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Counting Rectangles of Size $r \times s$ in Nondecreasing and Smirnov Words

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Abstract

The rectangle capacity, a word statistic that Mansour and the author recently introduced, counts, for two fixed positive integers r and s, the number of occurrences of a rectangle of size $r \times s$ in the bargraph representation of a word. In this work we find the bivariate generating function for the distribution on nondecreasing words of the number of rectangles of size $r \times s$ and the generating function for their total number over all nondecreasing words. We also obtain the analog results for Smirnov words, which are words that have no consecutive equal letters. This complements our recent results concerned with general words (i.e., not restricted) and Catalan words.

1 Introduction

Let n and k be two positive integers and set $[k] = \{1, 2, \ldots, k\}$. A word over k of length n is any element of the set $[k]^n$. Words have a visual representation in terms of bargraphs (see Figure 1), giving rise to many natural statistics on words, such as their water capacity, number of lit cells, and perimeter (see [12] for a comprehensive review of the subject).

In this work we continue the study of a new statistic, which Mansour and the author introduced in [8], called *rectangle capacity*. This statistic, for two fixed positive integers r and s, counts the number of occurrences of a rectangle of size $r \times s$ in the bargraph representation of a word (see Figure 2). Rectangle capacity is a special case of a much more general statistic, defined by Mansour and Shabani [11]: for two bargraphs B and C, a vertex (x, y) of B is said to be a C-vertex if C lies entirely in B, when positioned starting at (x, y). In their work, Mansour and Shabani studied the number of C-vertices in several special cases. Two of these correspond to rectangle capacity of size $1 \times s$ and $r \times 1$.

Rectangle capacity has also been studied in the context of set partitions by Cakić et al. [3], for $r \times s = 1 \times 2$ and by Archibald et al. [1], for $r \times s = 2 \times 2$. Notice that the rectangle capacity of size 1×1 coincides with the area of the bargraph, a statistic that is naturally well studied (e.g., [4, 12]).

In our work [8] we studied general words (i.e., not restricted) and Catalan words. The purpose of this work is to explore additional families of words, namely *nondecreasing words* and *Smirnov words*. By the former we mean words $w_1 \cdots w_n \in [k]^n$ such that $w_{i+1} \geq w_i$, for each $1 \leq i \leq n-1$, and, by the latter, words $w_1 \cdots w_n \in [k]^n$ such that $w_{i+1} \neq w_i$, for each $1 \leq i \leq n-1$ (cf. [5, Example 23, p. 193]).

Smirnov words proved to be applicable to different kinds of problems such as waiting time distributions of runs in [7] and diagonal coinvariant rings in [10]. Furthermore, Smirnov words are connected to q-Eulerian polynomials (e.g., [6]). Several results concerned with statistics on nondecreasing words appear in [9, Section 4.3].

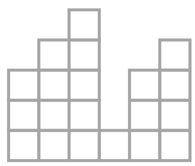

Figure 1: The bargraph representation of the word 345134.

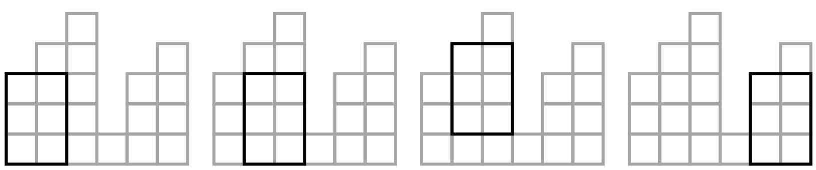

Figure 2: There are four occurrences of a rectangle of size 3×2 in the bargraph representation of the word 345134.

Before we begin, let us fix three positive integers r, s, and k and let n be a nonnegative integer. For a positive integer m, we denote by $[m]$ the set $\{1, 2, \ldots, m\}$.

2 Main results

2.1 Nondecreasing words

We denote by $\mathcal{W}^{\nearrow}_{n,k}$ the set of nondecreasing words over k of length n, i.e.

$$\mathcal{W}^{\nearrow}_{n,k} = \{w_1 \cdots w_n \in [k]^n \ : \ w_i \leq w_{i+1} \text{ for every } i \in [n-1]\}.$$

It is well known that

$$\left|\mathcal{W}^{\nearrow}_{n,k}\right| = \binom{n+k-1}{k-1}.$$

We distinguish between two cases, namely $r = 1$ and $r \geq 2$.

2.1.1 Rectangle capacity of size $1 \times s$

Denote by $a_{n,k} = a_{n,k}(t)$ the distribution on $\mathcal{W}^{\nearrow}_{n,k}$ of the number of $1 \times s$ rectangles and let $A_k(x,t)$ denote the generating function of the numbers $a_{n,k}$. We shall need the following restrictions of $\mathcal{W}^{\nearrow}_{n,k}$: Let

$$\mathcal{W}^{\nearrow,(0)}_{n,k} = \left\{w_1 \cdots w_n \in \mathcal{W}^{\nearrow}_{n,k} \ : \ w_j > 1 \text{ for every } j \in [n]\right\}$$

and, for $i \in [n]$, we define

$$\mathcal{W}^{\nearrow,(i)}_{n,k} = \left\{w_1 \cdots w_n \in \mathcal{W}^{\nearrow}_{n,k} \ : \ w_i = 1 \text{ and } w_j > 1 \text{ for every } i < j \leq n\right\}.$$

For $0 \leq i \leq n$, let $a^{(i)}_{n,k}$ be the restriction of $a_{n,k}$ to $\mathcal{W}^{\nearrow,(i)}_{n,k}$. Notice that if $w_1 \cdots w_n \in \mathcal{W}^{\nearrow,(i)}_{n,k}$, then $w_1 = \cdots = w_i = 1$.

