158	1459	14676	184854	2445918
$n = 5$	8	78	708	14676	340880	8999762	245619576
$n = 6$	13	237	4236	184854	8999762	478070832	26185264801
	18	687	26412	2445918	245619576	26185264801	2872221202512

This is given by OEIS sequence A222188.

Proposition 81. *When* $\mathcal{O}^{\langle r^2,f\rangle}_{\langle f\rangle} = 1$, *such as when*

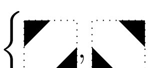
$$T = \Big\{$$  $$\Big\}$$

the number of tilings of the $n \times m$ *torus up to horizontal and vertical reflection by a tile horizontal (but not vertical) reflectional symmetry is given by the following table:*

$n = 1$	1	2	2	4	4	9	10
$n = 2$	2	5	9	26	62	205	623
$n = 3$	2	8	22	120	600	3936	25556
$n = 4$	4	22	126	1267	14164	181782	2437726
$n = 5$	4	44	592	13600	337192	8965354	245501608
$n = 6$	8	135	3936	178366	8980642	477655760	26184041441
$n = 7$	9	362	25314	2404372	245479140	26179947021	2872203226920

This has been added to the OEIS as sequence A368302.

Proposition 82. *When* $\mathcal{O}^{\langle r^2,f\rangle}_{\langle r^2\rangle} = 1$, *such as when*

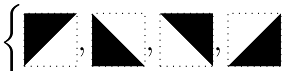
$$T = \Big\{$$  $$\Big\}$$

the number of tilings of the $n \times m$ torus up to horizontal and vertical reflection by a tile with $180°$ rotational symmetry is given by the following table:

$n=1$	1	2	2	4	4	8	9
$n=2$	2	5	8	22	44	135	362
$n=3$	2	8	24	120	612	3892	25482
$n=4$	4	22	120	1203	13600	177342	2404372
$n=5$	4	44	612	13600	337600	8962618	245492244
$n=6$	8	135	3892	177342	8962618	477371760	26179772237
$n=7$	9	362	25482	2404372	245492244	26179772237	2872202028544

This has been added to the OEIS as sequence A368303.

Proposition 83. *When $\mathcal{O}_{\mathbb{1}}^{\langle r^2,f\rangle} = 1$, such as when*

$$T = \{ \quad \}$$


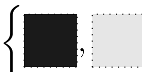
the number of tilings of the $n \times m$ torus up to horizontal and vertical reflection by a tile with no symmetry is given by the following table:

$n=1$	1	4	6	23	52
$n=2$	4	28	194	2196	26524
$n=3$	6	194	7296	350573	17895736
$n=4$	23	2196	350573	67136624	13744131446
$n=5$	52	26524	17895736	13744131446	11258999068672
$n=6$	194	351588	954495904	2932037300956	9607679419823148

This has been added to the OEIS as sequence A368304.

A.4.2 Under horizontal (equivalently vertical) reflection

Proposition 84. *When $\mathcal{O}_{\langle f\rangle}^{\langle f\rangle} = 2$, such as when*

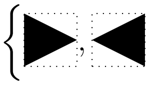
$$T = \{ \quad \} \qquad \text{or} \qquad T = \{ \quad \},$$



the number of tilings of the $n \times m$ torus up to horizontal reflection by two distinct tiles with horizontal reflectional symmetry is given by the following table:

$n=1$	2	3	4	6	8	14	20
$n=2$	3	7	14	40	108	362	1182
$n=3$	4	13	44	218	1200	7700	51112
$n=4$	6	34	226	2386	27936	361244	4869276
$n=5$	8	78	1184	26892	674384	17920876	491003216
$n=6$	13	237	7700	354680	17950356	955180432	52367383810
$n=7$	18	687	50628	4804062	490958280	52359294854	5744406453840

This has been added to the OEIS as sequence A368305.

Proposition 85. *When* $\mathcal{O}_{\mathbb{1}}^{\langle f \rangle} = 1$, *such as when*

$$T = \Big\{$$  $$\Big\},$$

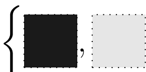
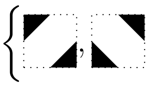
the number of tilings of the $n \times m$ *torus up to horizontal reflection by a tile that does not have horizontal reflectional symmetry is given by the following table:*

$n=1$	1	2	2	4	4	8	10
$n=2$	2	5	8	24	56	190	596
$n=3$	2	9	32	186	1096	7356	49940
$n=4$	4	26	182	2130	26296	350316	4794376
$n=5$	4	62	1096	26380	671104	17899020	490853416
$n=6$	9	205	7356	350584	17897924	954481360	52357796826
$n=7$	10	623	49940	4795870	490853416	52357896710	5744387279872

This has been added to the OEIS as sequence A368306.

A.4.3 Under 180° rotation

Proposition 86. *When* $\mathcal{O}_{\langle r^2 \rangle}^{\langle r^2 \rangle} = 2$, *such as when*

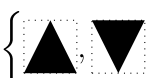
$$T = \Big\{$$  $$\Big\}$$ *or* $$T = \Big\{$$  $$\Big\},$$

the number of tilings of the $n \times m$ *torus up to* 180° *rotation by two distinct tiles with* 180° *rotational symmetry is given by the following table:*

$n=1$	2	3	4	6	8	13	18
$n=2$	3	7	13	34	78	237	687
$n=3$	4	13	48	224	1224	7696	50964
$n=4$	6	34	224	2302	27012	353384	4806078
$n=5$	8	78	1224	27012	675200	17920860	490984488
$n=6$	13	237	7696	353384	17920860	954677952	52359294790
$n=7$	18	687	50964	4806078	490984488	52359294790	5744404057088

This has been added to the OEIS as sequence A368307.

Proposition 87. *When* $\mathcal{O}_{\mathbb{1}}^{\langle r^2 \rangle} = 1$, *such as when*

$$T = \Big\{$$  $$\Big\}$$ *or* $$T = \Big\{$$  $$\Big\},$$

the number of tilings of the $n \times m$ torus up to $180°$ rotation by a tile without $180°$ rotational symmetry is given by the following table:

$n = 1$	1	2	2	4	4	9	10
$n = 2$	2	5	9	26	62	205	623
$n = 3$	2	9	32	192	1096	7440	49940
$n = 4$	4	26	192	2174	26500	351336	4797886
$n = 5$	4	62	1096	26500	671104	17904476	490853416
$n = 6$	9	205	7440	351336	17904476	954546880	52358246214
$n = 7$	10	623	49940	4797886	490853416	52358246214	5744387279872

This has been added to the OEIS as sequence A368308.

A.4.4 Under toroidal action only

Proposition 88. *When $\mathcal{O}_{\mathbb{1}}^{\mathbb{1}} = 2$, such as when*

$T = \Big\{$  $\Big\}$,

the number of tilings of the $n \times m$ grid up to cyclic shifting of rows and columns by any two distinct tile designs is given by the following table:

$n = 1$	2	3	4	6	8	14	20
$n = 2$	3	7	14	40	108	362	1182
$n = 3$	4	14	64	352	2192	14624	99880
$n = 4$	6	40	352	4156	52488	699600	9587580
$n = 5$	8	108	2192	52488	1342208	35792568	981706832
$n = 6$	14	362	14624	699600	35792568	1908897152	104715443852
$n = 7$	20	1182	99880	9587580	981706832	104715443852	11488774559744

This is OEIS sequence A184271.

A.5 The $n \times n$ torus

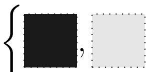
This section gives examples of every choice of symmetry of the $n \times n$ torus together with every essentially different set of tile designs that consists of a single orbit (or two orbits, in the case of a fully symmetric tile). Each sequence is annotated with its corresponding entry in the On-Line Encyclopedia of Integer Sequences. A table of all such sequences is given in Table 5.

	$\langle r, f\rangle$	$\langle r^2, rf\rangle$	$\langle r\rangle$	$\langle rf\rangle$
$\mathcal{O}_{\langle r,f\rangle}$	Sequence 89 A255016	—	—	—
$\mathcal{O}_{\langle r^2,f\rangle}$	Sequence 90 A367533	—	—	—
$\mathcal{O}_{\langle r^2,rf\rangle}$	Sequence 91 A295223	Sequence 97 A368139	—	—
$\mathcal{O}_{\langle r\rangle}$	Sequence 92 A367534	—	Sequence 101 A368143	—
$\mathcal{O}_{\langle f\rangle}$	Sequence 93 A367535	—	—	—
$\mathcal{O}_{\langle rf\rangle}$	Sequence 94 A367536	Sequence 98 A368140	—	Sequence 104 A255015
$\mathcal{O}_{\langle r^2\rangle}$	Sequence 95 A367537	Sequence 99 A368141	Sequence 102 A368144	—
$\mathcal{O}_{\mathbb{1}}$	Sequence 96 A367538	Sequence 100 A368142	Sequence 103 A368145	Sequence 105 A367530

Table 5: An index of tables that describe the number of tilings of the $n \times n$ torus.

A.5.1 Under the symmetries of the square

Proposition 89. *When* $\mathcal{O}^{\langle r,f\rangle}_{\langle r,f\rangle} = 2$*, such as when*

$T = \{$  $\}$,

the number of tilings of the $n \times n$ *torus up to symmetries of the square by two distinct tiles are fixed under all symmetries of the square is given by*

$$2, 6, 26, 805, 172112, 239123150, 1436120190288, 36028817512382026, \ldots$$

This is OEIS sequence A255016.

Proposition 90. *When* $\mathcal{O}^{\langle r,f\rangle}_{\langle r^2,f\rangle} = 1$*, such as when*

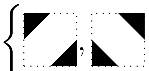
$T = \{$  $\}$,

the number of tilings of the $n \times n$ *torus up to symmetries of the square by a tile that is fixed under horizontal and vertical reflections is given by*

$$1, 4, 18, 733, 170440, 239035502, 1436110601256, 36028815364865610, \ldots$$

This has been added to the OEIS as sequence A367533.

Proposition 91. *When* $\mathcal{O}^{\langle r,f\rangle}_{\langle r^2,rf\rangle} = 1$*, such as when*

$T = \{$  $\}$,

the number of tilings of the $n \times n$ *torus up to symmetries of the square by a tile that is fixed under diagonal and antidiagonal reflections is given by*

$$1, 4, 18, 669, 170440, 238773358, 1436110601256, 36028800332480074, \ldots$$

This is OEIS sequence A295223.

Proposition 92. *When* $\mathcal{O}^{\langle r,f\rangle}_{\langle r\rangle} = 1$*, such as when*

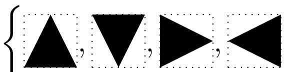
$T = \{$  $\}$,

the number of tilings of the $n \times n$ *torus up to symmetries of the square by a tile that is fixed under* 90° *rotations is given by*

$$1, 4, 14, 613, 168832, 238686222, 1436101016320, 36028798185029194, \ldots$$

This has been added to the OEIS as sequence A367534.

Proposition 93. *When* $\mathcal{O}^{\langle r,f\rangle}_{\langle f\rangle} = 1$*, such as when*

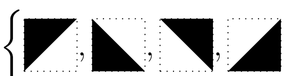
$T = \{$  $\}$,

the number of tilings of the $n \times n$ *torus up to symmetries of the square by a tile that is fixed under horizontal (respectively vertical) reflections is given by*

$$1, 16, 3692, 33570410, 5629501212064, 163971058561827918 56, \ldots$$

This has been added to the OEIS as sequence A367535.

Proposition 94. *When* $\mathcal{O}^{\langle r,f\rangle}_{\langle rf\rangle} = 1$*, such as when*

$T = \{$  $\}$,

the number of tilings of the $n \times n$ *torus up to symmetries of the square by a tile that is fixed under antidiagonal (respectively diagonal) reflections is given by*

$$1, 17, 3692, 33572458, 5629501212064, 16397105857614447792, \ldots$$

This has been added to the OEIS as sequence A367536.

Proposition 95. *When* $\mathcal{O}_{\langle r^2 \rangle}^{\langle r,f \rangle} = 1$*, such as when*

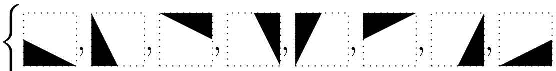
$T = \{$  $\}$,

the number of tilings of the $n \times n$ *torus up to symmetries of the square by a tile that is fixed under* 180° *rotations is given by*

$$1, 23, 3776, 33601130, 5629507922944, 16397105889110874288, \ldots$$

This has been added to the OEIS as sequence A368137.

Proposition 96. *When* $\mathcal{O}_{\mathbb{1}}^{\langle r,f \rangle} = 1$*, such as when*

$T = \{$  $\}$,

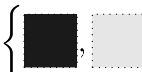
the number of tilings of the $n \times n$ *torus up to symmetries of the square by a tile that is fixed under only the identity is given by*

$$1, 154, 1864192, 2199026796168, 188894659314785812480, \ldots$$

This has been added to the OEIS as sequence A368138. The 2×2 case had been enumerated by Dan Davis [8].

A.5.2 Under diagonal and antidiagonal reflection

Proposition 97. *When* $\mathcal{O}_{\langle r^2, rf \rangle}^{\langle r^2, rf \rangle} = 2$*, such as when*

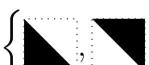
$T = \{$  $\}$ *or* $T = \{$  $\}$,

the number of tilings of the $n \times n$ *torus up to diagonal and antidiagonal rotations by two distinct tiles that are symmetric under both reflections is given by*

$$2, 6, 36, 1282, 340880, 477513804, 2872221202512, 72057600262282324, \ldots$$

This has been added to the OEIS as sequence A368139.

Proposition 98. *When* $\mathcal{O}_{\langle rf \rangle}^{\langle r^2, rf \rangle} = 1$*, such as when*

$T = \{$  $\}$

the number of tilings of the $n \times n$ *torus up to diagonal and antidiagonal rotations by a tile that is symmetric only under antidiagonal reflections is given by*

$$1, 4, 22, 1154, 337192, 477360876, 2872203226920, 72057597041056852, \ldots$$

This has been added to the OEIS as sequence A368140.

Proposition 99. *When* $\mathcal{O}_{\langle r^2\rangle}^{\langle r^2, rf\rangle} = 1$*, such as when*

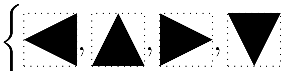
$T = \{$  $\}$

the number of tilings of the $n \times n$ *torus up to diagonal and antidiagonal rotations by a tile that is symmetric only under* $180°$ *rotations is given by*

$$1, 4, 24, 1154, 337600, 477339020, 2872202028544, 72057595967315028, \dots$$

This has been added to the OEIS as sequence A368141.

Proposition 100. *When* $\mathcal{O}_{\mathbb{1}}^{\langle r^2, rf\rangle} = 1$*, such as when*

$T = \{$  $\}$

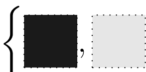
the number of tilings of the $n \times n$ *torus up to diagonal and antidiagonal rotations by a tile that is asymmetric is given by*

$$1, 23, 7296, 67124308, 11258999068672, 327942117009122706688, \dots$$

This has been added to the OEIS as sequence A368142.

A.5.3 Under $90°$ rotation

Proposition 101. *When* $\mathcal{O}_{\langle r\rangle}^{\langle r\rangle} = 2$*, such as when*

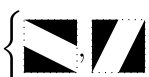
$T = \{$  $\}$ *or* $T = \{$  $\}$,

the number of tilings of the $n \times n$ *torus up to* $90°$ *rotations by two distinct tiles that are symmetric under* $90°$ *rotations is given by*

$$2, 6, 28, 1171, 337664, 477339616, 2872202032640, 72057595967392816, \dots$$

This has been added to the OEIS as sequence A368143.

