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Bijections Between Colored Compositions, Dyck Paths, and Polygon Partitions

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Abstract

We give part-preserving bijections between three fundamental families of objects that serve as natural framework for many problems in enumerative combinatorics. Specifically, we consider compositions, Dyck paths, and partitions of a convex polygon, and identify suitable building blocks that are then appropriately decorated to achieve matching cardinalities. Our bijections are constructive and apply for the general case where the building blocks are allowed to come in different colors.

1 Introduction

Motivated by the approach suggested by Birmajer, Gil, and Weiner [2], we use known enumeration formulas in terms of partial Bell polynomials to establish connections between colored compositions, colored Dyck paths, and colored partitions of convex polygons.

Let us start by recalling some basic definitions and known results.

1.1 Compositions

A *composition* of a positive integer n is an ordered k-tuple $(j_1, \ldots, j_k)$, for $k \geq 1$, of positive integers (parts) such that $j_1 + \cdots + j_k = n$. For example, $(1, 3, 3, 2)$ is a composition of $n = 9$

with $k = 4$ parts.

There are $\binom{n-1}{k-1}$ compositions of n with k parts. Moreover, given a sequence of nonnegative integers $\gamma = (\gamma_j)_{j\in\mathbb{N}}$, a γ-color composition of n is a composition such that part j can take on γ_j colors. If $\gamma_j = 0$, then j is not admissible as part of the composition. We let $c_{n,k}(\gamma)$ denote the number of γ-color compositions of n with exactly k parts.

Proposition 1 (Hoggatt & Lind [4])**.**

$$c_{n,k}(\gamma) = \frac{k!}{n!} B_{n,k}(1!\gamma_1, 2!\gamma_2, \dots) \quad \textit{for } n \geq 1,$$

where $B_{n,k}$ is the (n,k)-th partial Bell polynomial.[1]

1.2 Dyck paths

A *Dyck path* of semilength n (or n-Dyck path) is a lattice path from $(0,0)$ to $(2n,0)$, consisting of steps $(1,1) \rightsquigarrow \mathsf{U}$ and $(1,-1) \rightsquigarrow \mathsf{D}$, never going below the x-axis.

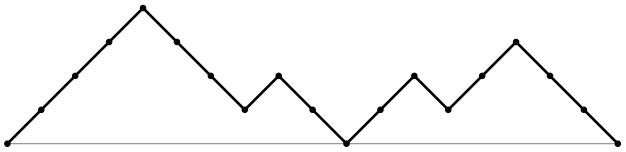


Figure 1: Dyck path of semilength $n = 9$ with 4 peaks.

Note that every Dyck path can be constructed using primitive blocks of the form

$$\mathsf{U}, \mathsf{UD}, \mathsf{UD}^2, \mathsf{UD}^3, \mathsf{UD}^4, \dots$$

or alternatively, using primitive blocks of the form

$$\mathsf{D}, \mathsf{UD}, \mathsf{U}^2\mathsf{D}, \mathsf{U}^3\mathsf{D}, \mathsf{U}^4\mathsf{D}, \dots$$

For instance, the path shown in Figure 1 could be written as $\mathsf{UUU}(\mathsf{UD}^3)(\mathsf{UD}^2)\mathsf{U}(\mathsf{UD})\mathsf{U}(\mathsf{UD}^3)$ or as $(\mathsf{U}^4\mathsf{D})\mathsf{DD}(\mathsf{UD})\mathsf{D}(\mathsf{U}^2\mathsf{D})(\mathsf{U}^2\mathsf{D})\mathsf{DD}$, depending on the chosen set of building blocks.

There are $\frac{1}{n}\binom{n}{k-1}\binom{n}{k}$ n-Dyck paths with exactly k peaks. Given a sequence of nonnegative integers $\gamma = (\gamma_j)_{j\in\mathbb{N}}$, we let $d_{n,k}(\gamma)$ denote the number of n-Dyck paths with k peaks such that the primitive component of length $j+1$ (e.g. UD^j) may be colored in γ_j different ways.

Proposition 2 (Mansour & Sun [5])**.**

$$d_{n,k}(\gamma) = \frac{1}{(n-k+1)!} B_{n,k}(1!\gamma_1, 2!\gamma_2, \dots) \textit{ for } n \geq 1.$$

[1] For the definition and properties of these polynomials, see the book by Comtet [3, Sec. 3.3].