Theorem 1. *We have*

$$A_k(x,t) = \sum_{i=0}^{k-1} \frac{\alpha_{k-i}(t^i x, t)}{t^{i(s-1)} \prod_{j=1}^{i}(1-t^j x)}, \tag{1}$$

where

$$\alpha_m(x,t) = \sum_{n=0}^{s-2} \binom{n+m-1}{m-1} x^n + \frac{x^{s-1}}{1-tx}\binom{s+m-3}{m-1} - \frac{1}{t^{s-1}(1-tx)} \sum_{n=0}^{s-2} \binom{n+m-2}{m-2} (tx)^n.$$

Proof. The set $\mathcal{W}^{\nearrow}_{n,1}$ consists solely of the word $1 \cdots 1$, containing $n-s+1$ rectangles of size $1 \times s$. Thus, $a_{n,1} = t^{\max\{0,n-s+1\}}$ and it is easily verified that the corresponding generating function is given by

$$A_1(x,t) = \alpha_1(x,t) = \frac{1-x^{s-1}}{1-x} + \frac{x^{s-1}}{1-tx}.$$

Assume now that $k \geq 2$. For $n \geq s$ we have

$$\begin{aligned} a_{n,k} &= a_{n,k}^{(0)} + \sum_{i=1}^{n} a_{n,k}^{(i)} \\ &= t^{n-s+1} a_{n,k-1} + t^{n-s+1} \sum_{i=1}^{n} a_{n-i,k-1}. \end{aligned}$$

Multiplying both sides of this equation by x^n, summing over $n \geq s$ and adding $\sum_{n=0}^{s-1} a_{n,k} x^n$ to both sides, we obtain

$$\begin{aligned} A_k(x,t) &= \sum_{n=0}^{s-1} \binom{n+k-1}{k-1} x^n - \binom{s+k-3}{k-2} x^{s-1} \\ &\quad + \frac{1}{t^{s-1}(1-tx)} \left(A_{k-1}(tx,t) + \sum_{n=0}^{s-2} \binom{n+k-2}{k-2} ((tx)^s - (tx)^n) \right), \end{aligned}$$

from which (1) follows by induction. □

Corollary 2. *The generating function for the total number $f_k(n)$ of $1 \times s$ rectangles over all words belonging to $\mathcal{W}_{n,k}^{\nearrow}$ is given by*

$$\frac{x^s \sum_{i=0}^{k-1} (-1)^i \binom{i+s-2}{i} \binom{k+s}{s+i+1} x^i}{(1-x)^{k+1}}.$$

Thus, for $n \geq s$,

$$f_k(n) = \sum_{i=0}^{\min\{k-1,n-s\}} (-1)^i \binom{i+s-2}{i} \binom{k+s}{s+i+1} \binom{n-s-i+k}{k}.$$

Proof. Let $0 \leq i \leq k-1$. We have

$$\begin{aligned} &\frac{\partial}{\partial t} \frac{\alpha_{k-i}(t^i x, t)}{t^{i(s-1)} \prod_{j=1}^{i} (1-t^j x)} \\ &= \frac{\left(\frac{\partial}{\partial t} \alpha_{k-i}(t^i x, t)\right) t^{i(s-1)} \prod_{j=1}^{i} (1-t^j x) - \alpha_{k-i}(t^i x, t) \left(\frac{\partial}{\partial t} t^{i(s-1)} \prod_{j=1}^{i} (1-t^j x)\right)}{\left(t^{i(s-1)} \prod_{j=1}^{i} (1-t^j x)\right)^2}. \end{aligned}$$

We have

$$\begin{aligned} \left[t^{i(s-1)} \prod_{j=1}^{i} (1-t^j x) \right]_{|t=1} &= (1-x)^i, \\ \left[\frac{\partial}{\partial t} t^{i(s-1)} \prod_{j=1}^{i} (1-t^j x) \right]_{|t=1} &= i(1-x)^{i-1} \left((s-1)(1-x) - \frac{x(i+1)}{2} \right). \end{aligned}$$

It is not hard to see that

$$\left[\alpha_{k-i}(t^i x,t)\right]_{|t=1} = \begin{cases} 0, & \text{if } i<k-1; \\ \frac{1}{1-x}, & \text{otherwise.} \end{cases}$$

$$\left[\frac{\partial}{\partial t}\alpha_{k-i}(t^i x,t)\right]_{|t=1} = \begin{cases} \sum_{n=0}^{s-2}(s-1-n)\binom{n+k-1-i}{k-1-i}x^n, & \text{if } i<k-1; \\ \frac{x(x^{s-1}+k-1)}{(1-x)^2}, & \text{otherwise.} \end{cases}$$

It follows that

$$\begin{aligned} &\left[\frac{\partial}{\partial t}A_k(x,t)\right]_{|t=1} \\ &= \sum_{i=0}^{k-1}\left[\frac{\partial}{\partial t}\frac{\alpha_{k-i}(t^i x,t)}{t^{i(s-1)}\prod_{j=1}^{i}(1-t^j x)}\right]_{|t=1} \\ &= \sum_{i=0}^{k-2}\frac{\sum_{n=0}^{s-2}(s-1-n)\binom{n+k-1-i}{k-1-i}x^n}{(1-x)^i} + \frac{x\left(x^{s-1}+\frac{(k-1)(2s+k)}{2}\right)-(k-1)(s-1)}{(1-x)^{k+1}}, \end{aligned}$$

from which the assertion follows. □

Example 3. For $k=2$ we have

$$f_2(n) = \frac{3n^2+(7-4s)n+(s-1)(s-4)}{2}$$

and Table 1 lists the corresponding sequences that we found in the On-Line Encyclopedia of Integer Sequences (OEIS) [14]. For $k=3$ we have

$$f_3(n) = \frac{2n^3-3(s-3)n^2+(s^2-10s+13)n+2(s-1)(s-3)}{2}$$

and it seems that no corresponding sequence is registered in the OEIS.