Proposition 102. *When* $\mathcal{O}_{\langle r^2\rangle}^{\langle r\rangle} = 1$*, such as when*

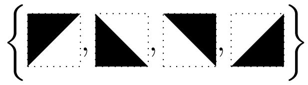
$T = \{$  $\}$,

the number of tilings of the $n \times n$ *torus up to* $90°$ *rotations by a tile that is symmetric under* $180°$ *rotations is given by*

$$1, 4, 24, 1155, 337600, 477339104, 2872202028544, 72057595967327280, \dots$$

This has been added to the OEIS as sequence A368144.

Proposition 103. *When $\mathcal{O}_{\mathbb{1}}^{\langle r \rangle} = 1$, such as when*

$$T = $$  ,

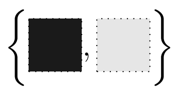
the number of tilings of the $n \times n$ torus up to $90°$ rotations by a tile asymmetric with respect to rotations is given by

$$1, 23, 7296, 67124336, 11258999068672, 3279421170091231436 8, \ldots$$

This has been added to the OEIS as sequence A368145, and the 2×2 case had previously been enumerated by hand by M. C. Escher [26].

A.5.4 Under diagonal (equivalently antidiagonal) reflection

Proposition 104. *When $\mathcal{O}_{\langle rf \rangle}^{\langle rf \rangle} = 2$, such as when*

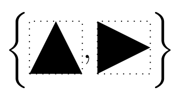
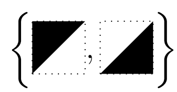
$$T = $$  ,

the number of tilings of the $n \times n$ torus up to transposition by two distinct tiles that are fixed under is given by

$$2, 6, 44, 2209, 674384, 954623404, 5744406453840, 144115192471496836, \ldots$$

This is OEIS sequence A255015.

Proposition 105. *When $\mathcal{O}_{\mathbb{1}}^{\langle rf \rangle} = 1$, such as when*

$$T = $$  , *or* $$T = $$  ,

the number of tilings of the $n \times n$ torus up to transposition by tiles that are asymmetric with respect to this transposition is given by

$$1, 4, 32, 2081, 671104, 954448620, 5744387279872, 144115188176529540, \ldots$$

This has been added to the OEIS as sequence A367530.

B Illustrations

This section of the appendix gives illustrations corresponding to all of the sequences and tables described in Appendix A, which shows an example of the tilings arising from all valid choices of $R \leq D_8$ and all sets of tile designs consisting of a single orbit.

B.1 The $n \times m$ grid

B.1.1 Under horizontal and vertical reflection

Illustration 106. [This is shown in Table 43.]

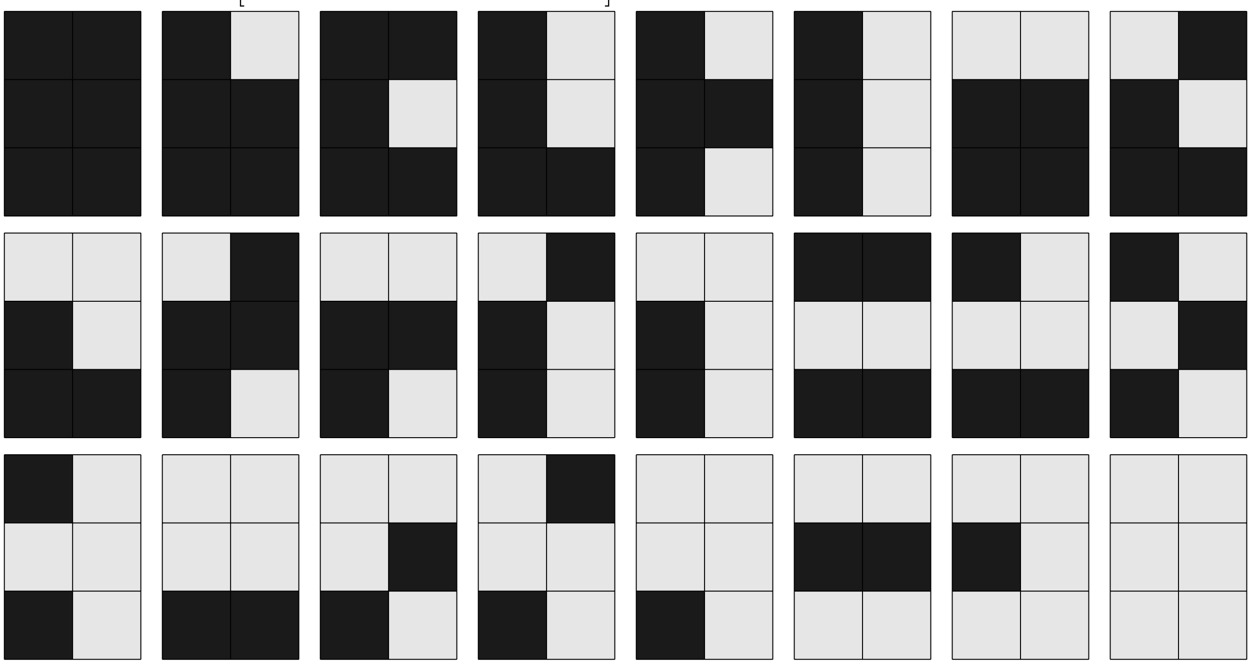


Figure 21: The 24 ways of tiling the 2×3 grid up to $D_4 = \langle r^2, f \rangle$ from a set of tile designs that consists of two orbits both of which contain an element with stabilizer subgroup D_4.

Illustration 107. [This is shown in Table 44.]

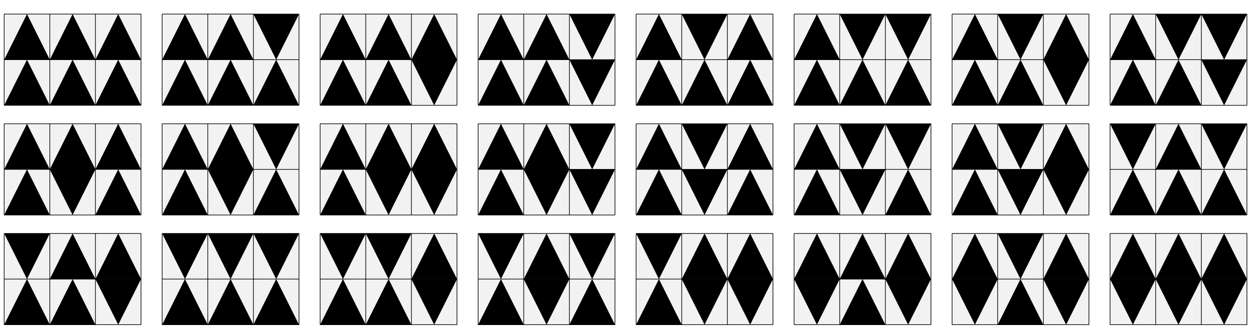


Figure 22: The 24 ways of tiling the 3×2 grid up to $D_4 = \langle r^2, f \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\langle f \rangle \leq D_4$.

Illustration 108. [This is shown in Table 45.]

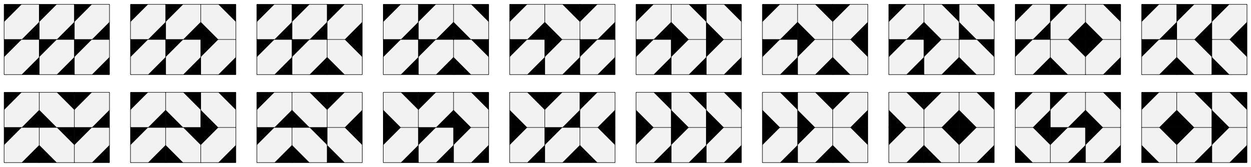


Figure 23: The 20 ways of tiling the 3×2 grid up to $D_4 = \langle r^2, f \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\langle r^2 \rangle \leq D_4$.

Illustration 109. [This is shown in Table 46.]

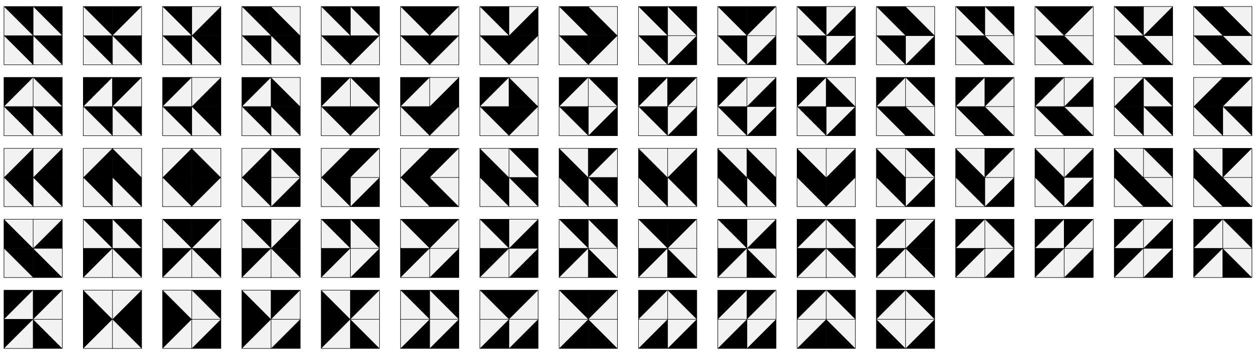


Figure 24: The 76 ways of tiling the 2×2 grid up to $D_4 = \langle r^2, f \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\mathbb{1} \leq D_4$.

B.1.2 Under horizontal (equivalently vertical) reflection

Illustration 110. [This is shown in Table 47.]

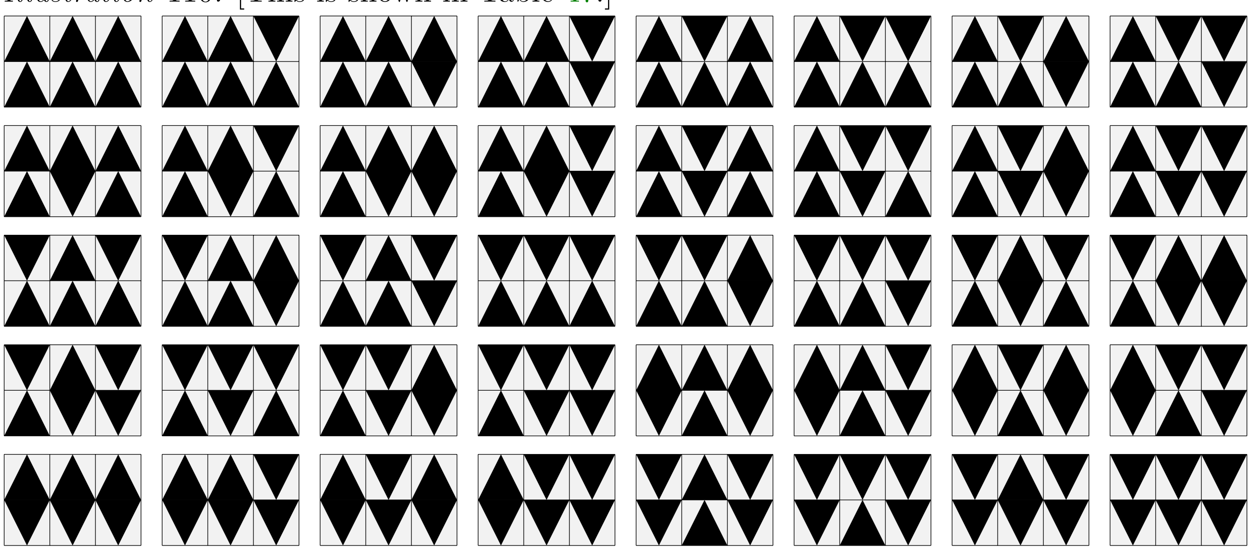


Figure 25: The 40 ways of tiling the 3×2 grid up to $\langle f \rangle$ from a set of tile designs that consists of two orbits both of which contain an element with stabilizer subgroup $\langle f \rangle$.

Illustration 111. [This is shown in Table 48.]

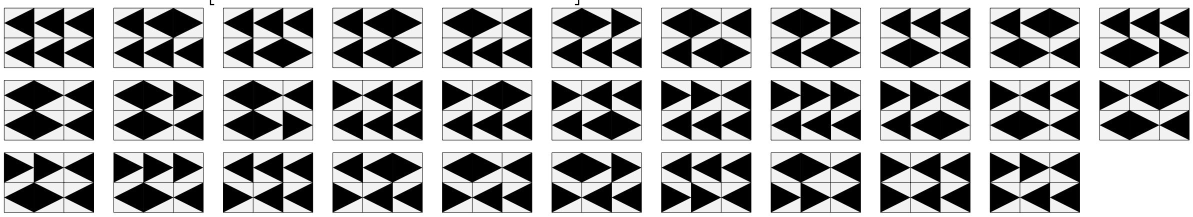


Figure 26: The 32 ways of tiling the 3×2 grid up to $\langle f \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\mathbb{1} \leq \langle f \rangle$.

B.1.3 Under $180°$ rotation

Illustration 112. [This is shown in Table 49.]

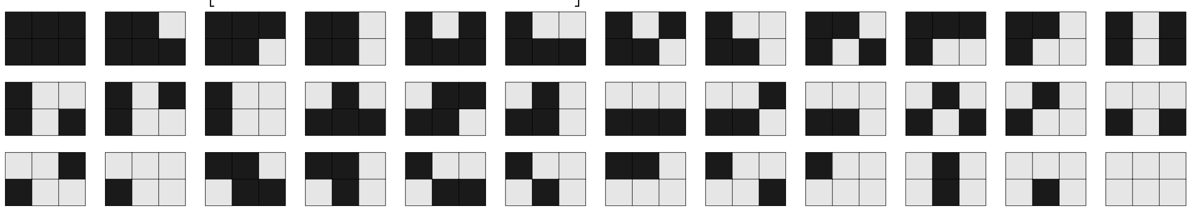


Figure 27: The 36 3×2 grids up to $\langle r^2 \rangle$ from a set of tile designs that consists of two orbits both of which contain an element with stabilizer subgroup $\langle r^2 \rangle$.

Illustration 113. [This is shown in Table 50.]

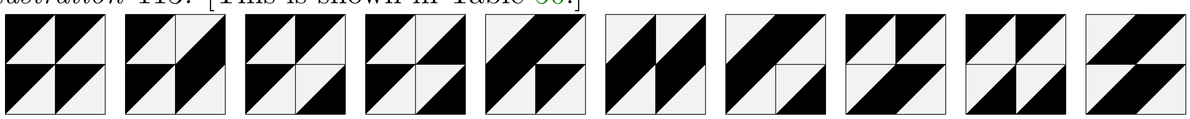


Figure 28: The 10 2×2 grids up to $\langle r^2 \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\mathbb{1} \leq \langle r^2 \rangle$.

B.2 The $n \times n$ grid

B.2.1 Under symmetries of the square

Illustration 114. [This is shown in Sequence 51.]

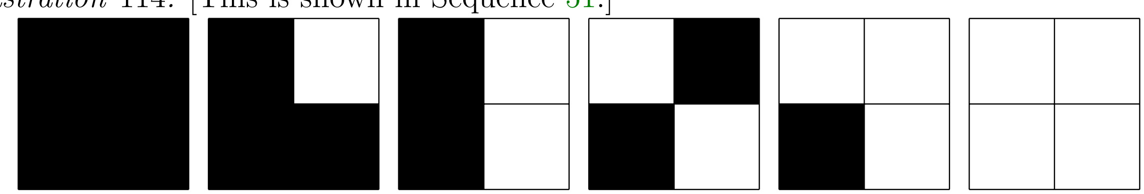


Figure 29: The 6 distinct ways of tiling the 2×2 grid up to $D_8 = \langle r, f \rangle$ from a set of tile designs that consists of two orbits both of which contain an element with stabilizer subgroup D_8.

