Journal of Integer Sequences, Vol. 27 (2024),
Article 24.8.6

On Some Discrete Statistics of Parking Functions

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Abstract

A parking function is a sequence $\alpha = (a_1, a_2, \ldots, a_n) \in [n]^n$ whose nondecreasing rearrangement $\beta = (b_1, b_2, \ldots, b_n)$ satisfies $b_i \leq i$ for all $1 \leq i \leq n$. We study parking

[1]P. E. Harris was partially supported by a Karen Uhlenbeck EDGE Fellowship.

functions by their ascents (indices at which $a_i < a_{i+1}$), descents (indices at which $a_i > a_{i+1}$), and ties (indices at which $a_i = a_{i+1}$). By using multiset Eulerian polynomials, we give a generating function for the number of parking functions of length n with i descents. We present a recursive formula for the number of parking functions of length n with descents at a specified subset of $[n-1]$. We establish the set of parking functions with descent set I and the set of parking functions with descent set $J = \{n-i : i \in I\}$ are in bijection, and hence these sets have the same cardinality. As a special case, we show that the number of parking functions of length n with descents at the first k indices is given by $\frac{1}{n}\binom{n}{k}\binom{2n-k}{n-k-1}$. We prove this by bijecting to the set of standard Young tableaux of shape $((n-k)^2, 1^k)$, which are enumerated by the same formula. We also study peaks and valleys of parking functions, which are indices at which $a_{i-1} < a_i > a_{i+1}$ and $a_{i-1} > a_i < a_{i+1}$, respectively. We show that the set of parking functions with no peaks and no ties is enumerated by the Catalan numbers, and the set of parking functions with no valleys and no ties is enumerated by the Fine numbers. We conclude our study by characterizing when a parking function is uniquely determined by its statistic encoding; a word indicating which indices in the parking function are ascents, descents, and ties. We provide open problems throughout.

1 Introduction

Throughout, we let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $[n] = \{1, 2, \ldots, n\}$ for $n \in \mathbb{N}$. We define $[0] = \emptyset$. Moreover, we let $\mathfrak{S}_n$ denote the set of permutations of $[n]$ and we use the one-line notation $\pi = \pi_1\ \pi_2\ \cdots\ \pi_n$ to denote elements of $\mathfrak{S}_n$. If $\pi_i > \pi_{i+1}$, then π has a *descent* at index i; and if $\pi_i < \pi_{i+1}$, then π has an *ascent* at index i. There is a long history of studying and enumerating permutations with certain descents and ascents. This dates back to the work of MacMahon, who established that for a fixed set I, with $n \geq \max(I)$ varying, the number of permutations in S_n with descent set I is a polynomial in n [12, Art. 157]. These polynomials are often referred to as *descent polynomials*, and the coefficients of these polynomials and their roots are studied in the literature [1, 5]. Moreover, there is ample work on generalizing these findings to multipermutations [7, 11, 13]. In the present paper, we are motivated by the work of Schumacher, who enumerated descents, ascents, and ties in parking functions [14].

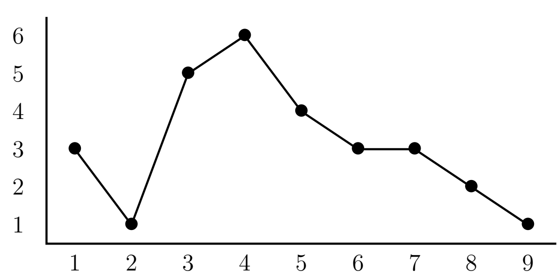

Figure 1: Graph of the parking function $(3, 1, 5, 6, 4, 3, 3, 2, 1) \in \mathrm{PF}_9$.

A parking function is a sequence $\alpha = (a_1, a_2, \ldots, a_n) \in [n]^n$ whose nondecreasing rearrangement $\beta = (b_1, b_2, \ldots, b_n)$ satisfies $b_i \leq i$ for all $1 \leq i \leq n$. Parking functions encode the parking preferences for n cars in a queue attempting to park on a one-way street with n parking spots numbered sequentially as follows. For $i \in [n]$, car i enters the street and attempts to park in spot a_i. If that parking spot is available, car i parks. If the parking spot is occupied, then the car moves forward attempting to park in the first available spot the car encounters. If no such a spot exists, we say the car fails to park. If all cars are able to park under this parking scheme, then α is a parking function.

We let PF_n denote the set of parking functions of length n. If $\alpha = (a_1, a_2, \ldots, a_n) \in \mathrm{PF}_n$, then α has descents and ascents defined in the analogous way as for permutations, and α has a *tie* at i if $a_i = a_{i+1}$. In Figure 1, we plot (i, a_i) to illustrate the parking function $\alpha = (3, 1, 5, 6, 4, 3, 3, 2, 1) \in \mathrm{PF}_9$ has descents at 1, 4, 5, 7, and 8, ascents at 2 and 3, and a tie at 6. Let $\mathrm{des}(\alpha)$ denote the number of descents in α. Schumacher showed that the number of descents among all parking functions of length n is $\binom{n}{2}(n+1)^{n-2}$ [14, Theorem 10]. Schumacher also showed that the set of parking functions with exactly i ties, denoted $\mathrm{PF}_{(n,i)}$, is enumerated by

$$|\mathrm{PF}_{(n,i)}| = \binom{n-1}{i} n^{n-1-i} \tag{1}$$

[14, Theorem 13] and

$$\sum_{\alpha \in \mathrm{PF}_{(n,i)}} \mathrm{des}(\alpha) = \frac{n-1-i}{2} \binom{n-1}{i} n^{n-1-i},$$

[14, Lemma 17]. Let $T_n(i, j)$ denote the number of parking functions of length n with i ties and j descents. If $i + j \geq n$, then $T_n(i, j) = 0$. Table 2 provides the values of $T_6(i, j)$ arranged in a triangular array such that i decreases from top to bottom and j decreases from left to right.[2] More information on this triangle can be found on the OEIS A333829.

For general n, the numbers along the diagonal edges of the triangular array in Figure 2 are the Narayana numbers [14, Theorem 12]

$$T_n(i, 0) = T_n(n-1-i, i) = \frac{1}{i+1} \binom{n}{i} \binom{n-1}{i},$$

and the row sums are given by (1). General closed formulas for the values of $T_n(i, j)$ remain unknown. However, in Section 2, we utilize multiset Eulerian polynomials to give a generating function for the values $\sum_{i=0}^{n} T_n(i, j)$. Our first result follows which we prove later in this section.

[2]Table 2 first appeared in a paper by Schumacher [14, Figure 2], however there was a typographical error for the value $T_6(3, 1)$, which should be 260.

$j=0$ $j=1$ $j=2$ $j=3$ $j=4$ $j=5$

$i=5$						1					
$i=4$					15		15				
$i=3$				50		260		50			
$i=2$			50		1030		1030		50		
$i=1$		15		1240		3970		1240		15	
$i=0$	1		407		3480		3480		407		1

Figure 2: Parking function distribution for $n = 6$.

Theorem 2. *Consider the generating function $\sum_{n=1}^{\infty}\sum_{j=1}^{n-1} d(n,j)y^j x^n$, where $d(n,j)$ is the number of parking functions in PF_n that have j descents, and let $\mathrm{PF}_n^{\uparrow}$ denote the set of nondecreasing parking functions of length n. Then*

$$\sum_{n=1}^{\infty}\sum_{\alpha\in\mathrm{PF}_n} x^n y^{\mathrm{des}(\alpha)} = \sum_{n=1}^{\infty}\sum_{j=1}^{n-1} d(n,j)x^n y^j = \sum_{n=1}^{\infty}\left(\sum_{\beta\in\mathrm{PF}_n^{\uparrow}} A_{M(\beta)}(y)\right)x^n,$$

where $A_\beta(y)$ denotes the multiset Eulerian polynomial in y on all multipermutations of β.

Now, we turn our attention to *descent sets* of parking functions. Given a multiset X of size n with elements in $[n]$, we let $W(X)$ denote the set of multiset permutations of X, written as words $w = w_1\, w_2\, \cdots\, w_n$. If $I \subseteq [n-1]$, then the set of words in $W(X)$ whose descent set is exactly I is denoted

$$D_X(I) = \{w \in W(X) \,:\, \mathrm{Des}(w) = I\},$$

where $\mathrm{Des}(w) = \{i \in [n-1] : w_i > w_{i+1}\}$. Let $D_X(I) = |D_X(I)|$. Let X be the set of nondecreasing parking functions where each $\beta \in \mathrm{PF}_n^{\uparrow}$ is considered as a multiset, denoted as $M(\beta)$. If $I \subseteq [n-1]$, then define

$$D(I;n) = \{\alpha \in \mathrm{PF}_n : \mathrm{Des}(\alpha) = I\} = \bigsqcup_{\beta\in\mathrm{PF}_n^{\uparrow}} D_{M(\beta)}(I),$$

where $\bigsqcup$ denotes the sets are disjoint. We let $d(I;n) = |D(I;n)| = \sum_{\beta\in\mathrm{PF}_n^{\uparrow}} d_{M(\beta)}(I)$. The set $D(\emptyset;n) = \mathrm{PF}_n^{\uparrow}$, and it is known that $|\mathrm{PF}_n^{\uparrow}| = C_n = \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number A000108. Hence, $d(\emptyset;n) = C_n$.

In Section 3, we consider nonempty descent sets $I \subseteq [n-1]$ and establish the following results:

- Theorem 16: The set of parking functions with descent set I and the set of parking functions with descent set $J = \{n-i : i \in I\}$ are in bijection, and hence these sets have the same cardinality. Namely, $d(I;n) = d(J;n)$.

- Theorem 18: Let $I \subseteq [n-1]$ be nonempty, $m = \max(I)$, and $I^- = I \setminus \{m\}$. Then

$$d(I;n) = \sum_{\beta \in \mathrm{PF}_n^{\uparrow}} \left(\sum_{X \in \mathcal{M}(\beta,m)} d_X(I^-) \right) - d(I^-;n), \tag{2}$$

  where, for $\beta \in \mathrm{PF}_n^{\uparrow}$, $\mathcal{M}(\beta,m)$ denotes the collection of multisets consisting of m elements of β.

  We remark that Equation (2) is a generalization of Diaz-Lopez, Harris, Insko, Omar, and Sagan's result [5, Proposition 2.1], which gives a recursion for the number of permutations with a given descent set.

- Proposition 24: Let $n \geq 1$ and $0 \leq k \leq n-1$. If $[k] \subseteq [n-1]$, then

$$d([k];n) = \frac{1}{n}\binom{n}{k}\binom{2n-k}{n-k-1}. \tag{3}$$

  The proof of Proposition 24 is given by a bijection between parking functions of length n with descent set $[k]$ and the set of standard Young tableaux of shape $\lambda = ((n-k)^2, 1^k)$. Hence, Equation (3) can be seen as the reindexing $d([k];n) = f(n, n-k-1)$.

Other permutation statistics of interest are peaks and valleys. Let $\pi = \pi_1\,\pi_2\,\cdots\,\pi_n \in \mathfrak{S}_n$. Recall that if $\pi_{i-1} < \pi_i$ and $\pi_i > \pi_{i+1}$, then i is a *peak* of π. Also recall that if $\pi_{i-1} > \pi_i$ and $\pi_i < \pi_{i+1}$, then i is a *valley* of π. Billey, Burdzy, and Sagan [2] showed that the number of elements in $\mathfrak{S}_n$ with peaks exactly at $I \subset [n-1]$ is given by $p(n)2^{|I|-1}$, where $p(n)$ is a polynomial in n whose degree is one less than the maximum of the set I. The polynomial $p(n)$ is called the *peak polynomial*. It was conjectured that $p(n)$ has a positive integer expansion in a binomial basis centered at the maximum of I and was proven in the affirmative by Diaz-Lopez, Harris, Insko, and Omar [4]. Motivated by these results, we study the set PTPF_n consisting of peakless-tieless parking functions of length n, which is the subset of parking functions that have no peaks and no ties.

In Section 4, we prove the following:

- Corollary 31: If $n \geq 1$, then $|\mathrm{PTPF}_n| = C_n$, the nth Catalan number.

- Corollary 32: If $\mathrm{PTPF}_n(i) = \{\alpha = (a_1, a_2, \ldots, a_n) \in \mathrm{PTPF}_n : a_n = i\}$, then

$$|\mathrm{PTPF}_n(i)| = C_{n,i}\ ,$$

  where $C_{n,i}$ denotes the ith entry in the nth row of the Catalan triangle A009766.