1.3 Polygon partitions

Finally, we consider the set of rooted convex polygons partitioned by noncrossing diagonals.

Given a sequence of nonnegative integers $\gamma = (\gamma_j)_{j\in\mathbb{N}}$, we let $p_{n,k}(\gamma)$ be the number of colored partitions of a rooted convex $(n+2)$-gon made by $k-1$ noncrossing diagonals into k polygons such that each $(j+2)$-gon may be colored in γ_j different ways.

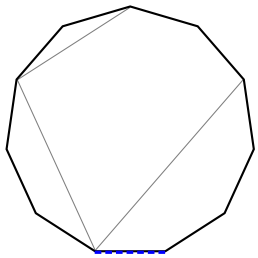


Figure 2: Partition of a rooted 11-gon.

Proposition 3 (Birmajer, Gil & Weiner [1])**.**

$$p_{n,k}(\gamma) = \frac{1}{n+1}\binom{n+k}{k}\frac{k!}{n!}B_{n,k}(1!\gamma_1, 2!\gamma_2, \dots) \quad \textit{for } n \geq 1.$$

It is clear that the aforementioned sets are not equinumerous, but with appropriate markings, they can be bijectively connected. In fact, based on the above three propositions, it is straightforward to see that for any coloring sequence $\gamma = (\gamma_j)_{j\in\mathbb{N}}$, we have

$$\begin{aligned}
\binom{n}{k-1}c_{n,k}(\gamma) &= k\, d_{n,k}(\gamma),\\
\binom{n+k}{k}d_{n,k}(\gamma) &= \binom{n+1}{k}p_{n,k}(\gamma),\\
(n+1)p_{n,k}(\gamma) &= \binom{n+k}{k}c_{n,k}(\gamma).
\end{aligned} \tag{1}$$

The goal of this paper is to provide constructive combinatorial proofs for these identities.

2 Map from Dyck paths to compositions

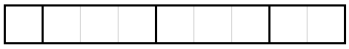
In this section, we give a bijective proof of the following result.

Theorem 4. *Let $1 \leq k \leq n$. For any coloring sequence $\gamma = (\gamma_j)_{j\in\mathbb{N}}$, we have*

$$\binom{n}{k-1}c_{n,k}(\gamma) = k\, d_{n,k}(\gamma).$$

We first assume $\gamma = (1, 1, 1, \dots)$ and represent every composition of n as a tiling of an $n \times 1$ rectangular board with n unit squares. For example,

$$(1,\ 3,\ 3,\ 2) \quad \longleftrightarrow \quad$$



Moreover, we can represent every lattice path with steps U and D as a word over the alphabet $\{\mathsf{U}, \mathsf{D}\}$ with valuations $\mathrm{val}(\mathsf{U}) = 1$ and $\mathrm{val}(\mathsf{D}) = -1$. A Dyck word w is then a word over that alphabet with $\mathrm{val}(w) = 0$ and such that, if $w = uv$, then $\mathrm{val}(u) \geq 0$.

Let $\mathcal{C}_{n,k}^{(k-1)}$ be the set of compositions of n with k parts and such that $k-1$ of the n unit squares are marked. Moreover, let $\mathcal{D}_{n,k}^{(\mathrm{peak})}$ be the set of n-Dyck paths with k peaks and such that one peak is marked.

We now proceed to give a bijective map $\varphi_{dc} : \mathcal{D}_{n,k}^{(\mathrm{peak})} \to \mathcal{C}_{n,k}^{(k-1)}$, illustrating each step of our construction with an example.

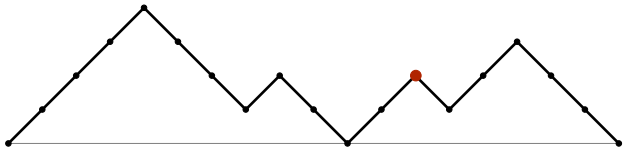


Figure 3: Dyck path with one peak marked.

- Given a Dyck path P with one peak marked, write it as a word w_P and extend it by adding an extra U to the beginning $\rightsquigarrow w = \mathsf{U}w_P$. Note that $\mathrm{val}(w) = 1$, and if $w = uv$ with a nonempty word u, then $\mathrm{val}(u) > 0$.