Illustration 115. [This is shown in Sequence 52.]

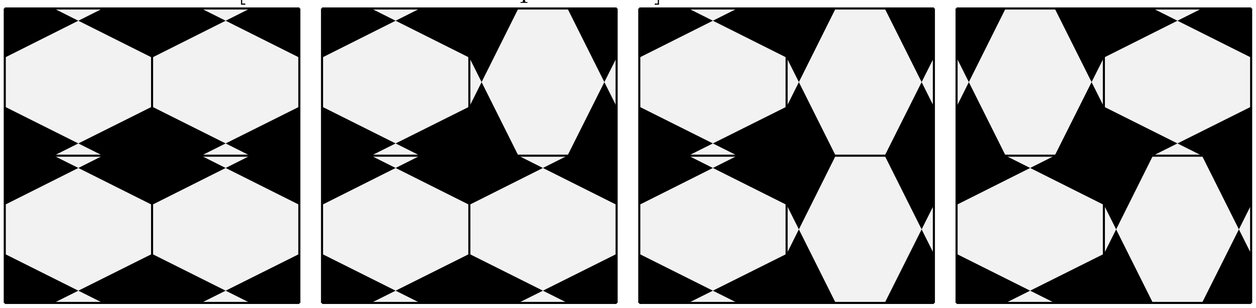


Figure 30: The 4 distinct ways of tiling the 2×2 grid up to $D_8 = \langle r, f \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\langle r^2, f \rangle \leq D_8$.

Illustration 116. [This is shown in Sequence 53.]

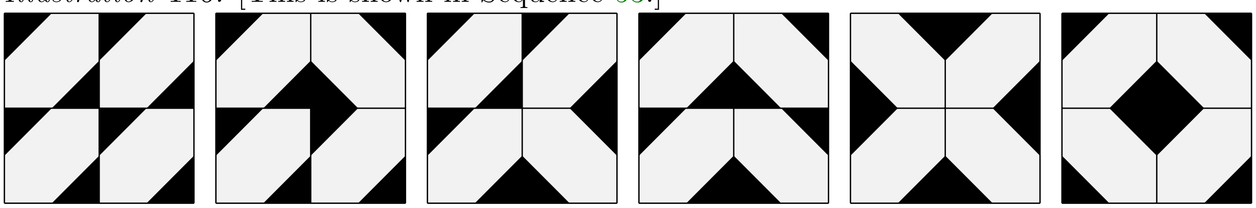


Figure 31: The 6 distinct ways of tiling the 2×2 grid up to $D_8 = \langle r, f \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\langle r^2, rf \rangle \leq D_8$.

Illustration 117. [This is shown in Sequence 54.]

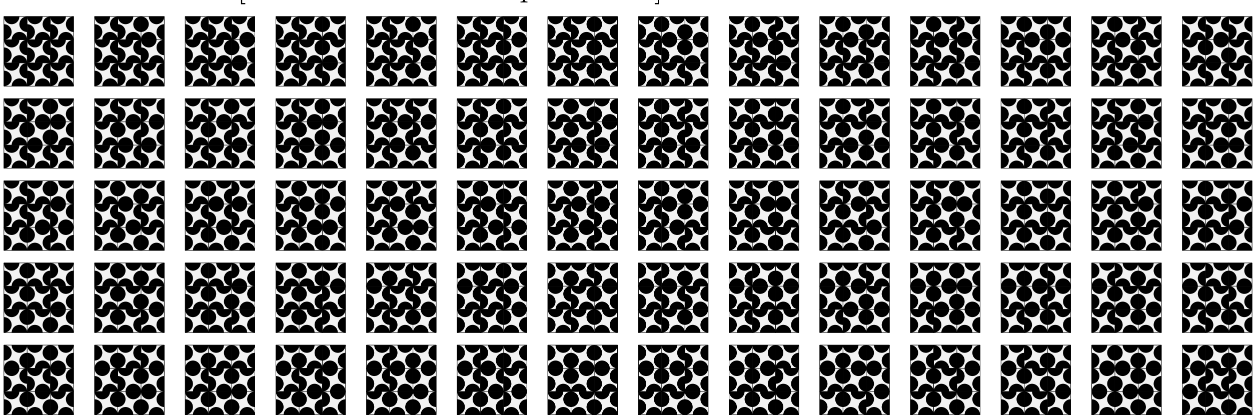


Figure 32: The 70 distinct ways of tiling the 3×3 grid up to $D_8 = \langle r, f \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\langle r \rangle \leq D_8$.

Illustration 118. [This is shown in Sequence 55.]

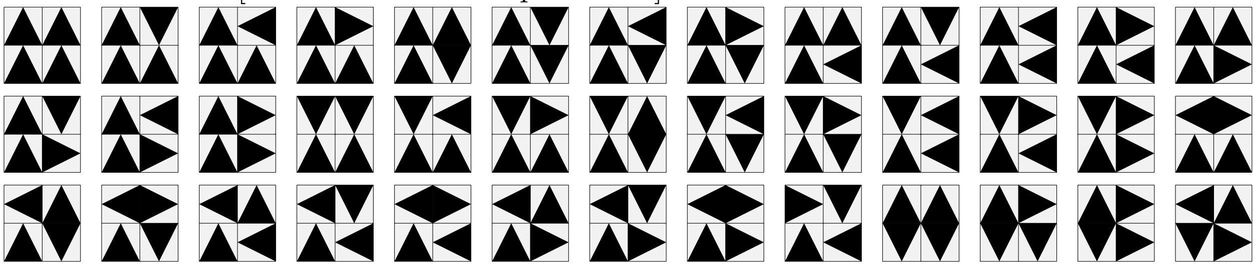


Figure 33: The 39 distinct ways of tiling the 2×2 grid up to $D_8 = \langle r, f \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\langle f \rangle \leq D_8$.

Illustration 119. [This is shown in Sequence 56.]

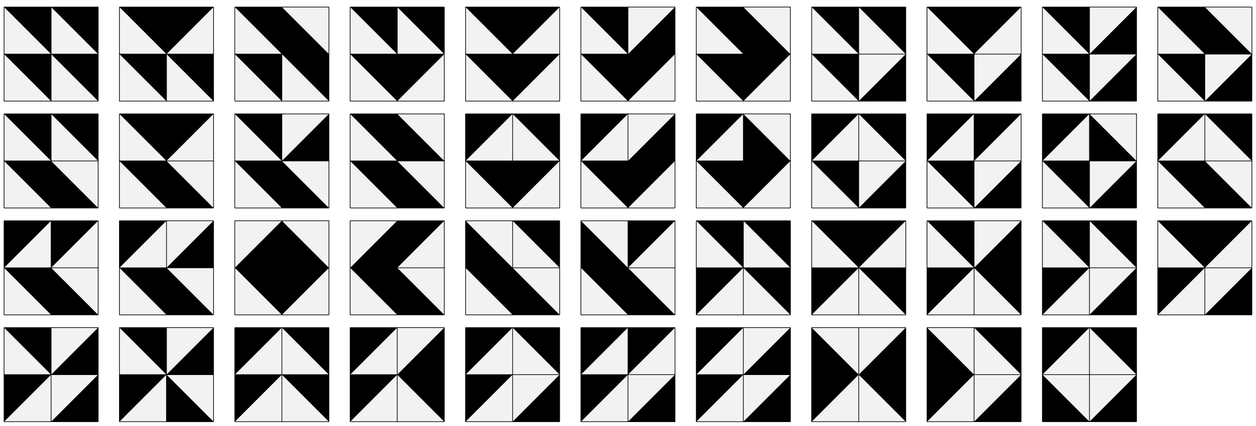


Figure 34: The 43 distinct ways of tiling the 2×2 grid up to $D_8 = \langle r, f \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\langle rf \rangle \leq D_8$.

Illustration 120. [This is shown in Sequence 57.]

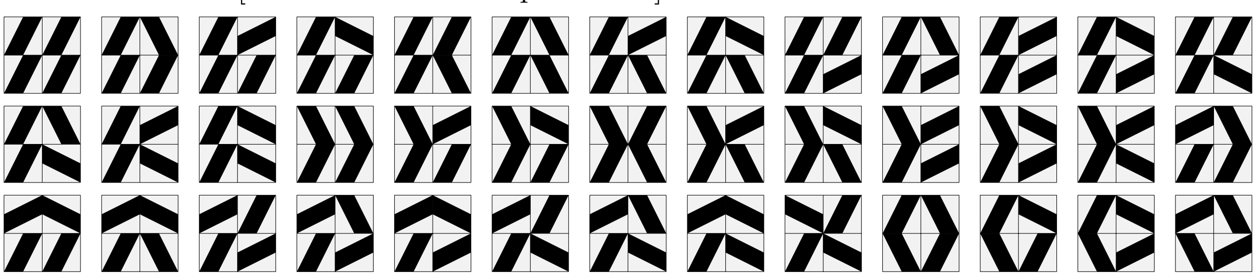


Figure 35: The 39 distinct ways of tiling the 2×2 grid up to $D_8 = \langle r, f \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\langle r^2 \rangle \leq D_8$.

Illustration 121. [This is shown in Sequence 58.]

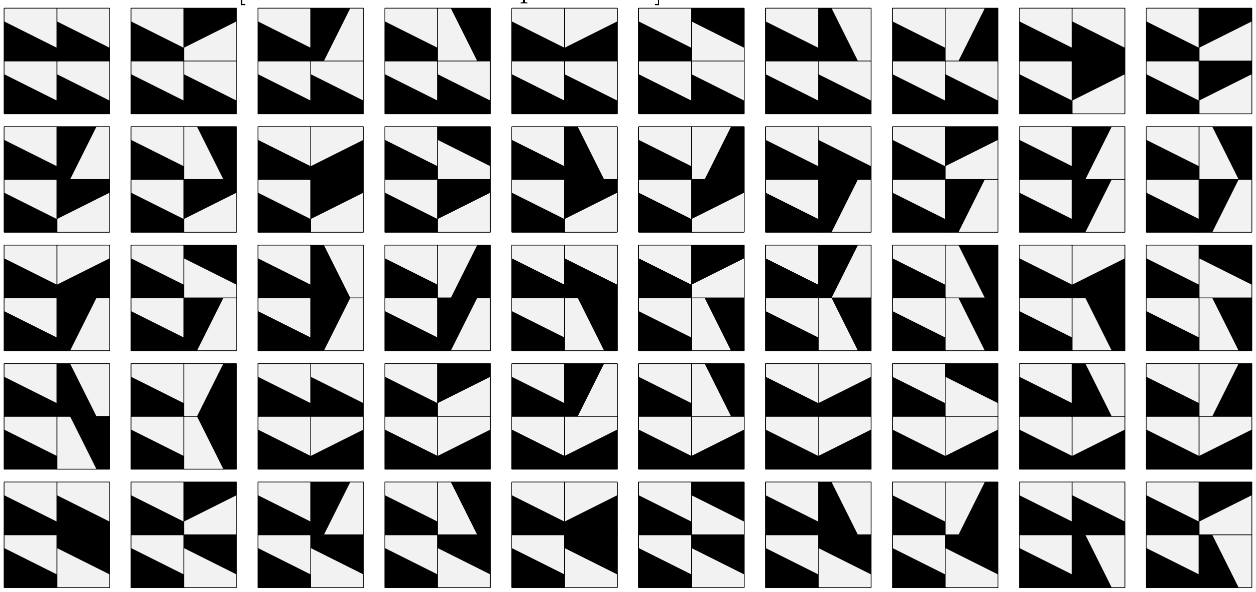


Figure 36: 50 of the 538 distinct ways of tiling the 2×2 grid up to $D_8 = \langle r, f \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\mathbb{1} \leq D_8$.

B.2.2 Under diagonal and antidiagonal reflection

Illustration 122. [This is shown in Sequence 59.]

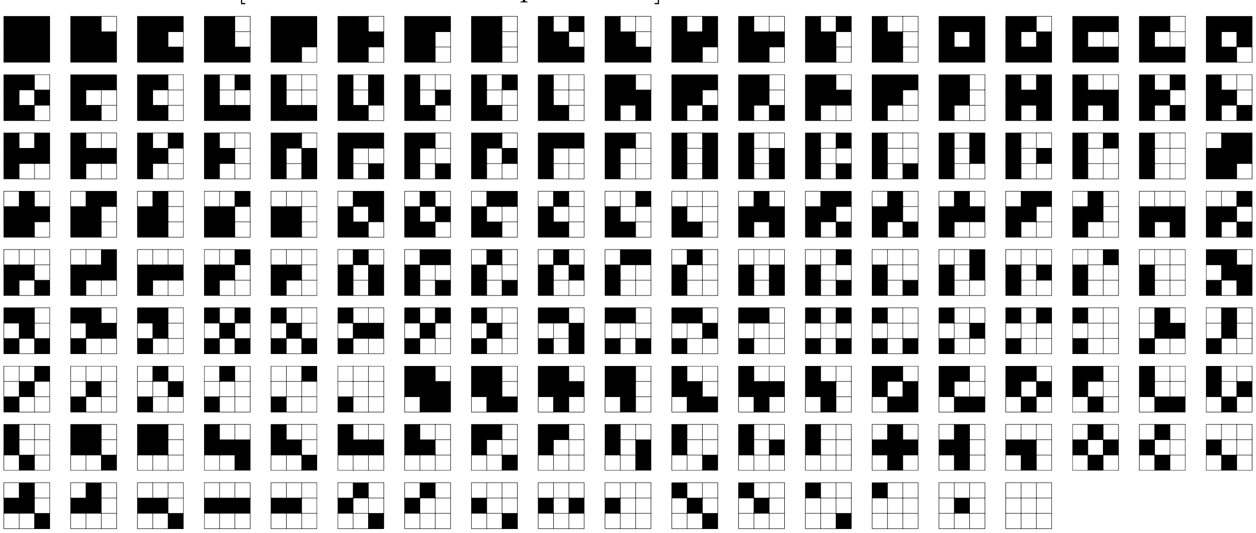


Figure 37: The 168 tilings of the 3×3 grid up to $\langle r^2, rf \rangle$ from a set of tile designs that consists of two orbits both of which contain an element with stabilizer subgroup $\langle r^2, rf \rangle$.

Illustration 123. [This is shown in Sequence 60.]

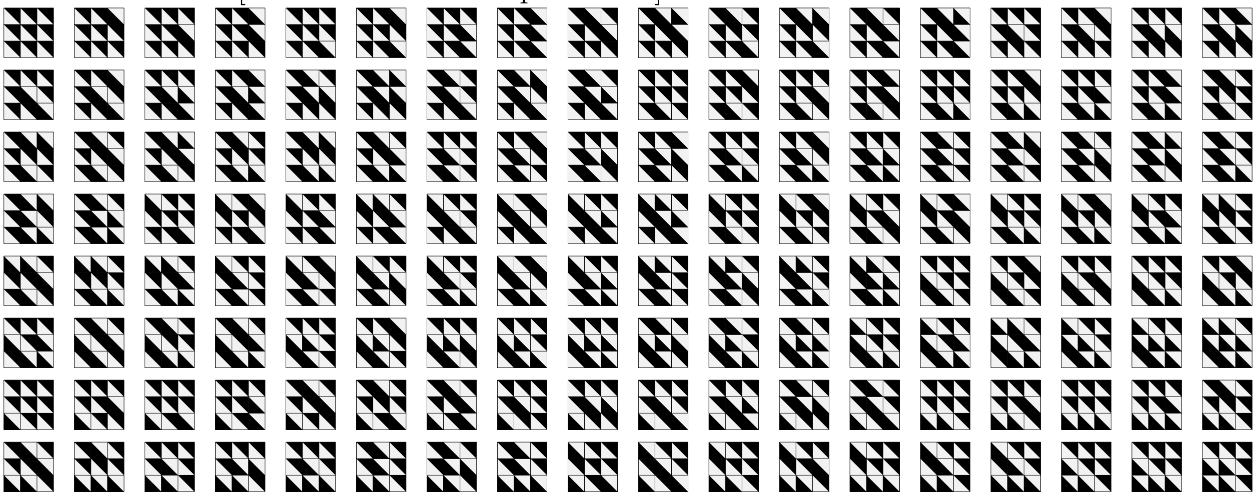


Figure 38: The 144 tilings of the 3×3 grid up to $\langle r^2, rf \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\langle rf \rangle \leq \langle r^2, rf \rangle$.

