Some Applications of the Lagrange Inversion Formula for the k-Fibonacci Numbers

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Abstract

We provide summation formulas for the k-Fibonacci numbers ($k \in \mathbb{Z}$, $k \geq 2$) and their asymptotic equivalents in terms of generalized binomial coefficients. Our main tools are the Lagrange inversion formula and one of its consequences due to Hermite.

1 Introduction and notation

Throughout this paper, we let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ the set of nonnegative integers. Next, the (compositional) inverse function of a function f is denoted by $f^{\langle -1 \rangle}$ and the coefficient of z^n ($n \in \mathbb{N}_0$) in a power series $\varphi(z)$ is denoted by $[z^n]\varphi(z)$. Further, we let $\binom{a}{b}$ ($a, b \in \mathbb{Z}$) denote the generalized binomial coefficient defined by

$$\binom{a}{b} := \begin{cases} \frac{a(a-1)\cdots(a-b+1)}{b!}, & \text{if } b \geq 0; \\ \binom{a}{a-b}, & \text{if } a \geq b; \\ 0, & \text{if } a < b < 0. \end{cases} \tag{1}$$

Using this definition, we easily check the following formulas:

$$\binom{a}{b} = \binom{a-1}{b-1} + \binom{a-1}{b} \qquad (\forall (a,b) \in \mathbb{Z}^2 \setminus \{(0,0)\}), \tag{2}$$

$$\binom{a}{b} = \begin{cases} (-1)^{b-1}\binom{b-a-1}{b}, & \text{if } b \le a < 0; \\ (-1)^b \binom{b-a-1}{b}, & \text{else.} \end{cases} \qquad (\forall (a,b) \in \mathbb{Z}^2). \tag{3}$$

For $x \in \mathbb{R}$ and $n \in \mathbb{Z}$, we let $x^{\underline{n}}$ denote the nth *falling factorial power* of x, defined by

$$x^{\underline{n}} := \begin{cases} x(x-1)\cdots(x-n+1), & \text{if } n > 0; \\ 1, & \text{if } n = 0; \\ ((x+1)(x+2)\cdots(x-n))^{-1}, & \text{if } n < 0. \end{cases}$$

(See [4, §2.6].) For a given integer $k \ge 2$, the k-Fibonacci sequence is the integer sequence $(F_n^{(k)})_{n \in \mathbb{N}_0}$ given by the recurrence relation

$$F_{n+k}^{(k)} = F_n^{(k)} + F_{n+1}^{(k)} + \cdots + F_{n+k-1}^{(k)} \qquad (\forall n \in \mathbb{N}_0), \tag{4}$$

with the initial values $F_0^{(k)} = F_1^{(k)} = \cdots = F_{k-2}^{(k)} = 0$ and $F_{k-1}^{(k)} = 1$. The special case $k = 2$ corresponds to the usual Fibonacci sequence, simply denoted by $(F_n)_{n \in \mathbb{N}_0}$ and given by

$$\begin{cases} F_0 = 0, F_1 = 1, \\ F_{n+2} = F_n + F_{n+1} \quad (\forall n \in \mathbb{N}_0) \end{cases}$$

(see the sequence [A000045](#) in the OEIS [10]). The above linear recurrence relation (4) of order k for the k-Fibonacci sequence easily implies the following simpler linear recurrence relation of order $(k+1)$:

$$F_{n+k+1}^{(k)} = 2F_{n+k}^{(k)} - F_n^{(k)} \qquad (\forall n \in \mathbb{N}_0) \tag{5}$$

(see [6] for the details). By relying on (5), we easily check that we have

$$\begin{aligned} F_n^{(k)} &= 2^{n-k} && \text{for } k \le n < 2k, \\ F_{2k}^{(k)} &= 2^k - 1. \end{aligned} \tag{6}$$

The k-Fibonacci sequence plays an important role in many areas of mathematics, including discrete mathematics and combinatorics. Here is a simple combinatorial interpretation: given an integer $k \ge 2$, the number of possible ways of writing a positive integer n as a sum of numbers, all belonging to the set $\{1, 2, \ldots, k\}$ (taking into account the permutations), is exactly $F_{n+k-1}^{(k)}$. To find a closed form of $F_n^{(k)}$, the theory of linear recurrence sequences associates the linear recurrence (4) with the polynomial equation (known as its *characteristic equation*)

$$x^k = x^{k-1} + x^{k-2} + \cdots + x + 1. \tag{7}$$

Multiplying the both sides of (7) by $(x-1)$ and rearranging, we obtain the simpler polynomial equation

$$x^{k+1} - 2x^k + 1 = 0, \tag{8}$$

which is nothing else than the characteristic equation associated to the linear recurrence (5). The closed form of $F_n^{(k)}$ then depends on the complex roots of Equation (7) and their multiplicities. Furthermore, in order to establish a simple asymptotic equivalent for $F_n^{(k)}$ (as $n \to +\infty$), it is necessary to locate the complex roots of Equation (7) in the complex plane, a topic that has been investigated by several authors. Using the famous Rouché theorem of complex analysis, Miles [8] proved that Equation (7) has exactly $k-1$ pairwise distinct complex roots in the open unit disk $D(0,1)$ of the complex plane and a unique complex root ρ_k of modulus greater than 1; more precisely, ρ_k is real and belongs to the interval $(1,2)$. Wolfram [14] showed that we have more precisely