2.1.2 Rectangle capacity of size $r \times s$, where $r \geq 2$

Denote by $b_{n,k} = b_{n,k}(t)$ the distribution on $\mathcal{W}_{n,k}^{\nearrow}$ of the number of $r \times s$ rectangles and by $B_k(x,t)$ the generating function of the numbers $b_{n,k}$. We shall need the following restriction of $\mathcal{W}_{n,k}^{\nearrow}$: for $m \in [k]$, let

$$\mathcal{W}_{n,k}^{\nearrow,(\geq m)} = \left\{w_1\cdots w_n \in \mathcal{W}_{n,k}^{\nearrow} \;:\; w_j \geq m \text{ for each } j\in[n]\right\},$$

and let $b_{n,k}^{(\geq m)}$ be the restriction of $b_{n,k}$ to $\mathcal{W}_{n,k}^{\nearrow,(\geq m)}$. Let $B_k^{(\geq m)}(x,t)$ be the generating function of the numbers $b_{n,k}^{(\geq m)}$.

s	$f_2(n)$	OEIS
1	$(3n^2-3n)/2$	A045943
2	$(3n^2-n-2)/2$	A115067
3	$(3n^2-5n-2)/2$	A140090
4	$(3n^2-9n)/2$	A140091
5	$(3n^2-13n+4)/2$	A059845
6	$(3n^2-17n+10)/2$	A140672
7	$(3n^2-21n+18)/2$	A140673
8	$(3n^2-25n+28)/2$	A140674
9	$(3n^2-29n+40)/2$	A140675
10	$(3n^2-33n+54)/2$	A151542
11	$(3n^2-37n+70)/2$	A370238

Table 1: The total number $f_2(n)$ of $1\times s$ rectangles over all words of length n, for several values of s.

Lemma 4. *Assume that $k\geq r-1$. Then*

$$B_k^{(\geq r-1)}(x,t)=\sum_{i=0}^{k-r+1}\frac{\alpha_{k-i}^{(\geq r-1)}(t^ix,t)}{t^{i(s-1)}\prod_{j=0}^{i-1}(1-t^jx)},\tag{2}$$

where

$$\alpha_m^{(\geq r-1)}(x,t)$$
$$=\sum_{n=0}^{s-1}\binom{n+m-r+1}{m-r+1}x^n+\frac{x^s}{1-x}\binom{s+m-r}{s-1}-\frac{1}{t^{s-1}(1-x)}\sum_{n=0}^{s-1}\binom{n+m-r}{m-r}(tx)^n.$$

Proof. The set $\mathcal{W}_{n,r-1}^{\nearrow;(\geq r-1)}$ consists solely of the word $(r-1)\cdots(r-1)$, containing no rectangles of size $r\times s$. Thus, $b_{n,r-1}^{(\geq r-1)}=1$ and the corresponding generating function is given by $B_{r-1}^{(\geq r-1)}(x,t)=1/(1-x)$. Assume now that $k\geq r$. For $n\geq s$ we have

$$\begin{aligned}b_{n,k}^{(\geq r-1)}&=b_{n,k}^{(\geq r)}+\sum_{i=1}^{n}b_{i,r-1}^{(\geq r-1)}b_{n-i,k}^{(\geq r)}\\&=t^{n-s+1}b_{n,k-1}^{(\geq r-1)}+\sum_{i=1}^{n-s}t^{n-i-s+1}b_{n-i,k-1}^{(\geq r-1)}+\sum_{i=n-s+1}^{n}b_{n-i,k-1}^{(\geq r-1)}.\end{aligned}$$

Multiplying both sides of this equation by x^n, summing over $n\geq s$ and adding $\sum_{n=0}^{s-1}b_{n,k}^{(\geq r-1)}x^n$

to both sides, we obtain the equation

$$B_k^{(\geq r-1)}(x,t) = \frac{1}{t^{s-1}(1-x)} B_{k-1}^{(\geq r-1)}(tx,t) + \sum_{n=0}^{s-1} \binom{n+k-r+1}{k-r+1} x^n$$
$$- \frac{1}{t^{s-1}(1-x)} \sum_{n=0}^{s-1} \binom{n+k-r}{k-r} (tx)^n + \frac{x^s}{1-x} \sum_{n=0}^{s-1} \binom{n+k-r}{k-r},$$

from which the statement immediately follows by induction. □

Theorem 5. *Assume that $k \geq r$. Then*

$$B_k(x,t) = \frac{1}{(1-x)^{r-1}} \sum_{i=1}^{k-r+1} \frac{\alpha_{k-i}^{(\geq r-1)}(t^i x,t)}{t^{i(s-1)} \prod_{j=1}^{i-1}(1-t^j x)} + \beta_k(x,t),$$

where

$$\beta_k(x,t) = \sum_{i=0}^{s-1} \binom{i+k-1}{k-1} x^i + \frac{1}{(1-x)^{r-1}} \sum_{i=0}^{s-1} \binom{i+k-r}{k-r} x^i$$
$$- \sum_{i=0}^{s-1} \binom{i+k-r}{k-r} x^i \sum_{n=0}^{s-1-i} \binom{n+r-2}{r-2} x^n - \frac{1}{t^{s-1}(1-x)^{r-1}} \sum_{i=0}^{s-1} \binom{i+k-r}{k-r} (tx)^i.$$

Proof. For $n \geq s$ we have

$$b_{n,k} = b_{n,k}^{(\geq r)} + \sum_{i=1}^{n} b_{i,r-1} b_{n-i,k}^{(\geq r)}$$
$$= t^{n-s+1} b_{n,k-1}^{(\geq r-1)} + \sum_{i=1}^{n-s} t^{n-i-s+1} b_{i,r-1} b_{n-i,k-1}^{(\geq r-1)} + \sum_{i=n-s+1}^{n} b_{i,r-1} b_{n-i,k-1}^{(\geq r-1)}.$$