Illustration 124. [This is shown in Sequence 61.]

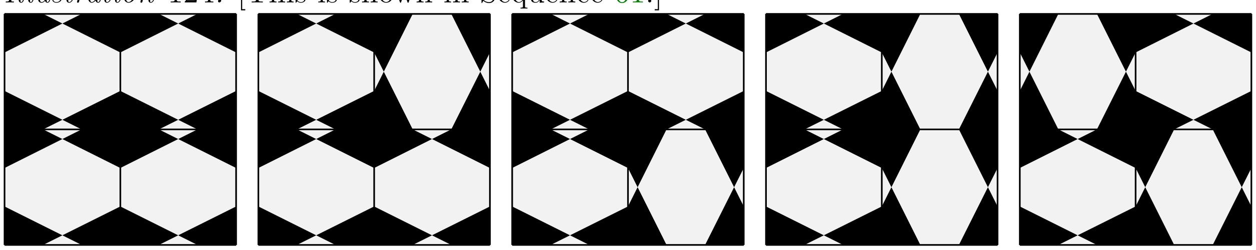


Figure 39: The 5 tilings of the 2×2 grid up to $\langle r^2, rf \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\langle r^2 \rangle \leq \langle r^2, rf \rangle$.

Illustration 125. [This is shown in Sequence 62.]

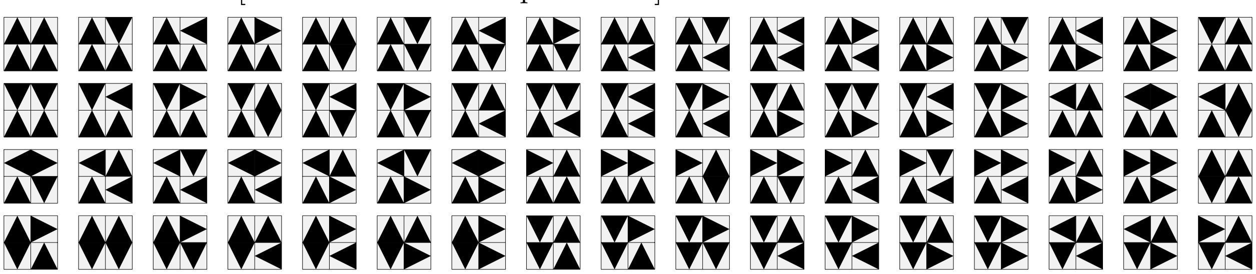


Figure 40: The 68 tilings of the 2×2 grid up to $\langle r^2, rf \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\mathbb{1} \leq \langle r^2, rf \rangle$.

B.2.3 Under 90° rotation

Illustration 126. [This is shown in Sequence 63.]

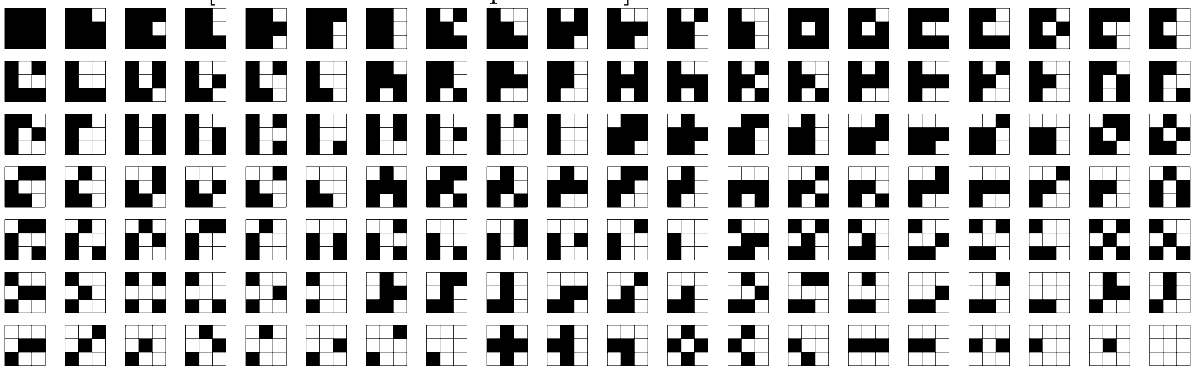


Figure 41: The 140 tilings of the 3×3 grid up to $\langle r \rangle$ from a set of tile designs that consists of two orbits both of which contain an element with stabilizer subgroup $\langle r \rangle$.

Illustration 127. [This is shown in Sequence 64.]

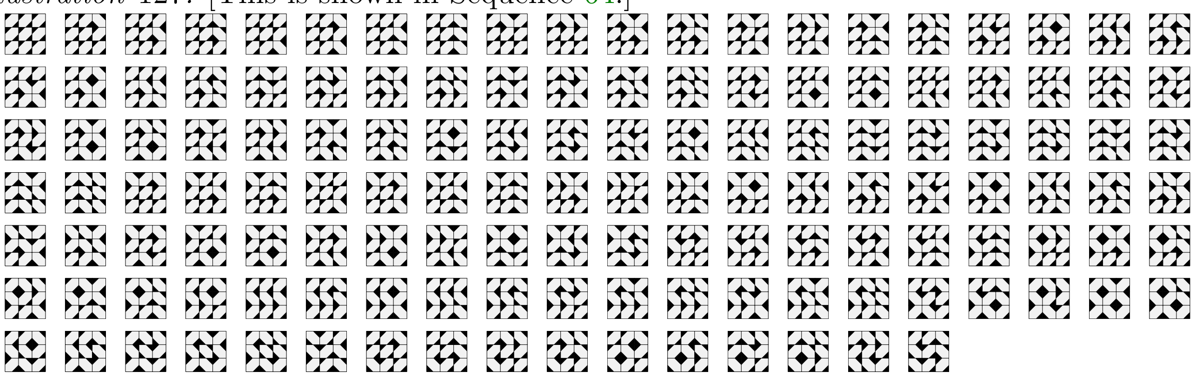


Figure 42: The 136 tilings of the 3×3 grid up to $\langle r \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\langle r^2 \rangle \leq \langle r \rangle$.

Illustration 128. [This is shown in Sequence 65.]

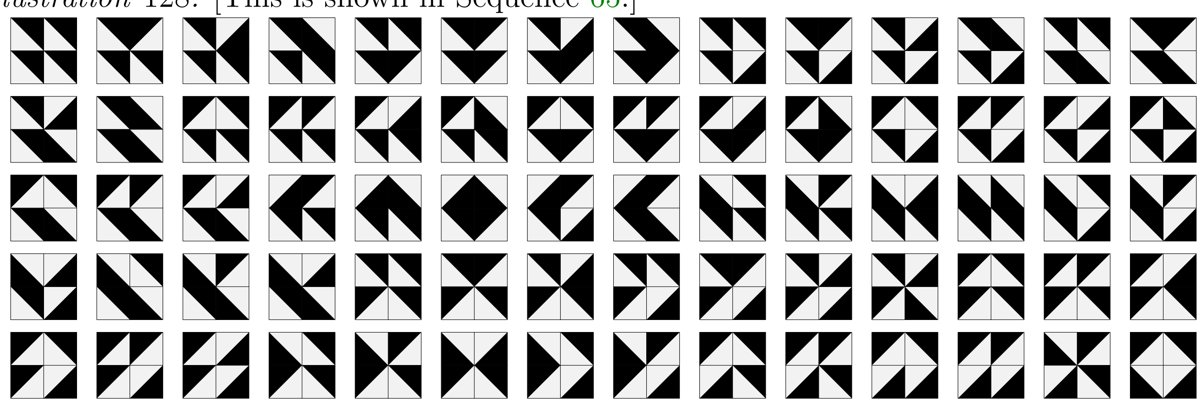


Figure 43: The 70 tilings of the 2×2 grid up to $\langle r \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\mathbb{1} \leq \langle r \rangle$.

B.2.4 Under diagonal (equivalently antidiagonal) reflection

Illustration 129. [This is shown in Sequence 66.]

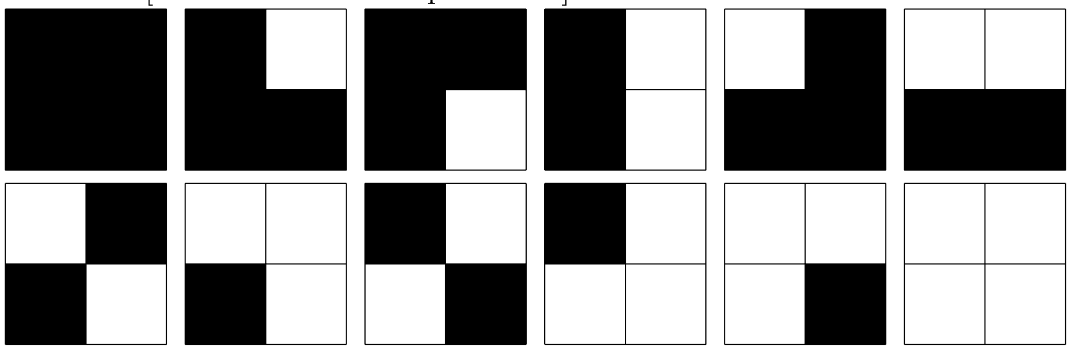


Figure 44: The 12 tilings of the 2×2 grid up to $\langle rf \rangle$ from a set of tile designs that consists of two orbits both of which contain an element with stabilizer subgroup $\langle rf \rangle$.

Illustration 130. [This is shown in Sequence 67.]

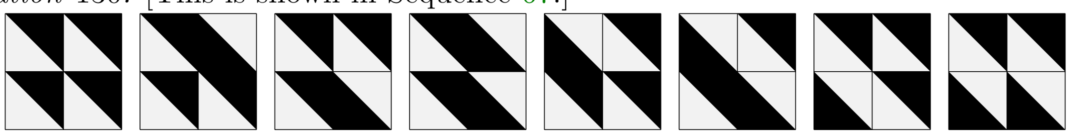


Figure 45: The 8 tilings of the 2×2 grid up to $\langle rf \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\mathbb{1} \leq \langle rf \rangle$.

B.3 The $n \times m$ cylinder

B.3.1 Under horizontal and vertical reflection

Illustration 131. [This is shown in Table 68.]

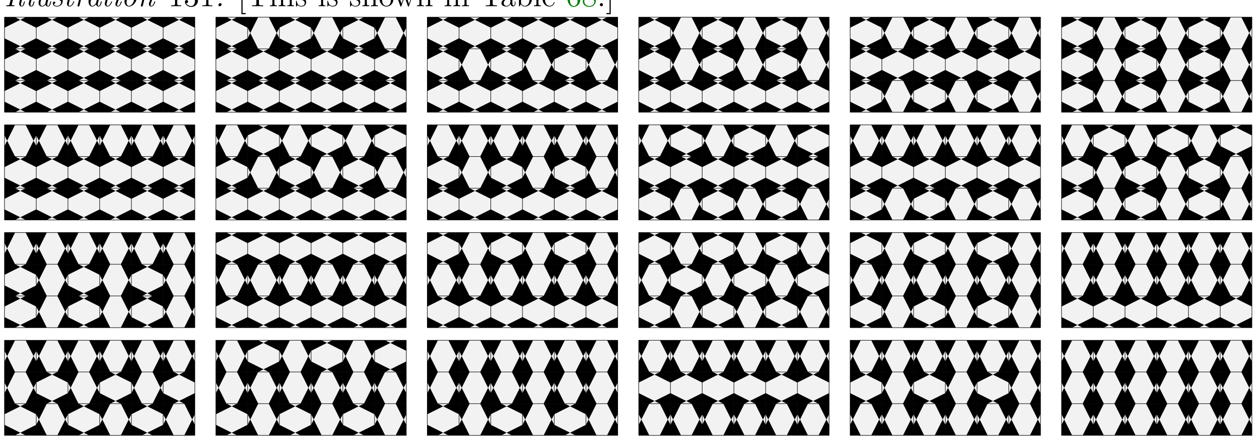


Figure 46: The 24 distinct ways of tiling the 2×3 cylinder up to $D_4 = \langle r^2, f \rangle$ from a set of tile designs that consists of two orbits both of which contain an element with stabilizer subgroup D_4.

Illustration 132. [This is shown in Table 69.]

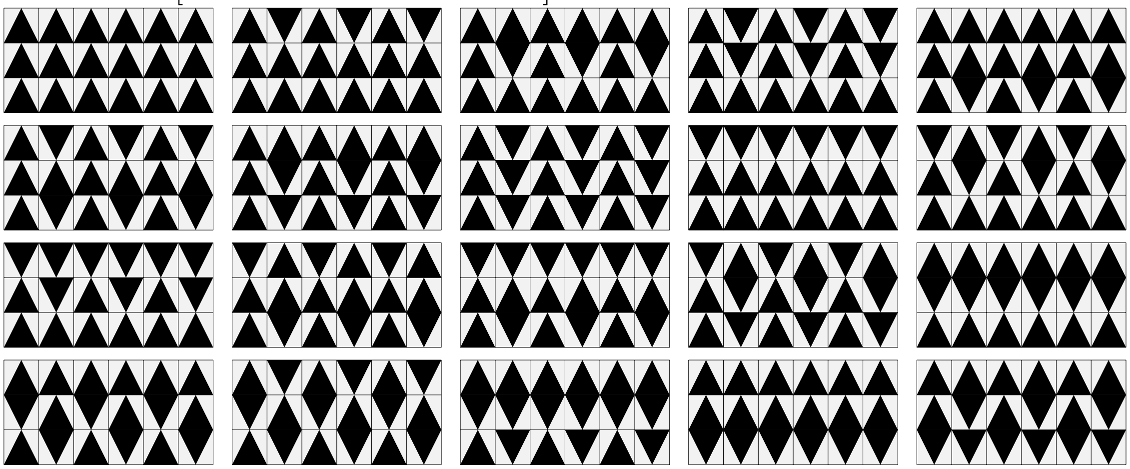


Figure 47: The 20 distinct ways of tiling the 2×3 cylinder up to $D_4 = \langle r^2, f \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\langle f \rangle \leq D_4$.

Illustration 133. [This is shown in Table 70.]

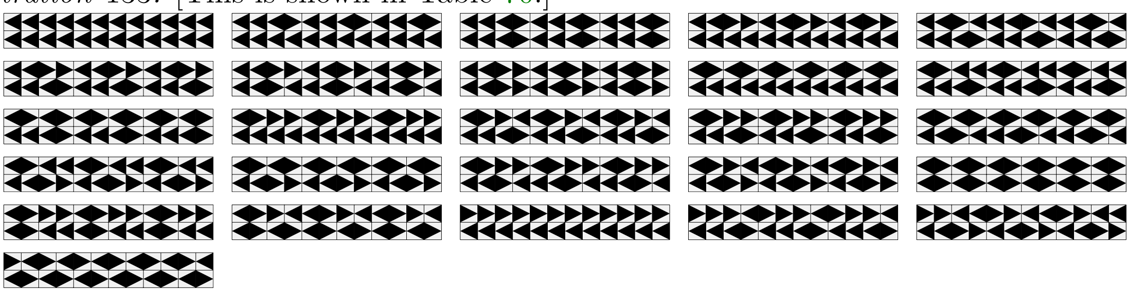


Figure 48: The 26 distinct ways of tiling the 4×2 cylinder up to $D_4 = \langle r^2, f \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\langle r^2 f \rangle \leq D_4$.

