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On the Variation of the Sum of Digits in the Zeckendorf Representation: An Algorithm to Compute the Distribution and Mixing Properties

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Abstract

We study probability measures defined by the variation of the sum of digits in the Zeckendorf representation. For $r \geq 0$ and $d \in \mathbb{Z}$, we consider $\mu^{(r)}(d)$, the density of integers $n \in \mathbb{N}$ for which the sum of digits increases by d when r is added to n. We give a probabilistic interpretation of $\mu^{(r)}$ via the dynamical system provided by the odometer of Zeckendorf-adic integers and its unique invariant measure. We give an algorithm for computing $\mu^{(r)}$ and we prove the exponential decay of $\mu^{(r)}(d)$ as $d \to -\infty$, as well as the formula $\mu^{(F_\ell)} = \mu^{(1)}$ where F_ℓ is a term of the Fibonacci sequence. Finally, we decompose the Zeckendorf representation of an integer r into so-called "blocks" and show that when added to an adic Zeckendorf integer, the successive actions of these blocks can be seen as a sequence of mixing random variables.

1 Introduction

1.1 Framework, notations and main results

Throughout this article, we let $\mathbb{N} := \{0, 1, 2, \ldots\}$ denote the set of *integers* and $\varphi := \frac{1+\sqrt{5}}{2}$ represent the *golden ratio*. We also define the well-known *Fibonacci sequence* as follows:

$$F_k := \begin{cases} 1, & \text{if } k = 1 \text{ or } k = 2; \\ F_{k-1} + F_{k-2}, & \text{if } k \geq 3. \end{cases} \tag{1}$$

By Zeckendorf's theorem (proved by Lekkerkerker [13, Theorem 1] and Zeckendorf [19, Theorem I.a, p. 179]), every integer can be uniquely written as a sum of non-consecutive Fibonacci terms. That is, for every $n \in \mathbb{N}$, there exists a unique sequence of *digits* $(n_k)_{k\geq 2} \in \{0,1\}^\infty$ without two consecutive 1's, finitely many of them being equal to 1 and such that

$$n = \sum_{k\geq 2} n_k F_k. \tag{2}$$

For $n \neq 0$ and $\ell := \max\{k : \; n_k \neq 0\}$, we introduce the notation $[n_\ell \cdots n_2] := n$ that generalizes the usual way we write numbers in an integer base, and that we refer to as the *(Zeckendorf) expansion* of n. This way to expand numbers is actually a particular case of an *Ostrowski numeration system* [15, 2, 1]. By convention, we set $[0] := 0$. Then we define the *(Zeckendorf-)sum-of-digits* function as

$$s(n) := \sum_{k\geq 2} n_k.$$

A central object in our paper is the *variation of the sum of digits* when we add a fixed integer r to n: for $r, n \in \mathbb{N}$, we set

$$\Delta^{(r)}(n) := s(n+r) - s(n). \tag{3}$$

The analogous variation in an integer base has been studied extensively. The first appearance of the case of an integer base is in a paper from Bésineau [3] in 1970. Using a statistical vocabulary, he showed the existence of the asymptotic density for the set of integers such that the variation is some integer $d \in \mathbb{Z}$. Given an integer r, these densities can be seen as a probability law. The variance of this law was studied, in the binary base, by Emme and Prikhod'ko [9] and Spiegelhofer and Wallner [18]. Emme and Hubert [8] proved a central limit theorem in the binary case that was improved by Spiegelhofer and Wallner [18] (still in binary) and by the author in collaboration with Janvresse and de la Rue [11] (in an arbitrary integer base). We also refer to [14, 16, 6, 17] for connected results. Some theorems were proved for the Zeckendorf expansion of an integer: for instance, Griffiths [10] about the digit proportions, and Labbé and Lepšovà [12] about addition in this numeration system. Drmota, Müllner and Spiegelhofer [7] obtained results about the existence of prime numbers with a fixed Zeckendorf sum-of-digits. However, not much has been done about this variation in the Zeckendorf representation except the work from Dekking [5] that characterizes the integers such that $\Delta^{(1)}$ is > 0 (or < 0 or $= 0$) and from Spiegelhofer [16, Lemma 1.30], which adapted Bésineau's techniques and proved that, for all $d \in \mathbb{Z}$, the following asymptotic density exists

$$\mu^{(r)}(d) := \lim_{N\to+\infty} \frac{1}{N} \left|\left\{n < N : \; \Delta^{(r)}(n) = d\right\}\right|.$$

In the present paper, we adapt the ergodic theory point of view introduced for the case of an integer base [11] to the Zeckendorf expansion, recovering Spiegelhofer's result and getting new results about $\mu^{(r)}$ and $\Delta^{(r)}$. In particular, we provide an algorithm that computes $\mu^{(r)}(d)$

and we represent $\Delta^{(r)}$ as the sum of a stochastic mixing process. A perspective we have with this result is to find a central limit theorem for $\Delta^{(r)}$.

To get our results, following the path initiated for the case of an integer base [11], we study the variations of the sum-of-digits function in an appropriate probability space given by the compact set $\mathbb{X}$ of *(Zeckendorf-)adic numbers*. We consider the action of the odometer on $\mathbb{X}$ (see Subsection 2.2), and endow $\mathbb{X}$ with its unique invariant probability measure $\mathbb{P}$.

We extend $\Delta^{(r)}$ almost everywhere on $\mathbb{X}$ and show in Section 4 (Proposition 23) that, for every $d \in \mathbb{Z}$

$$\mu^{(r)}(d) = \mathbb{P}\left(\left\{x \in \mathbb{X} \ : \ \Delta^{(r)}(x) = d\right\}\right).$$

Using the Rokhlin towers of the dynamical system (see Subsection 3.1), we provide an algorithm to compute $\mu^{(r)}(d)$. This algorithm and its consequences can be adapted to an integer base. One implication is the following corollary on the behavior of the (negative) tail of the distribution:

Corollary 1. *For d small enough in $\mathbb{Z}$, we have the formula*

$$\mu^{(r)}(d-1) = \mu^{(r)}(d) \cdot \frac{1}{\varphi^2}.$$

Remark 2. One can show that the analogous result in base $b \geq 2$ is the same replacing the formula by $\mu^{(r)}(d-(b-1)) = \mu^{(r)}(d) \cdot \frac{1}{b}$.

Another implication is the next theorem about the measure $\mu^{(F_\ell)}$.

Theorem 3. *For $\ell \geq 3$*

$$\mu^{(F_\ell)} = \mu^{(1)}. \tag{4}$$

The analogous result in an integer base b replaces F_ℓ by b^ℓ. Actually, it is a trivial result in base b. However, in the Zeckendorf decomposition, this result is much less obvious, due to the particular behavior of carry propagations that we describe in Subsection 2.1. The main difference with additions in base b is, here, that carries propagate in *both* directions.

Now to state our mixing result, we need to define the notion of *blocks* in the expansion of an integer r and to define a probabilistic notion of (α-)*mixing coefficients* (see the survey from Bradley [4] for others).

Definition 4. A *block* in the expansion of an integer $r \in \mathbb{N}$ is defined as a maximal sequence of the pattern [10]. (If $r_2 = 1$, we agree that a maximal sequence $[r_{2\ell} \cdots r_2]$ is a block if $r_{2k} = 1$ for $k = 1, \ldots, \ell$.) We define $\rho(r)$ as the number of blocks in the expansion of r.

$$\begin{array}{ll}
r = [\underbrace{1010}_{B_2}\,0\,\underbrace{10}_{B_1}] & \qquad r = [\underbrace{10}_{B_3}\,00\,\underbrace{10}_{B_2}\,000\,\underbrace{1010101}_{B_1}] \\
\rho(r) = 3 & \qquad \rho(r) = 2 \\
n(B_2) = 5 & \qquad n(B_1) = 2 \\
\ell(B_2) = 2 & \qquad \ell(B_1) = 3
\end{array}$$

Figure 1: Two examples of decomposition into blocks.

Definition 5. Let $(X_i)_{i\geq 1}$ be a (finite or infinite) sequence of random variables. The associated *α-mixing coefficients* $\alpha(k)$, $k \geq 1$, are defined by

$$\alpha(k) := \sup_{p\geq 1} \sup_{A,B} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|,$$

where the second supremum is taken over all events A and B such that

- $A \in \sigma(X_i : 1 \leq i \leq p)$ and
- $B \in \sigma(X_i : i \geq k + p)$.

By convention, if X_i is not defined when $i \geq k + p$ then the σ-algebra is trivial.

For an integer r, we enumerate the blocks of r from the units position and we define $X_i^{(r)}$ as the action of the i^{th} block once the previous blocks have already been taken into consideration (see Subsection 7.1 for more details). These actions are constructed in order to have the equality

$$\Delta^{(r)} = \sum_{i=1}^{\rho(r)} X_i^{(r)}. \tag{5}$$

We state the following theorem that gives an upper bound on the α-mixing coefficients that is independent of r.

Theorem 6. *The α-mixing coefficients of $(X_i^{(r)})_{i=1,\ldots,\rho(r)}$ satisfy*

$$\forall k \geq 1, \quad \alpha(k) \leq 12 \left(1 - \frac{1}{\varphi^8}\right)^{\frac{k}{6}} + \frac{1}{\varphi^{2k}}.$$

The author et al. [11] proved a central limit theorem for $\mu^{(r)}$ in the case of an integer base using a similar result together with some estimate of the variance depending on the number of blocks. Such an estimate is, here, more difficult than in base b since we do not have inductive relations on $\mu^{(r)}$.

1.2 Roadmap

Section 2 is devoted to understanding the effect of adding "1" and "F_k" on the digits of Zeckendorf-adic integers. We highlight stopping patterns in the carry propagation, which play an important role in the analysis.

In Section 3, we focus on the unique ergodic measure $\mathbb{P}$ of the odometer. We show that $\mathbb{P}$ satisfies some renewal properties (Proposition 16) and we estimate the ϕ-mixing coefficients for the coordinates of a Zeckendorf-adic integer (Proposition 19).

Then in Section 4, we place the study of the measures $\mu^{(r)}$ in the context of the odometer on $\mathbb{X}$. We extend $\Delta^{(r)}$ almost everywhere on $\mathbb{X}$ and we show that the convergence

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n<N} f(\Delta^{(r)}(n)) = \int_{\mathbb{X}} f(\Delta^{(r)}(x))\mathrm{d}\mathbb{P}(x)$$

is satisfied for functions $f : \mathbb{Z} \to \mathbb{C}$ of polynomial growth (Proposition 23) and, more generally, for functions f such that $f \circ \Delta^{(r)}$ is integrable (Proposition 28). We deduce from Proposition 23 that $\mu^{(r)}(d) = \mathbb{P}(\{x \in \mathbb{X} : \Delta^{(r)}(x) = d\})$ and that $\mu^{(r)}$ has finite moments. In particular, we show that $\mu^{(r)}$ is of zero-mean.

In Section 5, we construct an algorithm that computes $\mu^{(r)}$. Pseudocode is given in Subsection 5.2. In Subsection 5.4, we prove Corollary 1 that gives an estimate on the tail of $\mu^{(r)}$.

Section 6 is devoted to the proof of Theorem 3. The proof consists of applying the algorithm.

In the last section, we build a finite sequence of random variables associated with addition of an integer r that gives the decomposition of $\Delta^{(r)}$ mentioned in (5). Using that sequence, we prove Theorem 6 on the estimation of the α-mixing coefficients for this sequence.

2 How to do additions

2.1 How to add integers

Here, we describe the algorithm for addition using the Zeckendorf way to represent numbers. We start with the addition of 1. There are two cases.

1. Either there exists $\ell \geq 0$ such that the Zeckendorf decomposition of n is

$$n = \sum_{k=1}^{\ell} F_{2k+1} + \sum_{k \geq 2\ell+4} n_k F_k.$$

Then with the relations (1), we get

$$\begin{array}{ccccccc} (n) & & \cdots & n_{2\ell+4} & 0 & 0 & (10)^{\ell} \\ & + & & & & & 1 \\ \hline (n+1) & = & \cdots & n_{2\ell+4} & 0 & 1 & (00)^{\ell} \end{array}$$

Indeed, $F_3 + \cdots + F_{2\ell+1} + F_2 = F_{2\ell+2}$.

2. Or there exists $\ell \geq 0$ such that

$$n = \sum_{k=1}^{\ell+1} F_{2k} + \sum_{k \geq 2\ell+5} n_k F_k.$$

Then for the same reasons

$$\begin{array}{rccccccc}
(n) & & \cdots & n_{2\ell+5} & 0 & 0 & 1 & (01)^\ell \\
 & + & & & & & & 1 \\
\hline
(n+1) & = & \cdots & n_{2\ell+5} & 0 & 1 & 0 & (00)^\ell
\end{array}$$

We observe that adding 1 to the rightmost digit of a block as defined in (4) modifies that block into a chain of 0 digits of the same length and put a 1 in the first left position of the block.

Of course, in order to compute the sum of two integers, adding 1 as many times as needed is enough. However, we want to show the main difference with addition in an integer base. For instance, in an integer base, adding 1 at some position $k \geq 2$ to an integer n may change the digits of the expansion of n of higher indices, due to carry propagation. Here it can also change the digits of lower indices. We consider the addition $n + F_k$ where $k \geq 3$. We observe that a consequence of (1) is

$$2F_k = \begin{cases} F_2 + F_4, & \text{if } k = 3; \\ F_{k+1} + F_{k-2}, & \text{otherwise.} \end{cases} \tag{6}$$

Many cases appear. For simplicity, the digit in **color** represents the digit at position k and we do not represent digits that remain the same in the expansion of n and $n + F_k$. We start with the cases that change only digits of indices $\geq k$.

1. If $n = [\cdots 0\mathbf{0}0 \cdots]$ then

$$\begin{array}{rcccccc}
(n) & & \cdots & 0 & \mathbf{0} & 0 & \cdots \\
(+F_k) & + & & & \mathbf{1} & & \\
\hline
(n+F_k) & = & \cdots & 0 & \mathbf{1} & 0 & \cdots
\end{array}$$

2. If there exists $\ell \geq 0$ such that $n = \left[\cdots 001(01)^\ell \mathbf{0} \cdots\right]$ then

$$\begin{array}{rcccccccc}
(n) & & \cdots & 0 & 0 & 1 & (01)^\ell & \mathbf{0} & \cdots \\
(+F_k) & + & & & & & & \mathbf{1} & \\
\hline
(n+F_k) & = & \cdots & 0 & 1 & 0 & (00)^\ell & \mathbf{0} & \cdots
\end{array}$$

We continue with the first case where the digit of index $k-1$ is changed.

3. If there exists $\ell \geq 0$ such that $n = \left[\cdots 00(10)^\ell \mathbf{0}1 \cdots\right]$ then

$$\begin{array}{rcccccccc}
(n) & & \cdots & 0 & 0 & (10)^\ell & \mathbf{0} & 1 & \cdots \\
(+F_k) & + & & & & & \mathbf{1} & & \\
\hline
(n+F_k) & = & \cdots & 0 & 1 & (00)^\ell & \mathbf{0} & 0 & \cdots
\end{array}$$

Now we consider the cases where many digits of indices $< k$ are changed. It is due to (6).

4. If $k \geq 5$ and there exist $\ell, \ell' \geq 0$ such that $n = [\cdots 00(10)^\ell \mathbf{1}(01)^{\ell'}000\cdots]$ then

$$\begin{array}{ccccccccccc} (n) & & \cdots & 0 & 0 & (10)^\ell & \mathbf{1} & (01)^{\ell'} & 0 & 0 & 0 & \cdots \\ (+F_k) & + & & & & & \mathbf{1} & & & & & \\ \hline (n+F_k) & = & \cdots & 0 & 1 & (00)^\ell & \mathbf{0} & (10)^{\ell'} & 0 & 1 & 0 & \cdots \end{array}$$

5. If $k \geq 6$ and there exist $\ell, \ell' \geq 0$ such that $n = [\cdots 00(10)^\ell \mathbf{1}(01)^{\ell'}0010\cdots]$ then

$$\begin{array}{cccccccccccc} (n) & & \cdots & 0 & 0 & (10)^\ell & \mathbf{1} & (01)^{\ell'} & 0 & 0 & 1 & 0 & \cdots \\ (+F_k) & + & & & & & \mathbf{1} & & & & & & \\ \hline (n+F_k) & = & \cdots & 0 & 1 & (00)^\ell & \mathbf{0} & (10)^{\ell'} & 1 & 0 & 0 & 0 & \cdots \end{array}$$

Finally, we consider the "boundary" cases where all the digits of indices $< k$ are changed.