$$2 - \frac{2}{2^k} < \rho_k < 2. \tag{9}$$

For the special case $k = 2$, ρ_k is nothing else than the famous *golden ratio*, commonly denoted by ϕ ($\phi = \frac{\sqrt{5}+1}{2} = 1.618\ldots$). Miller [9] found an elementary way to prove the Miles result, which relies on Descartes's rules of signs. Letting $\alpha_{k,1}, \alpha_{k,2}, \ldots, \alpha_{k,k} = \rho_k$ denote the complex roots of Equation (7), the theory of linear recurrence sequences shows that $F_n^{(k)}$ has the closed form $F_n^{(k)} = \sum_{\ell=1}^{k} \lambda_{k,\ell} \alpha_{k,\ell}^n$ ($\forall n \in \mathbb{N}_0$), where the $\lambda_{k,\ell}$'s are complex numbers (not depending on n) which can be calculated by relying on the initial values of the sequence $(F_n^{(k)})_n$. Spickerman and Joyner [11] showed that

$$\lambda_{k,\ell} = \frac{\alpha_{k,\ell}^{k+1} - \alpha_{k,\ell}^k}{\alpha_{k,\ell}\left(2\alpha_{k,\ell}^k - (k+1)\right)} \qquad (\forall \ell \in \{1, 2, \ldots, k\}),$$

and Dresden [3] simplified this to

$$\lambda_{k,\ell} = \frac{\alpha_{k,\ell} - 1}{\alpha_{k,\ell}\left(2 + (k+1)(\alpha_{k,\ell} - 2)\right)} = \frac{\alpha_{k,\ell} - 1}{\alpha_{k,\ell}\left((k+1)\alpha_{k,\ell} - 2k\right)} \qquad (\forall \ell \in \{1, 2, \ldots, k\}).$$

So we have

$$F_n^{(k)} = \sum_{\ell=1}^{k} \frac{\alpha_{k,\ell} - 1}{(k+1)\alpha_{k,\ell} - 2k} \alpha_{k,\ell}^{n-1} \qquad (\forall n \in \mathbb{N}_0).$$

Then, since ρ_k is the largest root of (7) in modulus, we derive the following asymptotic equivalence (as $n \to +\infty$):

$$F_n^{(k)} \sim_{+\infty} \frac{\rho_k - 1}{(k+1)\rho_k - 2k} \rho_k^{n-1}. \tag{10}$$

Let us denote the expression on the right-hand side of (10) by $\mathrm{As}_n^{(k)}$; that is

$$\mathrm{As}_n^{(k)} := \frac{\rho_k - 1}{(k+1)\rho_k - 2k} \rho_k^{n-1} \tag{11}$$

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(for all integer $k \geq 2$ and all $n \in \mathbb{N}_0$). Next, for all integer $k \geq 2$, define

$$\varepsilon_k := 2 - \rho_k.$$

From (9), we have that $\varepsilon_k \in (0, 2^{1-k}) \subset (0, 1/2)$. Furthermore, from the fact that ρ_k is a root of Equation (8), we derive that

$$\varepsilon_k = \frac{1}{(2 - \varepsilon_k)^k}, \tag{12}$$

which is a crucial equality for our purpose.

The goal of this paper is first to represent ρ_k^n and $\mathrm{As}_n^{(k)}$ as series involving binomial coefficients. Then, by judiciously truncating the series related to $\mathrm{As}_n^{(k)}$, we derive a representation of $F_n^{(k)}$ as a finite sum involving binomial coefficients. Our results are essentially obtained by employing the Lagrange inversion formula and one of its consequences due to Hermite (see §2). While some of these results have been previously pointed out by other authors, they were often accompanied by *ad hoc* proofs, and in some instances, contained mistakes (see §3 for detailed discussion).

2   The Lagrange inversion formula

The Lagrange inversion formula is one of the most important formulas of combinatorics. In its general form, it can be stated as follows

Theorem 1 (The Lagrange inversion formula). *Let $\alpha, z_0 \in \mathbb{C}$ and f be a complex analytic function of z at a neighborhood of z_0. Let also $w = w(z)$ be a complex analytic function of z at a neighborhood of z_0, which is implicitly defined by*

$$w = z_0 + \alpha f(w). \tag{13}$$

Then for every complex analytic function g of z at a neighborhood of z_0, we have (for $|\alpha|$ sufficiently small)

$$g(w) = g(z_0) + \sum_{n=1}^{+\infty} \frac{\alpha^n}{n!} \left(\left(\frac{d}{dz}\right)^{n-1} (f^n g') \right) (z_0).$$