Illustration 134. [This is shown in Table 71.]

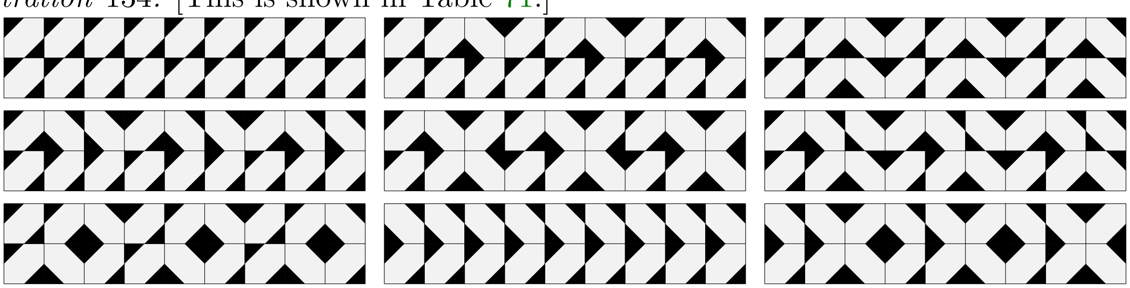


Figure 49: The 9 distinct ways of tiling the 3×2 cylinder up to $D_4 = \langle r^2, f \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\langle r^2 \rangle \leq D_4$.

Illustration 135. [This is shown in Table 72.]

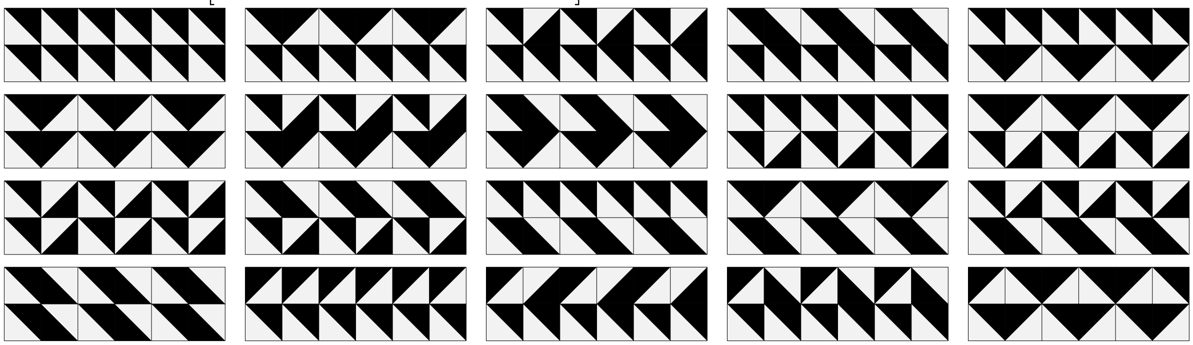


Figure 50: The 20 distinct ways of tiling the 2×2 cylinder up to $D_4 = \langle r^2, f \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\mathbb{1} \leq D_4$.

B.3.2 Under horizontal reflection

Illustration 136. [This is shown in Table 73.]

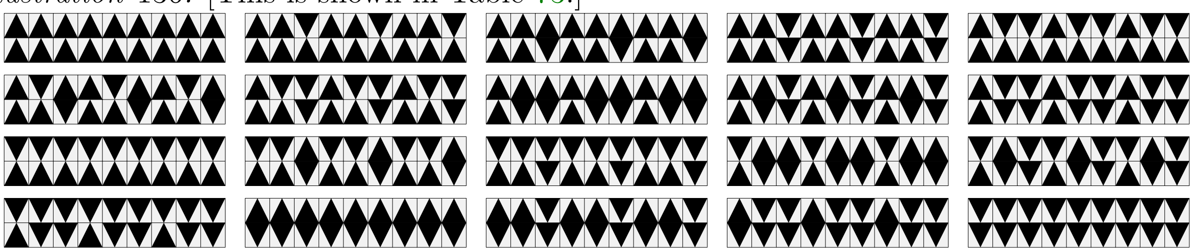


Figure 51: The 20 distinct ways of tiling the 3×2 cylinder up to $\langle f \rangle$ from a set of tile designs that consists of two orbits both of which contain an element with stabilizer subgroup $\langle f \rangle$.

Illustration 137. [This is shown in Table 74.]

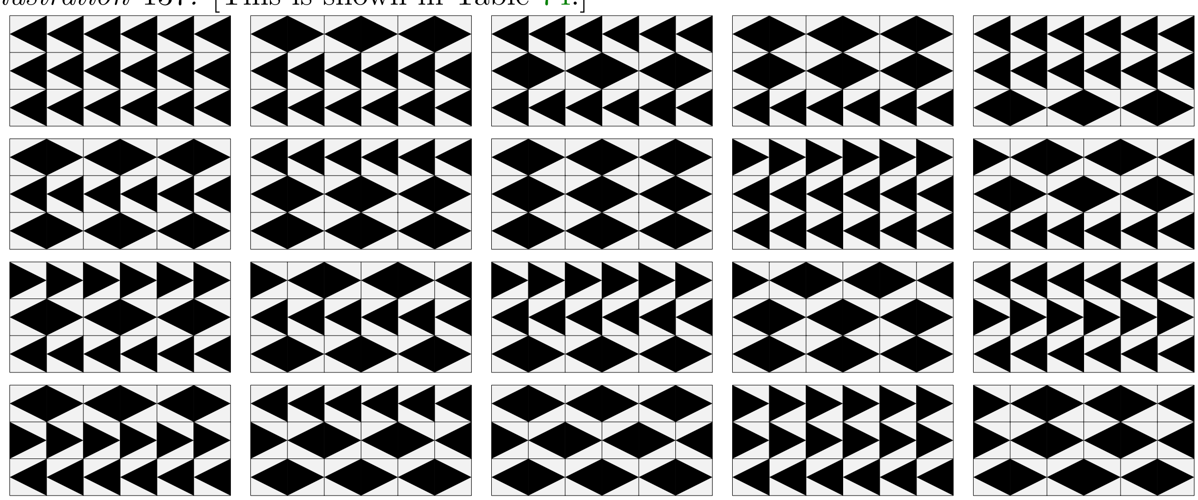


Figure 52: The 20 distinct ways of tiling the 2×3 cylinder up to $\langle f \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\mathbb{1} \leq D_4$.

B.3.3 Under vertical reflection

Illustration 138. [This is shown in Table 75.]

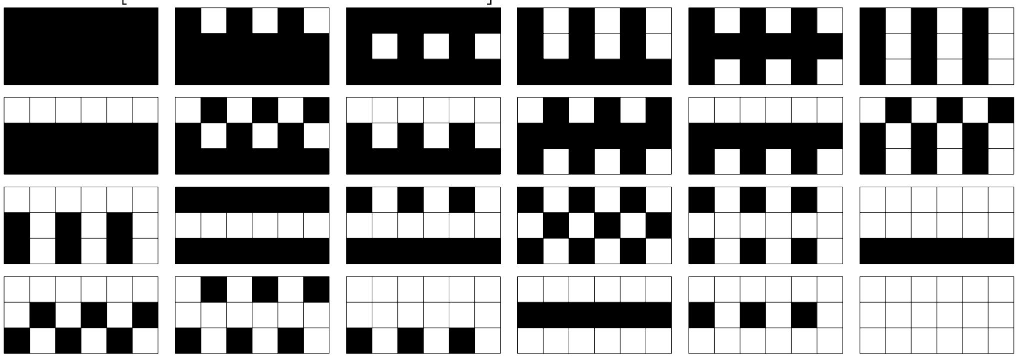


Figure 53: The 24 distinct ways of tiling the 2×3 cylinder up to $\langle r^2 f \rangle$ from a set of tile designs that consists of two orbits both of which contain an element with stabilizer subgroup $\langle r^2 f \rangle$.

Illustration 139. [This is shown in Table 76.]

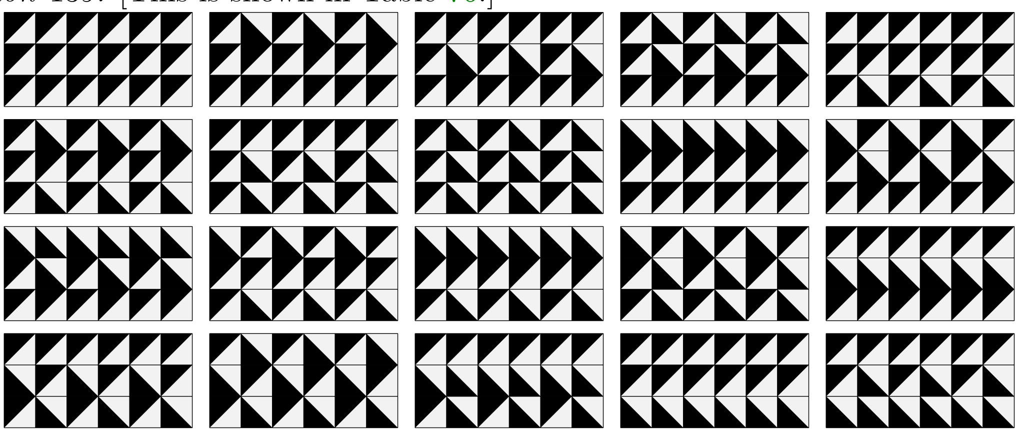


Figure 54: The 20 distinct ways of tiling the 2×3 cylinder up to $\langle r^2 f \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\mathbb{1} \leq D_4$.

Illustration 140. [This is shown in Table 77.]

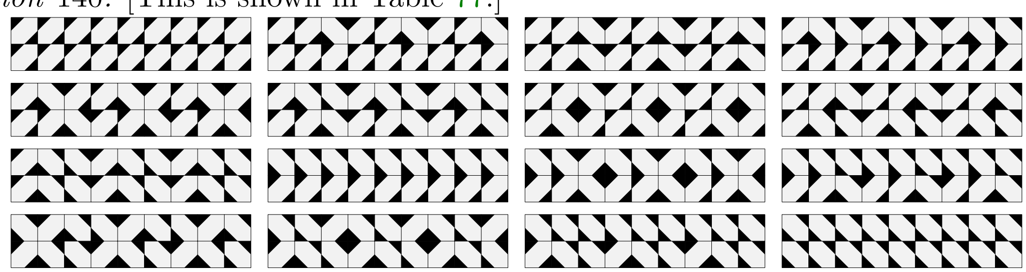


Figure 55: The 16 distinct ways of tiling the 3×2 cylinder up to $\langle r^2 \rangle$ from a set of tile designs that consists of two orbits both of which contain an element with stabilizer subgroup $\langle r^2 \rangle$.

Illustration 141. [This is shown in Table 78.]

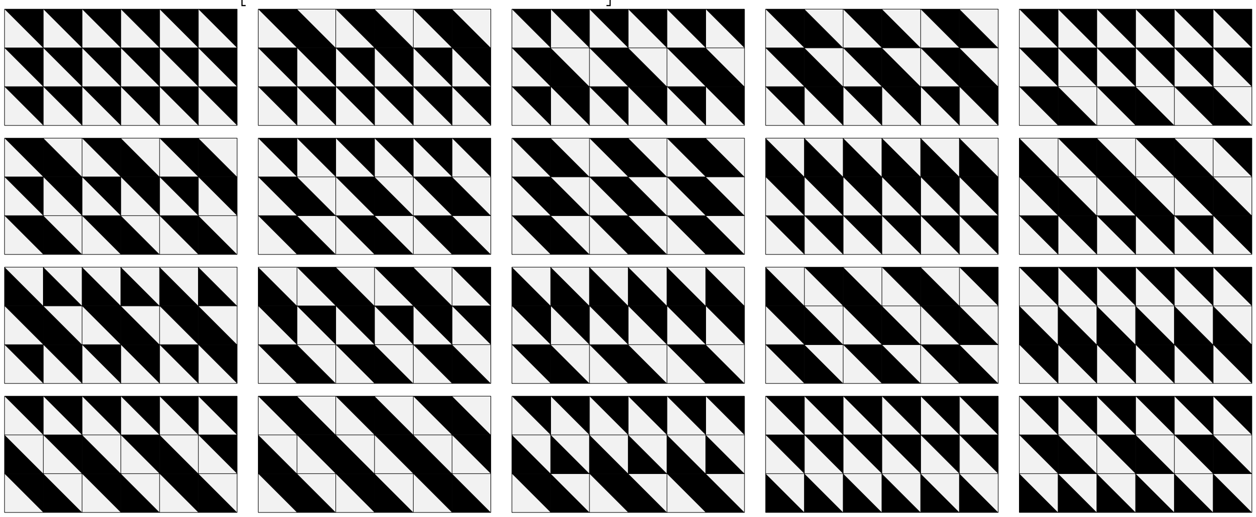


Figure 56: The 20 distinct ways of tiling the 2×3 cylinder up to $\langle r^2\rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\mathbb{1} \leq \langle r^2\rangle$.

B.3.4 Under cylindrical action only

Illustration 142. [This is shown in Table 79.]

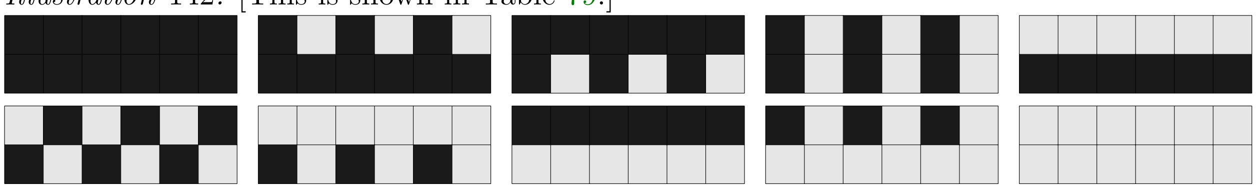


Figure 57: The 10 distinct ways of tiling the 2×2 cylinder from a set of tile designs that consists of two orbits, each containing a single tile design.

B.4 The $n \times m$ torus

B.4.1 Under horizontal and vertical reflection

Illustration 143. [This is shown in Table 80.]

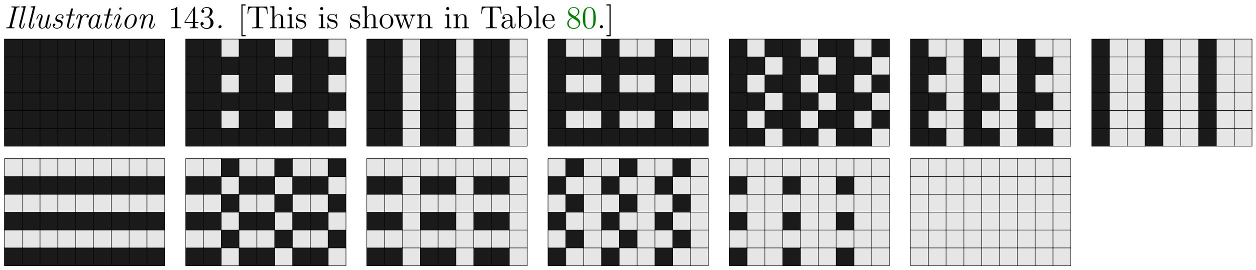


Figure 58: The 13 distinct ways of tiling the 3×2 torus up to $\langle r^2, f\rangle$ from a set of tile designs that consists of two orbits both of which contain an element with stabilizer subgroup D_4.

Illustration 144. [This is shown in Table 81.]