6. If $k \geq 4$, k is even and there exists $\ell \geq 0$ such that $n = [\cdots 00(10)^\ell \mathbf{1}(01)^{\frac{k-2}{2}}]$ then

$$\begin{array}{cccccccc} (n) & & \cdots & 0 & 0 & (10)^\ell & \mathbf{1} & (01)^{\frac{k-2}{2}} \\ (+F_k) & + & & & & & \mathbf{1} & \\ \hline (n+F_k) & = & \cdots & 0 & 1 & (00)^\ell & \mathbf{0} & (10)^{\frac{k-2}{2}} \end{array}$$

7. If $k \geq 5$, k is odd and there exists $\ell \geq 0$ such that $n = [\cdots 00(10)^\ell \mathbf{1}(01)^{\frac{k-3}{2}}0]$ then

$$\begin{array}{ccccccccc} (n) & & \cdots & 0 & 0 & (10)^\ell & \mathbf{1} & (01)^{\frac{k-3}{2}} & 0 \\ (+F_k) & + & & & & & \mathbf{1} & & \\ \hline (n+F_k) & = & \cdots & 0 & 1 & (00)^\ell & \mathbf{0} & (10)^{\frac{k-3}{2}} & 1 \end{array}$$

2.2 The Zeckendorf-adic integers and how to add one of them to an integer

We define the *Zeckendorf-adic integers* (or *Z-adic integers* for simplicity) as elements of

$$\mathbb{X} := \left\{ x \in \{0;1\}^{\mathbb{N}_{\geq 2}} : \forall k \geq 2 \ x_k x_{k+1} = 0 \right\}.$$

Also, coordinates of a Z-adic integer $x \in \mathbb{X}$ are interpreted as digits in the Zeckendorf representation: elements of $\mathbb{X}$ can be viewed as "generalized integers having possibly infinitely many nonzero digits in their Zeckendorf representation". An element $x = (x_k)_{k \geq 2} \in \mathbb{X}$ is represented as a left-infinite sequence $(\ldots, x_3, x_2)$, with x_2 being the unit digit. We endow $\mathbb{X}$ with the product topology that turns it into a compact metrizable space. The set $\mathbb{N}$ can be identified with the subset of sequences with finite support. More precisely, using the inclusion function

$$i : n = [n_\ell \cdots n_2] \in \mathbb{N} \longmapsto (\dots, 0, n_\ell, \dots, n_2) \in \mathbb{X}$$

we identify $\mathbb{N}$ and $i(\mathbb{N})$. We can also identify the notation

$$(\dots, x_3, x_2) = [\cdots x_3 x_2].$$

We take the opportunity to define $\mathbb{X}_f$ as the set of finite sequences of 0's and 1's without two consecutive 1's. For instance, the Zeckendorf expansion of a given integer is composed using a sequence in $\mathbb{X}_f$. Let us define, for $\ell \geq 2$ and $(n_k)_{k\geq 2} \in \mathbb{X}_f$, the *cylinder* $C_{n_\ell \cdots n_2}$ as the set of sequences $x \in \mathbb{X}$ such that $x_i = n_i$ for $i = 2, \dots, \ell$. We observe that $n_\ell \cdots n_2$ is not necessarily the Zeckendorf expansion of a given integer: the leftmost digit(s) can be 0('s).

We want to extend the transformation $n \mapsto n+1$, defined on $\mathbb{N}$, on $\mathbb{X}$. From the description given in Subsection 2.1, it is convenient to consider the transformation T defined on $\mathbb{X}$ by the following formula, where $\ell \in \mathbb{N}$

$$T(x) := \begin{cases} [\cdots x_{2\ell+4} 00(10)^\ell], & \text{if } x = [\cdots x_{2\ell+4} 01(00)^\ell]; \\ [\cdots x_{2\ell'+3} 001(01)^\ell], & \text{if } x = [\cdots x_{2\ell'+3} 010(00)^\ell]; \\ [(10)^\infty] & \text{if } x = [0^\infty]; \\ [(01)^\infty] & \text{if } x = [0^\infty]. \end{cases}$$

Indeed, by Subsection 2.1 we observe that $T_{|\mathbb{N}}(n) = n+1$. Now if we take $x \in \mathbb{X}$ whose expansion contains two consecutive 0's, we observe that the sequence $([x_\ell \cdots x_2] + 1)_{\ell \geq 2}$ converges to $T(x)$ because the digits are not changed eventually so we can define $x+1 := T(x)$ in that case. Otherwise, if $x \in \mathbb{X}$ does not have two consecutive 0's in its expansion, there are two cases: $[(10)^\infty]$ and $[(01)^\infty]$. Adding 1 to the truncated sequence $([(10)^\ell])_{\ell \in \mathbb{N}}$ converges to $[(10)^\infty] + 1 := [0^\infty]$. It is the same for $[(01)^\infty]$. Thus, the transformation T can be described in a simpler way as

$$T : \begin{cases} \mathbb{X} & \longrightarrow \mathbb{X} \\ x & \longmapsto x+1. \end{cases}$$

Due to the two pre-images of 0^∞, the transformation T is not a homeomorphism on $\mathbb{X}$, but remains continuous and surjective on $\mathbb{X}$. Thus, $(\mathbb{X}, T)$ is a topological dynamical system that we call the *odometer*.

We know how to add 1 to a Z-adic integer x. Repeating this operation enables us to add an integer r to x. For later purposes, we need to specify how to add F_k (≥ 3) directly. In Subsection 2.1, we have to compute $x + F_k$ for the $x \in \mathbb{X}$ with two consecutive 0's at indices $> k$. Thus, we only need to focus on what happens if x does not have two consecutive 0's at indices $> k$. There are only finitely many cases to consider, which we detail below. Again for simplicity, we write the digits at position k in **color**, and we do not represent digits that are not modified. We start with the cases where the only digits that change are those of indices $\geq k-1$.

1. If $x = [(10)^\infty \mathbf{0} 1 \cdots]$ then

$$\begin{array}{ccccc} (x) & & (10)^\infty & \mathbf{0} & 1 & \cdots \\ (F_k) & + & & \mathbf{1} & & \\ \hline (x+F_k) & = & (00)^\infty & \mathbf{0} & 0 & \cdots \end{array}$$

2. If $x = [(01)^\infty \mathbf{0} \cdots]$ then

$$\begin{array}{ccccc} (x) & & (01)^\infty & \mathbf{0} & \cdots \\ (F_k) & + & & \mathbf{1} & \\ \hline (x+F_k) & = & (00)^\infty & \mathbf{0} & \cdots \end{array}$$

Now we consider cases where some digits of small indices are changed, but not all them.

3. If $k \geq 5$ and there exists $\ell' \geq 0$ such that $x = [(10)^\infty \mathbf{1} (01)^{\ell'} 000 \cdots]$ then

$$\begin{array}{ccccccccc} (x) & & (10)^\infty & \mathbf{1} & (01)^{\ell'} & 0 & 0 & 0 & \cdots \\ (F_k) & + & & \mathbf{1} & & & & & \\ \hline (x+F_k) & = & (00)^\infty & \mathbf{0} & (10)^{\ell'} & 0 & 1 & 0 & \cdots \end{array}$$

4. If $k \geq 6$ and there exists $\ell' \geq 0$ such that $x = [(10)^\infty \mathbf{1} (01)^{\ell'} 0010 \cdots]$ then

$$\begin{array}{cccccccccc} (x) & & (10)^\infty & \mathbf{1} & (01)^{\ell'} & 0 & 0 & 1 & 0 & \cdots \\ (F_k) & + & & \mathbf{1} & & & & & & \\ \hline (x+F_k) & = & (00)^\infty & \mathbf{0} & (10)^{\ell'} & 1 & 0 & 0 & 0 & \cdots \end{array}$$

Finally, we consider cases where the whole prefix of x is modified.

5. If $k \geq 4$ and is even and $x = [(10)^\infty \mathbf{1} (01)^{\frac{k-2}{2}}]$ then

$$\begin{array}{ccccc} (x) & & (10)^\infty & \mathbf{1} & (01)^{\frac{k-2}{2}} \\ (F_k) & + & & \mathbf{1} & \\ \hline (x+F_k) & = & (00)^\infty & \mathbf{0} & (10)^{\frac{k-2}{2}} \end{array}$$

6. If $k \geq 3$ and is odd and $x = [(10)^\infty \mathbf{1} (01)^{\frac{k-3}{2}} 0]$ then

$$\begin{array}{cccccc} (x) & & (10)^\infty & \mathbf{1} & (01)^{\frac{k-3}{2}} & 0 \\ (F_k) & + & & \mathbf{1} & & \\ \hline (x+F_k) & = & (00)^\infty & \mathbf{0} & (10)^{\frac{k-3}{2}} & 1 \end{array}$$

We can sum up all these cases in the following proposition.

Proposition 7. *The table below summarizes the action of the addition of F_k ($k \geq 2$) on the digits of $x \in \mathbb{X}$. Here, $\ell, \ell' \in \mathbb{N}$ and we represent in* ***color****, the digits at position k. The digits that are not explicitly written remain untouched by the operation.*

$$T^{F_k} : \mathbb{X} \longrightarrow \mathbb{X}$$

$$[\cdots 0\mathbf{0}0 \cdots] \longmapsto [\cdots 0\mathbf{1}0 \cdots] \tag{7}$$

$$[\cdots 001(01)^{\ell}\mathbf{0} \cdots] \longmapsto [\cdots 010(00)^{\ell}\mathbf{0} \cdots] \tag{8}$$

$$[\cdots 00(10)^{\ell}\mathbf{0}1 \cdots] \longmapsto [\cdots 01(00)^{\ell}\mathbf{0}0 \cdots] \tag{9}$$

$$[\cdots 00(10)^{\ell}\mathbf{1}(01)^{\ell'}000 \cdots] \longmapsto [\cdots 01(00)^{\ell}\mathbf{0}(10)^{\ell'}010 \cdots] \tag{10}$$

$$[\cdots 00(10)^{\ell}\mathbf{1}(01)^{\ell'}0010 \cdots] \longmapsto [\cdots 01(00)^{\ell}\mathbf{0}(10)^{\ell'}1000 \cdots] \tag{11}$$

$$[\cdots 00(10)^{\ell}\mathbf{1}(01)^{\frac{k-5}{2}}001] \longmapsto [\cdots 01(00)^{\ell}\mathbf{0}(10)^{\frac{k-5}{2}}100] \tag{12}$$

$$[\cdots 00(10)^{\ell}\mathbf{1}(01)^{\frac{k-4}{2}}00] \longmapsto [\cdots 01(00)^{\ell}\mathbf{0}(10)^{\frac{k-4}{2}}01] \tag{13}$$

$$[\cdots 00(10)^{\ell}\mathbf{1}(01)^{\frac{k-2}{2}}] \longmapsto [\cdots 01(00)^{\ell}\mathbf{0}(10)^{\frac{k-2}{2}}] \tag{14}$$

$$[\cdots 00(10)^{\ell}\mathbf{1}(01)^{\frac{k-3}{2}}0] \longmapsto [\cdots 01(00)^{\ell}\mathbf{0}(10)^{\frac{k-3}{2}}1] \tag{15}$$

$$[(01)^{\infty}\mathbf{0} \cdots] \longmapsto [0^{\infty}\mathbf{0} \cdots] \tag{16}$$

$$[(10)^{\infty}\mathbf{0}1 \cdots] \longmapsto [0^{\infty}\mathbf{0}0 \cdots] \tag{17}$$

$$[(10)^{\infty}\mathbf{1}(01)^{\ell'}000 \cdots] \longmapsto [0^{\infty}\mathbf{0}(10)^{\ell'}010 \cdots] \tag{18}$$

$$[(10)^{\infty}\mathbf{1}(01)^{\ell'}0010 \cdots] \longmapsto [0^{\infty}\mathbf{0}(10)^{\ell'}1000 \cdots] \tag{19}$$

$$[(10)^{\infty}\mathbf{1}(01)^{\frac{k-5}{2}}001] \longmapsto [0^{\infty}\mathbf{0}(10)^{\frac{k-5}{2}}100] \tag{20}$$

$$[(10)^{\infty}\mathbf{1}(01)^{\frac{k-4}{2}}00] \longmapsto [0^{\infty}\mathbf{0}(10)^{\frac{k-4}{2}}01] \tag{21}$$

$$[(10)^{\infty}\mathbf{1}(01)^{\frac{k-2}{2}}] \longmapsto [0^{\infty}\mathbf{0}(10)^{\frac{k-2}{2}}] \tag{22}$$

$$[(10)^{\infty}\mathbf{1}(01)^{\frac{k-3}{2}}0] \longmapsto [0^{\infty}\mathbf{0}(10)^{\frac{k-3}{2}}1]. \tag{23}$$

(We specify that k is even in cases (13), (14), (21) *and* (22)*, whereas k is odd in cases* (12), (15), (20) *and* (23)*. Also, in some cases, k is assumed to be large enough for the operation to be possible.)*

2.3 Stopping conditions when adding an integer to an adic number

Through the cases described in Proposition 7, we observe that if there is a 1 at position k in the expansion of x, the addition of F_k yields a carry propagation in both directions:

- to the left, modifying digits of higher indices (as in an integer base),
- to the right, modifying digits of lower indices.

The propagation (in both directions) happens through a maximal sequence of alternative 0's and 1's and is stopped at the first occurrence of two consecutive 0's. But the modifications depend on the propagation direction.

- In propagation to the left, the maximal subword of alternative 1's and 0's is transformed into a subword of 0's of the same length (case (10) for instance), and the stopping pattern 00 is transformed into 01. Note that this propagation also happens if $x_k = 0$ (case (8) for instance).

- In propagation to the right, which only happens if $x_k = 1$, the maximal subword of alternative 1's and 0's is transformed into a symmetrical subword where the 1's become 0's and vice-versa (cases (10) and (11) for instance). Then the first occurrence of 00 (in the sense the largest index $\leq k$ such that the digits of x are 00) can either be part of the pattern $w_0 := [01000]$ or $w_1 := [10010]$. (We call w_0 and w_1 the *right-stopping pattern.*)

Depending on the right-stopping pattern ([01000] or [10010]), the modifications of digits at these indices are given by the next scheme.

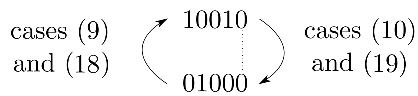

Figure 2: Modifications at the position of the right-stopping pattern.

Note that in both cases, we get a new right-stopping pattern in the same position as before the addition of F_k. This addition of F_k to $x \in \mathbb{X}$ modifies some digits at positions $\leq k-2$ only if $x_k = 1$, and, in this case, the modification of the digits takes place up to the first occurrence of one of the right-blocking patterns.

Formally: let $x \in \mathbb{X}$ and $k \geq 2$.

- If $x_k = 0$ then $(x + F_k)_n = x_n$ for all $n \leq k-2$.

- If $x_k = 1$ and if there exists $j \leq k+1$ such that $x_j x_{j-1} \cdots x_{j-4}$ is a right-stopping patterns w_i ($i \in \{0,1\}$)—we let j' denote the largest index with this property—then
  - $(x + F_k)_n = x_n$ for all $n \leq j' - 4$,
  - w_{1-i} appears in $(x + F_k)$ in the same position j', unless $w_i = w_0$ and $j' = k+1$, in which case we might have 00010 instead of w_1 at position j' in $x + F_k$ (case (10) with $\ell = \ell' = 0$).

A straightforward consequence is the following result.

Proposition 8. *Assume that the right-stopping pattern w_i ($i = 0,1$) appears in $x \in \mathbb{X}$ at position $j \geq 5$, that is, assume $x_j x_{j-1} \cdots x_{j-4} = w_i$. Let $k \geq j-1$. Then*

- *for all $n \leq j-4$, $(x + F_k)_n = x_n$, and*

- *w_0, w_1 or* 00010 *appears in $x + F_k$ at some position j' with $k+1 \geq j' \geq j$.*

We now state the following corollary that enhances that property when we add not only a Fibonacci term but an integer whose expansion involves Fibonacci numbers of high indices.

Corollary 9. *Let $x \in \mathbb{X}$. Assume that, for some $\ell \geq 2$, $x_\ell = x_{\ell+1} = 0$. Let $r \in \mathbb{N}$ be such that $r_j = 0$ for $j = 2, \dots, \ell+1$. Then for each $n \leq \ell - 2$, we have $(x+r)_n = x_n$.*

Proof. We are considering the following addition:

$$\begin{array}{ccccccccccc}
(x) & & \cdots & x_{\ell+3} & x_{\ell+2} & 0 & 0 & x_{\ell-1} & x_{\ell-2} & x_{\ell-3} & \cdots \\
(r) & + & \cdots & r_{\ell+3} & r_{\ell+2} & 0 & 0 & 0 & 0 & 0 & \cdots \\
\hline
(x+r) & = & \cdots & \star & \star & \star & \star & \star & x_{\ell-2} & x_{\ell-3} & \cdots
\end{array}$$

Let $r = F_{k_{s(r)}} + \cdots + F_{k_1}$ be the Zeckendorf decomposition of r with $k_{s(r)} > \cdots > k_2 > k_1 \geq \ell + 2$. We first consider the addition of F_{k_1} to x:

- if, for all j such that $k_1 \geq j \geq \ell$, we have $x_j = 0$, then for every $n \leq \ell + 1$ we have $(x + F_{k_1})_n = x_n$ (in particular 00 appears in the same place in $x + F_{k_1}$);
- otherwise, there exists a largest integer j' with $k_1 + 1 \geq j' \geq \ell + 2$, such that one of the right-stopping pattern appears in x at position j'. Then we can apply the above proposition, which proves that
  - either w_0 or w_1 appears at position j' in $x + F_{k_1}$,
  - or 00 appears at position j' in $x + F_{k_1}$ and $j' \geq \ell + 3$.

  In each case, we still have $(x + F_{k_1})_n = x_n$ for $n \leq \ell - 2$.

Then we prove by induction on t such that for each t, $1 \leq t \leq s(r)$, the above is true for $x + F_{k_1} + \cdots + F_{k_t}$. □

The following lemma ensures that the pattern 00 is a left-stopping condition: it stops the propagation of a carry coming from the right.