We also prove in Theorem 33 that the number of parking functions of length n with no valleys and no ties is the Fine numbers A000957. To those who are used to thinking about permutations, this asymmetry between peaks and valleys may be surprising. It remains an open problem to enumerate all parking functions with a particular peak or valley set.

In Section 5, we study the characterization of parking functions by the locations of their ascents, descents, and ties which we call the statistic encoding of a parking function. We establish the existence of parking functions for every statistic encoding and fully characterize the parking functions that are uniquely identified by their statistic encoding.

Remark 1. We present open problems, labeled "Problem n", throughout the paper.

2 Permutations of multisets

For a multiset M of positive integers, we let m_i denote the multiplicity of i in M for all $i \in [n]$. A *multipermutation* of M is a word $\pi = \pi_1 \, \pi_2 \, \cdots \, \pi_m$ where $m = m_1 + m_2 + \cdots + m_n$ and π contains i exactly m_i times for all $i \in [n]$. We let $\mathfrak{S}_M$ denote the set of multipermutations on M. A *descent* in a multipermutation π is an index j such that $\pi_j > \pi_{j+1}$. As before, we let $\operatorname{des}(\pi)$ be the number of descents of π. Then

$$A_M(t) = \sum_{\pi \in \mathfrak{S}_M} t^{\operatorname{des}(\pi)} \tag{4}$$

is the *multiset Eulerian polynomial*, where $\mathfrak{S}_M$ is the set of all multipermutations on M. The coefficients of $A_M(t)$ count the number of permutations of a given multiset with a certain number of descents and are known as the *Simon Newcomb numbers*. MacMahon [12, p. 211] showed that $A_M(t)$ occurs as the numerator of the following generating function:

$$\frac{A_M(t)}{(1-t)^{m+1}} = \sum_{\ell=1}^{\infty} \prod_{i=1}^{n} \binom{m_i + \ell}{\ell} t^{\ell}. \tag{5}$$

Equation (5) allows for the explicit computation of the coefficients as

$$[t^k]A_M(t) = |\{\pi \in \mathfrak{S}_M : \operatorname{des}(\pi) = k\}| = \sum_{\ell=0}^{k} (-1)^{\ell} \binom{m+1}{\ell} \prod_{i=1}^{n} \binom{m_i + k - \ell}{k - \ell}, \tag{6}$$

where $[t^k]A_M(t)$ denotes the coefficient of t^k in $A_M(t)$. Using (6), we enumerate descents in parking functions, a superset of permutations, and generalize the results of Schumacher [14].

Now consider the generating function

$$\sum_{n=1}^{\infty} \sum_{j=1}^{n-1} d(n,j) x^n y^j, \tag{7}$$

where $d(n,j)$ is the number of parking functions in PF_n that have j descents. We can now prove Theorem 2 and use it to compute (7). We restate the theorem more succinctly below.

Theorem 2. *Let $n \in \mathbb{N}$, then the generating function encoding the number of descents throughout all* PF_n *is given by*

$$\sum_{n=1}^{\infty} \sum_{\alpha \in \mathrm{PF}_n} x^n y^{\mathrm{des}(\alpha)} = \sum_{n=1}^{\infty} \sum_{j=1}^{n-1} d(n,j) x^n y^j = \sum_{n=1}^{\infty} \left(\sum_{\beta \in \mathrm{PF}_n^{\uparrow}} A_{M(\beta)}(y) \right) x^n.$$

Proof. The result follows from Equation (4). For more details, reference the work by Dillon and Roselle [6, Equation 3.7]. □

To better understand Theorem 2, we present the case where $n = 3$.

Example 3. Let $n = 3$. We have $\mathrm{PF}_3^{\uparrow} = \{(1,1,1),(1,1,2),(1,1,3),(1,2,2),(1,2,3)\}$. We can partition the set PF_3 into $\mathfrak{S}_3$-orbits of the elements of $\mathrm{PF}_3^{\uparrow}$, where $\mathfrak{S}_3$ acts by permuting preferences:

$$\mathrm{PF}_3 = \mathfrak{S}_3(1,1,1) \bigsqcup \mathfrak{S}_3(1,1,2) \bigsqcup \mathfrak{S}_3(1,1,3) \bigsqcup \mathfrak{S}_3(1,2,2) \bigsqcup \mathfrak{S}_3(1,2,3).$$

We can compute $A_\alpha(y)$ for each $\alpha \in \mathrm{PF}_3^{\uparrow}$ using (6). For example, we find

$$\mathfrak{S}_3(1,1,2) = \{(1,1,2),(1,2,1),(2,1,1)\},$$

and therefore $A_{(1,1,2)}(y) = 2y + 1$. Summing over all elements of $\mathrm{PF}_3^{\uparrow}$, we retrieve

$$\sum_{\alpha \in \mathrm{PF}_3^{\uparrow}} A_\alpha(y) = y^2 + 10y + 5.$$

Similarly, we can recover results from Schumacher [14, Figure 2] (see Figure 2), who provides counts for the numbers of parking functions with i ties and j descents. Setting $n = 6$, we find

$$\mathcal{A}_6 = \sum_{\alpha \in \mathrm{PF}_6^{\uparrow}} A_\alpha(y) = y^5 + 422y^4 + 4770y^3 + 8530y^2 + 2952y + 132.$$

Each coefficient in $\mathcal{A}_6$ agrees with the sum of a diagonal of Schumacher's triangle (from the top left to the bottom right): there is 1 element in PF_6 with 5 descents, there are 422 elements in PF_6 with 4 descents, and so on.

Remark 4. Recall that $|\mathrm{PF}_n^{\uparrow}| = C_n$, where C_n is the nth Catalan number. Therefore, partitioning PF_n into $\mathfrak{S}_n$-orbits means partitioning PF_n into C_n disjoint subsets. Setting $t = 1$ in $\sum_{\alpha \in \mathrm{PF}_n^{\uparrow}} A_\alpha(t)$ gives us the known enumeration for parking functions:

$$(n+1)^{n-1} = \sum_{\alpha \in \mathrm{PF}_n^{\uparrow}} A_\alpha(1).$$

Problem 5. Give formulas for $T_n(i,j)$ which denotes the number of parking functions of length n with i ties and j descents.

3 Descent sets of parking functions

We now consider parking functions with a specified descent or ascent set. We restate the definition of $\mathrm{Des}(\alpha)$ for clarity for the reader and define $\mathrm{Asc}(\alpha)$. For $\alpha = (a_1, a_2, \ldots, a_n) \in \mathrm{PF}_n$, we let $\mathrm{Des}(\alpha) = \{i \in [n-1] : a_i > a_{i-1}\}$ and $\mathrm{Asc}(\alpha) = \{i \in [n-1] : a_i < a_{i-1}\}$. For $I \subseteq [n-1]$, define

$$D(I;n) = \{\alpha \in \mathrm{PF}_n : \mathrm{Des}(\alpha) = I\}, \text{ and}$$
$$A(I;n) = \{\alpha \in \mathrm{PF}_n : \mathrm{Asc}(\alpha) = I\},$$

and denote their cardinalities by $d(I;n) = |D(I;n)|$ and $a(I;n) = |A(I;n)|$.

Recall that $D(\emptyset;n) = \mathrm{PF}_n^{\uparrow}$ and that $|\mathrm{PF}_n^{\uparrow}| = C_n = \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number A000108. Hence, $d(\emptyset;n) = C_n$. For $1 \leq n \leq 4$, Table 1 provides data on the number of parking functions with a given descent set.

Next, we prove some preliminary results which help establish that the number of parking functions with descents at the indices in $I \subseteq [n-1]$ is the same as the number of parking functions with descents at indices in $J = \{n - i : i \in I\}$.

Proposition 6. *Let $I \subseteq [n-1]$ and $J = \{n-i : i \in I\}$. Then $d(I;n) = a(J;n)$*

Proof. Given $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\mathrm{Des}(\alpha) = I$, then $(\alpha_n, \alpha_{n-1}, \ldots, \alpha_1)$ has $\mathrm{Asc}(\alpha) = J$. This defines a bijection between the sets $D(I;n)$ and $A(J;n)$. Thus, $d(I;n) = a(J;n)$, as claimed. □

Next, we define a function used in our subsequent results. To simplify notation, we denote tuples in one-line notation and refer to them as words, i.e., the tuple $(w_1, w_2, \ldots, w_n)$ is written as the word $w_1\ w_2\ \cdots\ w_n$.

Definition 7. Let $\mathfrak{S}_n(\mathbf{y})$ denote the orbit of $\mathbf{y} \in \mathbb{N}^n$ under the action of the symmetric group $\mathfrak{S}_n$ permuting indices. Given $\mathbf{x} \in \mathfrak{S}_n(\mathbf{y})$, let $\arg\min(\mathbf{x})$ denote the set of indices of minimal elements of $\mathbf{x}$ and $\arg\max(\mathbf{x})$ denote the set of indices of maximal elements of $\mathbf{x}$.

Define the map $\nu : \mathfrak{S}_n(\mathbf{y}) \to \mathfrak{S}_n(\mathbf{y})$ recursively as follows. For $\mathbf{x} \in \mathfrak{S}_n(\mathbf{y})$:

Case 0: If the length of $\mathbf{x}$ is 0 or 1, then $\nu(\mathbf{x}) = \mathbf{x}$.

Case 1: If the length of $\mathbf{x}$ is larger than 1 and $a \geq b$ for all $a \in \arg\min(\mathbf{x})$ and $b \in \arg\max(\mathbf{x})$, then let $i = \max(\arg\min(\mathbf{x}))$ and $j = \min(\arg\max(\mathbf{x}))$. Then

$$\nu(\mathbf{x}) = \nu(x_1\ x_2\ \cdots\ x_{j-1})\ x_i\ \nu(x_{j+1}\ \cdots\ x_{i-1})\ x_j\ \nu(x_{i+1}\ \cdots\ x_n).$$

Case 2: If the length of $\mathbf{x}$ is larger than 1 and there is some $a < b$ with $a \in \arg\min(\mathbf{x})$ and $b \in \arg\max(\mathbf{x})$, then let i be the greatest element of $\arg\min(\mathbf{x})$ that is smaller than some element of $\arg\max(\mathbf{x})$ and let j be the smallest element of $\arg\max(\mathbf{x})$ greater than i. Then

$$\nu(\mathbf{x}) = \nu(x_1\ x_2\ \cdots\ x_{i-1})\ x_j\ \nu(x_{i+1}\ \cdots\ x_{j-1})\ x_i\ \nu(x_{j+1}\ \cdots\ x_n).$$

n	$I \subseteq [n-1]$	$d(I,n)$	$D(I,n)$
1	$\emptyset$	$C_1 = 1$	1
2	$\emptyset$	$C_2 = 2$	11, 12
	$\{1\}$	1	21
3	$\emptyset$	$C_3 = 5$	111, 112, 113, 122, 123
	$\{1\}$	5	211, 311, 212, 213, 312
	$\{2\}$	5	131, 121, 221, 132, 231
	$\{1,2\}$	1	321
4	$\emptyset$	$C_4 = 14$	1111, 1112, 1113 1114, 1122, 1123, 1124, 1133, 1134, 1222, 1223, 1224, 1233, 1234
	$\{1\}$	21	2111, 3111, 4111, 2112, 2113, 3112, 2114, 4112, 3113, 3114, 4113, 2122, 2123, 3122, 4122, 2133, 3123, 2134, 3124, 4123, 2124
	$\{2\}$	31	1211, 1311, 1411, 1212, 2211, 1213, 1312, 2311, 1214, 1412, 2411, 1313, 3311, 1314, 1413, 3411, 2212, 1322, 2213, 2312, 1422, 2214, 2412, 1323, 2313, 3312, 1324, 1423, 2314, 2413, 3412
	$\{3\}$	21	1121, 1131, 1141, 1221, 1132, 1231, 1142, 1241, 1331, 1143, 1341, 2221, 1232, 2231, 1242, 2241, 1332, 2331, 1243, 1342, 2341
	$\{1,2\}$	9	3211, 4211, 4311, 3212, 4212, 3213, 3214, 4213, 4312
	$\{1,3\}$	19	2121, 2131, 3121, 4121, 3131, 2141, 3141, 4131, 2132, 3221, 2142, 3121, 3231, 4221, 2143, 3142, 3241, 4132, 4231
	$\{2,3\}$	9	1321, 1421, 1431, 2421, 3321, 1432, 2431, 2321, 3421
	$\{1,2,3\}$	1	4321

Table 1: Parking functions of length $1 \leq n \leq 4$ with a given descents set.