  For example, the path in Figure 3 leads to $w = \mathsf{U}\mathsf{UUUUDDDUDDUUDUUDDD}$.

- Split the word w at each valley (occurrence of DU) into subwords $w_1, \dots, w_k$ that start with a string of U's and end with a string of D's, and mark the subword that corresponds to the marked peak of the path.

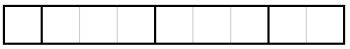
$$(\mathsf{UUUUUDDD})(\mathsf{UDD})(\overline{\mathsf{UUD}})(\mathsf{UUDDD})$$

- If needed, rotate the subwords $w_1, \dots, w_k$, repeatedly moving the last subword to the front of the word until the marked subword is the left most subword.

$$(\mathsf{UUUUUDDD})(\mathsf{UDD})(\overline{\mathsf{UUD}})(\mathsf{UUDDD})$$
$$\downarrow$$
$$(\mathsf{UUDDD})(\mathsf{UUUUUDDD})(\mathsf{UDD})(\overline{\mathsf{UUD}})$$
$$\downarrow$$
$$(\overline{\mathsf{UUD}})(\mathsf{UUDDD})(\mathsf{UUUUUDDD})(\mathsf{UDD})$$

Note that each of the rotated words will have a left factor with valuation 0.

- Let $\hat{w}$ denote the (rotated) word starting with the marked subword. The k parts of the composition associated to P are the number of D's in each of the k subwords of $\hat{w}$.

  In our example, this leads to the composition  .

- Next, we identify the first $k-1$ peaks (occurrences of UD) in $\hat{w}$ and mark the positions of the corresponding U-steps. If $k = 1$, we skip this step.

$$(\underline{\mathsf{U}}\mathsf{UD})(\mathsf{U}\underline{\mathsf{U}}\mathsf{DDD})(\mathsf{UUUU}\underline{\mathsf{U}}\mathsf{DDD})(\mathsf{UDD})$$

- Finally, identifying the first n U-steps of $\hat{w}$ with the n unit squares of the associated composition, we place $k-1$ dots at the unit squares that correspond to the positions of the marked peaks in $\hat{w}$. The resulting marked composition is denoted by $\varphi_{dc}(P)$.

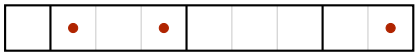


Figure 4: Marked composition corresponding to Dyck path in Figure 3.

Going back from a marked composition C to a word $\hat{w}$ over the alphabet $\{\mathsf{U}, \mathsf{D}\}$ is fairly simple. The length of each tile in C gives the distribution of the D's, and the location of the dots gives the placement of the U's (with an extra U added to the last group of D's). By construction, $\mathrm{val}(\hat{w}) = 1$.

Now, split $\hat{w}$ at its valleys to obtain subwords $\hat{w}_1, \ldots, \hat{w}_k$ that start with a string of U's and end with a string of D's. Mark the subword $\hat{w}_1$. If $\hat{w}$ has a left factor with valuation 0, we rotate the subwords repeatedly moving the first subword to the end of the word until the rotated subword has no left factor with valuation 0. Once that point is reached, we remove the first U from the rotated word and obtain a marked Dyck word, $\varphi_{dc}^{-1}(C)$. Note that the marked peak is the one inside the subword $\hat{w}_1$.

We finish by observing that, under our bijection, each part j of a given composition corresponds to a maximal descent UD^j of the associated Dyck path, so our proof also works for an arbitrary coloring sequence $\gamma = (\gamma_1, \gamma_2, \ldots)$. □

The rotation strategy we used to count the marked Dyck paths is known for proving Narayana's formula.

Remark 5. For $\gamma = (1, 1, \ldots)$ and $n \geq 1$, the sequence

$$a_n = \sum_{k=1}^{n} \binom{n}{k-1} c_{n,k} = \sum_{k=1}^{n} k\, d_{n,k}$$

gives $1, 3, 10, 35, 126, 462, 1716, 6435, 24310, 92378, \ldots$, cf. [6, A001700].

3 Map from Dyck paths to polygon partitions

In this section, we discuss a connection between colored Dyck paths and colored partitions of a convex polygon made by noncrossing diagonals.