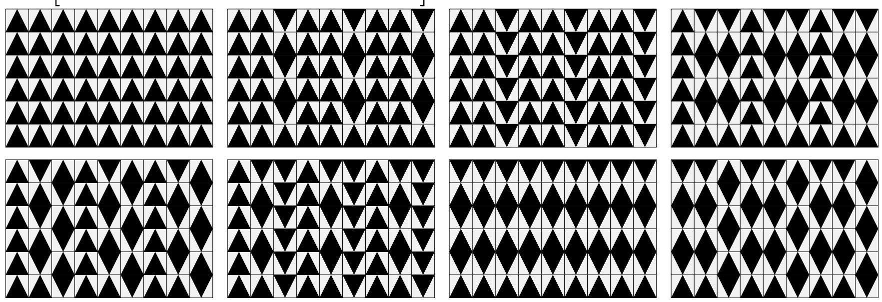


Figure 59: The 8 distinct ways of tiling the 3×2 torus up to $\langle r^2, f\rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\langle f\rangle \leq D_4$.

Illustration 145. [This is shown in Table 82.]

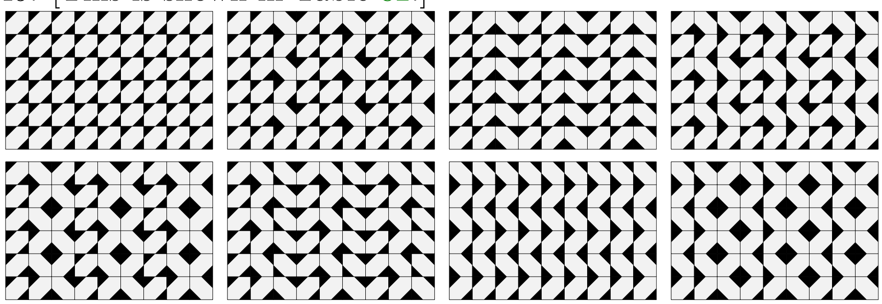


Figure 60: The 8 distinct ways of tiling the 3×2 torus up to $\langle r^2, f\rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\langle r^2\rangle \leq D_4$.

Illustration 146. [This is shown in Table 83.]

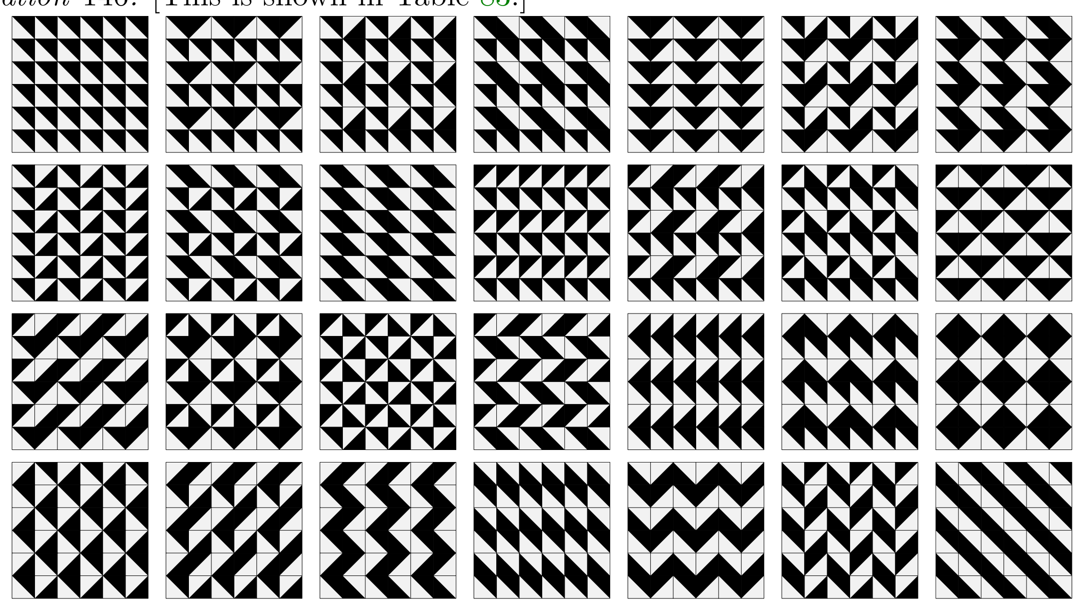


Figure 61: The 28 distinct ways of tiling the 2×2 torus up to $\langle r^2, f\rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\mathbb{1} \leq D_4$.

B.4.2 Under horizontal (equivalently vertical) reflection

Illustration 147. [This is shown in Table 84.]

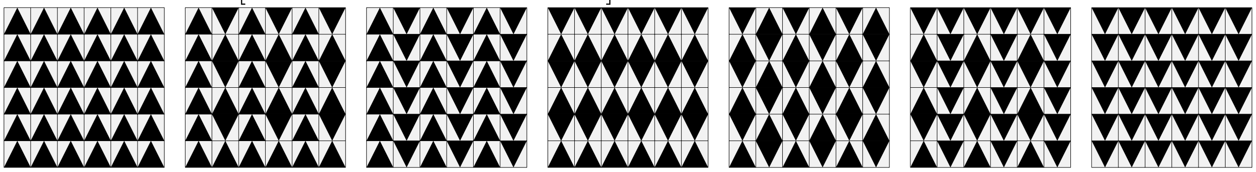


Figure 62: The 7 distinct ways of tiling the 2×2 torus up to $\langle f \rangle$ from a set of tile designs that consists of two orbits both of which contain an element with stabilizer subgroup $\langle f \rangle$.

Illustration 148. [This is shown in Table 85.]

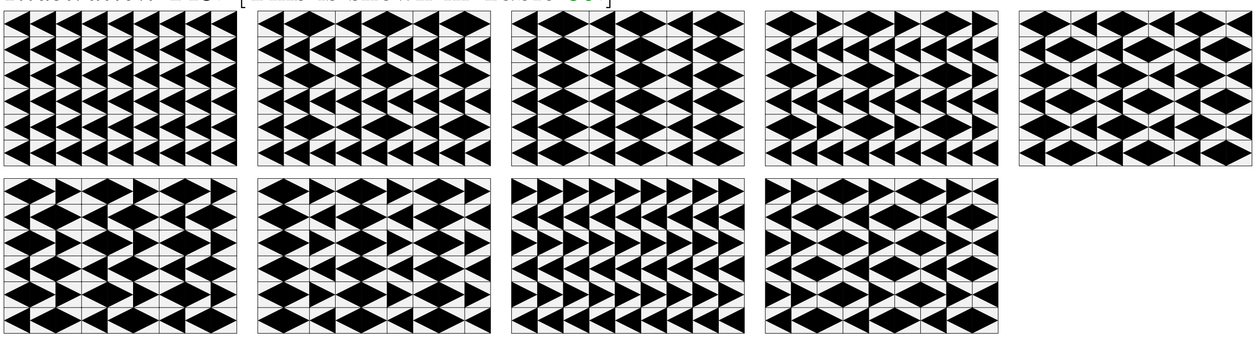


Figure 63: The 9 distinct ways of tiling the 3×2 torus up to $\langle f \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\mathbb{1} \leq \langle f \rangle$.

B.4.3 Under 180° rotation

Illustration 149. [This is shown in Table 86.]

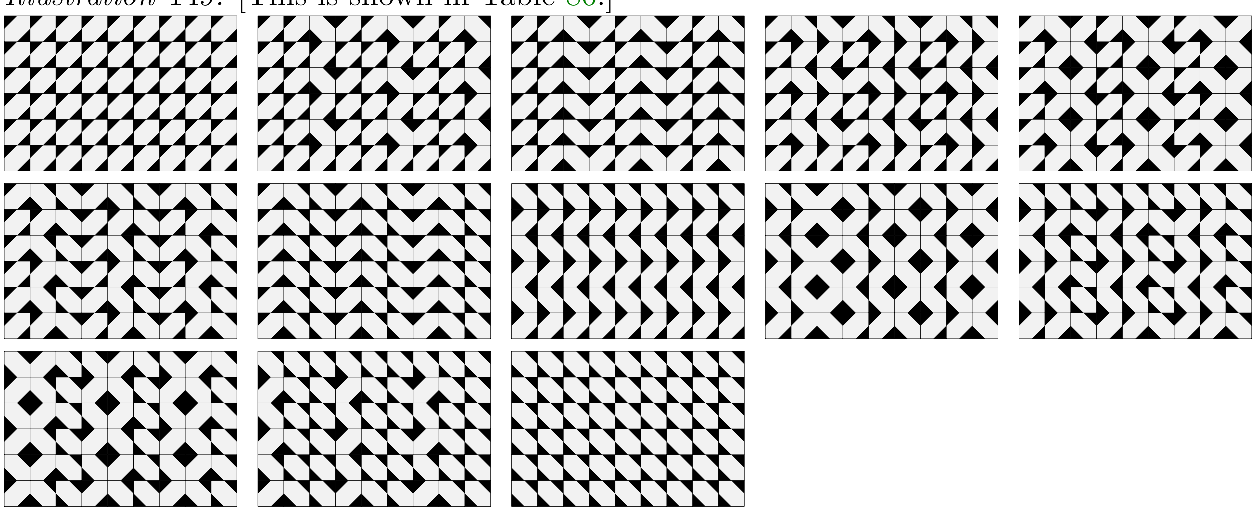


Figure 64: The 13 distinct ways of tiling the 3×2 torus up to $\langle f \rangle$ from a set of tile designs that consists of two orbits both of which contain an element with stabilizer subgroup $\langle f \rangle$.

Illustration 150. [This is shown in Table 87.]

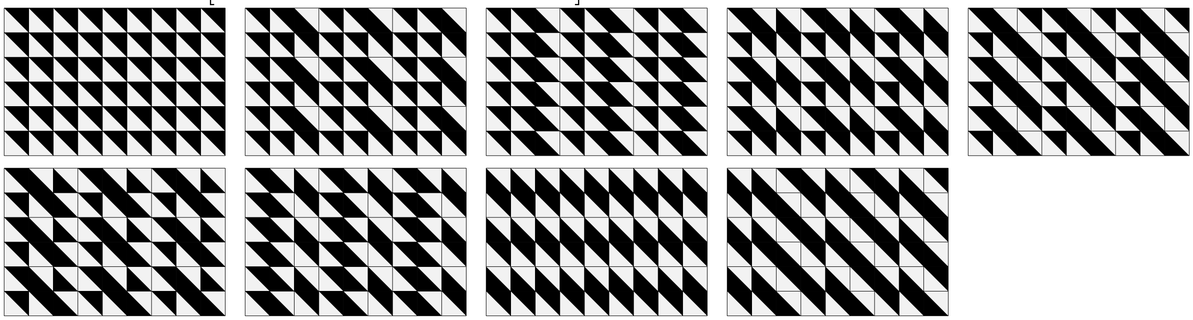


Figure 65: The 9 distinct ways of tiling the 3×2 torus $\langle r^2 \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\mathbb{1} \leq \langle r^2 \rangle$.

B.4.4 Under toroidal action only

Illustration 151. [This is shown in Table 88.]

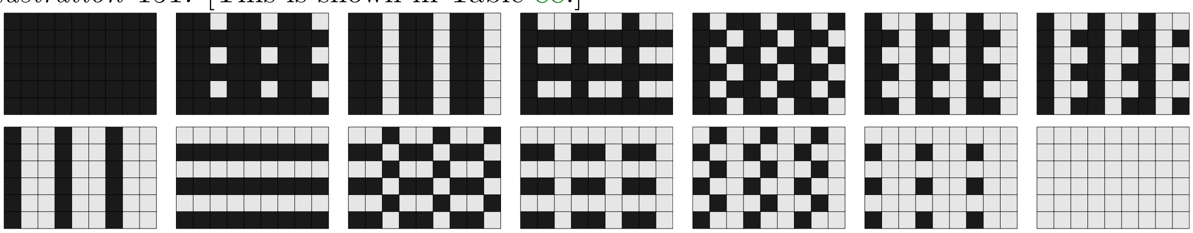


Figure 66: The 14 distinct ways of tiling the 3×2 torus from a set of tile designs that consists of two orbits, each containing a single tile design.

B.5 The $n \times n$ torus

B.5.1 Under the symmetries of the square

Illustration 152. [This is shown in Sequence 89.]

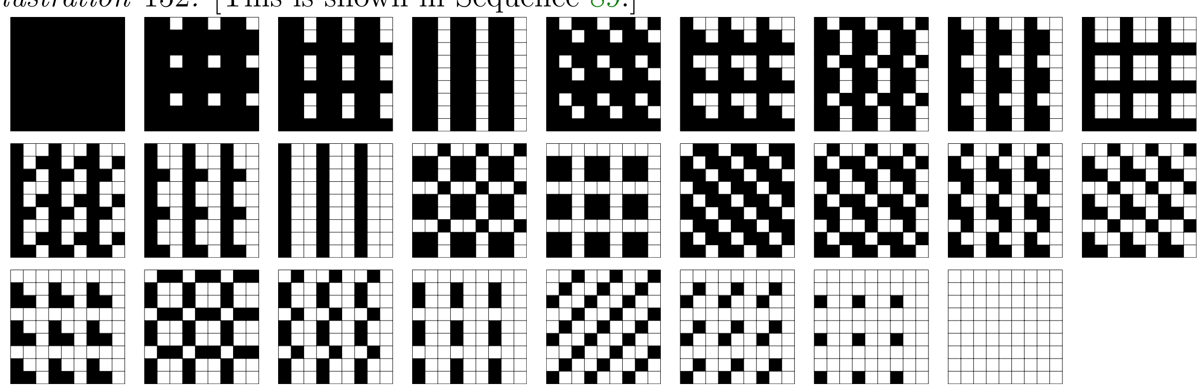


Figure 67: The 26 ways of tiling the 3×3 torus up to $D_8 = \langle r, f \rangle$ from a set of tile designs that consists of two orbits both of which contain an element with stabilizer subgroup D_8.

Illustration 153. [This is shown in Sequence 90.]

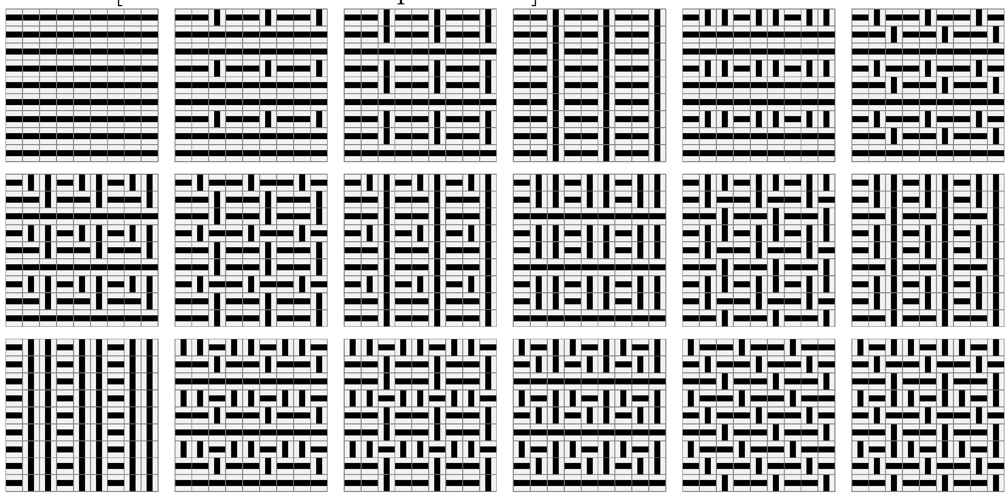


Figure 68: The 18 ways of tiling the 3×3 torus up to $D_8 = \langle r, f \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\langle r^2, f \rangle \leq D_8$.