Multiplying both sides of this equation by x^n, summing over $n \geq s$ and adding $\sum_{n=0}^{s-1} b_{n,k} x^n$ to both sides, we obtain the equation

$$B_k(x,t)$$
$$= \frac{1}{t^{s-1}(1-x)^{r-1}} B_{k-1}^{(\geq r-1)}(tx,t) + \sum_{i=0}^{s-1} \binom{i+k-1}{k-1} x^i + \frac{1}{(1-x)^{r-1}} \sum_{i=0}^{s-1} \binom{i+k-r}{k-r} x^i$$
$$- \sum_{i=0}^{s-1} \binom{i+k-r}{k-r} x^i \sum_{n=0}^{s-1-i} \binom{n+r-2}{r-2} x^n - \frac{1}{t^{s-1}(1-x)^{r-1}} \sum_{i=0}^{s-1} \binom{i+k-r}{k-r} (tx)^i,$$

from which, together with (2), the statement immediately follows. □

Corollary 6. *The generating function for the total number $g_k(n)$ of $r\times s$ rectangles over all words belonging to $\mathcal{W}_{n,k}^{\nearrow}$ is given by*

$$\frac{x^s\sum_{i=0}^{k-r+1}(-1)^i\binom{i+s-2}{i}\binom{k-r+1+s}{s+i+1}x^i}{(1-x)^{k+1}}.$$

Thus, for $n\geq s$,

$$g_k(n)=\sum_{i=0}^{\min\{k-r+1,n-s\}}(-1)^i\binom{i+s-2}{i}\binom{k-r+1+s}{s+i+1}\binom{n-s-i+k}{k}.$$

Proof. Let $i\in[k-r+1]$. We have

$$\begin{aligned}&\frac{\partial}{\partial t}\frac{\alpha_{k-i}^{(\geq r-1)}(t^ix,t)}{t^{i(s-1)}\prod_{j=1}^{i-1}(1-t^jx)}\\&=\frac{\left(\frac{\partial}{\partial t}\alpha_{k-i}^{(\geq r-1)}(t^ix,t)\right)t^{i(s-1)}\prod_{j=1}^{i-1}(1-t^jx)-\alpha_{k-i}^{(\geq r-1)}(t^ix,t)\left(\frac{\partial}{\partial t}t^{i(s-1)}\prod_{j=1}^{i-1}(1-t^jx)\right)}{\left(t^{i(s-1)}\prod_{j=1}^{i-1}(1-t^jx)\right)^2}.\end{aligned}$$

We have

$$\begin{aligned}\left[t^{i(s-1)}\prod_{j=1}^{i-1}(1-t^jx)\right]_{|t=1}&=(1-x)^{i-1},\\\left[\frac{\partial}{\partial t}t^{i(s-1)}\prod_{j=1}^{i-1}(1-t^jx)\right]_{|t=1}&=i(1-x)^{i-2}\left((s-1)(1-x)-\frac{x(i-1)}{2}\right).\end{aligned}$$

It is not hard to see that

$$\begin{aligned}\left[\alpha_{k-i}^{(\geq r-1)}(t^ix,t)\right]_{|t=1}&=\begin{cases}0, & \text{if } i<k-r+1;\\\frac{1}{1-x}, & \text{otherwise.}\end{cases}\\\left[\frac{\partial}{\partial t}\alpha_{k-i}^{(\geq r-1)}(t^ix,t)\right]_{|t=1}&=\begin{cases}\frac{1}{1-x}\sum_{n=0}^{s-2}(s-1-n)\binom{n+k-r-i}{k-r-i}x^n, & \text{if } i<k-r+1;\\\frac{x(k-r+1)}{(1-x)^2}, & \text{otherwise.}\end{cases}\\\left[\frac{\partial}{\partial t}\beta_k(x,t)\right]_{|t=1}&=\frac{1}{(1-x)^{r-1}}\sum_{n=0}^{s-2}(s-1-n)\binom{n+k-r}{k-r}x^n.\end{aligned}$$

It follows that

$$\left[\frac{\partial}{\partial t}A_k(x,t)\right]_{|t=1}$$

$$= \frac{1}{(1-x)^{r-1}}\sum_{i=1}^{k-r+1}\left[\frac{\partial}{\partial t}\frac{\alpha_{k-i}^{(\geq r-1)}(t^i x,t)}{t^{i(s-1)}\prod_{j=1}^{i-1}(1-t^j x)}\right]_{|t=1} + \left[\frac{\partial}{\partial t}\beta_k(x,t)\right]_{|t=1}$$

$$= \frac{1}{(1-x)^{r-1}}\left(\sum_{i=1}^{k-r}\frac{\sum_{n=0}^{s-2}(s-1-n)\binom{n+k-r-i}{k-r-i}x^n}{(1-x)^i} + (k-r+1)\frac{x(k-r+2)-2(s-1)(1-x)}{2(1-x)^{k-r+2}}\right)$$

$$+ \frac{1}{(1-x)^{r-1}}\sum_{n=0}^{s-2}(s-1-n)\binom{n+k-r}{k-r}x^n$$

$$= \frac{1}{(1-x)^{r-1}}\sum_{i=0}^{k-r}\frac{\sum_{n=0}^{s-2}(s-1-n)\binom{n+k-r-i}{k-r-i}x^n}{(1-x)^i} + (k-r+1)\frac{x(k-r+2)-2(s-1)(1-x)}{2(1-x)^{k+1}}$$

from which the assertion follows. □

Example 7. For $r = k = 2$ and arbitrary s we have

$$g_2(n) = \binom{n-s+2}{2}.$$

For $r = s = 2$ and arbitrary k we have

$$g_k(n) = \frac{n-1}{n+1}\binom{k}{2}\binom{n-1+k}{k},$$

which, for $k = 3, 4$ corresponds to A077414 and A105938 in the OEIS, respectively. For $s = 1$ and arbitrary k and r we have

$$g_k(n) = \binom{k-r+2}{2}\binom{n-1+k}{k}$$

and Table 2 lists the corresponding sequences that we found in OEIS.