We start by pointing out that polygon partitions are easily converted to rooted trees. In fact, a convex $(n+2)$-gon partitioned by $k-1$ noncrossing diagonals can be bijectively mapped to a rooted tree with $n+1$ leaves, $k-1$ internal nodes, and having no node of outdegree 1. An example of this known bijection is illustrated in Figure 5.

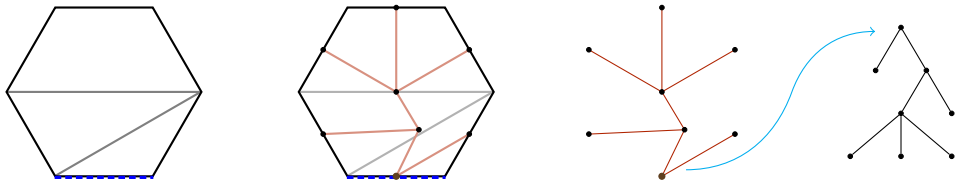


Figure 5: Bijection between polygon partitions and rooted trees

Theorem 6. *Let $1 \le k \le n$. For any coloring sequence $\gamma = (\gamma_j)_{j\in\mathbb{N}}$,*

$$\binom{n+k}{k} d_{n,k}(\gamma) = \binom{n+1}{k} p_{n,k}(\gamma).$$

As before, we start by assuming $\gamma = (1, 1, 1, \dots)$.

Every n-Dyck path with k peaks has n U-steps, and it must have k D-steps that are each part of a peak. We let $\mathcal{D}_{n,k}^{(k \text{ steps})}$ denote the set of such paths where k of these $n+k$ steps are marked. On the other hand, a rooted convex $(n+2)$-gon must have one side designated as the base. We let $\mathcal{P}_{n,k}^{(k)}$ be the set of partitions of a rooted convex $(n+2)$-gon made by $k-1$ noncrossing diagonals, where k of the $n+1$ non-base sides of the polygon are marked.

The above bijection between rooted polygon partitions and rooted trees (depicted in Figure 5) gives a bijection between $\mathcal{P}_{n,k}^{(k)}$ and the set $\mathcal{T}_{n,k}^{(k)}$ of rooted trees having $n+1$ leaves and k nodes of outdegree greater than 1 (for a total of $n+k+1$ nodes), where k of the leaves are marked. We will give an explicit bijective map $\varphi_{dp} : \mathcal{D}_{n,k}^{(k \text{ steps})} \to \mathcal{T}_{n,k}^{(k)}$ along with an illustrating example.

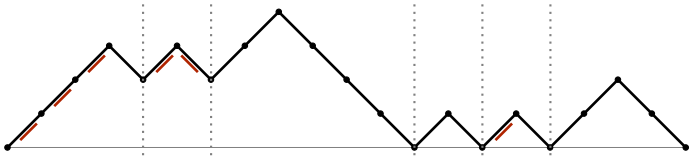


Figure 6: Marked Dyck path in $\mathcal{D}_{10,6}^{(6 \text{ steps})}$.

- Given a marked Dyck path P, break the path at each valley so that there are k shorter lattice paths with one peak each (as shown in Figure 6).

- For each segment $\mathsf{U}^a\mathsf{D}^b$, create a *primitive* rooted tree of the form

with $a+1$ leaves and the weight b (omitted if $b=1$) assigned to the right-most edge. Mark the leaves of any branches that correspond to a marked step on the Dyck path. This gives an ordered tuple of marked trees $(T_1, \ldots, T_k)$ corresponding to the given Dyck path.

Figure 7: Marked trees associated with the Dyck path in Figure 6.

- For every $j \in \{1, \ldots, k\}$, let ℓ_j be the number of unmarked leaves in T_j and let b_j be the weight of its right-most edge. Going left to right, we proceed with the following merging process:

  If $\ell_1 \geq b_1$, place the root of T_2 into T_1 on its b_1-st unmarked leaf from the right. We denote the merged tree by $T_{1,2}$ and declare the $b_1 - 1$ right-most leaves of T_1 as inactive. The tree $T_{1,2}$ now has $\ell_1 - b_1 + \ell_2$ unmarked active leaves, and we assign to it the weight b_2. If $\ell_1 < b_1$, do nothing and move to T_2.