We illustrate Definition 7 in the following example.

Example 8. Let $\mathbf{y} = (1, 1, 1, 1, 2, 3, 3, 4, 4) \in \mathbb{N}^9$ and consider $\mathbf{x} = (1, 4, 1, 3, 4, 3, 1, 2, 1) \in \mathfrak{S}_n(\mathbf{y})$. We write $\mathbf{x}$ in one-line notation as: $\mathbf{x} = 1\ 4\ 1\ 3\ 4\ 3\ 1\ 2\ 1$. Note $\mathrm{Asc}(\mathbf{x}) = \{1, 3, 4, 7\}$, $\arg\min(\mathbf{x}) = \{1, 3, 7, 9\}$, and $\arg\max(\mathbf{x}) = \{2, 5\}$. We use Definition 7 to give $\nu(\mathbf{x})$.

- Since the length of $\mathbf{x}$ is larger than 1 and there exists $a \in \arg\min(\mathbf{x})$ and $b \in \arg\max(x)$ with $a < b$, we are in Case 2. Then $i = 3$ and $j = 5$. Hence,

$$\nu(\mathbf{x}) = \nu(1\ 4)\ \underbrace{4}_{x_5}\ \nu(3)\ \underbrace{1}_{x_3}\ \nu(3\ 1\ 2\ 1). \tag{8}$$

  We now consider $\nu(1\ 4)$, $\nu(3)$ and $\nu(3\ 1\ 2\ 1)$ independently.

- For the subword 1 4, we are again in Case 2. So, $\nu(1\ 4) = 4\ 1$.
- Since 3 has length 1, by Case 0, $\nu(3) = 3$.
- For the subword 3 1 2 1, $\arg\min(3\ 1\ 2\ 1) = \{2, 4\}$ and $\arg\max(3\ 1\ 2\ 1) = \{1\}$. Hence by Case 1, $\nu(3\ 1\ 2\ 1) = 1\ \nu(1\ 2)\ 3$. Then by Case 2, $\nu(1\ 2) = 2\ 1$.

Substituting these findings into Equation (8) yields

$$\nu(1\ 4\ 1\ 3\ 4\ 3\ 1\ 2\ 1) = 4\ 1\ 4\ 3\ 1\ 1\ 2\ 1\ 3.$$

The following remark is helpful and uses wording we implement in subsequent proofs.

Remark 9. In Definition 7, Case 1 only considers words $\mathbf{x}$ whose minimal value(s) all appear to the right of all of its maximal value(s). Applying ν to $\mathbf{x}$, the result is a word $\nu(\mathbf{x})$ that has all maximal value(s) to the right of all minimal value(s). Whenever $\mathbf{x}$ has a minimal value to the left of a maximal value, then Case 2 in Definition 7 is applied. Applying ν to $\mathbf{x}$ results in $\nu(\mathbf{x})$ having a minimal value to the left of a maximal value.

Proposition 10. *Fix $\mathbf{y} \in \mathbb{N}^n$. The restriction of the map ν taking the set of elements of $\mathfrak{S}_n(\mathbf{y})$ with ascent set I to the set of elements of $\mathfrak{S}(\mathbf{y})$ with descent set I is a bijection.*

Before proving Proposition 10, we need the following lemmas.

Lemma 11. *Let $\mathbf{y} \in \mathbb{N}^n$ and $\mathbf{x} \in \mathfrak{S}_n(\mathbf{y})$. Then $\mathrm{Asc}(\mathbf{x}) = \mathrm{Des}(\nu(\mathbf{x}))$ for ν as defined in Definition 7.*

Example 12. Continuing Example 8, observe that $\mathrm{Asc}(\mathbf{x}) = \mathrm{Asc}(1\ 4\ 1\ 3\ 4\ 3\ 1\ 2\ 1) = \{1, 3, 4, 7\}$ and $\mathrm{Des}(\nu(\mathbf{x})) = \mathrm{Des}(4\ 1\ 4\ 3\ 1\ 1\ 2\ 1\ 3) = \{1, 3, 4, 7\}$, thus $\mathrm{Asc}(\mathbf{x}) = \mathrm{Des}(\nu(\mathbf{x}))$.

Proof of Lemma 11. Let x and y have length n. For the base case, if $n = 0$ or $n = 1$, then by Case 0 of Definition 7, $\nu(\mathbf{x}) = \mathbf{x}$ and $\mathrm{Asc}(\mathbf{x}) = \emptyset = \mathrm{Des}(\nu(\mathbf{x}))$.

Fix $n > 1$, $\mathbf{y} \in \mathbb{N}^n$, and $\mathbf{x} \in \mathfrak{S}_n(\mathbf{y})$. Assume, for strong induction, that for all $k < n$, $\mathbf{y}' \in \mathbb{N}^k$, and $\mathbf{x}' \in \mathfrak{S}_k(\mathbf{y}')$, we have $\mathrm{Asc}(\mathbf{x}') = \mathrm{Des}(\nu(\mathbf{x}'))$. We now consider $\nu(\mathbf{x}) = \nu(x_1\ x_2\ \cdots\ x_n)$ and proceed via a case-by-case analysis given by the cases in Definition 7.

Case 1: Suppose $a \geq b$ for all $a \in \arg\min(\mathbf{x})$ and $b \in \arg\max(\mathbf{x})$. Let $M = \max(\mathbf{x})$ and let $m = \min(\mathbf{x})$. Then

$$\mathbf{x} = x_1\ x_2\ \cdots\ x_{j-1}\ \underbrace{M}_{j}\ x_{j+1}\ \cdots\ x_{i-1}\ \underbrace{m}_{i}\ x_{i+1}\ \cdots\ x_n$$

where $j = \min(\arg\max(\mathbf{x}))$ and $i = \max(\arg\min(\mathbf{x}))$. By Case 1 in Definition 7,

$$\nu(\mathbf{x}) = \nu(x_1\ x_2\ \cdots\ x_{j-1})\ \underbrace{m}_{j}\ \nu(x_{j+1}\ \cdots\ x_{i-1})\ \underbrace{M}_{i}\ \nu(x_{i+1}\ \cdots\ x_n).$$

By the inductive hypothesis, for each

$$\mathbf{x}' \in \{x_1\ x_2\ \cdots\ x_{j-1},\ x_{j+1}\ x_{j+2}\ \cdots\ x_{i-1},\ x_{i+1}\ x_{i+2}\ \cdots\ x_n\}$$

we know $\mathrm{Asc}(\mathbf{x}') = \mathrm{Des}(\nu(\mathbf{x}'))$. Thus, we only need to examine $\nu(\mathbf{x})$ at indices $j-1$, j, $i-1$, and i. We proceed with a case-by-case analysis:

(a) Consider the value at index $j-1$:

If $j = 1$, then $j - 1 = 0$, and hence there is no ascent, descent, or tie in $\mathbf{x}$ at index 0. Thus, the ascent and descent sets are empty and ν simply maps the empty set to the empty set.

If $j > 1$, then $\mathbf{x}$ has an ascent at position $j-1$ because $M = \max(\mathbf{x})$ and $j = \min(\arg\max(\mathbf{x}))$. Since $\arg\min(\mathbf{x}) \subseteq \{j, j+1, \ldots, i-1, i\}$, the element at index $j-1$ in $\nu(x)$ is larger than m. Since ν places m in position j, $\nu(\mathbf{x})$ has a descent at position $j-1$.

(b) Consider the values at indices j and $i-1$:

Note, that $\mathbf{x}$ must have a non-ascent at position j because M is a maximal element causing either a tie or descent at j. Likewise, there is a non-ascent in position i because m is a minimal element.

Swapping m and M in $\nu(x)$ results in the following: First, since m is a minimal element, placing m at position j ensures that at this index we either have an ascent or tie. This means $\nu(x)$ has a non-descent at index j. Second, since M is a maximal element, placing M at position i ensures that the index prior is less than or equal to M. This implies that $\nu(\mathbf{x})$ has a non-descent at index $i-1$.

(c) Consider the value at index i:

If $i = n$, then there is no ascent, descent, or tie at position i in $\mathbf{x}$, and the proposition is vacuously true.

If $i < n$, the reasoning is symmetric to the case for $j-1$; $\mathbf{x}$ has an ascent at position i and $\nu(\mathbf{x})$ has a descent at the same position.

Case 2: Suppose there is some $a < b$ with $a \in \arg\min(\mathbf{x})$ and $b \in \arg\max(\mathbf{x})$. Let i be the greatest element of $\arg\min(\mathbf{x})$ that is smaller than some element of $\arg\max(\mathbf{x})$ and let j be the smallest element of $\arg\max(\mathbf{x})$ greater than i. Furthermore, let $M = \max(\mathbf{x})$ and let $m = \min(\mathbf{x})$. Then

$$\mathbf{x} = x_1\ x_2\ \cdots\ x_{i-1}\ \underbrace{m}_{i}\ x_{i+1}\ \cdots\ x_{j-1}\ \underbrace{M}_{j}\ x_{j+1}\ \cdots\ x_n$$

and from Case 2 in Definition 7,

$$\nu(\mathbf{x}) = \nu(x_1\ x_2\ \cdots\ x_{i-1})\ \underbrace{M}_{i}\ \nu(x_{i+1}\ \cdots\ x_{j-1})\ \underbrace{m}_{j}\ \nu(x_{j+1}\ \cdots\ x_n).$$

We again only need to examine positions $i-1$, i, $j-1$, and j. We proceed with a case-by-case analysis:

(a) Consider the value at index $i-1$:

If $i = 1$, then $i - 1 = 0$, and hence there is no ascent, descent, or tie in $\mathbf{x}$ at index 0. Thus, the ascent and descent sets are both necessarily empty, so ν simply maps the empty set to the empty set.

If $i > 1$, then in $\mathbf{x}$ there is a non-ascent at position $i - 1$. Then swapping m and M, ensures that there is a non-descent at position $i - 1$ in $\nu(\mathbf{x})$.

(b) Consider the values at positions i and $j - 1$:

There are no minimal or maximal elements in $x_{i+1}\ \cdots\ x_{j-1}$, but $x_i = m$ and $x_j = M$. Therefore, $\mathbf{x}$ has ascents in positions i and $j - 1$.

Then for the same reasons, because ν places m in position j and M in position i, $\nu(\mathbf{x})$ has descents at positions i and $j - 1$.

(c) Consider the value at index j:

If $j = n$, then there is no ascent, descent, or tie at position j; and the ascent and descent sets are empty and ν is again mapping the empty set to the empty set.

If $j < n$, the reasoning is symmetric to the case for $i - 1$; $\mathbf{x}$ must have a non-ascent at position j and $\nu(\mathbf{x})$ must have a non-descent at the same position.

Therefore, $\mathrm{Asc}(\mathbf{x}) = \mathrm{Des}(\nu(\mathbf{x}))$. □

Lemma 13. *The map ν of Definition 7 is injective.*

Proof. We now left-invert ν by inducting on length. In the base case, when the length is zero or one, then ν acts as the identity causing the left-inverse to also be the identity, hence injective. Assume for strong induction, that ν is left-invertible for all inputs of length less than n (i.e., of length $1 < k < n$). Now consider $\nu(\mathbf{x})$ of length n. We left invert $\nu(\mathbf{x})$ by uniquely identifying $\mathbf{x}$. Let $m = \min(\nu(\mathbf{x}))$ and $M = \max(\nu(\mathbf{x}))$. By Definition 7, ν does not change the content of the input. Hence, $m = \min(\nu(\mathbf{x})) = \min(\mathbf{x})$ and $M = \max(\nu(\mathbf{x})) = \max(\mathbf{x})$. Given that $n > 1$, then either:

1. $\mathbf{x}$ has the property of either Case 1 of Definition 7 and

$$\nu(\mathbf{x}) = \nu(x_1\ x_2\ \cdots\ x_{j-1})\ \underbrace{m}_{j}\ \nu(x_{j+1}\ \cdots\ x_{i-1})\ \underbrace{M}_{i}\ \nu(x_{i+1}\ \cdots\ x_n),$$

   or

2. $\mathbf{x}$ is of the from in Case 2 of Definition 7 and

$$\nu(\mathbf{x}) = \nu(x_1\ x_2\ \cdots\ x_{i-1})\ \underbrace{M}_{i}\ \nu(x_{i+1}\ \cdots\ x_{j-1})\ \underbrace{m}_{j}\ \nu(x_{j+1}\ \cdots\ x_n),$$

Thus, i and j are indices determined using Definition 7.