2.2 Smirnov words

We denote by $\mathcal{S}_{n,k}$ the set of Smirnov words (e.g., [5, Example 23 on p. 193]), i.e.,

$$\mathcal{S}_{n,k} = \{w_1 \cdots w_n \in [k]^n \ :\ w_i \neq w_{i+1} \text{ for every } i \in [n-1]\}.$$

Since there are no Smirnov words for $k = 1$, we assume that $k \geq 2$. We distinguish between two cases, namely $r = 1$ and $r \geq 2$.

k	r	OEIS
3	2	A027480
4	2	A033487
5	2	A266732
6	2	A240440
7	2	A266733
4	3	A050534
5	3	A253945
6	3	A271040

Table 2: The total number $g_k(n)$ of $r \times 1$ rectangles over all words of length n, for several values of k and r.

2.2.1 Rectangle capacity of size $1 \times s$

Denote by $c_{n,k} = c_{n,k}(t)$ the distribution on $\mathcal{S}_{n,k}$ of the number of $1 \times s$ rectangles and by $C_k(x,t)$ the generating function of the numbers $c_{n,k}$. We shall need the following restrictions of $\mathcal{S}_{n,k}$: Let

$$\mathcal{S}_{n,k}^{(0)} = \{w_1 \cdots w_n \in \mathcal{S}_{n,k} \ : \ w_j > 1 \text{ for every } j \in [n]\}$$

and, for $i \in [n]$, we define

$$\mathcal{S}_{n,k}^{(i)} = \{w_1 \cdots w_n \in \mathcal{S}_{n,k} \ : \ w_i = 1 \text{ and } w_j > 1 \text{ for every } j \in [i-1]\}\,.$$

For $0 \leq i \leq n$, let $c_{n,k}^{(i)}$ be the restriction of $c_{n,k}$ to $\mathcal{S}_{n,k}^{(i)}$. Clearly,

$$|\mathcal{S}_{n,k}| = \begin{cases} 1, & \text{if } n = 0; \\ k(k-1)^{n-1}, & \text{otherwise.} \end{cases}$$

$$\left|\mathcal{S}_{n,k}^{(1)}\right| = \begin{cases} 1, & \text{if } n = 0; \\ (k-1)^{n-1}, & \text{otherwise.} \end{cases}$$

Theorem 8. *We have*

$$C_2(x,t) = \frac{2(t-1)x^s}{(1-tx)(1-x)} + \frac{1+x}{1-x} \tag{3}$$

and, for $k \geq 3$,

$$C_k(x,t) = \frac{(1+tx)\left(\gamma_k(x,t) + \delta_k(x,t)C_{k-1}(tx,t)\right)}{1 + tx - txC_{k-1}(tx,t)}, \tag{4}$$

where

$$\gamma_k(x,t) = \frac{1+x-k(k-1)^{s-1}x^s}{1-(k-1)x} - \frac{1+tx-k(k-1)^{s-1}(tx)^s}{t^{s-1}(1-(k-1)tx)},$$
$$\delta_k(x,t) = \frac{1-(k-2)tx-(k-1)^{s-1}(tx)^s}{t^{s-1}(1-(k-1)tx)} - \frac{tx}{1+tx}\frac{(1+x)(1-((k-1)x)^{s-1})}{1-(k-1)x}.$$

Proof. For $n \geq 1$, the set $\mathcal{S}_{n,2}$ consists of two words, namely $1212\cdots$ and $2121\cdots$, containing exactly $n-s+1$ $1\times s$ rectangles, each. It is easily verified that the corresponding generating function is given by (3). Assume now that $k \geq 3$. For $n \geq s$ we have

$$\begin{aligned} c_{n,k} &= c_{n,k}^{(0)} + \sum_{i=1}^{n} c_{n,k}^{(i)} \\ &= c_{n,k-1} + \sum_{i=1}^{n-s+1} t^{i-1} c_{i-1,k-1} c_{n-i+1,k}^{(1)} + \sum_{i=n-s+2}^{n} t^{n-s+1} c_{i-1,k-1} c_{n-i+1,k}^{(1)}, \quad (5) \\ c_{n,k}^{(1)} &= t\left(c_{n-1,k} - c_{n-1,k}^{(1)}\right). \quad (6) \end{aligned}$$

Multiplying both sides of (6) by x^n, summing over $n \geq s$ and adding $\sum_{n=0}^{s-1} c_{n,k}^{(1)} x^n$ to both sides, we obtain

$$C_k^{(1)}(x,t) = \frac{1}{1+tx}\left(\sum_{n=0}^{s-1} c_{n,k}^{(1)} x^n + tx\left(C_k(x,t) - \sum_{n=0}^{s-2} c_{n,k} x^n\right) + tx \sum_{n=0}^{s-2} c_{n,k}^{(1)} x^n\right). \quad (7)$$

Multiplying both sides of (5) by x^n, summing over $n \geq s$ and adding $\sum_{n=0}^{s-1} c_{n,k} x^n$ to both sides, we obtain

$$\begin{aligned} C_k(x,t) = \sum_{n=0}^{s-1} c_{n,k} x^n &+ \frac{1}{t^{s-1}}\left(C_{k-1}(tx,t) - \sum_{n=0}^{s-1} c_{n,k-1}(tx)^n\right) \\ &+ C_{k-1}(tx,t)\left(C_k^{(1)}(x,t) - \sum_{n=0}^{s-1} c_{n,k}^{(1)} x^n\right) \\ &+ \frac{1}{t^{s-1}} \sum_{i=2}^{s} c_{s-i+1,k}^{(1)} (tx)^{s-i+1} \left(C_{k-1}(tx,t) - \sum_{n=0}^{i-2} c_{n,k-1}(tx)^n\right). \quad (8) \end{aligned}$$