  We repeat this merging procedure until we reach T_k and let $(F_{v_1}, \ldots, F_{v_m})$ be the resulting ordered forest of merged primitive trees.

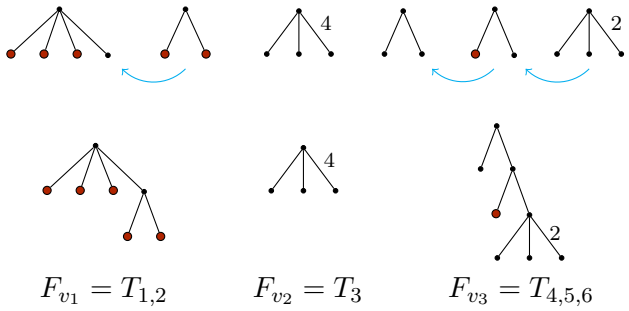


Figure 8: Ordered forest construction.

  If $m = 1$, this step gives a tree in $\mathcal{T}_{n,k}^{(k)}$ which we denote by $\varphi_{dp}(P)$.

- If $m > 1$, we define $\varphi_{dp}(P)$ by means of an additional merging procedure. Note that a completely merged tree of the form $T_{i,\ldots,j}$ has $\ell_i - b_i + \cdots + \ell_{j-1} - b_{j-1} + \ell_j$ unmarked active leaves and weight b_j. Moreover, for $j < k$, we must have

$$\ell_i - b_i + \cdots + \ell_{j-1} - b_{j-1} + \ell_j < b_j,$$

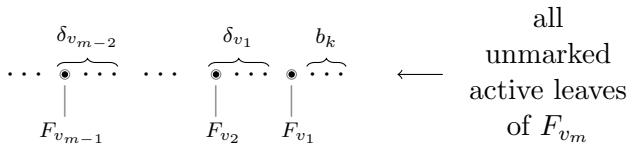
hence $\ell_i + \cdots + \ell_j + \delta_{i,\ldots,j} = b_i + \cdots + b_j$ for some $\delta_{i,\ldots,j} \geq 1$. Thus, if $F_{v_m} = T_{i_m,\ldots,k}$, the information from the trees $F_{v_1}, \ldots, F_{v_{m-1}}$ implies

$$\ell_1 + \cdots + \ell_{i_m - 1} + (\delta_{v_1} + \cdots + \delta_{v_{m-1}}) = b_1 + \cdots + b_{i_m - 1},$$

where δ_{v_i} is the difference between the weight of the tree F_{v_i} and the number of its unmarked active leaves. By definition, $\delta_{v_i} \geq 1$ for every $i \in \{1, \ldots, m-1\}$. In addition, since $\sum_{i=1}^{k} \ell_i = n = \sum_{i=1}^{k} b_i$, we have $\ell_{i_m} + \cdots + \ell_k - (\delta_{v_1} + \cdots + \delta_{v_{m-1}}) = b_{i_m} + \cdots + b_k$, and therefore,

$$\ell_{i_m} - b_{i_m} + \cdots + \ell_{k-1} - b_{k-1} + \ell_k = \delta_{v_1} + \cdots + \delta_{v_{m-1}} + b_k.$$

This means that F_{v_m} has exactly $\delta_{v_1} + \cdots + \delta_{v_{m-1}} + b_k$ unmarked active leaves. Finally, we construct the tree $\varphi_{dp}(P)$ by attaching the roots of the trees $F_{v_1}, \ldots, F_{v_{m-1}}$ to the unmarked active leaves of F_{v_m}, from right to left, according to the pattern:



In other words, the root of F_{v_1} is attached to the right-most unmarked leaf of F_{v_m} that allows the last b_k active leaves of F_{v_m} to stay unchanged. The remaining trees are attached in a way that $F_{v_{i+1}}$ is at the δ_{v_i}-th unmarked leaf of F_{v_m} to the left of F_{v_i}. Note that there will be $\delta_{v_{m-1}} - 1 \geq 0$ unmarked leaves to the left of $F_{v_{m-1}}$.