Given $\nu(\mathbf{x}) \in \nu(\mathfrak{S}_n(\mathbf{y}))$, if the indices i and j can be uniquely identified by the structure of the word $\nu(\mathbf{x})$, then we can inductively construct ν^{-1} by swapping the ith and jth entries of $\nu(\mathbf{x})$ and then applying ν^{-1} to each of the subwords of $\nu(\mathbf{x})$ with smaller length. This then would uniquely identify the input $\mathbf{x}$, and establish injectivity.

The key to creating the inverse relies on determining whether $\nu(\mathbf{x})$ satisfies either:

Case (a): all minimal elements are to the left of all maximal elements, or

Case (b): there exists a minimal element to the right of a maximal element.

We consider each of the above cases independently.

For Case (a) where all minimal elements are to the left of all maximal elements in $\nu(\mathbf{x})$, the first step in applying ν to $\mathbf{x}$ cannot have arisen from Case 2 in Definition 7, because Case 2 explicitly places a minimal element to the right of a maximal element. Therefore, the first step in applying ν to $\mathbf{x}$ arose from Case 1 in Definition 7. This implies that $\mathbf{x}$ has all maximal elements to the left of all minimal elements and in the subword $x_{j+1}\ \cdots\ x_{i-1}$, all instances of M are to the left of all instances of m. Therefore, if $x_{j+1}\ \cdots\ x_{i-1}$ has instances of both M and m, ν acts on $x_{j+1}\ \cdots\ x_{i-1}$ as in Case 1 of Definition 7. Then

$$\nu(\mathbf{x}) = \nu(x_1\ x_2\ \cdots\ x_{j-1})\ \underbrace{m}_{j}\ \nu(x_{j+1}\ \cdots\ x_{i-1})\ \underbrace{M}_{i}\ \nu(x_{i+1}\ \cdots\ x_n),$$

where index j is the smallest index that contains a minimal element and index i is the largest index that contains a maximal element. Therefore, $i = \max(\arg\max(\nu(\mathbf{x})))$ and $j = \min(\arg\min(\nu(\mathbf{x})))$. Then by induction hypothesis, we have uniquely identified $\mathbf{x}$, which gave rise to $\nu(\mathbf{x})$.

Now consider Case (b) where there exists a minimal element to the right of a maximal element in $\nu(\mathbf{x})$. Recall that $\nu(\mathbf{x}) \in \nu(\mathfrak{S}_n(\mathbf{y}))$. By induction on the minimum of the number of minimal and maximal elements in $\mathbf{y}$, if $\nu(\mathbf{x})$ was the output from Case 1 in Definition 7, then $\nu(\mathbf{x})$ would have all minimal elements to the left of all maximal elements. But by assumption $\nu(\mathbf{x})$ satisfies Case (b). Therefore, the first step in applying ν to $\mathbf{x}$ arose from Case 2 in Definition 7. Then i is the index in $\mathbf{x}$ containing the rightmost minimal element that has a maximal element to its right in $\mathbf{x}$, and j is the leftmost maximal element to the right of i. Thus,

$$\mathbf{x} = x_1 \; x_2 \; \cdots \; x_{i-1} \; \underbrace{m}_{i} \; x_{i+1} \; \cdots \; x_{j-1} \; \underbrace{M}_{j} \; x_{j+1} \; \cdots \; x_n.$$

By the maximality of i, all instances of m must appear to the right of all instances of M in the subword $x_{j+1} \; \cdots \; x_n$. Moreover, by our construction of i and j, there are no instances of m or M in the subword $x_{i+1} \; \cdots \; x_{j-1}$. Hence,

$$\nu(\mathbf{x}) = \nu(x_1 \; x_2 \; \cdots \; x_{i-1}) \; \underbrace{M}_{i} \; \nu(x_{i+1} \; \cdots \; x_{j-1}) \; \underbrace{m}_{j} \; \nu(x_{j+1} \; \cdots \; x_n),$$

and $\nu(x_{j+1} \; \cdots \; x_n)$ causes all instances of m to appear to the left of all instances of M according to Case 1 of Definition 7. It follows that i is the index of the rightmost maximal element of $\nu(\mathbf{x})$ with a minimal element to its right, and j is the index of the leftmost minimal element to the right of i in $\nu(\mathbf{x})$. Then by the induction hypothesis, we have uniquely identified $\mathbf{x}$, which gave rise to $\nu(\mathbf{x})$.

Thus, we have shown that whether $\nu(\mathbf{x})$ satisfies Case (a) or Case (b), we can uniquely recover $\mathbf{x}$ which gives rise to $\nu(\mathbf{x})$. This establishes that ν is an injection. □

We are now ready to prove Proposition 10.

Proof of Proposition 10. By Lemma 13 and Lemma 11, ν, as constructed in Definition 7, is an injective map $\mathfrak{S}_n(\mathbf{y}) \to \mathfrak{S}_n(\mathbf{y})$ for all $\mathbf{y} \in \mathbb{N}^n$ such that $\mathrm{Asc}(\mathbf{x}) = \mathrm{Des}(\nu(\mathbf{x}))$. Then since $|\mathfrak{S}(\mathbf{y})|$ is finite, ν must also be a surjection. Therefore, ν is the desired bijection. □

Remark 14. As defined in Proposition 10, ν maps the ascent set to the descent set but does not (and cannot possibly) make additional promises about mapping the descent set to the ascent set. In Example 8, $\mathrm{Asc}(\mathbf{x}) = \{1, 3, 4, 7\} = \mathrm{Des}(\nu(\mathbf{x}))$, but $\mathrm{Des}(\mathbf{x}) = \{2, 6, 8\}$ and $\mathrm{Asc}(\nu(\mathbf{x})) = \{2, 5, 6, 8\}$. The map that takes the descent set to the ascent set is ν^{-1}.

Corollary 15. *There is a bijection $D(I; n) \to A(I; n)$.*

Proof. Apply ν from Proposition 10 to a parking function. □

Theorem 16. *The set of parking functions with descent set I and the set of parking functions with descent set $J = \{n - i : i \in I\}$ are in bijection, and hence $d(I; n) = d(J; n)$.*

Proof. This result follows from Proposition 6 and Corollary 15. □

In what follows, we say the sets $I \subseteq [n-1]$ and $J = \{n-i : i \in I\}$ are *self-dual* if $I = J$. We now give a formula for the number of self-dual sets.

Our next objective is to give a recursive formula for the number of parking functions of length n with descent set $I \subseteq [n-1]$. We begin with some general definitions and notation. Given a multiset X of positive integers with size n, let $W(X)$ denote the set of multipermutations of X. For $I \subseteq [n-1]$, let

$$D_X(I) = \{w \in W(X) : \mathrm{Des}(w) = I\},$$

and $D_X(I) = |D_X(I)|$.

Lemma 17. *Given $\beta \in \mathrm{PF}_n^{\uparrow}$, let $M(\beta)$ be the multiset of entries of β. If $I \subseteq [n-1]$, then*

$$\sum_{\beta \in \mathrm{PF}_n^{\uparrow}} d_{M(\beta)}(I) = d(I; n).$$

Proof. This follows from definitions of $D_X(I)$ and $D(I;n)$, and that each parking function is a rearrangement of a unique increasing parking function. □

Recall that $d(\emptyset; n) = C_n$, the nth Catalan number. We now give a recursion for the number of parking functions with a nonempty descent set.

Theorem 18. *Let $I \subseteq [n-1]$ be nonempty, $m = \max(I)$, and $I^- = I \setminus \{m\}$. Then*

$$d(I; n) = \sum_{\beta \in \mathrm{PF}_n^{\uparrow}} \left(\sum_{X \in \mathcal{M}(\beta; m)} d_X(I^-; m) \right) - d(I^-; n),$$

where, for $\beta \in \mathrm{PF}_n^{\uparrow}$, $\mathcal{M}(\beta, m)$ denotes the collection of multisets consisting of m elements of β.

Proof. Consider the set P of parking functions $\alpha \in \mathrm{PF}_n$ that can be written as a concatenation $\alpha = \alpha' + \alpha''$ and satisfy

1. that the length of α' is m and the length of α'' is $n - m$ and
2. $\mathrm{Des}(\alpha') = I^-$ and $\mathrm{Des}(\alpha'') = \emptyset$.

We now count the elements of P in two ways.

First, observe that we can write P as the disjoint union of those α where $\alpha'_m > \alpha''_1$ and those where $\alpha'_m \leq \alpha''_1$. Hence,

$$|P| = d(I^-; n) + d(I; n). \tag{9}$$

On the other hand, the elements of P can be constructed as follows. Consider every $\beta \in \mathrm{PF}_n^{\uparrow}$ as a multiset with n elements. For a fixed multiset $\beta \in \mathrm{PF}_n^{\uparrow}$, $\mathcal{M}(\beta, m)$ is the collection of multisets of size m with entries in β.

For every $X \in \mathcal{M}(\beta, m)$, arrange the elements so that they have descent set I^-. This can be done in $d_X(I^-)$ ways. The remaining $n-m$ values in $\beta \setminus X$ are used to construct α'' by placing those remaining values in nondecreasing order, so as to have no descents. This can be done in a unique way. Thus, the number of elements of P is given by

$$|P| = \sum_{\beta \in \mathrm{PF}_n^{\uparrow}} \left(\sum_{X \in \mathcal{M}(\beta, m)} d_X(I^-) \right). \tag{10}$$

Solving Equation (9) for $d(I;n)$ and substituting the right hand-side of Equation (10) in for $|P|$ yields the desired result. □

Example 19. Let $I = \{1, 3\}$. Then by Theorem 18,

$$d(I;4) = \sum_{\beta \in \mathrm{PF}_4^{\uparrow}} \left(\sum_{X \in \mathcal{M}(\beta;3)} d_X(I^-) \right) - d(I^-;4).$$

To compute the sum, select subsets of size $3 = \max(I)$ for each multiset $\beta \in \mathrm{PF}_4^{\uparrow}$ and arrange the entries so that they have descents at $I^- = \{1\}$. Table 2 details the computations establishing

$$\sum_{\beta \in \mathrm{PF}_4^{\uparrow}} \left(\sum_{X \in \mathcal{M}(\beta,3)} d_X(I^-) \right) = 40.$$

Now $d(I^-;4) = 21$, from which we get $d(I;4) = 40 - 21 = 19$ as expected. In fact, Table 1 gives the elements of $D(J;4)$ for all subsets $J \subseteq [3]$; and in particular, we list the elements in $D(I;4)$ confirming that $d(I;4) = 19$.

Next, we enumerate parking functions with descents at the first k indices by bijecting onto the set of standard Young tableaux of shape $((n-k)^2, 1^k)$, which are known to be enumerated by $f(n, n-k-1) = \frac{1}{n}\binom{n}{k}\binom{2n-k}{n-k-1}$ (A033282).

For natural number n, a *partition* λ of n is a weakly decreasing tuple $(\lambda_1, \ldots, \lambda_k)$ such that $\sum_i \lambda_i = n$. Whenever λ is a partition of n we write $\lambda \vdash n$.

Definition 20 ([8, p. 1])**.** Given a partition $\lambda \vdash n$, a *Young diagram of shape* λ is a left-justified collection of boxes with λ_i boxes in row i and where the top row is row 1. A *standard Young tableau of shape* λ is a filling of the Young diagram of shape λ with the numbers $\{1, \ldots, n\}$ such that the rows and columns are strictly increasing left to right and top to bottom.

Definition 21. A *Dyck word* w of semilength n is a word that contains U's and D's where U and D each appear n times such that for all $1 \leq i \leq n$, the ith D in w appears after the ith U. If U is the ith U in w, then i is the *semi-index* of the specified U.