Lemma 10. *Let $x \in \mathbb{X}$, $r \in \mathbb{N}$. Let $\ell \geq 2$ be such that $r < F_{\ell+1}$ and assume $x_{\ell+2} = x_{\ell+3} = 0$. Then we have $(x+r)_k = x_k$ for all $k \geq \ell + 3$.*

Proof. The assumption $r < F_{\ell+1}$ implies $r = [r_\ell \cdots r_2]$ with $(r_\ell, \cdots, r_2) \in \mathbb{X}_f$. Since $[x_{\ell+1} \cdots x_2] + r < F_{\ell+2} + F_{\ell+1} = F_{\ell+3}$, a carry cannot propagate on digits of indices $\geq \ell + 3$: we have the addition

$$\begin{array}{cccccccccc}
(x) & & \cdots & x_{\ell+4} & 0 & 0 & x_{\ell+1} & x_\ell & \cdots & x_2 \\
(r) & + & & & & & & r_\ell & \cdots & r_2 \\
\hline
(x+r) & = & \cdots & x_{\ell+4} & 0 & (x+r)_{\ell+2} & (x+r)_{\ell+1} & (x+r)_\ell & \cdots & (x+r)_2
\end{array}$$

□

The next lemma enhances the previous one: it shows that, given x with some restrictions, right-stopping patterns can appear in the expansion $x + F_k$ when k is a small integer.

Lemma 11. *Let $x \in \mathbb{X}$ and $r \in \mathbb{N}$. Let $\ell \geq 2$ be such that $x_{\ell+1} = x_{\ell+2} = x_{\ell+3} = x_{\ell+4} = 0$ and $r_{\ell+2} = 1$. Assume $r < F_{\ell+3}$. Then for $k = 1, 2, 3, 4$ we have*

$$(x + r)_{\ell+k} = r_{\ell+k},$$

or

$$(x + r)_{\ell+1} = 0 \text{ and } (x + r)_{\ell+3} = 1.$$

Proof. We are actually considering the following addition:

$$\begin{array}{cccccccccc} (x) & & \cdots & x_{\ell+5} & 0 & 0 & 0 & 0 & x_\ell & \cdots & x_2 \\ (r) & + & \cdots & & & & 1 & 0 & r_\ell & \cdots & r_2 \end{array}$$

We want to show that the expansion of $x + r$ is either

$$[\cdots x_{\ell+5}0010(x + r)_\ell \cdots (x + r)_2]\,, \tag{C1}$$

or

$$[\cdots x_{\ell+5}0100(x + r)_\ell \cdots (x + r)_2]\,. \tag{C2}$$

Thanks to Lemma 10, we have $(x+r)_k = x_k$ for every $k \geq \ell+4$, which means that the digits of $x + r$ of indices $\leq \ell + 3$ are given by the addition $r + [x_\ell \cdots x_2]$. Now we claim

$$(x + r)_{\ell+3} \text{ or } (x + r)_{\ell+2} \text{ is } 1.$$

Indeed, if it is not the case then $[x_\ell \cdots x_2] + r < F_{\ell+2}$ while $r \geq F_{\ell+2}$ (since $r_{\ell+2} = 1$). We consider now both possibilities.

- If $(x + r)_{\ell+2} = 1$ then we obtain (C1).
- If $(x + r)_{\ell+3} = 1$ then we must have $(x + r)_{\ell+1} = 0$ since we have

$$\begin{aligned} [x_\ell \cdots x_2] + r - F_{\ell+3} &= [x_\ell \cdots x_2] + [r_\ell \cdots r_2] + F_{\ell+2} - F_{\ell+3} \\ &< F_{\ell+1}. \end{aligned}$$

  Thus, the expansion of $x + r$ is (C2).

□

We deduce the next corollary.

Corollary 12. *Let $x \in \mathbb{X}$ and $r \in \mathbb{N}$ such that it exists $\ell \geq 2$ with $x_\ell = \cdots = x_{\ell+5} = 0$, $r_{\ell+1} = r_{\ell+5} = 0$ and $r_{\ell+3} = 1$. Let $\widetilde{r} := [r_\ell \cdots r_2]$. Then for all $k \geq \ell + 4$ we have*

$$(x + \widetilde{r} + F_{\ell+3})_k = (x + \widetilde{r})_k = x_k, \tag{24}$$

and, for all $k \leq \ell + 2$

$$(x + r)_k = (x + \widetilde{r} + F_{\ell+3})_k = (x + \widetilde{r})_k. \tag{25}$$

Remark 13. The hypothesis means that we are considering the following addition:

$$\begin{array}{ccccccccccccc} (x) & & \cdots & x_{\ell+6} & 0 & 0 & 0 & 0 & 0 & 0 & x_{\ell-1} & \cdots & x_2 \\ (r) & + & \cdots & r_{\ell+6} & 0 & 0 & 1 & 0 & 0 & r_\ell & \cdots & \cdots & r_2 \end{array}$$

and the conclusion ensures that the digits of indices $\leq \ell+2$ (i.e., those on the right-hand side of the pattern we have imposed) of $x + r$ are the same if we compute this previous addition as if we compute

$$\begin{array}{ccccccccccccc} (x) & & \cdots & x_{\ell+6} & 0 & 0 & 0 & 0 & 0 & 0 & x_{\ell-1} & \cdots & x_2 \\ (\widetilde{r} + F_{\ell+3}) & + & & & & & 1 & 0 & 0 & r_\ell & \cdots & \cdots & r_2 \end{array}$$

or if we compute

$$\begin{array}{ccccccccccccc} (x) & & \cdots & x_{\ell+6} & 0 & 0 & 0 & 0 & 0 & 0 & x_{\ell-1} & \cdots & x_2 \\ (\widetilde{r}) & + & & & & & & & & r_\ell & \cdots & \cdots & r_2 \end{array}$$

In other words, the corollary states that the conditions we put on x and r stop the propagation of carries in both directions.

Proof. First, we have the following addition

$$\begin{array}{ccccccccccccc} (x) & & \cdots & x_{\ell+6} & 0 & 0 & 0 & 0 & 0 & 0 & x_{\ell-1} & \cdots & x_2 \\ (\widetilde{r}) & + & & & & & & & & r_\ell & \cdots & \cdots & r_2 \\ \hline (x + \widetilde{r}) & = & \cdots & x_{\ell+6} & 0 & 0 & 0 & 0 & (x+\widetilde{r})_{\ell+1} & \cdots & \cdots & \cdots & (x+\widetilde{r})_2 \end{array}$$

Indeed, due to Lemma 10, we have that $(x + \widetilde{r})_j = x_j$ for every $j \geq \ell + 2$. When we add $F_{\ell+3}$, it gives $x + \widetilde{r} + F_{\ell+3}$, whose Zeckendorf expansion is

$$(x + \widetilde{r} + F_{\ell+3}) = [\cdots x_{\ell+6} 0010 (x + \widetilde{r})_{\ell+1} \cdots (x + \widetilde{r})_2] \, .$$

We thus get the relation (24).

We now consider addition (where $\widetilde{x} := x + \widetilde{r} + F_{\ell+3}$)

$$\begin{array}{cccccccccc} (\widetilde{x}) & & \cdots & x_{\ell+6} & 0 & 0 & 1 & 0 & (x+\widetilde{r})_{\ell+1} & \cdots & (x+\widetilde{r})_2 \\ & + & \cdots & r_{\ell+6} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{array}$$

Now using Corollary 9, we obtain (25). $\square$

The statement of Corollary 12 means that, given a block of length 1 (in the sense that it has one pattern 10) in r, we are able to control the propagation of carries so that the left part of the addition does not change the expansion on the right part and vice versa. We now want to have the same kind of control for larger blocks. We could assume that x has many 0's facing the block in r that we want to control. However, this condition would be more and more "expensive" (in the sense that the probability for x to satisfy it would decrease to 0) as the length of the block increases. To avoid this, we are looking for conditions on x that affect only a bounded number of digits, regardless of the length of the block. In other words, we want our conditions to appear in "most" of $x \in \mathbb{X}$. We obtain the following corollary where the length of the block is $m+2$ and where we fixed 8 digits in x.

Corollary 14. *Let $x \in \mathbb{X}$, $r \in \mathbb{N}$ and $m \geq 0$. Assume that there exists $\ell \geq 2$ such that*

- $x_i = 0$ *where* $i \in \{\ell, \ell+1, \ell+2, \ell+3, \ell+2m+4, \ell+2m+5, \ell+2m+6, \ell+2m+7\}$,
- $r_{\ell+1} = r_{\ell+2m+7} = 0$ *and* $r_{\ell+2i+3} = 1$ *for* $i = 0, \dots, m+1$.

Let $\widetilde{r} := [r_\ell \cdots r_2]$. Then we have

$$(x + \widetilde{r} + \sum_{i=0}^{m+1} F_{\ell+2i+3})_k = (x + \widetilde{r})_k = x_k, \tag{26}$$

for all $k \geq \ell + 2m + 7$ and

$$(x + r)_k = (x + \widetilde{r} + \sum_{i=0}^{m+1} F_{\ell+2i+3})_k, \tag{27}$$

for all $k \leq \ell + 2m + 2$.

Proof. For simplicity, we write B as the block $\sum_{i=0}^{m+1} F_{\ell+2i+3}$. We decompose the sum $x+r$ in several steps. First we add $\widetilde{r}$. With the hypothesis on x, we actually consider the following addition:

$$\begin{array}{cccccccccccccccccc}
(x) & & \cdots & x_{\ell+8+2m} & 0 & 0 & 0 & 0 & x_{\ell+2m+3} & \cdots & x_{\ell+4} & 0 & 0 & 0 & 0 & x_{\ell-1} & \cdots & x_2 \\
(\widetilde{r}) & + & & & & & & & & & & & & & r_\ell & r_{\ell-1} & \cdots & r_2
\end{array}$$

Thanks to Lemma 10, we know that this addition can only modify digits of x of indices $\leq \ell+1$. We continue with the addition of B. For simplicity, we write $\widetilde{x} := x + \widetilde{r}$:

$$\begin{array}{cccccccccccccccccc}
(\widetilde{x}) & & \cdots & x_{\ell+8+2m} & 0 & 0 & 0 & 0 & x_{\ell+2m+3} & \cdots & x_{\ell+4} & 0 & 0 & \widetilde{x}_{\ell+1} & \widetilde{x}_\ell & \widetilde{x}_{\ell-1} & \cdots & \widetilde{x}_2 \\
(B) & + & & & & & 1 & 0 & 1 & \cdots & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0
\end{array}$$

Now Lemma 11 ensures that the expansion of $\widetilde{x}$ is not modified for digits of indices $\geq \ell+2m+7$. We thus prove (26). But more precisely, Lemma 11 concludes that the expansion of $\widetilde{x} + B$ is either

$$[\cdots x_{\ell+8+2m} 0100 (\widetilde{x}+B)_{\ell+2m+3} \cdots (\widetilde{x}+B)_2],$$

or

$$[\cdots x_{\ell+8+2m}0010(\widetilde{x}+B)_{\ell+2m+3}\cdots(\widetilde{x}+B)_2]\,.$$

In both case, a pattern 00 appears between the indices $\ell+2m+4$ and $\ell+2m+7$. Thus, we can apply Corollary 9: the final step of the addition, which consists of adding what remains in r, does not modify the digits of indices $\leq \ell+2m+2$. We obtain the conclusion (27). $\square$

3 Unique ergodicity of the odometer

3.1 Rokhlin towers and the ergodic measure

In this subsection, we focus on the Rokhlin towers of the odometer $(\mathbb{X}, T)$. A *Rokhlin tower* is a family of disjoint subsets $(A_0, A_1, \cdots, A_{k-1})$ such that $T(A_i) = A_{i+1}$ for $i < k-1$. We say that the subset A_i is a *level* of the Rokhlin tower and that k is the *height* of the tower. This family is usually represented by a tower (see Figure 3). In our case, we show that we can construct a partition of $\mathbb{X}$ using two Rokhlin towers.

More precisely, we focus on the action of T on cylinders. For each $k \geq 1$, we consider the partition of $\mathbb{X}$ into F_{k+2} cylinders corresponding to all possible blocks formed by the rightmost k digits of a Z-adic integers (we call them *cylinders of order* k). For example, at order 1, we partition $\mathbb{X}$ into C_0 and C_1. At order 2, we get $\mathbb{X} = C_{00} \sqcup C_{01} \sqcup C_{10}$. At order 3

$$\mathbb{X} = C_{000} \sqcup C_{001} \sqcup C_{010} \sqcup C_{100} \sqcup C_{101}.$$

In general, cylinders of order k are ordered lexicographically:

- first, those whose name has a 0 in the leftmost position (there are F_{k+1} of them),
- then those whose name has a 1 in the leftmost position (there are F_k of them).

We observe that each cylinders of order k with a 0 in the leftmost position, except the last one, is mapped by T onto the next one, giving rise to a Rokhlin tower of height F_{k+1}: we call it the *large* tower of order k. Similarly, each cylinders or order k with a 1 in the leftmost position, except the last one, is mapped by T onto the next one, giving rise to another Rokhlin tower of height F_k: we call it the *small* tower of order k. Thus, for each $k \geq 1$, we get a partition of $\mathbb{X}$ into two Rokhlin towers whose levels are all cylinders of order k. For instance, for $k = 4$, we get the partition depicted in Figure 3. It remains to describe the transition from the Rokhlin towers of order k to those of order $k+1$. First, note that all levels of the small tower of order k are also levels of the large tower of order $k+1$, since a cylinder of order k with a 1 in the leftmost position coincides with the cylinders of order $k+1$ with an additional 0 concatenated at the left of its name (e.g., $C_{101000} = C_{0101000}$).

Each level of the large tower of order k is partitioned into two cylinders of order $k+1$: one obtained by concatenating a 0 at the left of its name and the other obtained by concatenating a 1. Thus, the large Rokhlin tower of order k is cut into two subtowers:

- the first one is the bottom part of the large Rokhlin tower of order $k+1$ (the top part being nothing but the small Rokhlin tower of order k);
- the second one is the small Rokhlin tower of order $k+1$.

This transition is referred to as the *cutting-and-stacking process*. See Figure 4.

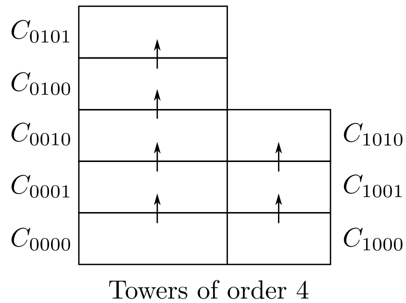

Figure 3: Rokhlin towers of order 4 (the action of T is represented by the arrows).

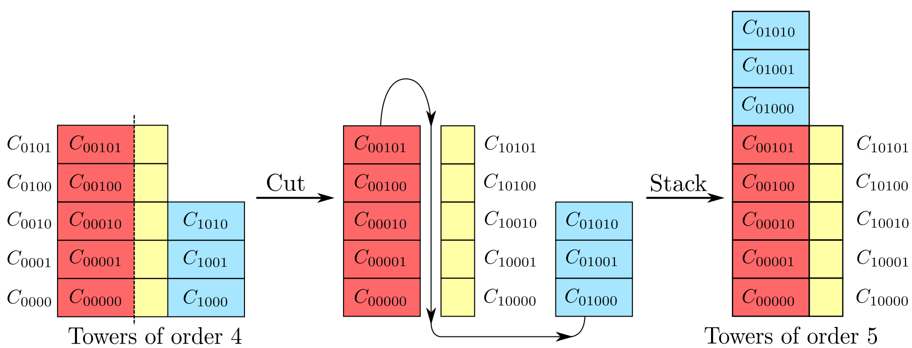

Figure 4: Visual description of the cutting-and-stacking process (for $k = 4$).

The analysis of the action of T on cylinders, yielding to the construction of the Rokhlin towers, enables us to describe the unique T-invariant probability measure.

Proposition 15. *$(\mathbb{X}, T)$ is uniquely ergodic and the unique T-invariant measure $\mathbb{P}$ satisfies*

$$\mathbb{P}(C_{r_\ell \cdots r_2}) = \begin{cases} \dfrac{1}{\varphi^{\ell-1}}, & \text{if } r_\ell = 0; \\ \dfrac{1}{\varphi^{\ell}}, & \text{otherwise.} \end{cases} \tag{28}$$

$\forall \ell \geq 2$ and $(r_\ell, \dots, r_2) \in \mathbb{X}_f$.