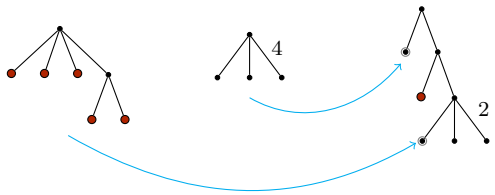


Figure 9: Right-to-left merging procedure.

For the marked Dyck path given in Figure 6, the above construction leads to the marked tree shown in Figure 10.

Conversely, there is an algorithm to label and decompose each element of $\mathcal{T}_{n,k}^{(k)}$ into an ordered forest of primitive rooted trees that leads to an element of $\mathcal{D}_{n,k}^{(k \text{ steps})}$.

Suppose we are given $T \in \mathcal{T}_{n,k}^{(k)}$. Such a tree consists of k primitive subtrees, $k-1$ internal nodes, $n+k$ edges, and $n+1-k$ unmarked leaves.

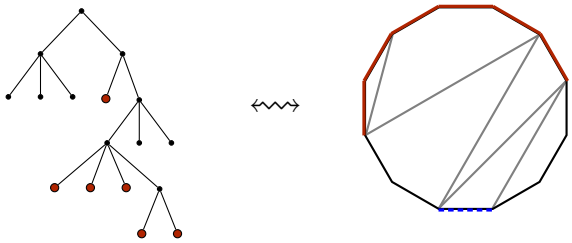


Figure 10: Marked tree and its corresponding polygon partition.

- Starting at the root of T, and going around clockwise, denote its primitive components by $T_1, \ldots, T_k$. For $i \in \{1, \ldots, k-1\}$, label the right-most edge of T_i with the number $\lambda_i \geq 1$ of unmarked nodes needed to arrive at the root of T_{i+1}. This gives a sequence of labels $\lambda_1, \ldots, \lambda_{k-1}$ with $\lambda_1 + \cdots + \lambda_{k-1} \leq n$. We let $\lambda_k = n - (\lambda_1 + \cdots + \lambda_{k-1})$.

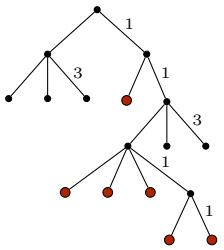


Figure 11: Labeled element of $\mathcal{T}^{(6)}_{10,6}$.

Let $a_i + 1$ be the number of edges of T_i, so $a_1 + \cdots + a_k = n$. Let $d_i = a_i - \lambda_i$.

- If $\sum_{i=1}^{j} d_i \geq 0$ for every $j \in \{1, \ldots, k-1\}$, then we let $\varphi_{dp}^{-1}(T)$ be the marked Dyck path $\mathsf{U}^{a_1}\mathsf{D}^{\lambda_1} \cdots \mathsf{U}^{a_k}\mathsf{D}^{\lambda_k}$, where a subpath $\mathsf{U}^{a_i}\mathsf{D}$ is marked at each step corresponding to the marked leaves of T_i. Otherwise, if the above path does not satisfy the Dyck path condition, we let $\ell > 1$ be the smallest index such that

$$\sum_{i=\ell}^{j} d_i \geq 0 \quad \text{for every } j \in \{\ell, \ldots, k-1\}.$$

In this case, $\varphi_{dp}^{-1}(T)$ is defined to be the Dyck path

$$\mathsf{U}^{a_\ell}\mathsf{D}^{\lambda_\ell} \cdots \mathsf{U}^{a_k}\mathsf{D}^{\lambda_k+1}\,\mathsf{U}^{a_1}\mathsf{D}^{\lambda_1} \cdots \mathsf{U}^{a_{\ell-1}}\mathsf{D}^{\lambda_{\ell-1}-1} \tag{2}$$

associated with the sequence $(T_\ell, \ldots, T_k, T_1, \ldots, T_{\ell-1})$, and marked accordingly.

For example, for the tree in Figure 11, we have $\ell = 4$ and the associated Dyck path is

$$(\underline{\mathsf{U}}\underline{\mathsf{U}}\underline{\mathsf{U}}\mathsf{D})(\underline{\mathsf{U}}\underline{\mathsf{D}})(\mathsf{UUDDDD})(\mathsf{UD})(\underline{\mathsf{U}}\mathsf{D})(\mathsf{UUDD}).$$

To verify that (2) is indeed a Dyck path, note that by definition, $d_{\ell-1} < 0$. If there is a $J < \ell - 1$ such that $d_{J+1} + \cdots + d_{\ell-1} < 0$ but $d_J + d_{J+1} + \cdots + d_{\ell-1} \geq 0$, then

$$d_J \geq -(d_{J+1} + \cdots + d_{\ell-1}) > 0,$$

and so $\sum_{i=J}^{j} d_i \geq 0$ for every $j \in \{J, \ldots, k-1\}$. This contradicts the minimality of ℓ.