$\beta \in \mathrm{PF}_4^{\uparrow}$	$\mathcal{M}(\beta,3)$	$\sum_{X\in\mathcal{M}(\beta;3)} d_X(I^-)$
$\{1,1,1,1\}$	$\{\{1,1,1\}\}$	0
$\{1,1,1,2\}$	$\{\{1,1,1\},\{1,1,2\}\}$	$0+1=1$
$\{1,1,1,3\}$	$\{\{1,1,1\},\{1,1,3\}\}$	$0+1=1$
$\{1,1,1,4\}$	$\{\{1,1,1\},\{1,1,4\}\}$	$0+1=1$
$\{1,1,2,2\}$	$\{\{1,1,2\},\{1,2,2\}\}$	$1+1=2$
$\{1,1,2,3\}$	$\{\{1,1,2\},\{1,1,2\},\{1,2,3\}\}$	$1+1+2=4$
$\{1,1,2,4\}$	$\{\{1,1,2\},\{1,1,4\},\{1,2,4\}\}$	$1+1+2=4$
$\{1,1,3,3\}$	$\{\{1,1,3\},\{1,3,3\}\}$	$1+1=2$
$\{1,1,3,4\}$	$\{\{1,1,3\},\{1,1,4\},\{1,3,4\}\}$	$1+1+2=4$
$\{1,2,2,2\}$	$\{\{1,2,2\},\{2,2,2\}\}$	$1+0=1$
$\{1,2,2,3\}$	$\{\{1,2,2\},\{1,2,3\},\{2,2,3\}\}$	$1+2+1=4$
$\{1,2,2,4\}$	$\{\{1,2,2\},\{1,2,4\},\{2,2,4\}\}$	$1+2+1=4$
$\{1,2,3,3\}$	$\{\{1,2,3\},\{1,3,3\},\{2,3,3\}\}$	$2+1+1=4$
$\{1,2,3,4\}$	$\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$	$2+2+2+2=8$

Table 2: Computations for Example 19.

Lemma 22. *[10], Theorem 6. There is a bijection f between Dyck words w of semilength n and standard Young tableaux of shape (n,n) which is given as follows: If $w_i = U$ (respectively, D), then i appears in the first (respectively, second) row of $f(w)$.*

For more on bijections between Dyck paths and standard Young tableaux, we point the interested reader to Gil, McNamara, Tirrell, and Weiner [9].

Lemma 23. *There is a bijection g between Dyck words w of semilength n and nondecreasing parking functions of length n which is given as follows: The ith entry in $g(w)$ is one plus the number of D's before the ith U in w.*

Proof. Because there are strictly less than i D's before the ith U in w, the ith entry in $g(w)$ never exceeds i. Then the map is well-defined. It is also injective because a Dyck word can be uniquely determined by the data of the number of D's before the ith U for all i. This fact also yields surjectivity, as $g^{-1}(\alpha)$ is the unique Dyck word that has $\alpha_i - 1$ D's before the ith U. □

Proposition 24. *Let $n \geq 1$ and $0 \leq k \leq n-1$. If $[k] \subseteq [n-1]$, then*

$$d([k];n) = \frac{1}{n}\binom{n}{k}\binom{2n-k}{n-k-1}.$$

Proof. Consider the standard Young tableau

$$T = \begin{array}{ccccc} a_1 & a_2 & a_3 & \cdots & a_{n-k} \\ b_1 & b_2 & b_3 & \cdots & b_{n-k} \\ c_1 & & & & \\ c_2 & & & & \\ \vdots & & & & \\ c_k & & & & \end{array}$$

whose entries strictly increase along the columns and rows. We construct a bijection τ mapping such a tableau to a parking function of length n with descent set k. Construct a word $w = w_1\ w_2\ \cdots\ w_{2n-k}$ for all $1 \le i \le 2n-k$ by letting

$$w_i = \begin{cases} U, & \text{if } i \text{ is in the first row;} \\ D, & \text{if } i \text{ is in the second row;} \\ X, & \text{if } i \text{ is in the third row or beyond.} \end{cases}$$

Because $b_i > a_i$, removing all X's from w would make w a Dyck word. Moreover, inserting a $U\ D$ into the middle of a Dyck path returns another Dyck path. We construct the word w' of length $2n$ by replacing every X in w with a $U\ D$. Let I consist of the semi-indices of the inserted U's. Using Lemma 23, let α be the nondecreasing parking function corresponding to w'. We then construct α' with descent set $[k]$ from α by moving the entries whose indices are in I to the front of the parking function α in decreasing order. By the construction of α, each such index in I is the last appearance of that index's values, hence the chosen entries are distinct. Then $\tau(T) = \alpha'$.

In the reverse direction, given a parking function $\alpha' \in \mathrm{PF}_n$ with descent set $[k]$, record the values in the first k entries in α' in the set V. Let α be the nondecreasing rearrangement of α', and let I be the set of indices of the last appearance of each element of V in α. Let w' be the Dyck word corresponding to α, and the U in w' with semi-index i has a D immediately to its right for all $i \in I$. Then for each $i \in I$, replace the subword $U\ D$, consisting of the ith U and the D immediately to its right, with an X. Then construct the corresponding standard Young tableau $T = \tau^{-1}(\alpha')$ by reversing the process from the Dyck word built from a standard Young tableau, as detailed above.

As τ and τ^{-1} are inverse maps of each other and each is defined on the entire desired domain, τ is the desired bijection. $\square$

We illustrate the bijection in the proof of Proposition 24 next.

Example 25. Consider $n = 7$, $k = 3$, and the standard Young tableau with shape $((7-3)^2, 1^3) = (4, 4, 1, 1, 1)$:

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 7 & 8 \\ \hline 2 & 5 & 9 & 10 \\ \hline 4 \\ \cline{1-1} 6 \\ \cline{1-1} 11 \\ \cline{1-1} \end{array}.$$

The corresponding word is $w = U\ D\ U\ X\ D\ X\ U\ U\ D\ D\ X$. For record-keeping, we underline elements instead of using the set I. Replacing X's with $\underline{U\ D}$ in w yields

$$w' = U\ D\ U\ \underline{U\ D}\ D\ \underline{U\ D}\ U\ U\ D\ D\ \underline{U\ D}.$$

The corresponding $\alpha \in \mathrm{PF}_n^{\uparrow}$ is given by $\alpha = (1, 2, \underline{2}, \underline{4}, 5, 5, \underline{7})$. Move the underlined numbers in α to the front (in decreasing order) to get $\alpha' = (7, 4, 2, 1, 2, 5, 5)$ which has the desired descent set $[3]$.

For an example of the reverse direction, consider $\alpha' = (5, 3, 2, 1, 1, 2, 2)$, which has descent set $[3]$. Underline the first 3 entries and write α' in nondecreasing order $(1, 1, 2, 2, \underline{2}, \underline{3}, \underline{5})$. The 2's are repeated and the right-most 2 is the underlined value. Then construct $w' = U\ U\ D\ U\ U\ \underline{U\ D}\ \underline{U\ D}\ D\ \underline{U\ D}\ D\ D$. Replacing $\underline{U\ D}$ with X in w' yields

$$w = U\ U\ D\ U\ U\ X\ X\ D\ X\ D\ D.$$

Then w corresponds to the standard Young tableau

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 5 \\ \hline 3 & 8 & 10 & 11 \\ \hline 6 \\ \cline{1-1} 7 \\ \cline{1-1} 9 \\ \cline{1-1} \end{array}.$$

Although we have a recursive formula for the number of parking functions of length n with descent set $I \subseteq [n-1]$ (Theorem 18), we pose the following problem.

Problem 26. Give non-recursive formulas for the number of parking functions with other interesting descent sets $I \subseteq [n-1]$?

4 Peaks of parking functions

The Catalan numbers are one of the most well-studied integer sequences, with 214 different combinatorial explanations in Stanley's work [15], and many more are found on the OEIS A000108. We now establish that the set of parking functions of length n that have no peaks and no ties, i.e., peakless-tieless parking functions, are a new set of Catalan objects.

We begin by setting some needed definitions and notation.

Definition 27. Let $\alpha = (a_1, a_2, \ldots, a_n) \in [n]^n$.

- Define the *tie set* of α as

  $$\mathrm{Tie}(\alpha) = \{j + 1 \in [n] : a_j = a_{j+1}\}$$

  and order the elements $\mathrm{Tie}(\alpha) = \{t_1 < t_2 < \cdots < t_j\}$, where $j = |\mathrm{Tie}(\alpha)|$.

- Define the *value set* of α as

  $$\mathrm{Val}(\alpha) = \{b'_1, b'_2, \ldots, b'_k\},$$

  such that set of elements of α are in increasing order $1 = b'_1 < b'_2 < \cdots < b'_k$, where k is the number of distinct elements of α.

Definition 28. Define the function $\varphi : \mathrm{PF}_n^{\uparrow} \to \mathbb{N}^n$ by

$$\varphi(\alpha) = (t_j, t_{j-1}, \ldots, t_2, t_1, b_1, b_2, \ldots, b_k), \tag{11}$$

where $\mathrm{Tie}(\alpha) = \{t_j > t_{j-1} > \cdots > t_1 > 1\}$ and $\mathrm{Val}(\alpha) = \{1 = b_1 < b_2 < \cdots < b_k\}$ are as in Definition 27.

In fact, we show $\varphi : \mathrm{PF}_n^{\uparrow} \to \mathrm{PTPF}_n$ and illustrate these definitions next.

Example 29. Consider the nondecreasing parking function $\alpha = (1, 1, 2, 3, 4, 4) \in \mathrm{PF}_6^{\uparrow}$. Then $\mathrm{Tie}(\alpha) = \{2, 6\}$ and $\mathrm{Val}(\alpha) = \{1, 2, 3, 4\}$. Now $\varphi(\alpha)$ returns the tuple whose first $|\mathrm{Tie}(\alpha)|$ elements are the integers in $\mathrm{Tie}(\alpha)$ arranged in decreasing order, while the remaining values are the elements of $\mathrm{Val}(\alpha)$ listed in increasing order. Namely, $\varphi(\alpha) = (6, 2, 1, 2, 3, 4)$. The result is a peakless-tieless parking function, as claimed.

Theorem 30. *The function φ in Definition 28 is a well-defined bijection $\varphi : \mathrm{PF}_n^{\uparrow} \to \mathrm{PTPF}_n$.*

Proof. Let $\alpha \in \mathrm{PF}_n^{\uparrow}$. In this construction, $|\mathrm{Val}(\alpha)| = k$ is the number of values appearing in α and $|\mathrm{Tie}(\alpha)| = j$ is the number of ties in α. Because $\alpha \in \mathrm{PF}_n^{\uparrow}$ and a value in α either repeats or appears exactly once, $k + j = n$. Moreover, the construction in Equation (11) ensures $b_1 = 1$ since the letter 1 must appear in $\mathrm{PF}_n^{\uparrow}$. Also, the letters before 1 (if they exist) decrease down to 1, and the letters after 1 (if they exist) increase. Hence, $\varphi(\alpha)$ has no peaks and no ties. Additionally, all cars are still able to park with preferences $\varphi(\alpha)$. To see this, observe that $\alpha = (a_1, a_2, \ldots, a_n) \in \mathrm{PF}_n^{\uparrow}$. The cars park in the order they arrive: car 1 parks in spot 1, car 2 in spot 2, and so on. To show that $\varphi(\alpha)$ parks, we define $\gamma = (g_1, \ldots, g_n)$ from α by replacing each a_i where $a_i = a_{i-1}$ with i. Because α parks, each unchanged preference satisfies $g_i \leq i$. If g_j is a changed preference, then $g_j = j \leq j$. Thus, we conclude that γ parks. Now, since γ and $\varphi(\alpha)$ may be permuted into each other, $\varphi(\alpha)$ must also park. Thus, $\varphi(\alpha) \in \mathrm{PTPF}_n$.

Injectivity: Let $\alpha,\ \beta \in \mathrm{PF}_n^{\uparrow}$, and let $\varphi(\alpha) = \varphi(\beta)$. This implies that $\mathrm{Tie}(\alpha) = \mathrm{Tie}(\beta)$ and $\mathrm{Val}(\alpha) = \mathrm{Val}(\beta)$. Notice that nondecreasing parking functions $x \in \mathrm{PF}_n^{\uparrow}$ are completely determined by their tie sets, $\mathrm{Tie}(x)$, and their value sets, $\mathrm{Val}(x)$. When reconstructing α and β from the value set and the tie set, the repeated values in x are the same for both $\varphi(\alpha)$ and $\varphi(\beta)$. Thus, $\alpha = \beta$.