Illustration 154. [This is shown in Sequence 91.]

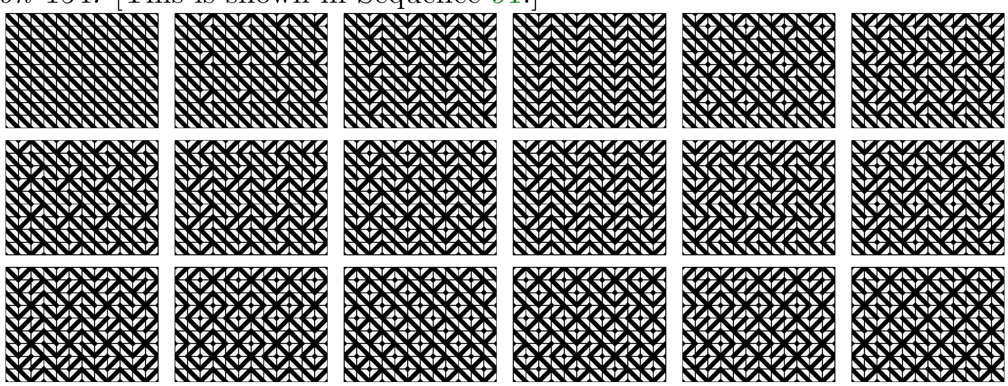


Figure 69: The 18 ways of tiling the 3×3 torus up to $D_8 = \langle r, f \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\langle r^2, rf \rangle \leq D_8$.

Illustration 155. [This is shown in Sequence 92.]

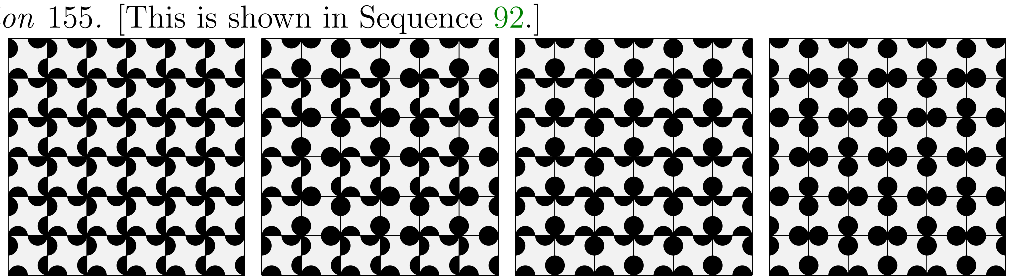


Figure 70: The 4 ways of tiling the 2×2 torus up to $D_8 = \langle r, f \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\langle r \rangle \leq D_8$.

Illustration 156. [This is shown in Sequence 93.]

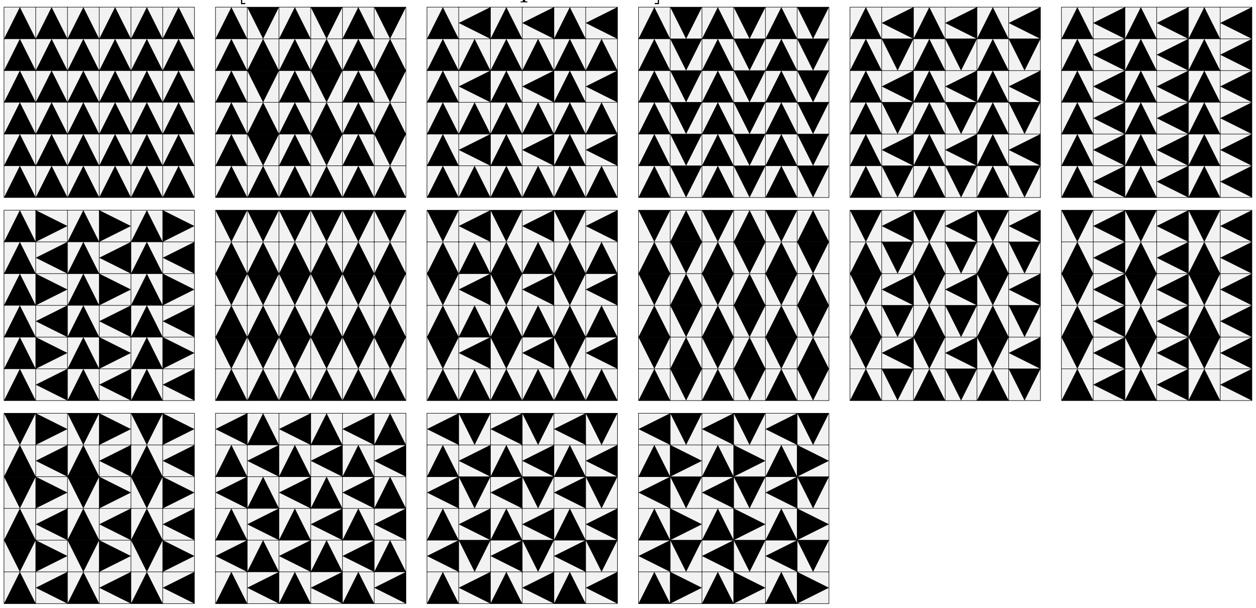


Figure 71: The 16 ways of tiling the 2×2 torus up to $D_8 = \langle r, f \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\langle f \rangle \leq D_8$.

Illustration 157. [This is shown in Sequence 94.]

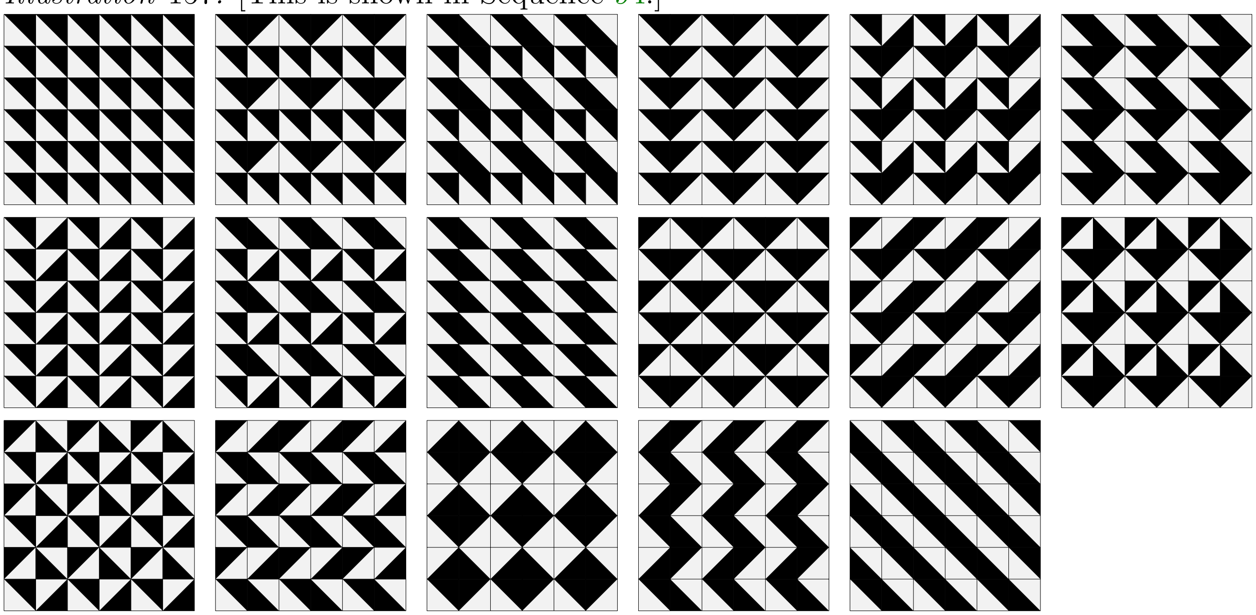


Figure 72: The 17 ways of tiling the 2×2 torus up to $D_8 = \langle r, f \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\langle rf \rangle \leq D_8$.

Illustration 158. [This is shown in Sequence 95.]

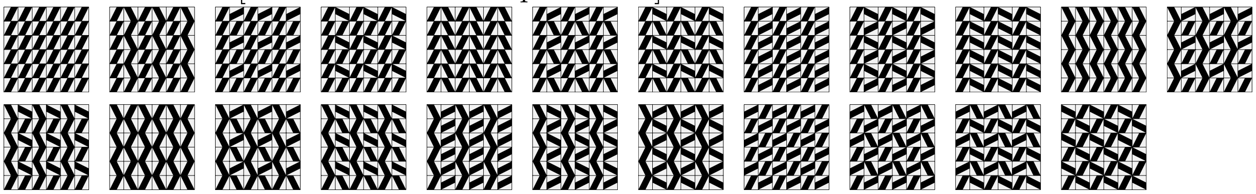


Figure 73: The 23 ways of tiling the 2×2 torus up to $D_8 = \langle r, f \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\langle r^2 \rangle \leq D_8$.

Illustration 159. [This is shown in Sequence 96.]

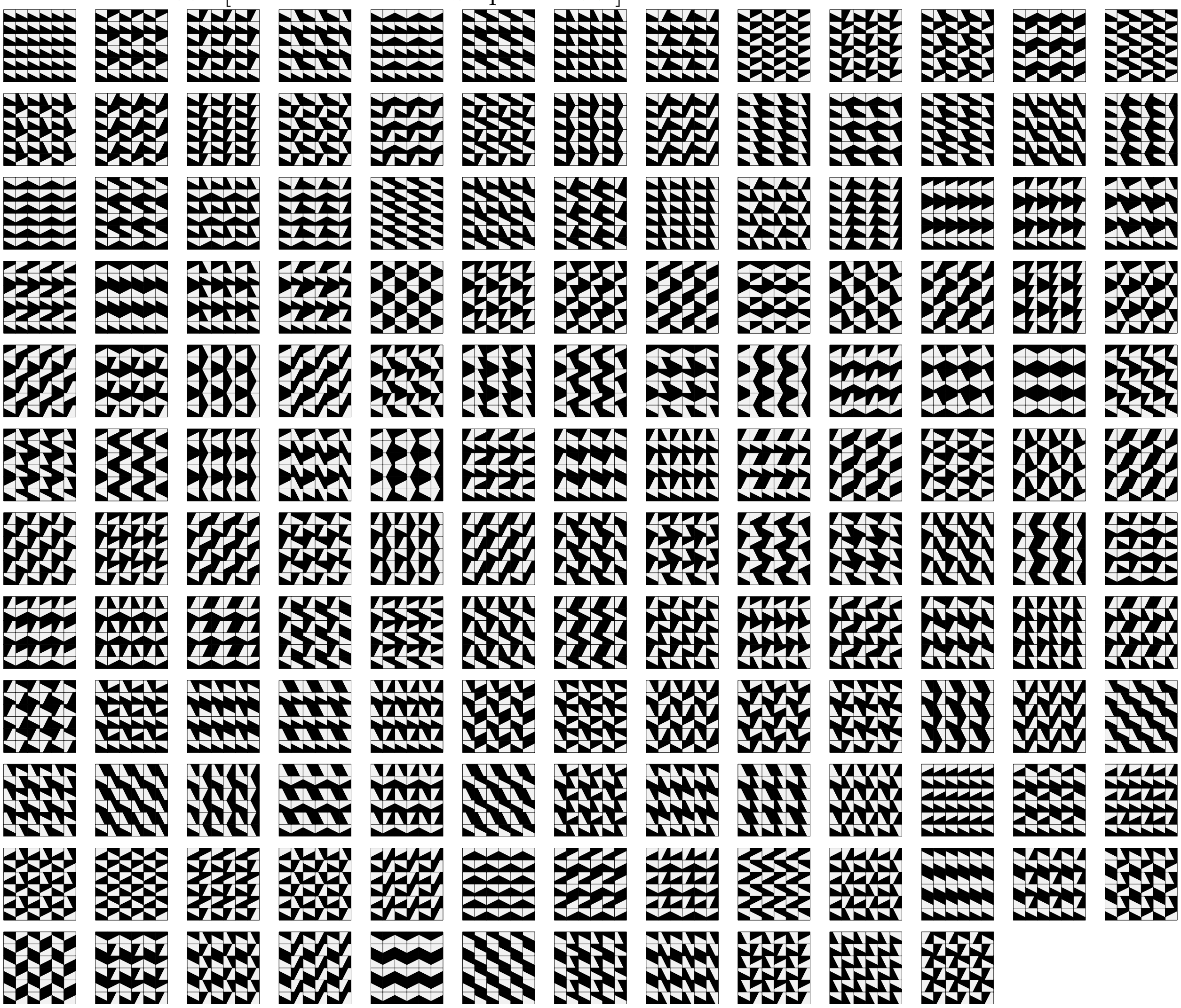


Figure 74: The 154 ways of tiling the 2×2 torus up to $D_8 = \langle r, f \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\mathbb{1} \leq D_8$.

B.5.2 Under diagonal and antidiagonal reflection

Illustration 160. [This is shown in Sequence 97.]

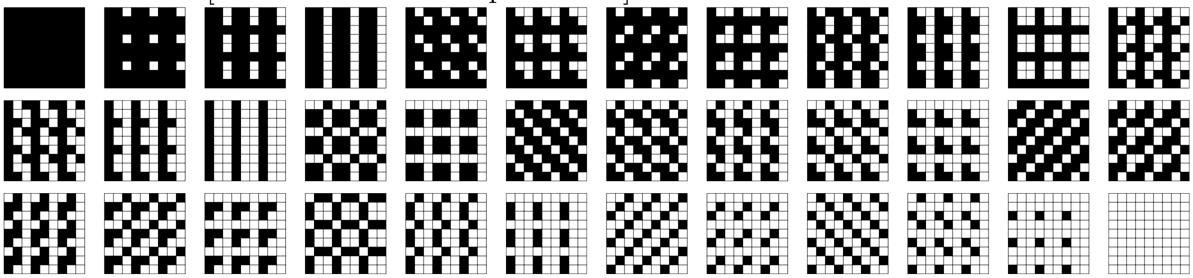


Figure 75: The 36 ways of tiling the 3×3 torus up to $\langle r^2, rf \rangle$ from a set of tile designs that consists of two orbits both of which contain an element with stabilizer subgroup $\langle r^2, rf \rangle$.

Illustration 161. [This is shown in Sequence 98.]

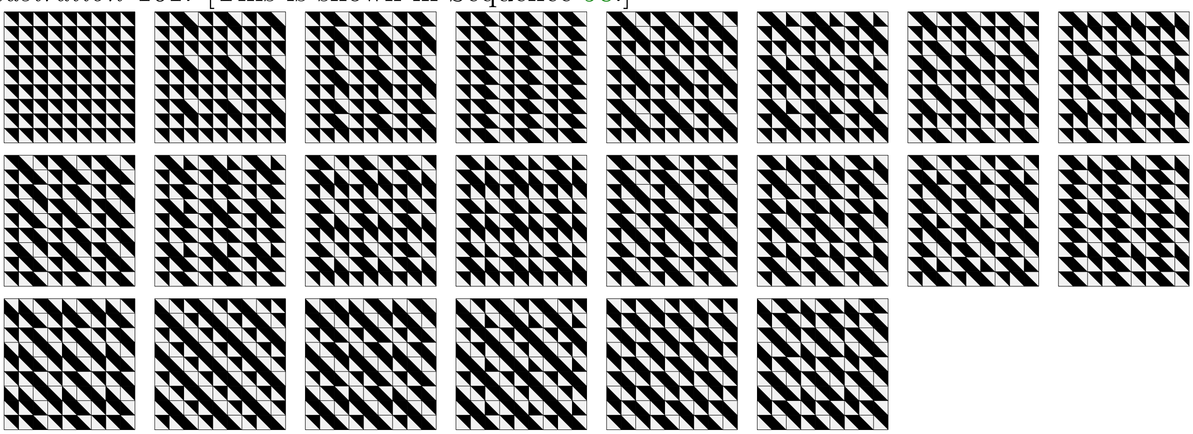


Figure 76: The 22 ways of tiling the 3×3 torus up to $\langle r^2, rf \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\langle rf \rangle \leq \langle r^2, rf \rangle$.