Proof. Let $\mathbb{P}$ be a T-invariant measure. For each order k we observe that, by T-invariance, all levels in the large tower have the same probability (and similarly in the small tower). We claim that

$$\mathbb{P}(C_1) = \frac{1}{\varphi^2}. \tag{29}$$

Indeed, let u_k (resp. v_k) denote the number of levels included in C_1 in the large (resp. small) tower of order k. From the cutting-and-stacking process, for $k \geq 1$ we get the following inductive equations

$$\begin{cases} u_{k+1} = u_k + v_k, \\ v_{k+1} = u_k, \end{cases}$$

with initial conditions $u_1 = 0$ and $v_1 = 1$. We deduce that $u_k = F_{k-1}$ and $v_k = F_{k-2}$ for $k \geq 2$. We also have the identity

$$\begin{aligned} \mathbb{P}(C_1) &= u_k\ \mathbb{P}\ (x \in \text{one fixed level in the large tower of order } k) \\ &\quad + v_k\ \mathbb{P}\ (x \in \text{one fixed level in the small tower of order } k) \\ &= \frac{u_k}{F_{k+1}}\ \mathbb{P}\ (x \in \text{the large tower of order } k) \\ &\quad + \frac{v_k}{F_k}\ \mathbb{P}\ (x \in \text{the small tower of order } k) \\ &= \frac{v_k}{F_k} + (\frac{u_k}{F_{k+1}} - \frac{v_k}{F_k})\ \mathbb{P}\ (x \in \text{the large tower of order } k). \end{aligned}$$

Taking the limit as $k \to \infty$ gives (29). As $\mathbb{X} = C_0 \sqcup C_1$, we deduce that $\mathbb{P}(C_0) = \frac{1}{\varphi}$.

Then assume that, at some order $k \geq 1$, each level in the large (resp. small) tower has measure $\frac{1}{\varphi^k}$ (resp. $\frac{1}{\varphi^{k+1}}$). Since a level in the small tower of order k is also a level in the large tower of order $k+1$, we deduce that each level in the large tower of order $k+1$ also has measure $\frac{1}{\varphi^{k+1}}$. Let $p \in [0,1]$ denote the common measure of all levels in the small tower of order $k+1$. Since there are F_{k+2} (resp. F_{k+1}) levels in the large (resp. small) tower of order $k+1$, we have the equation

$$\frac{F_{k+2}}{\varphi^{k+1}} + pF_{k+1} = 1,$$

which is equivalent to the relation

$$\varphi F_{k+2} + p\varphi^{k+2} F_{k+1} = \varphi^{k+2}.$$

Combining with the classical identity $\varphi F_{k+2} + F_{k+1} = \varphi^{k+2}$, we deduce that $p = \frac{1}{\varphi^{k+2}}$. By induction, we prove (28). □

3.2 Probabilistic interpretation of the measure

In the case of the b-adic odometer [11, p. 7] ($b \geq 2$), the T-invariant measure can be interpreted as an independent choice of each digit according to the uniform law on the set of possible digits. For the Z-adic odometer, it is not as easy to describe. First, the digits do not follow the same law. Indeed, for all $k \geq 2$, we have $\mathbb{P}(x_k = 1) = \frac{F_{k-1}}{\varphi^k}$, which depends on k. (This provides the classical asymptotic formula on the frequency of 1's in the Zeckendorf expansion: $\mathbb{P}(x_k = 1) \xrightarrow[k \to +\infty]{} \frac{1}{\varphi^2+1}$; see [13, 10] for other proofs.) Therefore, if σ is the shift on $\mathbb{X}$, the law of x is not the same as the law of $\sigma^k(x)$ for $k \geq 1$, which means $\mathbb{P}$ is not stationary. Furthermore, the choice of a digit is not independent of the other digits because a 1 must be followed by a 0. However, the lack of stationarity and independence of $\mathbb{P}$ is compensated by the following renewal property.

Proposition 16. *Let C be a cylinder, $k \geq 2$, and $(r_k, \dots, r_2) \in \mathbb{X}_f$. Then*

1. $\mathbb{P}\left(\sigma^k x \in C \mid x \in C_{0r_k \cdots r_2}\right) = \mathbb{P}(C)$,

and

2. $\mathbb{P}\left(\sigma^{k+1} x \in C \mid x \in C_{1r_k \cdots r_2}\right) = \mathbb{P}(C)$ *(if $r_k = 0$).*

Proof. Without loss of generality, we can assume that C is a cylinder of order k_0 with a 0 at the left side of its name for some $k_0 \geq 1$, so $\mathbb{P}(C) = \frac{1}{\varphi^{k_0}}$. Then

$$\mathbb{P}\left(\sigma^{k+2} x \in C \mid x \in C_{0r_k \cdots r_2}\right) = \frac{\mathbb{P}\left(x \in C_{0r_k \cdots r_2} \cap \sigma^{k+2} x \in C\right)}{\mathbb{P}(x \in C_{0r_k \cdots r_2})}.$$

We observe that the set $\{x \in C_{0r_k \cdots r_2} \cap \sigma^{k+2} x \in C\}$ is actually a cylinder of order $k + k_0$ with a leftmost 0 in its name. Its measure is therefore $\frac{1}{\varphi^{k+k_0}}$.

$$\mathbb{P}\left(\sigma^{k+2} x \in C \mid x \in C_{0r_k \cdots r_2}\right) = \frac{\varphi^k}{\varphi^{k+k_0}} = \frac{1}{\varphi^{k_0}} = \mathbb{P}(C).$$

We get the first point of the proposition. Then we observe that, since $C_{1r_k \cdots r_2} = C_{01r_k \cdots r_2}$, the second is a particular case of the first one (with $k+1$ instead of k). ☐

Once we know the value of x_k, the conditional law of x_{k+1} depends neither on k or $x_{k-1}, \dots, x_2$. In particular, we get that

$$\mathbb{P}(x_{k+2} = 1 \mid x_{k+1}, \dots, x_2) = \begin{cases} 1/\varphi^2, & \text{if } x_{k+1} = 0; \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\mathbb{P}(x_{k+2} = 0 \mid x_{k+1}, \dots, x_2) = \begin{cases} 1/\varphi, & \text{if } x_{k+1} = 0; \\ 1, & \text{otherwise.} \end{cases}$$

Therefore, under $\mathbb{P}$, the digits x_2, x_3, … form a Markov chain with transition probabilities given on Figure 5, starting with the initial law

$$\mathbb{P}(x_2 = 1) = \frac{1}{\varphi^2} \text{ and } \mathbb{P}(x_2 = 0) = \frac{1}{\varphi}.$$

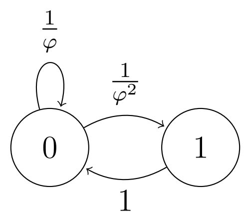

Figure 5: Transition probabilities of the Markov chain.

3.3 Reminders on some notions of mixing coefficients

In Definition 5, we introduced the notion of α-mixing coefficients. It happens that the distribution of the digits in a Z-adic integer satisfies a good inequality for another (better) notion of mixing coefficients: the ϕ-mixing coefficients. They are defined as follows.

Definition 17. Let $(X_j)_{j\geq 1}$ be a (finite or infinite) sequence of random variables. The associated *ϕ-mixing coefficients* $\phi(k)$, $k \geq 1$, are defined by

$$\phi(k) := \sup_{p\geq 1} \sup_{A,B} |\mathbb{P}_A(B) - \mathbb{P}(B)|,$$

where the second supremum is taken over all events A and B such that

- $A \in \sigma(X_j : 1 \leq j \leq p)$,
- $\mathbb{P}(A) > 0$ and,
- $B \in \sigma(X_j : j \geq k + p)$.

By convention, if X_j is not defined when $j \geq k + p$, then the σ-algebra is trivial.

In the case of a finite sequence $(X_1, \ldots, X_n)$, the convention implies that $\phi(k) = 0$ for $k \geq n$.

Both α and ϕ-mixing coefficients are linked together and, as written above, ϕ-mixing coefficients are "better" than α-mixing coefficients. Indeed, we have the following property from Bradley [4]:

Proposition 18 (Bradley)**.** *For every $k \geq 1$*

$$\alpha(k) \leq \frac{1}{2}\phi(k).$$

3.4 ϕ-mixing property for Z-adic digits

As mentioned in the previous subsection, there exists a good upper bound on the ϕ-mixing coefficients for the coordinates of $x \in \mathbb{X}$.

Proposition 19. *For $x = (x_j)_{j\geq 2} \in \mathbb{X}$ randomly chosen with law $\mathbb{P}$ and $k \geq 1$ we have*

$$\phi(k) \leq \frac{2}{\varphi^{2k}}.$$

Our way to prove this result needs to introduce a parametrization of the Markov chain. So, we define the function on $\{0,1\} \times [0,1]$ as follows:

$$\psi(\varepsilon, t) := \begin{cases} 1, & \text{if } \varepsilon = 0 \text{ and } t < \frac{1}{\varphi^2}; \\ 0, & \text{if } \varepsilon = 0 \text{ and } t \geq \frac{1}{\varphi^2}; \\ 0, & \text{otherwise.} \end{cases}$$

Now let $(U_j)_{j\geq 2}$ be independent and identically distributed random variables following a uniform law on the real unit interval and define the sequence $(y_j)_{j\geq 1}$ as follows:

$$y_j := \begin{cases} 0, & \text{if } j = 1; \\ \psi(y_{j-1}, U_j), & \text{otherwise.} \end{cases} \tag{30}$$

Lemma 20. *The law of $(y_j)_{j\geq 2}$ is $\mathbb{P}$.*

Proof of Lemma 20. The transition matrix is the same as for $\mathbb{P}$ and so is the initial law: the probability that y_2 is 0 is the same as the event $U_2 \geq \frac{1}{\varphi^2}$ that is $\frac{1}{\varphi} = \mathbb{P}(C_0)$. □

Proof of Proposition 19. Let $k \geq 1$ and $p \geq 2$. From Lemma 20, choosing x with law $\mathbb{P}$ is equivalent to constructing a sequence $y = (y_j)_{j\geq 2}$ using the process (30). Thus, we have a sequence of independent and identically distributed random variables $(U_j)_{j\geq 2}$ with a uniform law on the real unit interval. We consider the event

$$C := \left\{ \exists j \in]p, p+k] : U_j \geq \frac{1}{\varphi^2} \right\},$$

the probability of which is

$$\mathbb{P}(C) = 1 - \frac{1}{\varphi^{2k}} > 0.$$

If $y \in C$ then this implies that $y_j = 0$ because of (30). Then we take $A \in \sigma(x_j : 2 \leq j \leq p)$ and $B \in \sigma(x_j : j \geq k+p)$. We observe that, due to (30), $A \in \sigma(U_j : 2 \leq j \leq p)$ while $C \in \sigma(U_j : p < j \leq k+p)$ thus A and C are independent. Also, due to Proposition 16, we observe $B \cap C \in \sigma(U_j : j > p)$. So A and $B \cap C$ are independent. We also have

$$|\mathbb{P}_A(B) - \mathbb{P}(B)| \leq |\mathbb{P}_A(B) - \mathbb{P}_{A\cap C}(B)| + |\mathbb{P}_{A\cap C}(B) - \mathbb{P}_C(B)| + |\mathbb{P}_C(B) - \mathbb{P}(B)|.$$

But the independence between A and C and A and $B \cap C$ gives that

$$|\mathbb{P}_A(B) - \mathbb{P}(B)| \leq |\mathbb{P}_A(B) - \mathbb{P}_{A\cap C}(B)| + |\mathbb{P}_C(B) - \mathbb{P}(B)|. \tag{31}$$

As shown in the case of an integer base [11, Lemma 4.3], if we let $\overline{C}$ denote the complement of C, then for every event D we have the general inequality

$$|\mathbb{P}_C(D) - \mathbb{P}(D)| \le \mathbb{P}(\overline{C}), \tag{32}$$

which implies in (31) that

$$|\mathbb{P}_A(B) - \mathbb{P}(B)| \le 2\mathbb{P}(\overline{C}).$$

□

Another property about the measure is that if $\mathbb{P}(x_k = 1)$ indeed depends on k, then it is actually bounded between two positive values. We generalize this fact with the next proposition.

Proposition 21. *Let $(k_0, \dots, k_\ell)$ be a collection of integers such that*

$$2 \le k_0 < k_1 < \cdots < k_\ell.$$

Let A be a union of cylinders of order $k_0 - 1$ such that $x_{k_0} = 0$ if $x \in A$ and such that $\mathbb{P}(A) > 0$. Then

$$\frac{1}{\varphi^\ell} \le \mathbb{P}_A \left(\forall 1 \le i \le \ell \ : \ x_{k_i} = 0\right) \le \left(\frac{2}{\varphi^2}\right)^\ell .$$

Proof. Let $I \subset \mathbb{N}$. Since there is a finite number of cylinders of order $k_0 - 1$, we can write A as a disjoint union $\sqcup_{i\in I} C^{(i)}$ where $C^{(i)}$ is a cylinder of order $k_0 - 1$. We have the identity

$$\mathbb{P}_A \left(\forall 1 \le i \le \ell \ : \ x_{k_i} = 0\right) = \prod_{i=1}^{\ell} \mathbb{P}\left(x_{k_i} = 0 \mid A \cap \bigcap_{j=1}^{i-1} (x_{k_j} = 0)\right), \tag{33}$$

with the convention $\cap_{j=1}^{0} (x_{k_j} = 0) = \mathbb{X}$. But we have

$$\begin{aligned}
\mathbb{P}_A(x_{k_1} = 0) &= \frac{\mathbb{P}\left((x_{k_1} = 0) \cap \left(x \in \sqcup_{i\in I} C^{(i)}\right)\right)}{\mathbb{P}(A)} \\
&= \frac{1}{\mathbb{P}(A)} \sum_{i\in I} \mathbb{P}\left((x_{k_1} = 0) \cap x \in C^{(i)}\right).
\end{aligned}$$

We recall that $\mathbb{P}(A) = \frac{|I|}{\varphi^{k_0-1}}$, and that the summand is the probability of a disjoint union of $F_{k_1-k_0+1}$ cylinders of order $k_1 - 1$ such that $x_{k_1} = 0$, so its measure is $\frac{F_{k_1-k_0+1}}{\varphi^{k_1-1}}$. Thus, we obtain

$$\mathbb{P}_A(x_{k_1} = 0) = \mathbb{P}(x_{k_1-k_0+1} = 0).$$

We proceed similarly for the other terms in (33) and get

$$\mathbb{P}_A \left(\forall 1 \le i \le \ell \ : \ x_{k_i} = 0\right) = \prod_{i=1}^{\ell} \mathbb{P}\left(x_{k_i-k_{i-1}+1} = 0\right). \tag{34}$$

The last equality is given using the renewal of $\mathbb{P}$. Now we claim that for all $k \geq 2$ we have

$$\frac{1}{\varphi} \leq \mathbb{P}(x_k = 0) \leq \frac{2}{\varphi^2}.$$

Indeed, we recall that $\mathbb{P}(x_k = 0) = \frac{F_k}{\varphi^{k-1}}$, and we observe that the subsequences $\left(\frac{F_{2k}}{\varphi^{2k-1}}\right)$ and $\left(\frac{F_{2k+1}}{\varphi^{2k}}\right)$ are adjacent sequences. □

3.5 Ergodic convergence

For x in $\mathbb{X}$, we define the sequence of empirical probability measures along the (beginning of the) orbit of x: for every $N \geq 1$, we set

$$\epsilon_N(x) := \frac{1}{N} \sum_{0 \leq n < N} \delta_{T^n x}$$

, where δ_y is the Dirac measure on $y \in \mathbb{X}$.

Since the space of probability measures on $\mathbb{X}$ is compact for the weak-$*$ topology, we can always extract a convergent subsequence. Moreover, every subsequential limit of $(\epsilon_N(x))$ is a T-invariant probability measure. By the uniqueness of the T-invariant probability measure, for every $x \in \mathbb{X}$ we have $\epsilon_N(x) \to \mathbb{P}$. In other words, we have the convergence

$$\forall x \in \mathbb{X},\ \forall f \in \mathcal{C}(\mathbb{X}), \qquad \frac{1}{N} \sum_{0 \leq n < N} f(T^n x) \xrightarrow[N \to +\infty]{} \int_{\mathbb{X}} f \mathrm{d}\mathbb{P}. \tag{35}$$

Here, we are interested in the special case $x = 0$, because $\mathbb{N} = \{T^n 0 : n \in \mathbb{N}\}$. Then (35) becomes

$$\forall f \in \mathcal{C}(\mathbb{X}), \qquad \frac{1}{N} \sum_{0 \leq n < N} f(n) \xrightarrow[N \to +\infty]{} \int_{\mathbb{X}} f \mathrm{d}\mathbb{P}. \tag{36}$$

Equation (36) shows that, for a continuous function f, averaging f over $\mathbb{N}$ (for the natural density) amounts to averaging over $\mathbb{X}$ (for $\mathbb{P}$). The next section shows how this convergence can be extended to some non-continuous functions related to the sum-of-digits function.