Surjectivity: Let $\beta = (t_j, t_{j-1}, \ldots, t_2, t_1, b_1, b_2, \ldots, b_k) \in \mathrm{PTPF}_n$, where $T = \{1 < t_1 < t_2 < \cdots < t_{j-1} < t_j\}$ and $B = \{1 = b_1 < b_2 < \cdots < b_k\}$. It is sufficient to find $\alpha \in \mathrm{PF}_n^{\uparrow}$ with $\mathrm{Tie}(\alpha) = T$ and $\mathrm{Val}(\alpha) = B$. Order the elements of $X = [n] \setminus T = \{x_1 < x_2 < \cdots < x_{n-j}\}$. Construct $\alpha = (a_1, a_2, \ldots, a_n)$ as follows.

1. In α, place the elements of B in order $1 = b_1 < b_2 < \cdots < b_k$ at the indices indexed by X in order $x_1 < x_2 < \cdots < x_{n-j} = x_k$. Namely, let $a_{x_i} = b_i$ for all $1 \leq i \leq k$.

2. For every $j \in T$, set $a_j = a_{j-1}$.

Since $\alpha = (a_1, a_2, \ldots, a_n)$ is a nondecreasing tuple, α parks if and only if $a_i \leq i$ for all $1 \leq i \leq n$. If $a_i = a_{i-1}$ and $a_{i-1} \leq i-1$, then $a_i \leq i$. Therefore, it is sufficient to check that $a_{x_i} = b_i \leq x_i$ for all $1 \leq i \leq n-j$.

Let $P = (p_1, p_2, \ldots, p_n)$ be β in nondecreasing order. This is a parking function, so $p_i \leq i$.

1. If there is no duplicate in $(a_1, \ldots, a_i)$: $t_1 > i$. Then in P, t_1 has index at lest $i+1$, implying $\mathrm{Val}(a_1, \ldots, a_i) \subset \mathrm{Val}(p_1, \ldots, p_i)$.

2. If there is a duplicate, without loss of generality, we assume $a_{i-1} = a_i$.
   Then $\mathrm{Val}(a_1, \ldots, a_{i-1}) \subset \mathrm{Val}(p_1, \ldots, p_{i-1})$. Notice then, that since $a_{i-1} = a_i$, we have

$$\mathrm{Val}(a_1, \ldots, a_i) = \mathrm{Val}(a_1, \ldots, a_{i-1}) \subset \mathrm{Val}(p_1, \ldots, p_{i-1}) \subseteq \mathrm{Val}(p_1, \ldots, p_i).$$

Because of this, we satisfy the claim. If we take $a_i \in \alpha$, we can find some $\ell_i \leq i$ such that $a_i = p_{\ell_i} \leq \ell_i \leq i$. Then $a_i \leq i$ and $\alpha \in \mathrm{PF}_n^{\uparrow}$. □

The bijection in Theorem 30 immediately implies the following.

Corollary 31. *If $n \geq 1$, then $|\mathrm{PTPF}_n| = C_n$, the nth Catalan number.*

We now provide a connection between peakless-tieless parking functions and the entries of the Catalan triangle A009766, which are defined by

$$C_{n+1,i} = \sum_{j=0}^{i} C_{n,j}. \tag{12}$$

Table 3 provides some of the initial values in the Catalan triangle. The sum along the nth row of the Catalan triangle is given by C_n.

$n \setminus k$	1	2	3	4	5	6	7
1	1						
2	1	1					
3	1	2	2				
4	1	3	5	5			
5	1	4	9	14	14		
6	1	5	14	28	42	42	
7	1	6	20	48	90	132	132

Table 3: The entries of the Catalan triangle for $1 \leq n, i \leq 7$.

Corollary 32. *If* $\mathrm{PTPF}_n(i) = \{\alpha = (a_1, a_2, \ldots, a_n) \in \mathrm{PTPF}_n : a_n = i\}$, *then* $|\mathrm{PTPF}_n(i)| = C_{n,i}$.

Proof. We proceed by induction on n and $i \in [n]$. When $n = 1$, the set $|\mathrm{PTPF}_1(1)| = |\{(1)\}| = 1 = C_{1,1}$. When $n = 2$, the set $|\mathrm{PTPF}_2(1)| = |\{(2,1)\}| = 1 = C_{2,1}$ and $|\mathrm{PTPF}_2(2)| = |\{(1,2)\}| = 1 = C_{2,2}$.

Assume for induction that for all $n \leq k$ and $i \leq n$, $|\mathrm{PTPF}_n(i)| = C_{n,i}$. Let $\beta = (b_1, b_2, \ldots, b_{n-1}, i) \in \mathrm{PTPF}_n(i)$. Prepending $n+1$ to β yields $(n+1, b_1, b_2, \ldots, b_{n-1}, i) \in \mathrm{PTPF}_{n+1}(i)$. By induction, we construct $C_{n,i}$ many elements of $\mathrm{PTPF}_{n+1}(i)$. For every element $\gamma = (b_1, b_2, \ldots, b_{n-1}, j) \in \mathrm{PTPF}_n(j)$ with $1 \leq j \leq i-1$, appending i to γ yields $(b_1, b_2, \ldots, b_{n-1}, j, i) \in \mathrm{PTPF}_{n+1}(i)$, and we construct $\sum_{j=1}^{i-1} C_{n,j}$ elements. These constructions yield distinct elements of $\mathrm{PTPF}_{n+1}(i)$ since the construction creates tuples beginning with $n+1$, while the second begins with $b_1 \in [n]$. As these are the only possible values with which the tuple may begin, applying the induction hypothesis yields

$$|\mathrm{PTPF}_{n+1}(i)| = C_{n,i} + \sum_{j=1}^{i-1} C_{n,j} = \sum_{j=1}^{i} C_{n,j} = C_{n+1,i}$$

where the last equality holds by Equation (12). □

4.1 Valleys of parking functions

In permutations, the map $i \to n-i+1$ gives a bijection on permutations, which sends peaks to valleys. This establishes that the number of permutations with k peaks is the same as the number of permutations with k valleys. However, this map is not well-defined for parking functions. For example, $(1,1,1)$ goes to $(3,3,3)$ which is not even a parking function. Moreover, the number of parking functions with k peaks is not the same as the number with k valleys. For example, when $n=3$, there are four parking functions with one peak:

$$(1,2,1),(1,3,1),(1,3,2),(2,3,1).$$

Whereas, there are three parking functions with one valley:

$$(2,1,2),(2,1,3),(3,1,2).$$

We now consider the set of valleyless-tieless parking functions; parking functions for which there is no index $1 \leq i \leq n-1$ such that $a_i = a_{i+1}$ nor index $2 \leq i \leq n-1$ such that $a_{i-1} > a_i < a_{i+1}$. We let VTPF_n denote the set of valleyless-tieless parking functions of length n. Our main result is as follows.

Theorem 33. *Valleyless-tieless parking functions of length n are enumerated by $|\mathrm{VTPF}_n| = F_{n+2}$, where F_n is the nth Fine number (as defined by Deutsch and Shapiro [3]) with the first ten values of the sequence being $(1 \leq n \leq 10)$:*

$$1,2,6,18,57,186,622,2120,7338,25724.$$

To prove Theorem 33, in Subsection 4.2, we establish a bijection between the set of valleyless-tieless parking functions and certain types of peakless-tieless parking functions. In Subsection 4.3, we show that the special set of peakless-tieless parking functions are counted by the Fine numbers (A000957). This result implies Theorem 33.

4.2 Bijective maps

We let $\mathrm{VTPF}_{n+1}(a_1 = 1, a_{n+1} > 1)$ be the set of valleyless-tieless parking functions of length $n+1$ in which the first value is one and the last is larger than one. A more formal definition of peakless-tieless parking functions of length $n+1$ is the set of parking functions of length $n+1$ for which there is no index $i \in [n]$ such that $a_i = a_{i+1}$ or $a_{i-1} < a_i > a_{i+1}$.

We let $\mathrm{PTPF}_{n+1}(a_1 > a_{n+1})$ be the set of peakless-tieless parking functions of length $n+1$ in which the first value is larger than the last.

Lemma 34. *The function $\psi : \mathrm{VTPF}_n \to \mathrm{VTPF}_{n+1}(a_1 = 1, a_{n+1} > 1)$ defined by*

$$(a_1, a_2, \ldots, a_n) \to (1, a_1+1, a_2+1, \ldots, a_n+1)$$

is a bijection.

Proof. First, we show injectivity. If $\alpha = (a_1, a_2, \ldots, a_n), \beta = (b_1, b_2, \ldots, b_n) \in \mathrm{VTPF}_n$, and $\psi(\alpha) = \psi(\beta)$, then $a_i + 1 = b_i + 1$ for all $i \in [n]$. Then $a_i = b_i$ for all $i \in [n]$, so $\alpha = \beta$.

For surjectivity, suppose $\alpha = (1, a_2, \ldots, a_n) \in \mathrm{VTPF}_{n+1}(a_1 = 1, a_{n+1} > 1)$. We wish to show that $\beta = (a_1 - 1, a_2 - 1, \ldots, a_n - 1) \in \mathrm{VTPF}_n$. For $i \in [n]$, $a_i > 1$ and in general an element of VTPF_n can only have a 1 in the first and/or last position. If an instance of 1 has a neighbor both to its left and right and there are no ties, then both of 1's neighbors are greater than 1, creating a valley and giving rise to a contradiction. Recall α satisfies $a_n > 1$, so $a_i > 1$ for all $i \in [n]$. Moreover, $a_i - 1 \leq n$ for all $i \in [n]$. Hence, $\beta \in [n]^n$.

Next, we show that β is a parking function. Let $\alpha^{\uparrow} = (1, x_1, x_2, \ldots, x_n)$ be the nondecreasing rearrangement of α. Since α is a parking function, we know that $1 < x_i \leq i + 1$ for all $i \in [n]$. Then the nondecreasing rearrangement $\beta^{\uparrow}$ of β is $(x_1 - 1, x_2 - 1, \ldots, x_n - 1)$, which satisfies $0 < x_n - 1 \leq i$, so β parks. Removing the first element of a parking function, in particular of α, does not create a valley or a tie that was not there before. This ensures that β also does not have ties nor valleys. This then establishes that $\beta \in \mathrm{VTPF}_n$ and satisfies $\psi(\beta) = \alpha$. □

Remark 35. In a valleyless-tieless parking function there is a unique maximal entry. If this were not the case and the maximum value k appeared twice. Then either they are consecutive entries in the tuple, creating a tie (a contradiction), or they are nonadjacent entries in the tuple and thus the value(s) between them would either be larger or create a valley, giving a contradiction.

In the following, we reverse parts of a tuple. To this end, we define the following.

Definition 36. Let $(x_1, x_2, \ldots, x_n) \in [n]^n$. For $i \in [n-1]$ let

$$\mathrm{flick}_i(x) = (x_i, x_{i-1}, \ldots, x_1, x_n, x_{n-1}, \ldots, x_{i+1})$$

and

$$\mathrm{flick}_n(x) = (x_n, x_{n-1}, \ldots, x_2, x_1).$$

For example, if $\mathbf{x} = (1, 2, 1, 3, 5, 6, 3, 2)$, then $\mathrm{flick}_4(\mathbf{x}) = (3, 1, 2, 1, 2, 3, 6, 5)$ and $\mathrm{flick}_6(\mathbf{x}) = (6, 5, 3, 1, 2, 1, 2, 3)$.