Substituting (7) into (8), we obtain (4). □

Corollary 9. *The generating function for the total number $h_k(n)$ of $1\times s$ rectangles over all words belonging to $\mathcal{S}_{n,k}$ is given by*

$$\frac{x^s \sum_{i=1}^{k-1} i^{s-1}(i+1)}{(1-(k-1)x)^2}. \quad (9)$$

Thus, for $n \geq s$,

$$h_k(n) = (k-1)^{n-s}(n-s+1)\sum_{i=1}^{k-1} i^{s-1}(i+1).$$

Proof. We have

$$\left[\frac{\partial}{\partial t}C_2(x,t)\right]_{|t=1} = \frac{2x^s}{(1-tx)^2}.$$

Thus, (9) holds for $k=2$. Assume now that $k \geq 3$ and that (9) holds for $k-1$. It is not hard to see that

$$\begin{aligned}
[\gamma_k(x,t)]_{|t=1} &= 0,\\
\left[\frac{\partial}{\partial t}\gamma_k(x,t)\right]_{|t=1} &= \frac{x(s-1)+s}{1-(k-1)x} - \frac{1+x-k(k-1)^{s-1}x^s}{(1-(k-1)x)^2},\\
[\delta_k(x,t)]_{|t=1} &= 1,\\
\left[\frac{\partial}{\partial t}\delta_k(x,t)\right]_{|t=1} &= 1-s-\frac{x}{1+x}\left[\frac{\partial}{\partial t}\gamma_k(x,t)\right]_{|t=1},\\
[C_k(x,t)]_{|t=1} &= \frac{1+x}{1-(k-1)x},\\
\left[\frac{\partial}{\partial x}C_k(x,t)\right]_{|t=1} &= \frac{k}{(1-(k-1)x)^2},\\
\left[\frac{\partial}{\partial t}C_k(tx,t)\right]_{|t=1} &= x\left[\frac{\partial}{\partial x}C_k(x,t)\right]_{|t=1} + \left[\frac{\partial}{\partial t}C_k(x,t)\right]_{|t=1}.
\end{aligned}$$

Differentiating (4) with respect to t and substituting $t=1$, we obtain

$$\begin{aligned}
\left[\frac{\partial}{\partial t}C_k(x,t)\right]_{|t=1} &= \left[\frac{(1+tx)\left(\gamma_k(x,t)+\delta_k(x,t)C_{k-1}(tx,t)\right)}{1+tx-txC_{k-1}(tx,t)}\right]_{|t=1}\\
&= \frac{(s-1)(k-1)x^2+(2+(k-2)s)x-s+1}{(1-(k-1)x)^2}\\
&\quad + \frac{(1-(k-2)x)^2}{(1-(k-1)x)^2}\left[\frac{\partial}{\partial t}C_{k-1}(x,t)\right]_{|t=1} + \left[\frac{\partial}{\partial t}\gamma_k(x,t)\right]_{|t=1}\\
&= \frac{k(k-1)^{s-1}x^s}{(1-(k-1)x)^2} + \frac{x^s\sum_{i=1}^{k-2} i^{s-1}(i+1)}{(1-(k-1)x)^2}\\
&= \frac{x^s\sum_{i=1}^{k-1} i^{s-1}(i+1)}{(1-(k-1)x)^2}.
\end{aligned}$$

□

Example 10. For $k=2$ and arbitrary s we have

$$h_2(n) = 2(n-s+1).$$

For $k=3$ and arbitrary s we have

$$h_3(n) = 2^{n-s}(n-s+1)(2+3\cdot 2^{s-1}),$$

which, for $s=2,3$ corresponds to A241204 and A281200 in the OEIS, respectively.

2.2.2 Rectangle capacity of size $r\times s$, where $r\geq 2$

Denote by $d_{n,k}=d_{n,k}(t)$ the distribution on $\mathcal{S}_{n,k}$ of the number of $r\times s$ rectangles and by $D_k(x,t)$ the generating function of the numbers $d_{n,k}$. We shall need the following restrictions of $\mathcal{S}_{n,k}$: for $m\in[k]$ let

$$\begin{aligned}
\mathcal{S}_{n,k}^{(\geq m)} &= \{w_1\cdots w_n \in \mathcal{S}_{n,k} \ : \ w_i \geq m \text{ for every } i\in[n]\},\\
\bar{\mathcal{S}}_{n,k}^{(\geq m)} &= \left\{w_1\cdots w_n \in \mathcal{S}_{n,k}^{(\geq m)} \ : \ w_1\neq m\right\}.\\
\bar{\mathcal{S}}_{n,k} &= \{w_1\cdots w_n \in \mathcal{S}_{n,k} \ : \ w_1 \neq r-1\}.
\end{aligned}$$

Denote by $d_{n,k}^{(\geq m)}$ (resp., $\bar{d}_{n,k}^{(\geq m)}$, $\bar{d}_{n,k}$) the restriction of $d_{n,k}$ to $\mathcal{S}_{n,k}^{(\geq m)}$ (resp., to $\bar{\mathcal{S}}_{n,k}^{(\geq m)}$, $\bar{\mathcal{S}}_{n,k}$). Let $D_k^{(\geq m)}(x,t)$ (resp., $\bar{D}_k^{(\geq m)}(x,t)$, $\bar{D}_k(x,t)$) denote the generating function of the numbers $d_{n,k}^{(\geq m)}$ (resp., $\bar{d}_{n,k}^{(\geq m)}$, $\bar{d}_{n,k}$). Clearly,

$$\begin{aligned}
\left|\mathcal{S}_{n,k}^{(\geq m)}\right| &= \begin{cases} 1, & \text{if } n=0;\\ (k-m+1)(k-m)^{n-1}, & \text{otherwise.}\end{cases}\\
\left|\bar{\mathcal{S}}_{n,k}^{(\geq m)}\right| &= (k-m)^n.
\end{aligned}$$