4 Sum of digits on the odometer

In this section, we adapt the techniques used for the case of an integer base [11]. The main difference is the need of the new Lemma 27. For every integer $k \geq 2$, we define the continuous map $s_k : \mathbb{X} \to \mathbb{Z}$ as the sum of the digits of indices $\leq k$, that is to say

$$s_k(x) := x_k + \cdots + x_2.$$

Let $r \in \mathbb{N}$. We define the functions $\Delta_k^{(r)} : \mathbb{X} \to \mathbb{Z}$ by

$$\Delta_k^{(r)}(x) := s_k(x+r) - s_k(x).$$

The functions $\Delta_k^{(r)}$ are well-defined, continuous (and bounded) on $\mathbb{X}$. By (36), we have

$$\frac{1}{N} \sum_{n<N} \Delta_k^{(r)}(n) = \frac{1}{N} \sum_{n<N} \Delta_k^{(r)}(T^n 0) \xrightarrow[N \to +\infty]{} \int_{\mathbb{X}} \Delta_k^{(r)} \mathrm{d}\mathbb{P}. \tag{37}$$

Although the sum-of-digits function s is not well defined on $\mathbb{X}$, we can extend the function $\Delta^{(r)}$ defined by (3) on the set of $x \in \mathbb{X}$ for which the number of different digits between x and $x+r$ is finite. This subset contains the Z-adic integers x such that there exists an index $k \geq 2 + \max(\{\ell : r_\ell \neq 0\})$ such that $x_k = x_{k+1} = 0$ (see Lemma 10). So, except for a finite number of Z-adic integers, we can define

$$\Delta^{(r)}(x) := \lim_{k \to \infty} \Delta_k^{(r)}(x).$$

Remark 22. Let t, u be two integers. For every integer k we have the decomposition formula

$$\Delta_k^{(t+u)} = \Delta_k^{(t)} + \Delta_k^{(u)} \circ T^t. \tag{38}$$

So, taking $\mathbb{P}$-almost everywhere the limit as $k \to \infty$, we get

$$\Delta^{(t+u)} = \Delta^{(t)} + \Delta^{(u)} \circ T^t \ \ (\mathbb{P}\text{-a-s.}). \tag{39}$$

Then by induction on t, we deduce

$$\Delta^{(t)} = \Delta^{(1)} + \Delta^{(1)} \circ T + \cdots + \Delta^{(1)} \circ T^{t-1} \ \ (\mathbb{P}\text{-a-s.}). \tag{40}$$

The function $\Delta^{(r)}$ is not bounded on $\mathbb{X}$, and therefore it is not continuous. So Eq. (36) is not applicable for $f \circ \Delta^{(r)}$ with f a continuous map on $\mathbb{X}$. However, it is possible to get the same convergence as in (36) with weaker assumptions on f than continuity.

Proposition 23. *Let $r \geq 1$ and $f : \mathbb{Z} \to \mathbb{C}$. Assume that there exist $\alpha \geq 1$ and C in $\mathbb{R}_+^*$ such that for every $n \in \mathbb{Z}$ we have*

$$|f(n)| \leq C|n|^\alpha + |f(0)|. \tag{41}$$

Then $f \circ \Delta^{(r)} \in L^1(\mathbb{P})$ and we have the convergence

$$\begin{aligned} \lim_{N \to \infty} \frac{1}{N} \sum_{n<N} f(\Delta^{(r)}(n)) &= \int_{\mathbb{X}} f(\Delta^{(r)}(x)) \mathrm{d}\mathbb{P}(x) \\ &= \lim_{k \to \infty} \int_{\mathbb{X}} f(\Delta_k^{(r)}(x)) \mathrm{d}\mathbb{P}(x). \end{aligned}$$

Corollary 24. *For every $d \in \mathbb{Z}$ we have*

$$\begin{aligned}\mu^{(r)}(d) &:= \lim_{N\to\infty} \frac{1}{N} \left|\left\{n < N : \Delta^{(r)}(n) = d\right\}\right| \\ &= \mathbb{P}\left(\left\{x \in \mathbb{X} : \Delta^{(r)}(x) = d\right\}\right). \end{aligned} \tag{42}$$

Moreover, $\Delta^{(r)}$ has zero-mean and has finite moments.

The proof of this corollary is exactly the same as for the case of an integer base [11]. In particular, we just prove the existence of the asymptotic densities of the sets $\{n \in \mathbb{N} : \Delta^{(r)}(n) = d\}$, where $d \in \mathbb{Z}$. In the case of an integer base, this is easy to prove (see [3, 11]). Here, it is harder to follow Bésineau's proof because it uses the arithmetic properties of the sum-of-digits (in an integer base) function that we do not have any longer.

Remark 25. Using trivial arguments, Proposition 23 and Corollary 24 are also true when $r = 0$. We observe that $\mu^{(0)} = \delta_0$.

Before proving this proposition and its corollary, we need the following lemma that looks like Lemma 1.29 from Spiegelhofer [16].

Lemma 26. *Let $r \geq 1$. For $N \geq 1$, $k \geq 2$ and $d, d' \in \mathbb{Z}$, we have the inequality*

$$\begin{aligned}&\frac{1}{N}\left|\{n < N : (\Delta^{(r)}(n), \Delta_k^{(r)}(n)) = (d, d')\}\right| \\ &\qquad\qquad \leq r\varphi^3\, \mathbb{P}\Big(\{x \in \mathbb{X} : (\Delta^{(r)}(x), \Delta_k^{(r)}(x)) = (d, d')\}\Big). \end{aligned} \tag{43}$$

In particular, we have

$$\frac{1}{N}\left|\{n < N : \Delta^{(r)}(n) = d\}\right| \leq r\varphi^3\, \mathbb{P}\big(\{x \in \mathbb{X} : \Delta^{(r)}(x) = d\}\big). \tag{44}$$

Proof of Lemma 26. We adapt the proof of [11, Lemma 2.3] for the case of an integer base. First, (43) implies (44) so we just prove (43). We fix $k \geq 2$. For $\ell \geq 1$, let V_ℓ be the set of the values reached by the pair $(\Delta^{(r)}, \Delta_k^{(r)})$ on the $F_{\ell+1} - r$ (resp. $F_\ell - r$) first levels of the large (resp. small) tower of order ℓ. Of course, if $F_{\ell+1} - r \leq 0$ then $V_\ell := \emptyset$. In particular, for every $r \geq 1$, $V_1 = \emptyset$. Also, if $F_\ell \leq r < F_{\ell+1}$ then V_ℓ is defined considering only the large tower of order ℓ. For every ℓ, we observe that V_ℓ is a finite set. On each level that is not in the r top levels of the large or small tower, the first ℓ digits of both x and $x+r$ are constant, and digits of higher order are the same. Therefore, $\Delta^{(r)}$ and $\Delta_k^{(r)}$ are constant on such a level. We observe that the sequence $(V_\ell)_{\ell\geq 1}$ is increasing for the inclusion.

Now for $d, d' \in \mathbb{Z}$, there are two cases.

1. If $(d, d') \notin \cup_{\ell\geq 2} V_\ell$, then for each $n \in \mathbb{N}$ we have $(\Delta^{(r)}(n), \Delta_k^{(r)}(n)) \neq (d, d')$. Indeed, for each $n \in \mathbb{N}$, there exists a smallest integer $\ell \geq 2$ such that n is in the first $F_{\ell+1} - r$ levels of the large tower of order ℓ, hence $(\Delta^{(r)}(n), \Delta_k^{(r)}(n)) \in V_\ell$. In this case, (43) is trivial.

2. If $(d, d') \in \cup_{\ell \geq 1} V_\ell$ then there exists a unique $\ell \geq 2$ such that $(d, d') \in V_\ell \backslash V_{\ell-1}$. We observe that, due to the cutting-and-stacking process, the value (d, d') must appear firstly in the large tower of order ℓ. Since $(\Delta^{(r)}, \Delta_k^{(r)})$ is constant on each of the first $F_{\ell+1} - r$ levels of the large tower, it takes the value (d, d') on at least one whole such level, which is of measure $\frac{1}{\varphi^\ell}$. So, we have

$$\mathbb{P}\Big(\{x \in \mathbb{X} : (\Delta^{(r)}(x), \Delta_k^{(r)}(x)) = (d, d')\}\Big) \geq \frac{1}{\varphi^\ell}. \tag{45}$$

Also, since the pair $(d, d') \notin V_{\ell-1}$, we claim that for every $N \geq 1$

$$\frac{1}{N} \left| \{n < N : (\Delta^{(r)}(n), \Delta_k^{(r)}(n)) = (d, d')\} \right| \leq \frac{r}{F_{\ell-1}}.$$

Indeed,

(a) If $r \geq F_{\ell-1}$ then the inequality is trivial.

(b) If $r < F_{\ell-1}$ then since (d, d') is not in $V_{\ell-1}$, the pair (d, d') can only appear inside a part of the r highest levels of the big or small towers of order $\ell - 1$. Let C denote the union of these r highest levels. Since 0 lies in the bottom level of the large tower of order $\ell - 1$, the set S of integers $n \geq 0$ such that $T^n 0 \in C$ has the following properties:

- $\{0, \ldots, F_\ell - r - 1\} \cap S = \emptyset$ and
- S is the union of subsets formed by r consecutive integers separated by gaps of length $F_\ell - r$ or $F_{\ell-1} - r$.

So, we have

$$\begin{aligned} \frac{1}{N} \left| \{0 \leq n < N : (\Delta^{(r)}(n), \Delta_k^{(r)}(n)) = (d, d')\} \right| \\ &\leq \frac{1}{N} \left| \{0 \leq n < N : T^n 0 \in C\} \right| \\ &\leq \frac{r}{F_{\ell-1}}. \end{aligned}$$

Since $F_{\ell-1} \geq \varphi^{\ell-3}$ (by double induction), we get

$$\frac{1}{N} \left| \{n < N : (\Delta^{(r)}(n), \Delta_k^{(r)}(n)) = (d, d')\} \right| \leq \frac{r}{\varphi^{\ell-3}}. \tag{46}$$

Combining inequalities (45) and (46) gives (43).

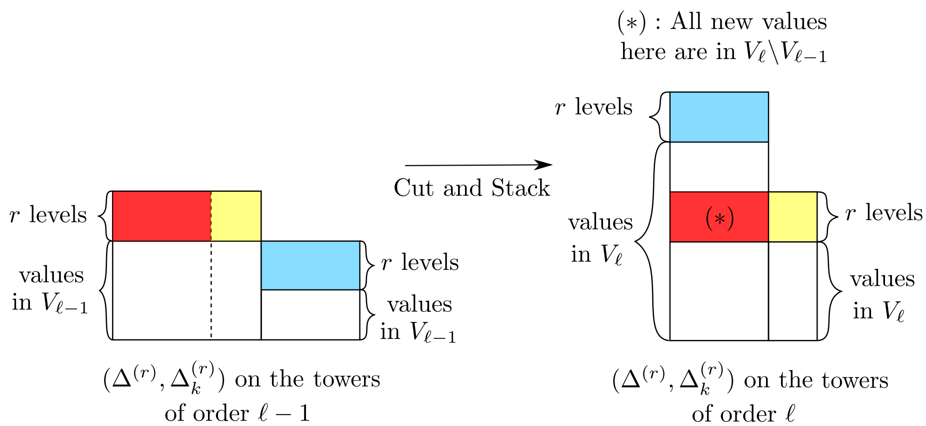

Figure 6: Visual description of $V_{\ell-1}$, V_ℓ and $V_\ell \backslash V_{\ell-1}$.

□

As shown in (40), the understanding of $\Delta^{(1)}$ (and so $\Delta_k^{(1)}$ for $k \geq 2$) is fundamental to understand $\Delta^{(r)}$, so we also state the following lemma.

Lemma 27. *The function $\Delta^{(1)}$ is well-defined for $x \in \mathbb{X}$ if and only if $x \in C_{00(10)^d} \cup C_{001(01)^d}$ for some $d \geq 0$.*

Furthermore, if $x \in C_{00(10)^d}$ then $\Delta^{(1)}(x) = 1 - d$ and

$$\Delta_k^{(1)}(x) = \begin{cases} \frac{2-k}{2}, & \text{if } k \equiv 0 \pmod 2 \text{ and } k \leq 2d; \\ \frac{1-k}{2}, & \text{if } k \equiv 1 \pmod 2 \text{ and } k \leq 2d+1; \\ 1-d, & \text{if } k \geq 2d+2. \end{cases}$$

Also, if $x \in C_{001(01)^d}$ then $\Delta^{(1)}(x) = -d$ and

$$\Delta_k^{(1)}(x) = \begin{cases} \frac{-k}{2}, & \text{if } k \equiv 0 \pmod 2 \text{ and } k \leq 2d+2; \\ \frac{1-k}{2}, & \text{if } k \equiv 1 \pmod 2 \text{ and } k \leq 2d+1; \\ \frac{3-k}{2}, & \text{if } k = 2d+3; \\ -d, & \text{if } k \geq 2d+4. \end{cases}$$

Proof of Lemma 27. If $x \in C_{00(10)^d}$ for some $d \geq 0$, the definition of T gives that $x + 1 \in C_{01(00)^d}$ with the other digits unchanged. Thus the sequence $(\Delta_k^{(1)}(x))$ is stationary, $\Delta^{(1)}(x)$ is well defined, and $\Delta^{(1)}(x) = 1 - d$. Also, if $x \in C_{001(01)^d}$ for some $d \geq 0$, then $x + 1 \in C_{010(00)^d}$, $\Delta^{(1)}(x)$ is well defined and $\Delta^{(1)}(x) = -d$. If x does not belong to those cylinders then $x = (01)^\infty$ or $x = (10)^\infty$. In that case, $x + 1 = 0$ and the sequence $(\Delta_k^{(1)}(x))$ diverges to $-\infty$. This proves equivalence. The values of $\Delta_k^{(1)}(x)$ can be easily found by looking at the addition we write in Subsection 2.1. □

Proof of Proposition 23. We adapt the proof of [11, Prop. 2.1] for the case of an integer base. Let $\varepsilon > 0$. For every integer $k \geq 2$, we have

$$\left|\frac{1}{N}\sum_{n<N} f(\Delta^{(r)}(n)) - \int_{\mathbb{X}} f(\Delta^{(r)}(x))\mathrm{d}\mathbb{P}(x)\right| \leq A_1 + A_2 + A_3,$$

where

$$\begin{aligned}
A_1 &:= \frac{1}{N}\sum_{n<N}\left|f(\Delta^{(r)}(n)) - f(\Delta_k^{(r)}(n))\right|,\\
A_2 &:= \left|\frac{1}{N}\sum_{n<N} f(\Delta_k^{(r)}(n)) - \int_{\mathbb{X}} f(\Delta_k^{(r)}(x))\mathrm{d}\mathbb{P}(x)\right|,\\
A_3 &:= \int_{\mathbb{X}}\left|f(\Delta^{(r)}(x)) - f(\Delta_k^{(r)}(x))\right|\mathrm{d}\mathbb{P}(x).
\end{aligned}$$

However, we have

$$\begin{aligned}
A_1 &= \frac{1}{N}\sum_{n<N}\sum_{j,j'\in\mathbb{Z}}\left|f(j) - f(j')\right|\mathbb{1}_{(j,j')}\left(\Delta^{(r)}(n), \Delta_k^{(r)}(n)\right)\\
&= \sum_{j,j'\in\mathbb{Z}}\left|f(j) - f(j')\right|\frac{1}{N}\sum_{n<N}\mathbb{1}_{(j,j')}\left(\Delta^{(r)}(n), \Delta_k^{(r)}(n)\right)\\
&\leq r\varphi^3\sum_{j,j'\in\mathbb{Z}}\left|f(j) - f(j')\right|\mathbb{P}\Big(\Delta^{(r)}(n) = j, \Delta_k^{(r)}(n) = j'\Big)\\
&= r\varphi^3\int_{\mathbb{X}}\left|f(\Delta^{(r)}(x)) - f(\Delta_k^{(r)}(x))\right|\mathrm{d}\mathbb{P}(x) = r\varphi^3 A_3.
\end{aligned}$$

Then

$$\left|\frac{1}{N}\sum_{n<N} f(\Delta^{(r)}(n)) - \int_{\mathbb{X}} f(\Delta^{(r)}(x))\mathrm{d}\mathbb{P}(x)\right| \leq A_2 + (1 + r\varphi^3)A_3.$$

We want to apply the dominated convergence theorem to deal with A_3 (observe that we have the convergence $f(\Delta_k^{(r)}(x)) \xrightarrow[k\to\infty]{} f(\Delta^{(r)}(x))$ $\mathbb{P}$-almost-surely). For this, we need to find a good dominant function. We define $g_i := \sup_{k\geq 2}|\Delta_k^{(1)} \circ T^i(x)|$ for $i = 0, \ldots, r-1$ and by (41) we get the inequalities

$$\left|f \circ \Delta_k^{(r)}(x)\right| \leq C \sum_{j_0+\cdots+j_{r-1}=\alpha}\sum_{i=0}^{r-1}\frac{\binom{\alpha}{j_0,\ldots,j_{r-1}}}{r} g_i(x)^{rj_i} + \left|f(0)\right|$$

and

$$\left|f \circ \Delta^{(r)}(x)\right| \leq C \sum_{j_0+\cdots+j_{r-1}=\alpha}\sum_{i=0}^{r-1}\frac{\binom{\alpha}{j_0,\ldots,j_{r-1}}}{r} g_i(x)^{rj_i} + \left|f(0)\right|.$$

We need to prove that $g_i^{rj_i}$ is integrable for the measure $\mathbb{P}$. It is equivalent to show that $\sum_m \mathbb{P}(\{x \in \mathbb{X} : g_i^{rj_i}(x) > m\})$ is a convergent series. We have

$$\begin{aligned} g_i(x) > m^{\frac{1}{rj_i}} &\Leftrightarrow \sup_{k\geq 2} |\Delta_k^{(1)} \circ T^i(x)| > m^{\frac{1}{rj_i}} \\ &\Leftrightarrow \exists k \geq 2,\ \ |\Delta_k^{(1)}(T^i x)| > m^{\frac{1}{rj_i}}. \end{aligned}$$

From Lemma 27

$$\exists k \geq 2,\ \ |\Delta_k^{(1)}(T^i x)| > m^{\frac{1}{rj_i}} \Leftrightarrow T^i x \in C_{00(10)^{\lfloor m^{\frac{1}{rj_i}} \rfloor}} \cup C_{0(01)^{\lfloor m^{\frac{1}{rj_i}} \rfloor}}.$$

It follows, by T-invariance, that

$$\mathbb{P}(\{x \in \mathbb{X} : g_i^{rj_i}(x) > m\}) \leq \left(\frac{1}{\varphi}\right)^{2+2\lfloor m^{\frac{1}{rj_i}} \rfloor} + \left(\frac{1}{\varphi}\right)^{1+2\lfloor m^{\frac{1}{rj_i}} \rfloor} = \left(\frac{1}{\varphi}\right)^{2\lfloor m^{\frac{1}{rj_i}} \rfloor}.$$

The quantity on the RHS is the general term of a convergent series, which shows that $g_i^{rj_i}$ is integrable for the measure $\mathbb{P}$. The dominated convergence theorem can be applied and, for k large enough, we have $(1 + r\varphi^3)A_3 \leq \frac{\varepsilon}{2}$ for every $N \geq 1$. Now once we have fixed such a k, for N large enough, A_2 is bounded by $\frac{\varepsilon}{2}$ because of (36) and the continuity of $\Delta_k^{(r)}$ and f. The convergence in the statement is thus proved.