Proposition 37. *Define the map* $\varphi : \mathrm{VTPF}_{n+1}(a_1 = 1, a_{n+1} > 1) \to \mathrm{PTPF}_{n+1}(a_1 > a_{n+1})$ *as follows: If* $v = (v_1, v_2, \ldots, v_{n+1}) \in \mathrm{VTPF}_{n+1}(a_1 = 1, a_{i+1} > 1)$ *and* $i \in [n+1]$ *is the unique index containing the maximal entry of* v*, then*

$$\varphi(v) = \mathrm{flick}_i(v).$$

The map φ *is a bijection.*

Proof. By definition, flick_i is an involution that, as a rearrangement, maps parking functions to parking functions. Let $\alpha = (a_1, a_2, \ldots, a_{n+1}) \in \text{PTPF}_{n+1}(a_1 > a_{n+1})$ with i being the index of the unique instance of $1 \in \alpha$. Then it is sufficient to show that $\beta = (b_1, b_2, \ldots, b_{n+1}) = \text{flick}_i(\alpha)$ is an element of $\text{VTPF}_{n+1}(a_1 = 1, a_{n+1} > 1)$. Recall that since $\alpha \in \text{PTPF}_{n+1}(a_1 > a_{n+1})$, we have that

$$a_1 > a_2 > \cdots > a_{i-1} > a_i < a_{i+1} < \cdots < a_{n+1} \tag{13}$$

and $a_1 > a_{n+1}$. Then by definition of $\text{flick}_i(\alpha)$ and the inequalities in Expression (13), we have that

$$1 = a_i < a_{i-1} < \cdots < a_2 < a_1 > a_{n+1} > a_n > \cdots > a_{i+1}.$$

So $\beta = (b_1, b_2, \ldots, b_{n+1}) = \text{flick}_i(\alpha)$ ensures that

$$1 = b_1 < b_2 < \cdots < b_{i-1} < b_i > b_{i+1} > b_{i+2} > \cdots > b_{n+1} \tag{14}$$

with $b_1 = 1$ and $b_{n+1} > b_1 = 1$. The inequalities in Expression (14) imply that $\beta \in \text{VTPF}_{n+1}(b_1 = 1, b_{n+1} > 1)$.

Then flick_i is a well-defined involution between the desired domains. Therefore, we get the bijection as desired. □

4.3 Valleyless-tieless parking functions and the Fine numbers

We now define a set partition which is useful in proving Theorem 33.

Definition 38. For $n \geq 0$, we partition the set PTPF_{n+1} as follows:

- $\mathscr{G}_{n+1} = \{\alpha \in \text{PTPF}_{n+1} : a_1 > a_{n+1}\}$,
- $\mathscr{E}_{n+1} = \{\alpha \in \text{PTPF}_{n+1} : a_1 = a_n\}$, and
- $\mathscr{L}_{n+1} = \{\alpha \in \text{PTPF}_{n+1} : a_1 < a_{n+1}\}$.

For convenience $\mathscr{G}_1 = \mathscr{L}_1 = \emptyset$ and $\mathscr{E}_1 = \{(1)\}$. Let $G_{n+1} = |\mathscr{G}_{n+1}|$, $E_{n+1} = |\mathscr{E}_{n+1}|$, and $L_{n+1} = |\mathscr{L}_{n+1}|$.

As expected,

$$G_{n+1} + E_{n+1} + L_{n+1} = |\text{PTPF}_{n+1}| = C_{n+1}, \tag{15}$$

the $(n+1)$th Catalan number, and where the last equality holds by Corollary 31.

Remark 39. Let $\alpha = (a_1, a_2, \ldots, a_n)$ be a peakless-tieless parking function of length $n \geq 2$. If $a_i = n$ for some index i, then $i = 1$ or $i = n$, but not both. This is because if both $a_1 = n$ and $a_n = n$, then α is not a parking function. If $a_i = n$ for $i \in [2, n-1]$, then either α has a peak at i or there is a tie in position $i - 1$ or i, contradicting that α is peakless and tieless.

Lemma 40. *If* $n \geq 1$, *then* $E_{n+1} = L_n$, *and if* $n \geq 0$, *then* $G_{n+1} = L_{n+1}$.

Proof. For the first equality, we establish a bijection between $\mathscr{E}_{n+1}$ and $\mathscr{L}_n$, from which the result follows.

Let $e = (e_1, e_2, \ldots, e_{n+1}) \in \mathscr{E}_{n+1}$, so that $e_1 = e_{n+1}$. Notice that e is peakless-tieless. Since $e_1 = e_{n+1}$ and $e_2 < e_1$, then we have that $e_2 < e_{n+1}$. Furthermore, removing e_1 from e does not create a peak, nor a tie. By Remark 39, the largest possible value in e is $n - 1$ and would occur at the endpoints. This implies that $e_1 = e_{n+1} \leq n - 1$. Hence, the tuple $(e_2, e_3, \ldots, e_{n+1})$ would allow the cars to park. Thus, $(e_2, e_3, \ldots, e_{n+1}) \in \mathscr{L}_n$.

In the reverse direction, if $\ell = (\ell_1, \ell_2, \ldots, \ell_n) \in \mathscr{L}_n$, then the tuple $\ell' = (\ell_n, \ell_1, \ell_2, \ldots, \ell_n)$ has no peaks and no ties as $\ell_n > \ell_1$. Moreover, we know $\ell_n \leq n$. Hence, the largest value in ℓ' is n, and all $n + 1$ cars can park. Therefore, we get $\ell' \in \mathscr{E}_{n+1}$.

If $\alpha = (a_1, a_2, \ldots, a_{n+1}) \in \mathscr{G}_{n+1}$, the map $\mathrm{flick}_n(\alpha) = (a_{n+1}, a_n, \ldots, a_1)$ gives a bijection from $\mathscr{G}_{n+1}$ to $\mathscr{L}_{n+1}$. Hence, $G_{n+1} = L_{n+1}$. □

Theorem 41. *If* $n \geq 0$, *then* $|\mathrm{PTPF}_{n+1}(a_1 < a_{n+1})| = F_{n+1}$, *the* $(n+1)$*th Fine number.*

Proof. Note, that $\mathrm{PTPF}_{n+1} = \mathscr{C}_{n+1} \bigsqcup \mathscr{E}_{n+1} \bigsqcup \mathscr{L}_{n+1}$, so by Equation (15), we have that

$$C_{n+1} = G_{n+1} + E_{n+1} + L_{n+1}.$$

By Lemma 40, $E_{n+1} = L_n$ and $G_{n+1} = L_{n+1}$, implying

$$C_{n+1} = 2L_{n+1} + L_n.$$

The initial values are $L_1 = 0$; and since $\mathscr{L}_2 = \{(1, 2)\}$, we have $L_2 = 1$. Deutsch and Shapiro [3, p. 8] proved the following identity relating the Fine numbers to the Catalan numbers

$$C_{n+1} = 2F_{n+2} + F_{n+1}$$

where $F_1 = 0$ and $F_2 = 1$. Together, this implies that $L_{n+1} = F_{n+2}$. □

The bijections in Lemma 34 and Proposition 37, along with the the enumerative result of Theorem 41 together imply Theorem 33, establishing that the number of valleyless-tieless parking functions is given by a Fine number.

4.4 Open problems

There are many open problems remaining when considering the set of parking functions of length n with j peaks and k ties, which we denote by $\mathrm{PF}_n(j, k)$. Table 4 provides the cardinality of the set $\mathrm{PF}_n(0, k)$ for small values of n and k. The column corresponding to $k = 0$ gives the Catalan numbers, which we prove in Corollary 31.

We prove the following two results related to the value of $\mathrm{PF}_n(0, k)$ with $k = n - 2, n - 1$.

Lemma 42. *If* $n \geq 1$, *then* $|\mathrm{PF}_n(0, n - 1)| = 1$.

$n \setminus k$ ties	0	1	2	3	4	5
1	1	0	0	0	0	0
2	2	1	0	0	0	0
3	5	6	1	0	0	0
4	14	32	12	1	0	0
5	42	178	110	20	1	0
6	132	1078	978	280	30	1

Table 4: Number of parking functions of length n with 0 peaks and k ties.

Proof. A tuple in $[n]^n$ with $n-1$ ties must have the same value at every entry. The only such parking functions is the all ones tuple. $\square$

Lemma 43. *If $n \geq 1$, then $|\mathrm{PF}_n(0, n-2)| = n(n+1)$, the nth Oblong number A002378.*

Proof. Such a tuple has exactly one non-tie. If there is exactly one ascent, the tuple begins with $1 \leq i \leq n$ many ones, followed by $n-i$ many repeated values k where $2 \leq k \leq i+1$. So the number of possibilities is given by $\sum_{i=1}^{n} i = n(n+1)/2$. We can reverse the tuple to account for the case where it has exactly one descent. This yields a total of $n(n+1)$, as claimed. $\square$

A general open problem follows.

Problem 44. For $n, j, k \in \mathbb{N}$, give recursive or closed formulas for the value of $|\mathrm{PF}_n(j, k)|$.

One could also consider parking functions of length n with j peaks. This set is given by

$$\mathrm{PF}_n^j = \bigcup_{k=0}^{n-1-2j} \mathrm{PF}_n(j, k).$$

Table 5 gives the cardinality of $|\mathrm{PF}_n^j|$ for $0 \leq j \leq n \leq 6$. The first column corresponding to $j = 0$ peaks is also given by the row sums in Table 4. We can now pose another open problem.

Problem 45. Characterize and enumerate the set PF_n^k for general values of n and k.

5 Statistic encoding

We now study a collection of parking functions with a prescribed pattern at every index.

$n \setminus j$ peaks	0	1	2	3
1	1	0	0	0
2	3	0	0	0
3	12	4	0	0
4	59	66	0	0
5	351	825	120	0
6	2499	9704	4604	0
7	20823	115892	115959	9470
8	197565	145478	2479110	651816

Table 5: Number of parking functions of length n with j peaks.

Definition 46. Every $\alpha = (a_1, a_2, \ldots, a_n) \in \mathrm{PF}_n$ gives rise to a word $w = w_1\, w_2\, \cdots\, w_{n-1} \in \{\mathtt{A}, \mathtt{D}, \mathtt{T}\}^{n-1}$, where for each $i \in [n-1]$ we let

$$w_i = \begin{cases} \mathtt{A}, & \text{if } a_i < a_{i+1}; \\ \mathtt{D}, & \text{if } a_i > a_{i+1}; \\ \mathtt{T}, & \text{if } a_i = a_{i+1}. \end{cases}$$

We call w the *statistic encoding of* α and denote it as $\mathrm{stat}(\alpha)$.

In defining a statistic encoding, we use the letter A to denote an ascent, D to denote a descent, and T to denote a tie. The vast majority of statistic encodings are non-unique. For example, $\alpha = (1, 1, 2, 3, 4)$ and $\beta = (1, 1, 3, 4, 5)$ both have $w = \mathtt{T\,A\,A\,A}$ as their statistic encoding.

In this section, we answer the following questions:

1. Does there exist a parking function with statistic encoding w for arbitrary w?

2. When is a statistic encoding determined by a unique parking function?

To begin, we set the following notation. Let W_{n-1} denote the set of all statistic encodings of length $n-1$, which implies $W_{n-1} = \{\mathtt{A}, \mathtt{D}, \mathtt{T}\}^{n-1}$. Our first result establishes that every word of length $n-1$ in the letters $\mathtt{A}, \mathtt{D}, \mathtt{T}$ arises as a statistic encoding for some parking function. Before proving the result, we illustrate this notation with an example.

Example 47. To construct a parking function for the word D A T A, we begin with a parking function whose statistic encoding is D A T, such as $\alpha = (3, 1, 2, 2)$. To account for the added ascent at the end of the word D A T A, we simply append 5 to α and obtain $(3, 1, 2, 2, 5) \in \mathrm{PF}_5$. Notice that $(3, 1, 2, 2, 5)$ is not the only parking function with statistic encoding D A T A as we could have also appended 3 or 4.

If instead we want to construct a parking function with statistic encoding A T A DD, we begin with the parking function $\beta = (1, 2, 2, 3, 1)$ with a corresponding statistic encoding

$\texttt{A T A D}$. To construct a parking function with statistic encoding $\texttt{A T A D D}$, we must create a new descent at the end of β. To do this, we begin by incrementing every entry in β by one, resulting in the tuple $(2,3,3,4,2)$. Then append 1 to the end of that tuple creating $(2,3,3,4,2,1)$, which is an element of PF_6 and has the desired statistic encoding $\texttt{A T A D D}$.

We are now ready to settle Question (1).

Theorem 48. *If $w \in W_{n-1}$, then there exists $\alpha \in \mathrm{PF}_n$ which has w as its statistic encoding.*

Proof. We proceed by induction on n, the length of the parking function. We begin with the base case where $n = 1$ and observe that the empty statistic encoding arises from the parking function (1).

Assume for induction that for $n > 1$ and for every $w \in \{\texttt{A}, \texttt{D}, \texttt{T}\}^{n-1}$ there is a parking function $\alpha = (a_1, a_2, \ldots, a_n) \in \mathrm{PF}_n$ with statistic encoding w.