Therefore, $\sum_{i=j}^{\ell-1} d_i < 0$ for every $j \in \{1, \ldots, \ell-1\}$. By definition, $\sum_{i=1}^{k} d_i = 0$, so if we let $v = \sum_{i=\ell}^{k} d_i$, then $v > 0$ and $v + \sum_{i=1}^{j} d_i > 0$ for every $j \in \{1, \ldots, \ell-2\}$. In addition, note that since $v - 1 \geq 0$, we have $(a_\ell - \lambda_\ell) + \cdots + (a_k - (\lambda_k + 1)) \geq 0$.

Under our bijection, a primitive subtree with $a+1$ leaves ($(a+2)$-gon on the polygon) corresponds to a primitive block $\mathsf{U}^a\mathsf{D}$ on the Dyck path. Therefore, our proof applies verbatim to the case of an arbitrary coloring sequence $\gamma = (\gamma_1, \gamma_2, \ldots)$. □

Remark 7. For $\gamma = (1, 1, \ldots)$ and $n \geq 1$, the sequence

$$a_n = \sum_{k=1}^{n} \binom{n+k}{k} d_{n,k} = \sum_{k=1}^{n} \binom{n+1}{k} p_{n,k}$$

gives $2, 9, 54, 375, 2848, 22981, 193742, 1688427, 15101778, 137930199, \ldots$ [6, A368178].

4 Map from compositions to polygon dissections

We now aim at providing a combinatorial proof for the last identity in (1):

Theorem 8. *Let $1 \leq k \leq n$. For any coloring sequence $\gamma = (\gamma_j)_{j \in \mathbb{N}}$,*

$$(n+1)p_{n,k}(\gamma) = \binom{n+k}{k} c_{n,k}(\gamma).$$

As in the previous two sections, our bijection will be part preserving, so without loss of generality we assume $\gamma = (1, 1, 1, \ldots)$. The general case follows verbatim.

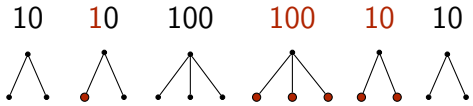
To prove the above identity, we will replace polygon dissections with rooted trees (as done in the previous section, see Figure 5), and it is more convenient to think of a composition as a binary word starting with 1 and having no consecutive 1's. More precisely, a composition of n with k parts, say $(j_1, \ldots, j_k)$, can be represented by the binary word $w_{j_1} \cdots w_{j_k}$ of length $n + k$, where w_{j_i} consists of a one followed by j_i zeros. For example, the composition $(1, 3, 2, 4)$ corresponds to the binary word 10100010010000.

With this in mind, we let $\mathcal{T}^{(1)}_{n,k}$ be the set of rooted trees having $n+1$ leaves and k nodes of outdegree greater than one, where one of the leaves is marked. Moreover, we let $\mathcal{B}^{(k)}_{n,k}$ be the set of binary words of length $n+k$ starting with 1, having n 0's and k 1's, no two consecutive 1's, and where k of the letters are marked. Clearly,

$$|\mathcal{T}^{(1)}_{n,k}| = (n+1)p_{n,k} \quad \text{and} \quad |\mathcal{B}^{(k)}_{n,k}| = \binom{n+k}{k} c_{n,k}.$$

Our goal is to give a bijective map $\varphi_{bt} : \mathcal{B}^{(k)}_{n,k} \to \mathcal{T}^{(1)}_{n,k}$ along with illustrating examples.