Note that the argument of the dominated convergence theorem also proves that $f \circ \Delta^{(r)} \in L^1(\mathbb{P})$ and $\int_{\mathbb{X}} f \circ \Delta^{(r)} \mathrm{d}\mathbb{P} = \lim\limits_{k\to\infty} \int_{\mathbb{X}} f \circ \Delta_k^{(r)} \mathrm{d}\mathbb{P}$. □

More generally, we have the following convergence.

Proposition 28. *Let $r \geq 1$ and $f : \mathbb{Z} \to \mathbb{C}$ be such that $f \circ \Delta^{(r)} \in L^1(\mathbb{P})$. Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n<N} f(\Delta^{(r)}(n)) = \int_{\mathbb{X}} f(\Delta^{(r)}(x)) \mathrm{d}\mathbb{P}(x).$$

The proof is exactly the same as Proposition 2.4 in [11]. We have shown that the random variable $\Delta^{(r)}$ satisfies some good properties such as the finiteness of moments of every order of $\mu^{(r)}$, the law of $\Delta^{(r)}$. The next section focuses on another natural question: how to compute the law $\mu^{(r)}$?

5 How to compute $\mu^{(r)}$

In this section, we present an algorithm that exactly computes the measure $\mu^{(r)}$ for r in $\mathbb{N}$ and its consequences.

5.1 Description of the algorithm

This algorithm is completely different from the classical way to compute $\mu^{(r)}$ in an integer base, as developed by Bésineau [3, p. 14]. In an integer base $b \geq 2$, the computation of $\mu^{(r)}$ relies on inductive relations for the expansion of r, which are easy to prove in that case. For instance, one of the relation states that if r is a multiple of b then the variation when adding r to x is the same as the variation when adding $\frac{r}{b}$ to σx. So the laws of the variation when adding r or $\frac{r}{b}$ are the same. In the Zeckendorf representation, this trivial argument is false because the carries can modify the digits on both sides (unlike the case of an integer base, where only the left side can be affected). However, we find an algorithm to compute $\mu^{(r)}$, in this Zeckendorf system that relies on Rokhlin towers. This algorithm can be adapted to an integer base.

First, let $r \geq 1$ ($r = 0$ is irrelevant) and let $\ell \geq 2$ be the unique integer such that $F_\ell \leq r < F_{\ell+1}$. At a given order $k \geq 1$, we define CST_{k} as the union of the levels of the large and small towers of order k, except the r highest levels. We observe $\mathrm{CST}_{1} = \emptyset$. On each level of CST_{k}, $\Delta^{(r)}$ is constant (see the proof of Lemma 26). But, due to the cutting-and-stacking process, the value of $\Delta^{(r)}$ on these "constant" levels of CST_{k} may be deduced from those taken in CST_{k-1}. That is why we also introduce, for $k \geq 2$, the *new information zone of the Rokhlin tower of order* k as the set

$$\mathrm{NIZ}_{k} := \mathrm{CST}_{k} \backslash \mathrm{CST}_{k-1}.$$

Of course, everything depends on r but we do not emphasize on that in the notation of ℓ, NIZ_{k} and CST_{k} for simplicity. Let us define clearly this set NIZ_{k}.

Proposition 29. *In a large tower, we enumerate, starting by* 1*, the levels from the base of the tower to the top of it. We have the following description of* NIZ_{k}*.*

- *If* $2 \leq k \leq \ell - 1$*,* $\mathrm{NIZ}_{k} = \emptyset$*.*
- NIZ_{ℓ} *is the union of the first* $F_{\ell+1} - r$ *levels of the large tower of order* ℓ*.*
- $\mathrm{NIZ}_{\ell+1}$ *is the union of the* F_ℓ *levels between the* $F_{\ell+1} - r + 1^{\text{th}}$ *and the* $F_{\ell+2} - r^{\text{th}}$ *levels of the large tower of order* $\ell + 1$*.*
- *If* $k > \ell + 1$*,* NIZ_{k} *is the union of the* r *levels between the* $F_k - r + 1^{\text{th}}$ *and the* F_k^{th} *levels of the large tower of order* k*.*

Remark 30. By definition, for every $k \geq 1$, the set NIZ_{k} is either empty or a disjoint union of cylinders of order k (exactly $F_{\ell+1} - r$ if $k = \ell$, F_ℓ if $k = \ell + 1$ or r if $k \geq \ell + 2$). Moreover, we observe that $(\mathrm{NIZ}_{k})_{k \geq \ell}$ is a partition of $\mathbb{X} \backslash \{(01)^\infty, (10)^\infty\}$.

Example 31. Figure 7 illustrates an example with $r = 4 = [101]$.

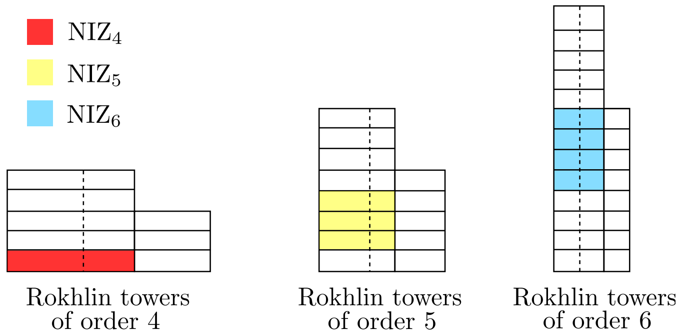

Figure 7: Visualization of $(\mathrm{NIZ}_k)_{k\geq 1}$ when $r = 4 = [101]$.

Figure 8 is a more general figure to aid understanding.

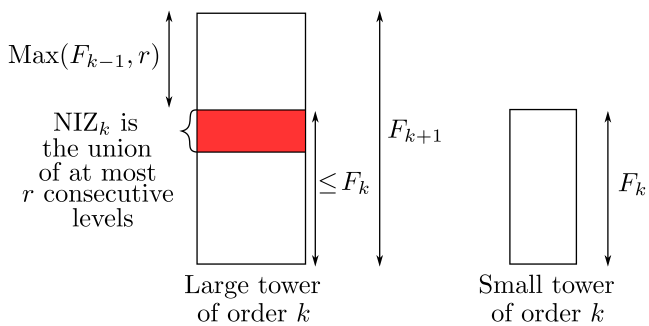

Figure 8: Visualization of NIZ_k, $k \geq \ell + 2$.

Through the following lemmas, we state some observations on this sequence $(\mathrm{NIZ}_k)_{k\geq 1}$. We start by a description of the cylinders of order k that appear in NIZ_k for $k \geq \ell + 2$.

Lemma 32. *If $k \geq 2$, the set NIZ_k is composed of cylinders of order k whose name has a pattern 00 in the leftmost positions.*

Proof. Let $k \geq \ell + 2$, consider a cylinder $C_{n_{k+1}\cdots n_2}$ of order k that belongs in NIZ_k and where $(n_{k+1}, \ldots, n_2) \in \mathbb{X}_f$. Since this cylinder is part of the large Rokhlin tower of order k, $n_{k+1} = 0$. Moreover, this cylinder is not in the F_{k-1} highest cylinders of the same tower, so $n_k = 0$, too. $\square$

Now we show a relation between cylinders of order k in NIZ_k and those of order $k+2$ in NIZ_{k+2} for $k \geq \ell$.

Lemma 33. *Let $k \geq 2$ and let $(n_{k-1}, \ldots, n_2) \in \mathbb{X}_f$. We have the implication*

$$C_{00n_{k-1}\cdots n_2} \subset \mathrm{NIZ}_k \implies C_{0010n_{k-1}\cdots n_2} \subset \mathrm{NIZ}_{k+2} \tag{47}$$

Remark 34. If $k = 2$, we agree that the word $n_{k-1}\cdots n_2$ is the empty word.

Proof. For $r = 1$, we let the reader check that $C_{00} \subset \mathrm{NIZ}_2$ and $C_{0010} \subset \mathrm{NIZ}_4$. For $r \geq 2$, we have $\mathrm{NIZ}_2 = \emptyset$. So, we can assume $k \geq 3$. Let $C_{00n_{k-1}\cdots n_2} \subset \mathrm{NIZ}_k$ and $n := \sum_{i=2}^{k-1} n_i F_i$. Then $n \in \mathrm{NIZ}_k \cap [0, F_k[$ where the interval is an integer interval and considered as being part of $\mathbb{X}$.

1. If $k \geq \ell + 2$, we observe that $\mathrm{NIZ}_k \cap [0, F_k[= [F_k - r, F_k - 1[$. It follows that $F_{k+1} + n$, whose Zeckendorf expansion is $[10n_{k-1} \cdots n_2]$, belongs to $[F_{k+2} - r, F_{k+2} - 1] \subset \mathrm{NIZ}_{k+2}$. Furthermore $F_{k+1} + n \in C_{0010n_{k-1}\cdots n_2}$, which is a level of the large tower of order $k+2$. So $C_{0010n_{k-1}\cdots n_2} \subset \mathrm{NIZ}_{k+2}$.

2. For $k = \ell$, we observe that $\mathrm{NIZ}_\ell \cap [0, F_\ell[= [0, F_{\ell+1} - r - 1[$. It follows that $n + F_{\ell+1} \in [F_{\ell+1}, 2F_{\ell+1} - r - 1]$. We claim that

$$F_{\ell+2} - r \leq F_{\ell+1} \leq 2F_{\ell+1} - r - 1 \leq F_{\ell+2} - 1. \tag{48}$$

   Indeed, since $F_\ell \leq r < F_{\ell+1}$;

   - $F_{\ell+2} - r \leq F_{\ell+2} - F_\ell = F_{\ell+1}$,
   - $F_{\ell+1} - r - 1 \geq 0$ and we deduce $2F_{\ell+1} - r - 1 \geq F_{\ell+1}$,
   - $2F_{\ell+1} - r - 1 \leq F_{\ell+1} + F_{\ell-1} - 1 \leq F_{\ell+2} - 1$.

   From (48), we get $n + F_{\ell+1} \in [F_{\ell+2} - r, F_{\ell+2} - 1]$. Since $[F_{\ell+2} - r, F_{\ell+2} - 1] \subset \mathrm{NIZ}_{\ell+2}$, we conclude as in the first point.

3. If $k = \ell + 1$, we get $\mathrm{NIZ}_k \cap [0, F_\ell[= [F_{\ell+1} - r, F_{\ell+2} - r - 1]$. It follows that $n + F_{\ell+2} \in [F_{\ell+3} - r, 2F_{\ell+2} - r - 1]$. Since $r \geq F_\ell$, $2F_{\ell+2} - r - 1 \leq F_{\ell+3} - 1$, and so $n + F_{\ell+2} \in [F_{\ell+3} - r, F_{\ell+3} - 1] \subset \mathrm{NIZ}_{\ell+2}$. We conclude as in the first point.

□

Lemma 35. *Let $k \geq \ell$ and a collection $(n_{k-1}, \ldots, n_2) \in \mathbb{X}_f$ such that $C_{00n_{k-1}\cdots n_2}$ is a cylinder of NIZ_k. Let $d \in \mathbb{Z}$ such that*

$$\Delta^{(r)}_{|C_{00n_{k-1}\cdots n_2}} = d,$$

then

$$\Delta^{(r)}_{|C_{0010n_{k-1}\cdots n_2}} = d - 1.$$

These lemmas imply the following corollary on which our algorithm is based.

Corollary 36. *Let $k \geq \ell + 4$, the values taken by $\Delta^{(r)}$ on the cylinders of order $k+2$ that compose NIZ_{k+2} are exactly the values taken on cylinders of order k in NIZ_k shifted by -1.*

Proof of Lemma 35. Let $n := \sum_{i=2}^{k-1} n_i F_i$ such that $C_{00n_{k-1}\cdots n_2} \subset \mathrm{NIZ}_k$. We have $C_{00n_{k-1}\cdots n_2} \subset C_{0n_{k-1}\cdots n_2}$, which is, due to the cutting-and-stacking process, one of the r top levels of the large tower of order $k-1$. Also, due to the cutting-and-stacking process, $n + r$ must lies in the small tower of order $k-1$ (see Figure 9) so $(n + r)_k$ must be 1.

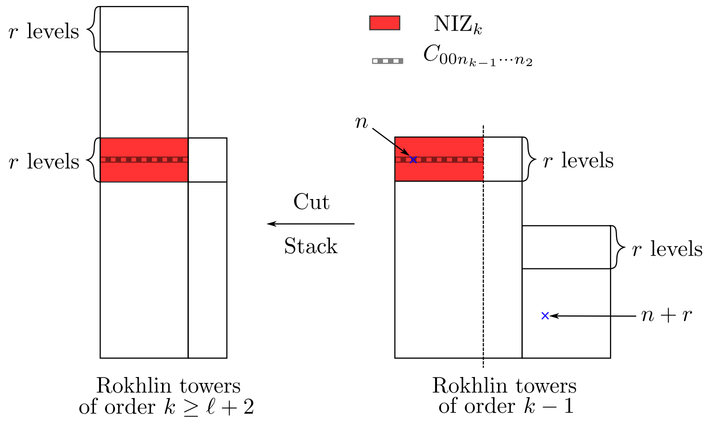

Figure 9: Main argument to justify $(n+r)_k = 1$.

So, we have the following addition

$$\begin{array}{ccccccccc}
(n) & & 0 & 0 & n_{k-1} & \cdots & n_\ell & \cdots & n_2 \\
(r) & + & & & & & r_\ell & \cdots & r_2 \\
\hline
(n+r) & = & 0 & 1 & 0 & \cdots & (n+r)_\ell & \cdots & (n+r)_2
\end{array}$$

with, by hypothesis, $\Delta^{(r)}(n) = d$. Then $n + F_{k+1} + r$ is calculated as follows

$$\begin{array}{ccccccccccc}
(n+F_{k+1}) & 0 & 0 & 1 & 0 & n_{k-1} & \cdots & n_\ell & \cdots & n_2 & \\
(r) & + & & & & & & & r_\ell & \cdots & r_2 \\
\hline
(n+F_{k+1}+r) & = & 0 & 0 & 1 & 1 & 0 & \cdots & (n+r)_\ell & \cdots & (n+r)_2 \\
(n+F_{k+1}+r) & = & 0 & 1 & 0 & 0 & 0 & \cdots & (n+r)_\ell & \cdots & (n+r)_2
\end{array}$$

The sum of digits of $n + F_{k+1} + r$ is the same as those of $n + r$ shifted by -1 due to the correction of the expansion made. Hence, we get $\Delta^{(r)}(n + F_{k+1}) = \Delta^{(r)}(n) - 1$, i.e.,

$$\Delta^{(r)}_{|C_{0010n_{k-1}\cdots n_2}} = \Delta^{(r)}_{|C_{00n_{k-1}\cdots n_2}} - 1.$$

□

Proof of Corollary 36. Definition 29 ensures that the same number of cylinders of order $k+2$ in NIZ_{k+2} is equal to the number of cylinders of order k in NIZ_k: there are r cylinders of each order. Lemma 33 and Lemma 35 enable us to conclude. □

These results enable us to give an algorithm that computes the value of $\mu^{(r)}(d)$ for every $d \in \mathbb{Z}$. This algorithm has a low complexity (polynomial). Indeed, Corollary 36 implies that we can guess the values that appear in the new information zone of order k. So we just need to compute $\Delta^{(r)}$ on a finite number of integers to compute exactly $\mu^{(r)}(d)$ ($d \in \mathbb{Z}$). We explain this in details further in this section. Before, we give pseudocode and provide an example of computation.