Now consider a word $w \in W_n = \{\texttt{A}, \texttt{D}, \texttt{T}\}^n$ such that $w = w'x$ where $w' \in \{\texttt{A}, \texttt{D}, \texttt{T}\}^{n-1}$ and $x \in \{\texttt{A}, \texttt{D}, \texttt{T}\}$. By the inductive step, we can find some $P' = (p'_1, p'_2, \ldots, p'_n) \in \mathrm{PF}_n$ whose statistic encoding is w'. To obtain a parking function $\alpha \in \mathrm{PF}_{n+1}$ with statistic encoding $w = w'x$, we append a new preference to $P' \in \mathrm{PF}_n$ based on the letter x by the following criteria:

C1: If $x = \texttt{T}$, we append p'_n to P' constructing

$$\alpha = (p'_1, p'_2, \ldots, p'_n, p'_n).$$

Note, α is a parking function of length $n+1$ as P' is a parking function of length n, and appending the value p'_n ensures that car $n+1$ parks in spot $n+1$. Moreover, this ensures that the parking function ends with a tie.

C2: If $x = \texttt{A}$, we append $n+1$ to P' to get

$$\alpha = (p'_1, p'_2, \ldots, p'_n, n+1).$$

Note, α is a parking function of length $n+1$ and P' is a parking function of length n. Appending the value $n+1$ to α ensures that car $n+1$ parks in spot $n+1$. Moreover, this ensures that the parking function ends with an ascent.

C3: If $x = \texttt{D}$, we modify P' by incrementing each p'_i by one and then appending the value 1 at the end of P' to get

$$\alpha = (p'_1 + 1, p'_2 + 1, \ldots, p'_n + 1, 1).$$

Note, P' is a parking function of length n, and by incrementing its values by one, the cars 1 through n in α park in spots 2 through $n+1$. Then car $n+1$ with preference 1 parks in spot 1. Thus, α is a parking function of length $n+1$. Moreover, this ensures that the parking function ends with a descent. $\square$

In the next result, we use the notation $\texttt{A}^* \texttt{T}^*$ to describe a word with some nonnegative integer number of $\texttt{A}$'s followed by a nonnegative integer number of $\texttt{T}$'s. When we specify that $\texttt{A}^* \texttt{T}^* \in W_{n-1}$, then the total number of $\texttt{A}$'s and $\texttt{T}$'s must be equal to $n-1$. Whenever we want to specify the full set of such words, we write $\{\texttt{A}^* \texttt{T}^*\}$. Likewise for $\texttt{T}^*\texttt{D}^*$ and $\{\texttt{T}^*\texttt{D}^*\}$.

With this notation in mind, we now settle Question (2).

Lemma 49. *Let $w \in W_{n-1}$. If there is a unique $\alpha \in \mathrm{PF}_n$ such that $\mathrm{stat}(\alpha) = w$, then for all indices $i \in [n-1]$, $|\alpha_{i+1} - \alpha_i| \leq 1$. Moreover, if $m_p(\alpha) > 0$, then $m_q(\alpha) > 0$ for all $q < p$.*

Proof. Let $w \in W_{n-1}$. Assume towards a contradiction that α is a parking function with an index i such that $|\alpha_{i+1} - \alpha_i| > 1$. Without loss of generality, suppose $p = \alpha_{i+1} > \alpha_i$. Let j be the first index after i such that $\alpha_j \neq \alpha_{j+1}$. In other words, let α have the following structure:

$$(\alpha_1, \ldots, \alpha_i, \underbrace{p}_{i+1}, p, \ldots, \underbrace{p}_{j}, \alpha_{j+1}, \ldots, \alpha_n),$$

where $\alpha_i, \alpha_{j+1} \neq p$. Then the tuple

$$\alpha' = (\alpha_1, \ldots, \alpha_i, \underbrace{p-1}_{i+1}, p-1, \ldots, \underbrace{p-1}_{j}, \alpha_{j+1}, \ldots, \alpha_n)$$

has the same statistic encoding as α. We know that α' is a parking function because reducing the value of an element in a parking function always yields a parking function. Then $\mathrm{stat}(\alpha') = w$, yielding a contradiction. Thus, α is not the unique parking function such that $\mathrm{stat}(\alpha) = w$.

The second part of the lemma results from induction on the maximal value p in α. First, observe that 1 appears at least once in every parking function. Then if $p = 1$ or $p = 2$ the proposition is straightforward. Now assume the inductive hypothesis on a parking function α with maximum value p. Let i be the first index such that exactly one of α_i and α_{i+1} is equal to p. Without loss of generality, assume $\alpha_i = p$ and $\alpha_{i+1} \neq p$. Then by the first part of the lemma, $\alpha_{i+1} = p-1$. Then $m_{p-1}(\alpha) > 0$, so by $m_q(\alpha) > 0$ for all $q < p$ by induction. □

Lemma 50. *Let $w \in W_{n-1}$. If there is a unique $\alpha \in \mathrm{PF}_n$ with $\mathrm{stat}(\alpha) = w$, then the statistic encoding of the nondecreasing rearrangement β of α is of the form $\texttt{A}^* \texttt{T}^*$.*

Proof. For the sake of contradiction, suppose that α is the unique parking function with $\mathrm{stat}(\alpha) = w$ but the statistic encoding of the nondecreasing rearrangement β of α is not of the form $\texttt{A}^* \texttt{T}^*$. Then let i be the index of the first $\texttt{A}$ in $\mathrm{stat}(\beta)$ that comes after a $\texttt{T}$. Then $\alpha_i < i$. By Lemma 49, $\alpha_{j+1} \leq \alpha_j + 1$ for all $j \in [n-1]$. Then using induction, $\alpha_j < j$ for all $j > i$. Now let p be the maximal value of α. In β, all instances of p are strictly less than their index. Then let α' be the tuple obtained by increment all instances of p in α. Then α' is still a parking function and has the same statistic encoding of α. This contradicts α being the unique parking function with statistic encoding w. □

Lemma 51. *Let $w \in W_{n-1}$ with unique $\alpha \in \mathrm{PF}_n$ such that $\mathrm{stat}(\alpha) = w$. If there exist indices $1 \leq i \leq j \leq n$ such that $\alpha_{i-1} > \alpha_i = \alpha_{i+1} = \cdots = \alpha_j < \alpha_{j+1}$ where α_{i-1} and α_{j+1} exist, then $\alpha_i = \cdots = \alpha_j = 1$.*

Proof. Let $w \in W_{n-1}$ with unique $\alpha \in \mathrm{PF}_n$ such that $\mathrm{stat}(\alpha) = w$. For the sake of contrapositive, let the indices satisfy $1 \leq i < j \leq n$ such that $\alpha_{i-1} > \alpha_i = \alpha_{i+1} = \cdots = \alpha_j < \alpha_{j+1}$ with $\alpha_i > 1$. The parking function α' obtained by replacing $\alpha_i, \ldots, \alpha_j$ with $1, \ldots, 1$ has the same statistic encoding as α. Thus, α is not the unique parking function with $\mathrm{stat}(\alpha) = w$. □

In other words, if α is the unique parking function with statistic encoding w, then the local minimum in α has value 1.

Theorem 52. *Let $w \in W_{n-1}$. Then there is a unique $\alpha \in \mathrm{PF}_n$ with statistic encoding w if and only if $w \in \{\mathtt{A}^* \mathtt{T}^*\} \cup \{\mathtt{T}^*\mathtt{D}^*\}$.*

Proof. ($\Rightarrow$) First we show that if w contains both A and D, then there are multiple parking functions α with $\mathrm{stat}(\alpha) = w$. Suppose w contains both A and D. Then there is some $n \geq 0$ such that either $\mathtt{A}\,\mathtt{T}^n\mathtt{D}$ or $\mathtt{D}\mathtt{T}^n\,\mathtt{A}$ is a contiguous subword of w.

- Suppose $v = \mathtt{A}\,\mathtt{T}^n\mathtt{D}$ is a contiguous subword of w. Where v first appears in w, let i be the index A and let j be the index of D. Let p be the minimum value in α with index weakly less than i and let q be the minimum value in α with index strictly greater than j. By Lemma 51, $p = q = 1$. Then the nondecreasing rearrangement β of α starts with two 1's, but is not itself all ones. This contradicts Lemma 50.

- Suppose $v = \mathtt{D}\mathtt{T}^n\,\mathtt{A}$ is a contiguous subword of w. For the sake of contradiction, suppose α is the unique parking function with $\mathrm{stat}(\alpha) = w$. By the previous bullet point, there is no A that comes before D in w. Then α can be broken into a nonincreasing part α^*, some amount of 1's, and a nondecreasing part α^{**}. We choose α^* and α^{**} such that neither contain 1's. By Lemma 49, α^* ends with a 2 and α^{**} begins with a 2. Then the nondecreasing rearrangement β of α is of the form $(1, \ldots, 1, 2, 2, \ldots)$. By Lemma 50, β contains only one instance of the value 1 and has no ascents after the first position. Then α is of the form $(2, \ldots, 2, 1, 2, \ldots, 2)$. However, the parking function $\alpha' = (3, \ldots, 3, 1, 2, \ldots, 2)$ has the same statistic encoding as α, contradicting α being the unique parking function with statistic encoding w.

Now we consider the case where w does not have both A and D.

- Suppose that w does not contain D. If $\mathrm{stat}(\alpha) = w$, α is nondecreasing. By Lemma 50, w is of the form $\mathtt{A}^*\,\mathtt{T}^*$.

- Suppose that w does not contain A. If $\mathrm{stat}(\alpha) = w$, α is nonincreasing. The nondecreasing rearrangement β of α is exactly the reverse of α. The statistic encoding of β is of the form $\mathtt{A}^*\,\mathtt{T}^*$, so the statistic encoding of α is of the form $\mathtt{T}^*\mathtt{D}^*$.

($\Leftarrow$) Let $w \in \{\mathtt{A}^* \, \mathtt{T}^*\} \cup \{\mathtt{T}^*\mathtt{D}^*\}$. We consider the case where $w \in \{\mathtt{A}^* \, \mathtt{T}^*\}$ and $w \in \{\mathtt{T}^*\mathtt{D}^*\}$, separately and show that α with statistic encoding w is unique.

- If $w = \mathtt{A}^i \, \mathtt{T}^{n-1-i}$ for some $i \in [n-1]$, then $a_1 = 1$. We claim $a_j = j$ for all $2 \leq j \leq i$. For the sake of contraction, suppose that there exists $1 < j' \leq i$ such that $a_{j'} \neq j'$. Then the number j' does not appear in α as its statistic encoding is $w = \mathtt{A}^i \, \mathtt{T}^{n-1-i}$. This would imply that no car parks in spot j and hence α is not a parking function giving us a contradiction. Thus, $a_j = j$ for all $1 \leq j \leq i$. Since $w = \mathtt{A}^i \, \mathtt{T}^{n-1-i}$, we now have that $a_{i+1} = a_{i+2} = \cdots = a_n = i$. Thus, if $w = \mathtt{A}^i \, \mathtt{T}^{n-1-i}$, then α is unique and has the form

$$\alpha = (1, 2, \ldots, i-1, i, i+1, i+1, \ldots, i+1) \in \mathrm{PF}_n. \tag{16}$$

- If $w = \mathtt{T}^{n-1-i}\mathtt{D}^i$ for some $i \in [n-1]$, then $a_n = 1$. By a similar argument in the bullet above, we can show w is a unique parking function. We can reverse the parking function α given in Equation (16) to create

$$\alpha^* = (i+1, i+1, \ldots, i+1, i, i-1, \ldots, 3, 2, 1),$$

which is a parking function and has w as its statistic encoding. As α was unique, so is α^*. □

We conclude by posing the following open problems.

Problem 53. Fix a statistic encoding $w \in W_{n-1}$. Characterize and enumerate the subset of parking functions of length n which have w as their statistic encoding.

Problem 54. For which statistic encoding $w \in W_{n-1}$ is the subset of parking functions of length n which have w as their statistic encoding the largest?

6 Acknowledgments

The authors thank the anonymous referee for an exceptionally detailed review process. We appreciate their feedback and care.

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2020 *Mathematics Subject Classification*: Primary 05A05; Secondary 05A15.
Keywords: parking function, descent, ascent, tie, multiset, Eulerian polynomial, Catalan number, Fine number.

(Concerned with sequences A000108, A000957, A002378, A009766, A033282, and A333829.)

Received December 28 2023; revised versions received May 11 2024; October 31 2024; November 28 2024; November 29 2024. Published in *Journal of Integer Sequences*, November 30 2024.

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