Illustration 162. [This is shown in Sequence 99.]

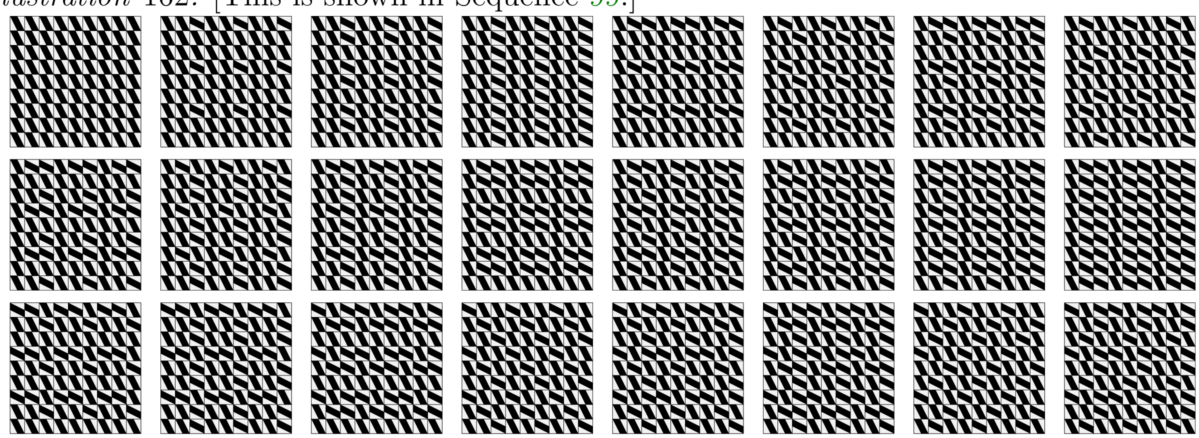


Figure 77: The 24 ways of tiling the 3×3 torus up to $\langle r^2, rf \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\langle r^2 \rangle \leq \langle r^2, rf \rangle$.

Illustration 163. [This is shown in Sequence 100.]

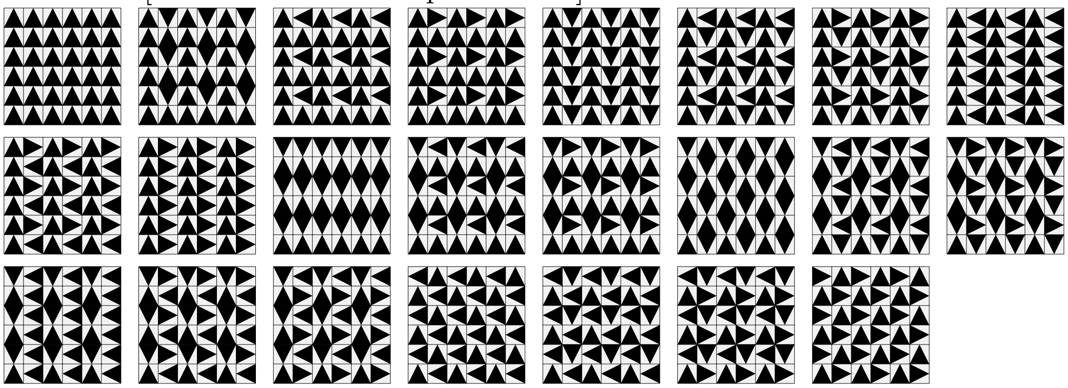


Figure 78: The 23 ways of tiling the 2×2 torus up to $\langle r^2, rf \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\mathbb{1} \leq \langle r^2, rf \rangle$.

B.5.3 Under 90° rotation

Illustration 164. [This is shown in Sequence 101.]

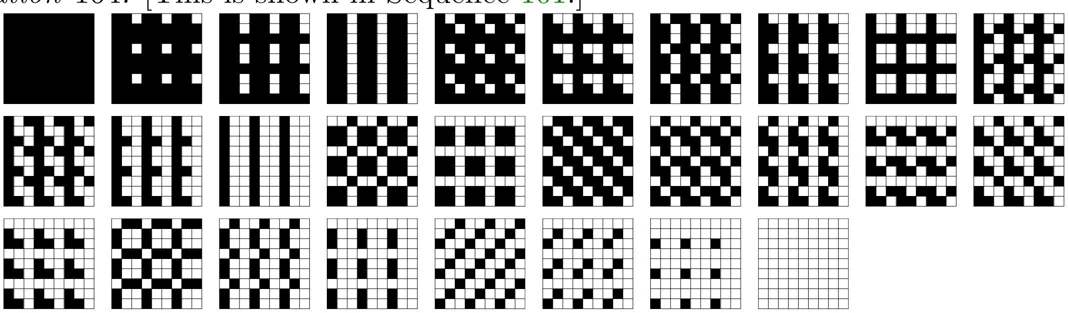


Figure 79: The 28 ways of tiling the 3×3 torus up to $\langle r \rangle$ from a set of tile designs that consists of two orbits both of which contain an element with stabilizer subgroup $\langle r \rangle$.

Illustration 165. [This is shown in Sequence 102.]

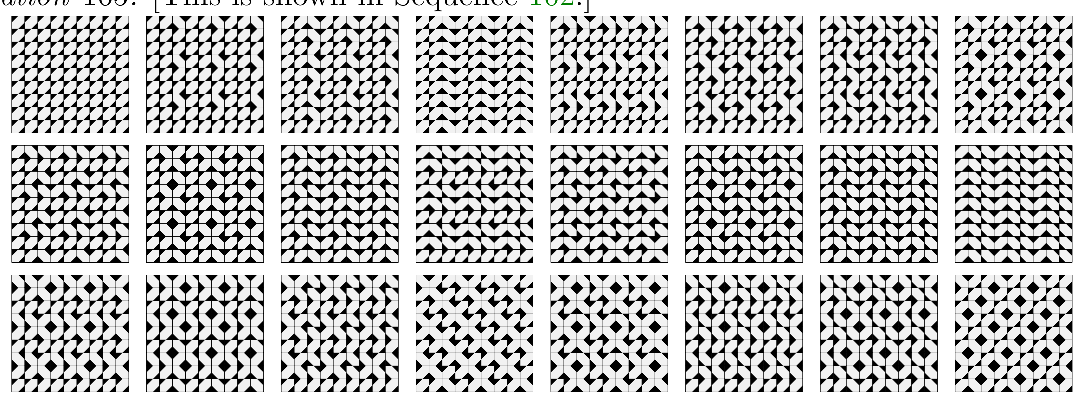


Figure 80: The 24 ways of tiling the 3×3 torus up to $\langle r \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\langle r^2 \rangle \leq \langle r \rangle$.

Illustration 166. [This is shown in Sequence 103.]

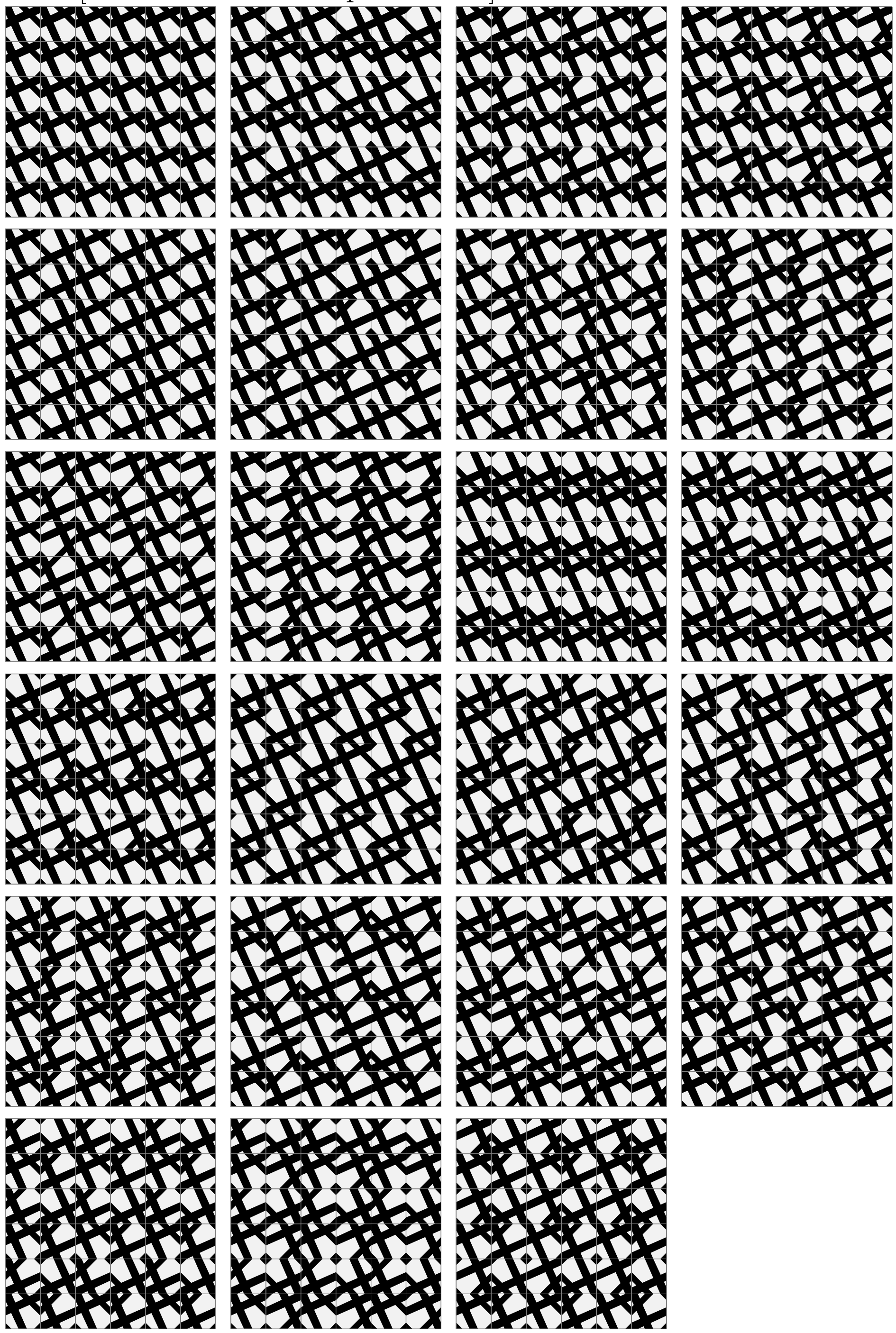


Figure 81: The 23 ways of tiling the 2×2 torus up to $\langle r \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\mathbb{1} \leq \langle r \rangle$. (This was first enumerated by M. C. Escher in May 1942, and the tile designs illustrated here are based on Escher's designs [25, p. 44].)

B.5.4 Under diagonal (equivalently antidiagonal) reflection

Illustration 167. [This is shown in Sequence 104.]

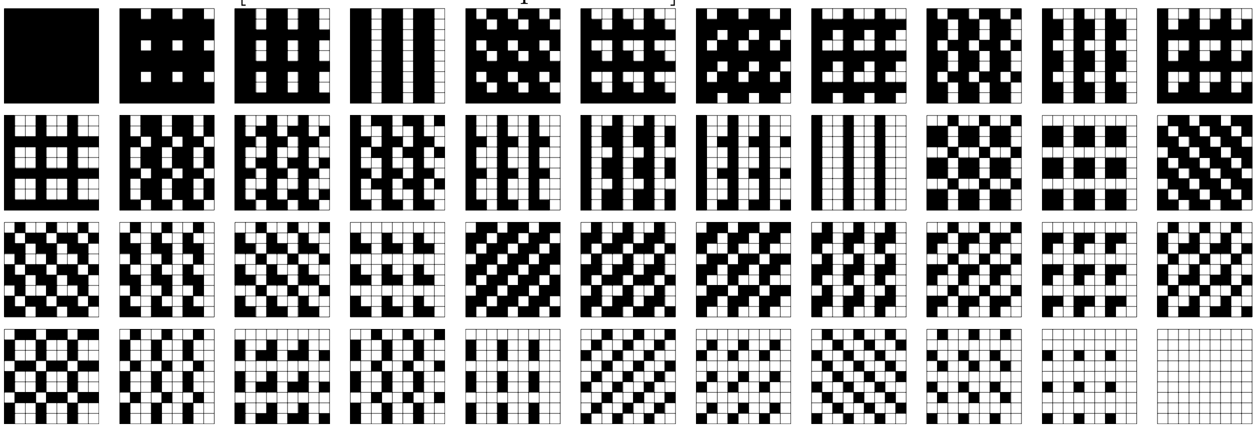


Figure 82: The 44 ways of tiling the 3×3 torus up to $\langle rf \rangle$ from a set of tile designs that consists of two orbits both of which contain an element with stabilizer subgroup $\langle rf \rangle$.

Illustration 168. [This is shown in Sequence 105.]

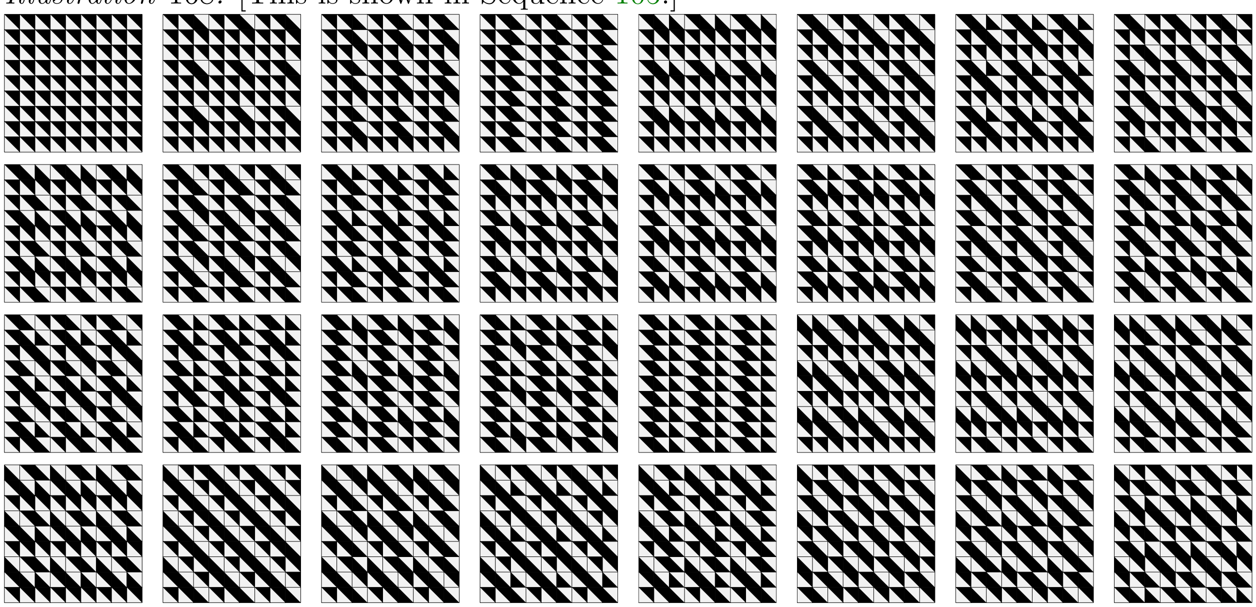


Figure 83: The 32 ways of tiling the 3×3 torus up to $\langle rf \rangle$ from a set of tile designs that consists of one orbit containing an element whose stabilizer subgroup is $\mathbb{1} \leq \langle rf \rangle$.

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