5.2 Pseudocode

Algorithm 1 Compute $\mu^{(r)}(d)$

Require: $r \geq 1$ and $d \in \mathbb{Z}$
Ensure: $\mu^{(r)}(d)$

```
STEP 0: Find the unique ℓ ≥ 1 such that F_ℓ ≤ r < F_{ℓ+1}
STEP 1: Passage in NIZ_ℓ
Define App_ℓ ← 0 (number of apparition of d during the for-loop)
for n = 0, ..., F_{ℓ+1} − r − 1 do
  if Δ^(r)(n) = d then
    App_ℓ ← App_ℓ + 1
  end if
end for
STEP 2: Passage in NIZ_{ℓ+1}
Define App_{ℓ+1} ← 0
for n = F_{ℓ+1} − r ..., F_{ℓ+2} − r − 1 do
  if Δ^(r)(n) = d then
    App_{ℓ+1} ← App_{ℓ+1} + 1
  end if
end for
STEP 3: Passage in NIZ_{ℓ+2}
Define App_{ℓ+2} ← 0
for n = F_{ℓ+2} − r, ..., F_{ℓ+2} − 1 do
  Store Δ^(r)(n) in an array ARR1 (with multiplicity)
  if Δ^(r)(n) = d then
    App_{ℓ+2} ← App_{ℓ+2} + 1
  end if
end for
```

STEP 4: Passage in $\mathrm{NIZ}_{\ell+3}$
Define $\mathrm{App}_{\ell+3} \leftarrow 0$
for $n = F_{\ell+3} - r, \ldots, F_{\ell+3} - 1$ **do**
  Store $\Delta^{(r)}(n)$ in an array **ARR2** (with multiplicity)
  if $\Delta^{(r)}(n) = d$ **then**
    $\mathrm{App}_{\ell+3} \leftarrow \mathrm{App}_{\ell+3} + 1$
  end if
end for
STEP 5: Search for d in higher orders
for $i \in$ **ARR1 do**
  if $i > d$ **then**
    d appears at the order $\ell + 2 + 2(i - d)$ so
    if $\mathrm{App}_{\ell+2+2(i-d)}$ is defined **then**
      $\mathrm{App}_{\ell+2+2(i-d)} \leftarrow \mathrm{App}_{\ell+2+2(i-d)} + 1$
    else
      $\mathrm{App}_{\ell+2+2(i-d)} \leftarrow 1$
    end if
  end if
end for
for $i \in$ **ARR2 do**
  if $i > d$ **then**
    d appears at the order $\ell + 3 + 2(i - d)$ so
    if $\mathrm{App}_{\ell+3+2(i-d)}$ is defined **then**
      $\mathrm{App}_{\ell+3+2(i-d)} \leftarrow \mathrm{App}_{\ell+3+2(i-d)} + 1$
    else
      $\mathrm{App}_{\ell+3+2(i-d)} \leftarrow 1$
    end if
  end if
end for
STEP 6: Compute $\mu^{(r)}(d)$
$\mu^{(r)}(d) \leftarrow \sum_{k \geq \ell} \frac{\mathrm{App}_k}{\varphi^k}$

5.3 An example: computation of $\mu^{(4)}$

Let us compute the measure $\mu^{(r)}$ for $r = 4$. We have $\ell = 4$. Looking at the Rokhlin towers (see Example 31) gives that

- $\mathrm{NIZ}_k = \emptyset$ if $k = 1, 2, 3$,
- $\mathrm{NIZ}_4 = C_{0000}$,

- $\mathrm{NIZ}_5 = C_{00001} \sqcup C_{00010} \sqcup C_{00100}$,
- $\mathrm{NIZ}_6 = C_{000101} \sqcup C_{001000} \sqcup C_{001001} \sqcup C_{001010}$,
- $\mathrm{NIZ}_7 = C_{0010000} \sqcup C_{0010001} \sqcup C_{0010010} \sqcup C_{0010100}$.

The next terms of NIZ_k are found using Lemma 33. Using Corollary 36, one can construct the table in FigureValue of Delta r that gives the values taken by $\Delta^{(r)}$ (with multiplicity) on each cylinder of NIZ_k.

Order k	4	5	6	7	8	9	10	11	12	13
Values of			0	-1	-1	-2	-2	-3	-3	-4
$\Delta^{(4)}$ on		1	0	0	-1	-1	-2	-2	-3	-3
the levels	2	1	1	0	0	-1	-1	-2	-2	-3
of NIZ_k		0	-1	-1	-2	-2	-3	-3	-4	-4

...

Values in the columns on both sides of the arrows are shifted by -1

Figure 10: Table of the values of $\Delta^{(4)}$ on NIZ_k.

Thus, it is possible to know how many times some integer $d \in \mathbb{Z}$ appear in each column, i.e., at each order. Since we know the measure of a cylinder at each order, then it is possible to deduce the value of $\mu^{(r)}(d)$. For instance

- The value 2 appears once at order 4, so $\mu^{(4)}(2) = \frac{1}{\varphi^4}$.
- The value 1 appears twice at order 5, once at order 6, so $\mu^{(4)}(1) = \frac{2}{\varphi^5} + \frac{1}{\varphi^6}$.
- The value 0 appears once at order 5 and 8. It also appears twice at order 6 and 7, we deduce that $\mu^{(4)}(0) = \frac{1}{\varphi^5} + \frac{2}{\varphi^6} + \frac{2}{\varphi^7} + \frac{1}{\varphi^8}$.
- The value -1 appears once at order 6 and 10. It appears twice at order 7, 8 and 9. We deduce $\mu^{(4)}(-1) = \frac{1}{\varphi^6} + \frac{2}{\varphi^7} + \frac{2}{\varphi^8} + \frac{2}{\varphi^9} + \frac{1}{\varphi^{10}}$.
- Since -1 is a value at order $\ell + 2 = 6$ that does not appear in the previous order, we deduce by Corollary 36 that the value -2 appears as many times as -1 appears, but at orders incremented by 2. In other words, the value -2 appears once at order 8 and 12. It appears twice at order 9, 10 and 11. Thus $\mu^{(4)}(-2) = \mu^{(4)}(-1) \cdot \frac{1}{\varphi^2}$. We observe that -2 does not appear in the lower orders. This observation is explained in Proposition 37 and Corollary 1.

So, after simplification, we get

$$\mu^{(4)}(d) = \begin{cases} 0, & \text{if } d > 2; \\ \frac{1}{\varphi^4}, & \text{if } d = 2; \\ \frac{1}{\varphi^3}, & \text{if } d = 1; \\ \frac{2}{\varphi^4}, & \text{if } d = 0; \\ \mu^{(4)}(-1) \cdot \varphi^{2d+2} & \text{if } d < 0, \end{cases}$$

(where $\mu^{(4)}(-1) = \frac{1}{\varphi^4} + \frac{1}{\varphi^6}$).

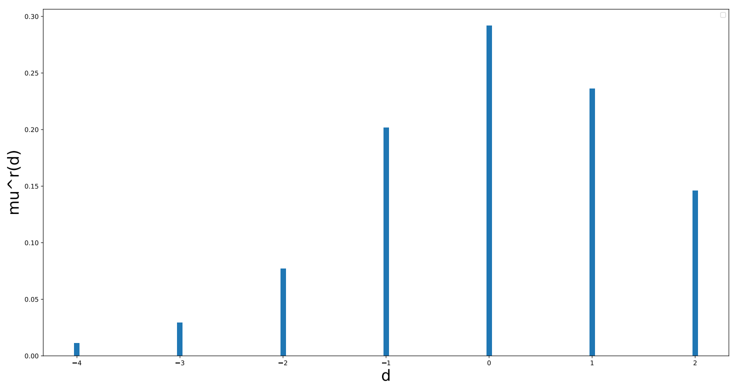

Figure 11: Bar chart of $\mu^{(4)}$.

As an exercise, the reader can check that $\mu^{(1)} = \mu^{(F_3)} = \mu^{(F_4)}$. In the next section, we are going to prove that, for $k \geq 2$ we have

$$\mu^{(F_k)} = \mu^{(1)}.$$

5.4 Remarks on the algorithm

Once again, the algorithm described in Subsections 5.1 and 5.2 can be adapted to an integer base, so the following observations can also be adapted to an integer base. Also, we write an algorithm that returns, for a given $r \geq 1$ and $d \in \mathbb{Z}$, the value $\mu^{(r)}(d)$. Of course, the algorithm can also be adapted to return $\mu^{(r)}(d)$ for every $d \in \mathbb{Z}$ or, at least to be executed on a computer, for a finite number of integers d without more computations (the adaptation only consists of manipulations of arrays).

In this subsection, let $r \geq 1$. Let also $\ell \geq 1$ be the unique integer such that $F_\ell \leq r < F_{\ell+1}$. The first important observation due to the algorithm is that we only need to compute $\Delta^{(r)}(n)$ for a finite number of n's to know exactly the quantity $\mu^{(r)}$. More precisely, we need to compute $\Delta^{(r)}$ for

- $F_{\ell+1} - r$ integers during **STEP 1**,
- F_ℓ integers during **STEP 2**,
- r integers during **STEP 3** and
- r integers during **STEP 4**.

So, we need to compute the image of $F_{\ell+2} + r$ different integers to compute $\mu^{(r)}$ exactly. One can prove that, when $r = F_\ell$, it is possible to adapt the algorithm to reduce that number to $F_{\ell+2}$. The adaptation consists of forgetting **STEP 4** and store the F_ℓ values in **STEP 2**.

The algorithm implies the following properties.

Proposition 37. *If d is small enough in $\mathbb{Z}$ then there exist exactly $2r$ cylinders of (non-necessarily distinct) orders $k_1, \ldots, k_{2r}$, respectively in $\mathrm{NIZ}_{k_1}, \ldots, \mathrm{NIZ}_{k_{2r}}$ on which $\Delta^{(r)}$ takes the value d.*

Remark 38. The "small enough" assumption is specified in the proof.

It leads to Corollary 1 we state again here.

Corollary. *For d small enough in $\mathbb{Z}$, we have the formula*

$$\mu^{(r)}(d-1) = \mu^{(r)}(d) \cdot \frac{1}{\varphi^2}.$$

Proof of Proposition 37. For $k \geq \ell + 2$, there are r cylinders of order k that compose NIZ_k so $\Delta^{(r)}$ takes, at most, r different values on those cylinders. For $k \geq \ell + 2$, let $(d_i^{(k)})_{1 \leq i \leq r}$ be the collection of values taken by $\Delta^{(r)}$ on the cylinders of order k in NIZ_k, repeated with multiplicity. Let

$$m := \min \left\{ d_i^{(k)} \mid i = 1, \ldots, r \text{ and } k = \ell + 2, \ell + 3 \right\}.$$

Now let $d \leq m$. We observe that due to Lemma 32 and Lemma 35, d is actually smaller than every value taken by $\Delta^{(r)}$ on NIZ_ℓ or $\mathrm{NIZ}_{\ell+1}$. Furthermore, for $i \in [1, r]$ and $k \in \{\ell+2, \ell+3\}$, there exists a unique $j_i^{(k)} \in \mathbb{N}$ such that $d = d_i^{(k)} - j_i^{(k)}$. Due to Corollary 36, d is a value reached by $\Delta^{(r)}$ on the corresponding cylinder of order $k + 2j_i^{(k)}$ (by repeated use of Lemma 48 and 35). There are $2r$ cylinders at order $\ell + 2$ and $\ell + 3$ so d appears exactly on $2r$ cylinders. □

Proof of Corollary 1. With the same notation as in the proof of Proposition 37, let $d \leq m$. Then d is a value taken by exactly $2r$ different cylinders of some orders. But, due to Proposition 37 and Corollary 36, $\Delta^{(r)}$ takes the value $d - 1$ on the same number of cylinders but with the orders of those cylinders shifted by $+2$ so their measures are divided by φ^2. □

6 How to prove $\mu^{(F_\ell)} = \mu^{(1)}$

There is an analogous relation in the integer base b case. The law of adding b^ℓ is the same as the one of adding 1. It is trivial in base b since the addition $x + b^\ell$ does not change the first few digits of x, the addition really consists of adding 1 from a certain position. In the Zeckendorf representation case, it is not trivial since carries propagate in both directions. Here, the proof consists of looking at our algorithm described in Section 5 in the special case $r = F_\ell$. However, before starting, we need to compute $\mu^{(1)}$.

Proposition 39. *For $d \in \mathbb{Z}$, we have*

$$\mu^{(1)}(d) = \begin{cases} 0, & \text{if } d \geq 2; \\ \frac{1}{\varphi^2}, & \text{if } d = 1; \\ \frac{1}{\varphi^{2-2d}}, & \text{otherwise.} \end{cases}$$

Proof. We consider the following partition:

$$\mathbb{X} \backslash \{(01)^\infty, (10)^\infty\} = C_{00} \bigsqcup_{d \leq 0} \left(C_{0(01)^{1-d}} \bigsqcup C_{00(10)^{1-d}} \right). \tag{49}$$

We observe that $\Delta^{(1)}(C_{00}) = 1$ and, for every $d \leq 0$,

$$\Delta^{(1)}(C_{0(01)^{1-d}}) = \Delta^{(1)}(C_{00(10)^{1-d}}) = d.$$

Using Proposition 15, we conclude the proof. □

Now let $\ell \geq 3$. We follow the path given by the algorithm. We can rewrite Proposition 29 in our special case $r = F_\ell$:

1. if $2 \leq k \leq \ell - 1$, $\mathrm{NIZ}_k = \emptyset$,
2. NIZ_ℓ is the union of the first $F_{\ell-1}$ levels of the large tower of order ℓ and
3. if $k \geq \ell + 1$, NIZ_k is the union of the F_ℓ levels between the $F_k - F_\ell^{\text{th}}$ and the F_k^{th} levels of the large tower of order k.

Executing the algorithm, we want to compute $\Delta^{(F_\ell)}$ on the levels of NIZ_ℓ. We obtain the following lemma.

Lemma 40.

$$\Delta^{(F_\ell)}(\mathrm{NIZ}_\ell) = 1.$$

Proof. Indeed, consider a cylinder of order ℓ in NIZ_ℓ. Its name has for leftmost digits 000 since the cylinders contains one element of the integer interval $[0, F_{\ell-1} - 1]$. So the addition with F_ℓ considered in these cylinders are

$$\begin{array}{ccccccc} (x) & & \cdots & 0 & 0 & 0 & \cdots \\ (F_\ell) & + & & & 1 & & \\ \hline (x+F_\ell) & = & \cdots & 0 & 1 & 0 & \cdots \end{array}$$

Thus the variation of the sum of digits is 1. $\square$

We now look at $\mathrm{NIZ}_{\ell+1}$, which is composed of F_ℓ cylinders of order $\ell + 1$. We have the following proposition.

Proposition 41. *In* $\mathrm{NIZ}_{\ell+1}$*, there are*

- $F_{\ell-1}$ *levels on which* $\Delta^{(F_\ell)} = 0$ *and*
- $F_{\ell-2}$ *levels on which* $\Delta^{(F_\ell)} = 1$.

For technical reasons, we need to separate the proof in two parts: one when ℓ is odd and the other if ℓ is even. However, since both are treated the same way (the differences are only technicalities), we only consider the case where ℓ is odd. We are going to prove that the values taken by $\Delta^{(F_\ell)}$ are the first F_ℓ terms of the sequence

- $1, 0, 1, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, \ldots$ (if ℓ is odd) or
- $0, 1, 0, 1, 1, 0, 0, 0, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, \ldots$ (if ℓ is even),

where both sequences start with 0 or 1 depending on the parity and then the construction consists of concatenating alternatively 0's or 1's a Fibonacci number of times.

Proof of Proposition 41 if ℓ is odd. Let $\ell' \geq 1$ and $\ell := 2\ell' + 1 \geq 3$. The highest level in $\mathrm{NIZ}_{\ell+1}$ is the cylinder $C_{00(10)^{\ell'}}$ since it contains $F_{\ell+1} - 1$; the lowest is the cylinder $C_{00010^{\ell-3}}$ since it contains $F_{\ell-1}$.

The cylinder of order $\ell + 1$ just below $C_{00(10)^{\ell'}}$ is $C_{00(10)^{\ell'-1}01}$. We observe that, if $\ell = 3$ (or, equivalently, $\ell' = 1$ or $F_\ell = 2$), this last cylinder $C_{00(10)^{\ell'-1}01}$ is actually the same as $C_{00010^{\ell-3}}$. In general, there are two possibilities:

- Either there are no other cylinders of order $\ell + 1$, which means $\ell = 3$ (or, equivalently, $\ell' = 1$ or $F_\ell = 2$). So far, we have mentioned 2 cylinders of order 4 in NIZ_4 and the values of $\Delta^{(F_3)}$ on these cylinders are 0 and 1 (with an easy computation): it is the conclusion of the proposition if $\ell = 3$ (if we agree $F_0 := 1$).
- Or there are other cylinders to find in $\mathrm{NIZ}_{\ell+1}$, which means $F_\ell > 2$ (or, equivalently, $\ell \geq 5$ or $F_\ell \geq 5$) and implies that there are $F_\ell - F_2 \geq 3$ levels to identify.

To continue, we assume $\ell \geq 5$ (or, equivalently, $\ell' \geq 2$) and identify the three cylinders that are below $C_{00(10)^{\ell'-1}01}$: they are

- $C_{00(10)^{\ell'-1}00}$ on which $\Delta^{(F_\ell)}$ equals 1,
- $C_{00(10)^{\ell'-2}0101}$ on which $\Delta^{(F_\ell)}$ equals 0 and
- $C_{00(10)^{\ell'-2}0100}$ on which $\Delta^{(F_\ell)}$ equals 0.

Once again, we observe that, if $\ell = 5$ (or, equivalently, $\ell' = 2$), $C_{00(10)^{\ell'-2}0100} = C_{00010^{\ell-3}}$. There are again two possibilities.