Lemma 11. *We have*

$$D_r^{(\geq r-1)}(x,t) = \frac{1+x}{1-x} \tag{10}$$

and, for $k\geq r+1$,

$$D_k^{(\geq r-1)}(x,t) = (1+x)\frac{\frac{1}{t^{s-1}}D_{k-1}^{(\geq r-1)}(tx,t)+\sigma_k(x,t)}{-\frac{x}{t^{s-1}}D_{k-1}^{(\geq r-1)}(tx,t)+\rho_k(x,t)},$$

where

$$\begin{aligned}
\sigma_k(x,t) &= \frac{1+x-(k-r+1)(k-r)^{s-1}x^s}{1-(k-r)x} - \frac{1+tx-(k-r+1)(k-r)^{s-1}(tx)^s}{t^{s-1}(1-(k-r)tx)},\\
\rho_k(x,t) &= 1-(k-r+1)x^2\frac{1-((k-r)x)^{s-1}}{1-(k-r)x} + x\frac{1+tx-(k-r+1)(k-r)^{s-1}(tx)^s}{t^{s-1}(1-(k-r)tx)}.
\end{aligned}$$

Proof. For $n \geq 1$, the set $\mathcal{S}_{n,r}^{(\geq r-1)}$ consists of two words, namely $r(r-1)r(r-1)\cdots$ and $(r-1)r(r-1)r(r-1)\cdots$, containing no $r \times s$ rectangles. Thus, (10) holds true in this case. Assume now that $k \geq r+1$. For $n \geq s$ we have

$$\begin{aligned} d_{n,k}^{(\geq r-1)} &= d_{n,k}^{(\geq r)} + \sum_{i=1}^{n} d_{i-1,k}^{(\geq r)} \bar{d}_{n-i,k}^{(\geq r-1)} \\ &= t^{n-s+1} d_{n,k-1}^{(\geq r-1)} + \sum_{i=1}^{s} d_{i-1,k-1}^{(\geq r-1)} \bar{d}_{n-i,k}^{(\geq r-1)} + \sum_{i=s+1}^{n} t^{i-s} d_{i-1,k-1}^{(\geq r-1)} \bar{d}_{n-i,k}^{(\geq r-1)}, \qquad (11) \\ \bar{d}_{n,k}^{(\geq r-1)} &= d_{n,k}^{(\geq r)} + \sum_{i=2}^{n} d_{i-1,k}^{(\geq r)} \bar{d}_{n-i,k}^{(\geq r-1)} \\ &= t^{n-s+1} d_{n,k-1}^{(\geq r-1)} + \sum_{i=2}^{s} d_{i-1,k-1}^{(\geq r-1)} \bar{d}_{n-i,k}^{(\geq r-1)} + \sum_{i=s+1}^{n} t^{i-s} d_{i-1,k-1}^{(\geq r-1)} \bar{d}_{n-i,k}^{(\geq r-1)}. \qquad (12) \end{aligned}$$

Multiplying both sides of (11) by x^n, summing over $n \geq s$ and adding $\sum_{n=0}^{s-1} d_{n,k}^{(\geq r-1)} x^n$ to both sides, we obtain the equation

$$\begin{aligned} & D_k^{(\geq r-1)}(x,t) \\ &= \sum_{n=0}^{s-1} d_{n,k}^{(\geq r-1)} x^n + \frac{1}{t^{s-1}} \left(D_{k-1}^{(\geq r-1)}(tx,t) - \sum_{n=0}^{s-1} d_{n,k-1}^{(\geq r-1)} (tx)^n \right) - \sum_{i=1}^{s} d_{i-1,k-1}^{(\geq r-1)} x^i \sum_{n=0}^{s-i-1} \bar{d}_{n,k}^{(\geq r-1)} x^n \\ &+ \bar{D}_k^{(\geq r-1)}(x,t) \left(\sum_{i=1}^{s} d_{i-1,k-1}^{(\geq r-1)} x^i + \frac{x}{t^{s-1}} \left(D_{k-1}^{(\geq r-1)}(tx,t) - \sum_{n=0}^{s-1} d_{n,k-1}^{(\geq r-1)} (tx)^n \right) \right). \qquad (13) \end{aligned}$$

Multiplying both sides of (12) by x^n, summing over $n \geq s$ and adding $\sum_{n=0}^{s-1} \bar{d}_{n,k}^{(\geq r-1)} x^n$ to both sides, we obtain the equation

$$\begin{aligned} & \bar{D}_k^{(\geq r-1)}(x,t) \left(1 - \frac{x}{t^{s-1}} D_{k-1}^{(\geq r-1)}(tx,t) - \sum_{i=2}^{s} d_{i-1,k-1}^{(\geq r-1)} x^i + \frac{x}{t^{s-1}} \sum_{n=0}^{s-1} d_{n,k-1}^{(\geq r-1)} (tx)^n \right) \\ &= \frac{1}{t^{s-1}} D_{k-1}^{(\geq r-1)}(tx,t) + \sum_{n=0}^{s-1} \bar{d}_{n,k}^{(\geq r-1)} x^n - \frac{1}{t^{s-1}} \sum_{n=0}^{s-1} d_{n,k-1}^{(\geq r-1)} (tx)^n \\ &- \sum_{i=2}^{s} d_{i-1,k-1}^{(\geq r-1)} x^i \sum_{n=0}^{s-i-1} \bar{d}_{n,k}^{(\geq r-1)} x^n. \end{aligned}$$

Substituting this into (13), the assertion follows. □

Theorem 12. *We have*

$$D_r(x,t) = \frac{1+x}{1-x}$$

and, for $k \geq r+1$,

$$D_k(x,t) = (1+x)\frac{\frac{1}{t^{s-1}}D_{k-1}^{(\geq r-1)}(tx,t) + \sigma_k(x,t)}{-\frac{(r-1)x}{t^{s-1}}D_{k-1}^{(\geq r-1)}(tx,t) - (r-2)(1+x) + (r-1)\rho_k(x,t)}. \qquad (14)$$