- Every binary word in $\mathcal{B}^{(k)}_{n,k}$ is of the form $w = w_{j_1} \cdots w_{j_k}$, where $w_j = 10^j$ (a one followed by j zeros) and k of the letters in w are marked. Split w into its k primitive components $(w_{j_1}, \ldots, w_{j_k})$, and for every component w_{j_i} make a primitive rooted tree T_i with $j_i + 1$ leaves. Mark the leaves that correspond to marked letters in the binary word. For example, the word 10101001001010 in $\mathcal{B}^{(6)}_{8,6}$ leads to the decomposition



(3)

- Combine the sequence of primitive trees $(T_1, \ldots, T_k)$ by shifting each component to the left (starting with T_2) so that its root is placed on the right-most marked leaf of the tree to its immediate left. If there is no marked leaf to place T_i, move to T_{i+1} and continue the process until you reach T_k. This step gives an ordered forest $(F_1, \ldots, F_m)$ of merged primitive trees with a total of m marked leaves, all of them on F_m.

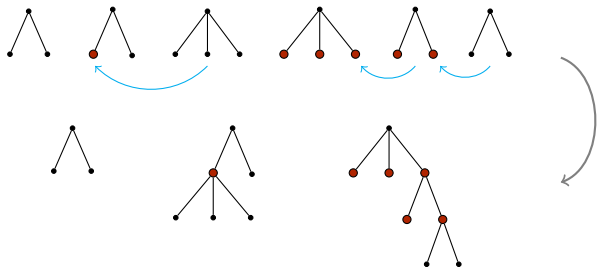


Figure 12: Left shifting procedure

- If $m = 1$, let $\varphi_{bt}(w)$ be the tree obtained from F_1 by removing the marks from its internal nodes. If $m > 1$, attach the trees $F_1, \ldots, F_{m-1}$ (from right to left) to the $m-1$ left-most marked leaves on F_m and remove the internal marks from the merged tree. The resulting tree will be $\varphi_{bt}(w) \in \mathcal{T}^{(1)}_{n,k}$.

The inverse map is straightforward. Let $T \in \mathcal{T}^{(1)}_{n,k}$. Such a tree consists of k primitive subtrees, $k-1$ internal nodes, and one marked leaf.

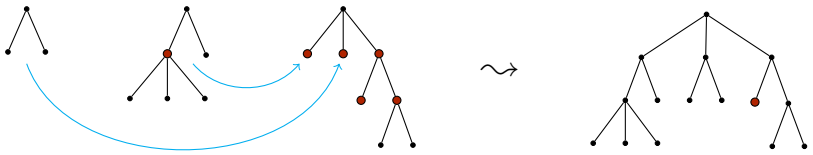


Figure 13: Right shifting procedure

- Starting at the root of T, and going around clockwise, denote its primitive components by $T_1, \ldots, T_k$. Let T_m be the subtree with the marked leaf.
- If the nodes of T_m to the right of the marked leaf have a total of j descendants, consider the ordered forest $(T_{m+j+1}, \ldots, T_k, T_1, \ldots, T_{m+j})$ and mark the leaves that correspond to internal nodes of T.

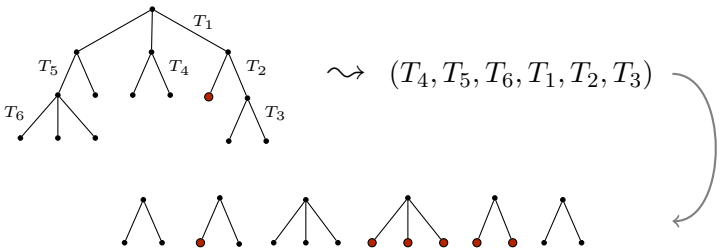


Figure 14: Right shifting procedure

- For a primitive tree with $j+1$ leaves, assign the word $\mathsf{10}^j$ and mark the letters that correspond to the marked leaves of the tree (like in (3)). Finally, $\varphi_{bt}^{-1}(T)$ is obtained by concatenating the k words associated with $(T_{m+j+1}, \ldots, T_k, T_1, \ldots, T_{m+j})$ into a single binary word. □

Remark 9. For $\gamma = (1, 1, \ldots)$ and $n \geq 1$, the sequence

$$a_n = \sum_{k=1}^{n} (n+1) p_{n,k} = \sum_{k=1}^{n} \binom{n+k}{k} c_{n,k}$$

gives $2, 9, 44, 225, 1182, 6321, 34232, 187137, 1030490, 5707449, \ldots$, cf. [6, A176479].

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