- Either there is no other cylinders of order $\ell + 1$, which means $\ell = 5$ (or, equivalently, $\ell' = 2$ or $F_\ell = 5$). So far again, we have mentioned 5 cylinders of order 6 in NIZ_6 and $\Delta^{(F_5)}$ takes thrice the value 0 and twice the value 1 on these cylinders: it is the conclusion of the proposition if $\ell = 5$.
- Or there are other cylinders to find in $\mathrm{NIZ}_{\ell+1}$, which means $F_\ell > 5$ (or, equivalently, $\ell \geq 7$ or $F_\ell \geq 13$) and implies that there are $F_\ell - F_5 \geq 8$ levels to identify.

We continue the same way. If we assume there are, for some $k \in [1, \ell-1]$, $F_\ell - F_{2k-1} \geq F_{2k}$ cylinders. There are two kind of cylinder.

- Those whose names are $C_{00(10)^{\ell'-k+1}00n_{2k-3}\cdots n_2}$ for all $(n_{2k-3}\cdots n_2) \in \mathbb{X}_f$: there are F_{2k-2} such cylinders and one can compute that $\Delta^{(F_\ell)}$ is 1.
- Those whose names are $C_{00(10)^{\ell'-k}010n_{2k-2}\cdots n_2}$ for all $(n_{2k-2}\cdots n_2) \in \mathbb{X}_f$: there are F_{2k-1} such cylinders and one can compute that $\Delta^{(F_\ell)}$ is 0.

We observe that if $\ell = 2k+1$ (or, equivalently, $\ell' = k$) the last cylinder identified is $C_{00(10)^{\ell'-k}010^{2k-2}}$ and is actually the same as $C_{00010^{\ell-3}}$. So far, we have identified $1 + F_1 + (F_2 + F_3) + (F_4 + F_5) + \cdots + (F_{2k-2} + F_{2k-1}) = F_{2k+1}$ cylinders and

- on $1 + F_2 + F_4 + F_{2k-2} = F_{2k-1}$ of them, $\Delta^{(F_\ell)}$ is 1;
- on $F_1 + F_3 + F_5 + \cdots + F_{2k-1} = F_{2k}$ of them, $\Delta^{(F_\ell)}$ is 0.

It is exactly the conclusion when $\ell = 2k+1$. $\square$

Finally, we look at $\mathrm{NIZ}_{\ell+2}$, which also contains F_ℓ cylinders of order $\ell + 2$.

Proposition 42.

$$\Delta^{(F_\ell)}(\mathrm{NIZ}_{\ell+2}) = 0.$$

Proof. Consider a cylinder of order $\ell + 2$ in $\mathrm{NIZ}_{\ell+2}$. Since it contains exactly one integer of the interval $[F_{\ell+1}, F_{\ell+2} - 1]$, its name is $C_{0010n_{\ell-1}\cdots n_2}$ for every $(n_{\ell-1}\cdots n_2) \in \mathbb{X}_f$. We compute that, on such a cylinder, $\Delta^{(F_\ell)}$ is null. $\square$

We do not need to prove anything for $\mathrm{NIZ}_{\ell+3}$, the algorithm teaches us that $\Delta^{(F_\ell)}$ is going to take the same values, but shifted by -1, on the same number of levels as in $\mathrm{NIZ}_{\ell+1}$. Now it is time to prove our theorem.

Theorem. *For every $\ell \geq 2$, $\mu^{(F_\ell)} = \mu^{(1)}$.*

Proof. We recall

$$\mu^{(1)}(d) = \begin{cases} 0, & \text{if } d \geq 2; \\ \frac{1}{\varphi^2}, & \text{if } d = 1; \\ \frac{1}{\varphi^{2-2d}}, & \text{otherwise.} \end{cases}$$

For $\ell = 2$, this is trivial. Let $\ell \geq 3$ and $d \in \mathbb{Z}$. It is also trivial when $d \geq 2$ since $\Delta^{(F_\ell)}$ does not take this value. Assume that $d = 1$, this value appears in $F_{\ell-1}$ levels in NIZ_ℓ, $F_{\ell-2}$ levels in $\mathrm{NIZ}_{\ell+1}$ and nowhere else. So

$$\mu^{(F_\ell)}(1) = \frac{F_{\ell-1}}{\varphi^\ell} + \frac{F_{\ell-2}}{\varphi^{\ell+1}} = \frac{\varphi^{\ell-1}}{\varphi^{\ell+1}} = \frac{1}{\varphi^2} = \mu^{(1)}(1).$$

If $d \leq 0$, the value d appears, due to Corollary 36, on $F_{\ell-1}$ levels at order $\ell+1-2d$, F_ℓ levels at order $\ell+2-2d$, $F_{\ell-2}$ levels at order $\ell+3-2d$ and nowhere else. Thus

$$\begin{aligned} \mu^{(F_\ell)}(d) &= \frac{F_{\ell-1}}{\varphi^{\ell+1-2d}} + \frac{F_\ell}{\varphi^{\ell+2-2d}} + \frac{F_{\ell-2}}{\varphi^{\ell+3-2d}} \\ &= \frac{\varphi^{\ell+1}}{\varphi^{\ell+3-2d}} \\ &= \frac{1}{\varphi^{2-2d}} \\ &= \mu^{(1)}(d). \end{aligned}$$

□

7 $\Delta^{(r)}$ as a mixing process

We work on the probability space $(\mathbb{X}, \mathscr{B}(\mathbb{X}), \mathbb{P})$. For a given integer r, $\Delta^{(r)}$ is viewed as a random variable with law $\mu^{(r)}$ by Corollary 24 (the randomness comes from the argument x of $\Delta^{(r)}$, considered as a random outcome in $\mathbb{X}$ with law $\mathbb{P}$). In this section, we decompose $\Delta^{(r)}$ as a sum composed of a finite number of random variables satisfying a universal inequality on their mixing coefficients.

7.1 The process

The case $r = 0$ is irrelevant so we assume $r \geq 1$. For $1 \leq i \leq \rho(r)$, we write B_i as the i^{th} block present in the expansion of r, starting from the unit digit. We define $r[i]$ as the integer whose expansion is given by the first i blocks of the expansion of r (see Figure below). We observe that $r[\rho(r)] = r$.

$$
\begin{array}{ll}
r = & [\underbrace{1010}_{B_4}\ 0\ \underbrace{10}_{B_3}\ 0\ \underbrace{1010}_{B_2}\ 000\ \underbrace{101}_{B_1}] \\
r[1] = & \qquad\qquad\qquad\qquad\qquad\quad [101] \\
r[2] = & \qquad\qquad\qquad [1010\ 000\ 101] \\
r[3] = & \qquad\quad [10\ 0\ 1010\ 000\ 101] \\
r[4] = & [1010\ 0\ 10\ 0\ 1010\ 000\ 101]
\end{array}
$$

Figure 12: Example of the construction of $(r[i])_{i=0,\dots,\rho(r)}$.

With the convention $r[0] := 0$, we observe the trivial equality

$$r = \sum_{i=1}^{\rho(r)} (r[i] - r[i-1]).$$

For $1 \leq i \leq \rho(r)$, we define almost everywhere on $\mathbb{X}$ (see Subsection 4)

$$X_i^{(r)} := \Delta^{(r[i]-r[i-1])} \circ T^{r[i-1]}.$$

Since $r[i] - r[i-1] = [B_i 0 \cdots 0]$, the function $X_i^{(r)}$ is a random variable corresponding to the action of the i^{th} block B_i once the previous blocks have already been taken into consideration. From (39), we get

$$\Delta^{(r)} = \sum_{i=1}^{\rho(r)} X_i^{(r)}.$$

In particular, if $x \in \mathbb{X}$ is randomly chosen with law $\mathbb{P}$, then $\displaystyle\sum_{i=1}^{\rho(r)} X_i^{(r)}(x)$ follows the law $\mu^{(r)}$.

7.2 The α-mixing coefficients on the actions of blocks

This part is devoted to the proof of one of the main theorem stated in the introduction: we are going to show that the α-mixing coefficients for the process $(X_i^{(r)})_{i=1,\dots,\rho(r)}$ satisfy a universal upper bound that is independent of r. In particular, these coefficients exponentially decrease to 0.

Theorem. *The α-mixing coefficients of $(X_i^{(r)})_{i=1,\dots,\rho(r)}$ satisfy*

$$\forall k \geq 1, \quad \alpha(k) \leq 12 \left(1 - \frac{1}{\varphi^8}\right)^{\frac{k}{6}} + \frac{1}{\varphi^{2k}}.$$

Proof. Let $k, p \geq 1$. Let $A \in \sigma(X_i^{(r)} : 1 \leq i \leq p)$ and $B \in \sigma(X_i^{(r)} : i \geq k+p)$. Let $C \in \sigma(X_i^{(r)} : p < i < k+p)$ such that $\mathbb{P}(C) > 0$. Then we have the following inequality:

$$\begin{aligned} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| \leq & \ |\mathbb{P}(A \cap B) - \mathbb{P}_C(A \cap B)| \\ & + |\mathbb{P}_C(A \cap B) - \mathbb{P}_C(A)\mathbb{P}_C(B)| \\ & + |\mathbb{P}_C(A)\mathbb{P}_C(B) - \mathbb{P}(A)\mathbb{P}(B)|. \end{aligned} \tag{50}$$

Using (32), we get

$$|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| \leq 3\mathbb{P}(\overline{C}) + |\mathbb{P}_C(A \cap B) - \mathbb{P}_C(A)\mathbb{P}_C(B)|. \tag{51}$$

Following the idea used in the case of an integer base [11], we will consider an event C that has a probability close to 1 and such that, conditionally to C, the events A and B are "almost" independent. A difference with the integer-base case is that, as the digits of a random Zeckendorf-adic numbers are not independent, the last term on the right-hand side of (51) does not totally vanish and, thus, some extra work is needed. However, we can anticipate that this problem of "almost" independence will be solved using the inequality on the ϕ-mixing coefficients (on the law of coordinates) stated in Proposition 19.

To define the event C, we use the ideas developed in Subsection 2.3 (especially Corollaries 12 and 14). Since we are working with blocks, we introduce ℓ_i as the number of patterns 10 in B_i, the i^{th} block of r. We also introduce n_i as the minimal index of digits in B_i. Then for $i = 1, \ldots, \rho(r)$ we define the set $\mathrm{Adm}(i)$ as the set of indices corresponding to the digits of x involved in the assumptions of Corollaries 12 and 14 if we want to apply them in the area of the block B_i in r. More precisely, if $\ell_i = 1$, then

$$\mathrm{Adm}(i) := [n_i - 2, n_i + 3].$$

And, if $\ell_i \geq 2$, then

$$\mathrm{Adm}(i) := [n_i - 2, n_i + 1] \sqcup [n_i - 2 + 2\ell_i, n_i + 1 + 2\ell_i].$$

Observe that $|\mathrm{Adm}(i)| \leq 8$.

$$r = [\cdots\ 00100\ \cdots] \qquad r = [\cdots\ 0010(10)^{\ell(i)-2}100\ \cdots]$$

$$x = [\cdots\ 000000 \cdots] \qquad x = [\cdots\ 0000 \quad \cdots \quad 0000 \cdots]$$

Figure 13: Indices of $\mathrm{Adm}(i)$ and conditions put on x.

Then for $i = 1, \ldots, \rho(r)$ we define

$$C^{(i)} := \{x \in \mathbb{X} \ : \ \forall j \in \mathrm{Adm}(i),\ x_j = 0\}.$$

We also define the events

$$C_1 := \bigcup_{\substack{i=p+2 \\ \text{even}}}^{p+\frac{k}{3}} C^{(i)} \quad \text{and} \quad C_2 := \bigcup_{\substack{i=p+\frac{2k}{3} \\ \text{even}}}^{p+k-2} C^{(i)}$$

and, finally

$$C := C_1 \cap C_2.$$

Blocks of r $\rightarrow \big[B_{\rho(r)} \cdots B_{p+k} \,\vdots\, \cdots B_{\lfloor p+\frac{2k}{3} \rfloor} \,\vdots\, \cdots B_{\lfloor p+\frac{k}{3} \rfloor} \cdots \vdots B_p \cdots B_1 \big]$

C_2 depends on these blocks

C_1 depends on these blocks

Figure 14: Location of the events C_1 and C_2 (dots represent blocks or possible zeros).

Thus, x belongs to C if and only if the expansion of x satisfies the hypotheses of Corollary 12 or 14 for, at least, two blocks placed on the leftmost third of the window (for at least one block) and the rightmost third of the window (for at least one block).

We prove now that C is an event that has a high probability to happen. We claim that

$$\mathbb{P}(C) \geq 1 - 2(1 - \frac{1}{\varphi^8})^{\frac{k}{6}}. \tag{52}$$

Indeed, if we let $\overline{C_1}$ (resp. $\overline{C_2}$) denote the complement of C_1 (resp. C_2) in $\mathbb{X}$, then we have

$$\mathbb{P}(C) = 1 - \mathbb{P}(\overline{C_1}) - \mathbb{P}(\overline{C_2}) + \mathbb{P}(\overline{C_1} \cap \overline{C_2}) \geq \mathbb{P}(C_1) + \mathbb{P}(C_2) - 1.$$

Then we obtain from Lemma 21

$$\begin{aligned}
\mathbb{P}(C_1) &= 1 - \mathbb{P}\left(\bigcap_{\substack{i=p+2 \\ \text{even}}}^{p+\frac{k}{3}} \overline{C^{(i)}} \right) \\
&= 1 - \prod_{\substack{i=p+2 \\ \text{even}}}^{p+\frac{k}{3}} \mathbb{P}\left(\overline{C^{(i)}} \;\middle|\; \bigcap_{\substack{j=p+2 \\ \text{even}}}^{i-2} \overline{C^{(j)}} \right) \\
&\geq 1 - (1 - \frac{1}{\varphi^8})^{\frac{k}{6}},
\end{aligned}$$

because the product contains $\lfloor \frac{p+\frac{k}{3}-p+2-2}{2} \rfloor = \lfloor \frac{k}{6} \rfloor$ terms. We show that $\mathbb{P}(C_2)$ satisfies the same inequality. We thus obtain (52). Observe that, if $k \geq 194$, $\mathbb{P}(C) > 0$.

Now we want to estimate the term $|\mathbb{P}_C(A \cap B) - \mathbb{P}_C(A)\mathbb{P}_C(B)|$ that appears in (51). We want to use Proposition 19, and for that we have to clarify which digits of x the events A and B depends on.

But, conditionally to C, from Corollaries 12 and 14 we know that the actions of the first p blocks only modify digits of indices $\leq N_1 + 2$ where N_1 is the index of digit of the leftmost 1 of the block $B_{\lfloor p+\frac{k}{3}\rfloor}$ of r. Thus, there exists $A' \in \sigma(x_i \ : \ i \leq N_1 + 2)$ such that $A \cap C = A' \cap C$. Likewise, the actions of the $\rho(r) - k - p + 1$ last blocks only modify digits of indices $\geq N_2$ where N_2 the index of the rightmost 0 of the block $B_{\lfloor p+\frac{2k}{3}\rfloor}$. Thus, there also exists $B' \in \sigma(x_i \ : \ i \geq N_2)$ such that $B' \cap C = B \cap C$.

$$r = [\cdots 00\overbrace{(10)^{\ell-1}1\underset{\substack{\uparrow\\ r_{N_2}}}{0}}^{B_{\lfloor p+\frac{2k}{3}\rfloor}}0 \ \cdots \ 00\overbrace{\underset{\substack{\uparrow\\ r_{N_1}}}{1}0(10)^{\ell'-1}}^{B_{\lfloor p+\frac{k}{3}\rfloor}}0 \ \cdots]$$

$$\text{For simplicity, } \ell := \ell(\lfloor p + \tfrac{2k}{3}\rfloor)\,. \qquad \ell' := \ell(\lfloor p + \tfrac{k}{3}\rfloor)$$

Figure 15: Location of indices N_1 and N_2 in the expansion of r.

Then

$$|\mathbb{P}_C(A \cap B) - \mathbb{P}_C(A)\mathbb{P}_C(B)| = |\mathbb{P}_C(A' \cap B') - \mathbb{P}_C(A')\mathbb{P}_C(B')|. \tag{53}$$

Then we get

$$\begin{aligned} |\mathbb{P}_C(A' \cap B') - \mathbb{P}_C(A')\mathbb{P}_C(B')| \leq\ & |\mathbb{P}_C(A' \cap B') - \mathbb{P}(A' \cap B')| \\ & + |\mathbb{P}(A' \cap B') - \mathbb{P}(A')\mathbb{P}(B')| \\ & + |\mathbb{P}(A')\mathbb{P}(B') - \mathbb{P}_C(A')\mathbb{P}_C(B')|. \end{aligned}$$

The term $|\mathbb{P}(A' \cap B') - \mathbb{P}(A')\mathbb{P}(B')|$ can be estimated using the ϕ-mixing coefficient on the law of coordinates. Indeed, observe that $N_2 - N_1$ is at least about order k. Then using (32) and Propositions 18 and 19, we obtain

$$|\mathbb{P}_C(A' \cap B') - \mathbb{P}_C(A')\mathbb{P}_C(B')| \leq 3\mathbb{P}(\overline{C}) + \frac{1}{2}\phi(k). \tag{54}$$

Finally, combining (51), (53) and (54), we obtain

$$|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| \leq 6\mathbb{P}(\overline{C}) + \frac{1}{2}\phi(k).$$

Taking the supremum on A and B, we conclude the proof. □

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