Proof. We have

$$\begin{aligned} d_{n,k} &= d_{n,k}^{(\geq r)} + (r-1)\sum_{i=1}^{n} d_{i-1,k}^{(\geq r)}\bar{d}_{n-i,k} \\ &= t^{n-s+1}d_{n,k-1}^{(\geq r-1)} + (r-1)\sum_{i=1}^{s} d_{i-1,k-1}^{(\geq r-1)}\bar{d}_{n-i,k} + (r-1)\sum_{i=s+1}^{n} t^{i-s}d_{i-1,k-1}^{(\geq r-1)}\bar{d}_{n-i,k}, \qquad (15) \\ \bar{d}_{n,k} &= d_{n,k}^{(\geq r)} + (r-1)\sum_{i=2}^{n} d_{i-1,k}^{(\geq r)}\bar{d}_{n-i,k} + (r-2)\bar{d}_{n-1,k} \\ &= t^{n-s+1}d_{n,k-1}^{(\geq r-1)} + (r-1)\sum_{i=2}^{s} d_{i-1,k-1}^{(\geq r-1)}\bar{d}_{n-i,k} \\ &+ (r-1)\sum_{i=s+1}^{n} t^{i-s}d_{i-1,k-1}^{(\geq r-1)}\bar{d}_{n-i,k} + (r-2)\bar{d}_{n-1,k}. \qquad (16) \end{aligned}$$

Multiplying both sides of (15) by x^n, summing over $n \geq s$ and adding $\sum_{n=0}^{s-1} d_{n,k}x^n$ to both sides, we obtain

$$\begin{aligned} &D_k(x,t) \\ &= \sum_{n=0}^{s-1} d_{n,k}x^n + \frac{1}{t^{s-1}}\left(D_{k-1}^{(\geq r-1)}(tx,t) - \sum_{n=0}^{s-1} d_{n,k-1}^{(\geq r-1)}(tx)^n\right) - (r-1)\sum_{i=1}^{s} d_{i-1,k-1}^{(\geq r-1)}x^i \sum_{n=0}^{s-i-1}\bar{d}_{n,k}x^n \\ &+ \bar{D}_k(x,t)\left((r-1)\sum_{i=1}^{s} d_{i-1,k-1}^{(\geq r-1)}x^i + \frac{(r-1)x}{t^{s-1}}\left(D_{k-1}^{(\geq r-1)}(tx,t) - \sum_{n=0}^{s-1} d_{n,k-1}^{(\geq r-1)}(tx)^n\right)\right). \qquad (17) \end{aligned}$$

Multiplying both sides of (16) by x^n, summing over $n \geq s$ and adding $\sum_{n=0}^{s-1} \bar{d}_{n,k}x^n$ to both sides, we obtain

$$\begin{aligned} &\bar{D}_k(x,t) \\ &= \sum_{n=0}^{s-1} \bar{d}_{n,k}x^n + \frac{1}{t^{s-1}}\left(D_{k-1}^{(\geq r-1)}(tx,t) - \sum_{n=0}^{s-1} d_{n,k-1}^{(\geq r-1)}(tx)^n\right) \\ &+ (r-1)\sum_{i=2}^{s} d_{i-1,k-1}^{(\geq r-1)}x^i\left(\bar{D}_k(x,t) - \sum_{n=0}^{s-i-1}\bar{d}_{n,k}x^n\right) \\ &+ \frac{(r-1)x}{t^{s-1}}\bar{D}_k(x,t)\left(D_{k-1}^{(\geq r-1)}(tx,t) - \sum_{n=0}^{s-1} d_{n,k-1}^{(\geq r-1)}(tx)^n\right) + (r-2)x\left(\bar{D}_k(x,t) - \sum_{n=0}^{s-2}\bar{d}_{n,k}x^n\right). \end{aligned}$$

Solving for $\bar{D}_k(x,t)$ and substituting it into (17), the assertion follows. □

Corollary 13. *The generating function for the total number $i_k(n)$ of $r \times s$ rectangles over all words belonging to $\mathcal{S}_{n,k}$ is given by*

$$\frac{x^s \sum_{i=1}^{k-r} i^{s-1}(i+1)}{(1-(k-1)x)^2}.$$

Thus, for $n \geq s$,

$$i_k(n) = (k-1)^{n-s}(n-s+1)\sum_{i=1}^{k-r} i^{s-1}(i+1).$$

Proof. It is not hard to see that

$$\begin{aligned}
\left[D_k^{(\geq r-1)}(x,t)\right]_{|t=1} &= \frac{1+x}{1-(k-r+1)x},\\
\left[\frac{\partial}{\partial x}D_k^{(\geq r-1)}(x,t)\right]_{|t=1} &= \frac{k-r+2}{\left(1-(k-r+1)\,x\right)^2},\\
\left[\frac{\partial}{\partial t}D_k^{(\geq r-1)}(x,t)\right]_{|t=1} &= \frac{x^s\sum_{n=0}^{k-r} n^{s-1}(n+1)}{(1-(k-r+1)x)^2}.
\end{aligned}$$

We omit the rest of the details, which are very similar to those in the proof of Corollary 9. □

3 Discussion

We mentioned in the introduction the works of Cakić et al. [3] and Archibald et al [1], who studied rectangle capacity of size 1×2 and 2×2 in set partitions, respectively. Another family of restricted words, which draws much attention in recent years, is inversion sequences (e.g., [2] and [13]). To the best of our knowledge, this family was not studied at all in terms of rectangle capacity. Thus, a possible direction for future work would be generalizing previous results on rectangle capacity in set partitions and initiating the study thereof in inversion sequences.

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