Taking in particular $g(z) = z$, we get (for $|\alpha|$ sufficiently small)

$$w = z_0 + \sum_{n=1}^{+\infty} \frac{\alpha^n}{n!} \left(\left(\frac{d}{dz}\right)^{n-1} (f^n) \right) (z_0).$$

Note that the last part of this important theorem was proved in 1770 by Lagrange [7], who applied it to solve the Kepler equation $y = x + \alpha \sin y$, while its generalization (i.e., the first part of Theorem 1) was obtained by Bürmann [1] in 1798. One simple way to prove Theorem

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[1] uses contour integration in the complex plane, while other more elementary approaches employ algebraic techniques on formal series (see, e.g., [2, 12]) or utilize induction (see [13]). From Theorem 1, we derive the following interesting corollary, which is due to Hermite (see Comtet's book [2, §3.8]):

Corollary 2 (Hermite). *Let φ be a complex analytic function at a neighborhood of 0 such that $\varphi(0) \neq 0$ and let ψ be the compositional inverse function of the function $z \to \frac{z}{\varphi(z)}$ at a neighborhood of 0 (ψ exists because $\left(\frac{z}{\varphi(z)}\right)'(0) = \frac{1}{\varphi(0)} \neq 0$). Then for every complex function g, which is analytic at a neighborhood of 0, we have*

$$\frac{z \cdot (g \circ \psi)(z)}{\psi(z) \cdot \left(\left(\frac{z}{\varphi(z)}\right)' \circ \psi(z)\right)} = \sum_{n=0}^{+\infty} z^n [z^n] \left(g(z)\varphi(z)^n\right).$$

Since this last result of Hermite is unfamiliar, we have preferred to include its proof in this paper.

Proof of Corollary 2. Let f be a primitive of the function $\frac{g}{\varphi}$ at a neighborhood of 0. By definition of ψ, we have $\left(\frac{z}{\varphi(z)} \circ \psi\right)(z) = z$; that is ψ is implicitly given by

$$\psi(z) = 0 + z \cdot (\varphi \circ \psi)(z). \tag{14}$$

So by applying the first part of Theorem 1 with (14) filling the role of (13); that is with w replaced by ψ, f by φ, α by z, z_0 by 0, and g by f, we get

$$(f \circ \psi)(z) = f(0) + \sum_{n=1}^{+\infty} \frac{z^n}{n!} \left(\left(\frac{d}{dz}\right)^{n-1} (\varphi(z)^n f'(z))\right)(0)$$

$$= f(0) + \sum_{n=1}^{+\infty} \frac{z^n}{n} \left[z^{n-1}\right] \left(g(z)\varphi(z)^{n-1}\right) \qquad \left(\text{since } f' = \frac{g}{\varphi}\right)$$

$$= f(0) + \sum_{n=0}^{+\infty} \frac{z^{n+1}}{n+1} \left[z^n\right] \left(g(z)\varphi(z)^n\right).$$

By differentiating with respect to z, we get

$$\psi'(z) \cdot (f' \circ \psi)(z) = \sum_{n=0}^{+\infty} z^n \left[z^n\right] \left(g(z)\varphi(z)^n\right).$$

But since

$$\psi'(z) = \left(\left(\frac{z}{\varphi(z)}\right)^{\langle -1 \rangle}\right)' = \frac{1}{\left(\frac{z}{\varphi(z)}\right)' \circ \psi(z)}$$

and

$$(f' \circ \psi)(z) = \left(\frac{g}{\varphi} \circ \psi\right)(z) = \frac{(g \circ \psi)(z)}{(\varphi \circ \psi)(z)} = \frac{1}{\psi(z)} \cdot \frac{\psi(z)}{(\varphi \circ \psi)(z)} \cdot (g \circ \psi)(z)$$

$$= \frac{1}{\psi(z)} \cdot \underbrace{\left(\frac{z}{\varphi(z)} \circ \psi(z)\right)}_{=z} \cdot (g \circ \psi)(z) = \frac{z}{\psi(z)} \cdot (g \circ \psi)(z),$$

it follows that

$$\frac{z \cdot (g \circ \psi)(z)}{\psi(z) \cdot \left(\left(\frac{z}{\varphi(z)}\right)' \circ \psi(z)\right)} = \sum_{n=0}^{+\infty} z^n [z^n] \left(g(z)\varphi(z)^n\right),$$

as required. $\qquad\qquad\square$

3   The results and the proofs

We begin with the following result, which expresses the powers of ρ_k $(k \geq 2)$ as real series involving binomial coefficients.

Theorem 3. *For all positive integers k and n, with $k \geq 2$, we have*

$$\rho_k^n = 2^n - n2^{n-k-1} \sum_{\ell=0}^{+\infty} \frac{1}{\ell+1} \binom{k(\ell+1)+\ell-n}{\ell} \frac{1}{2^{(k+1)\ell}}.$$

In particular, we have

$$\rho_k = 2 - \frac{1}{2^k} \sum_{\ell=0}^{+\infty} \frac{1}{\ell+1} \binom{k(\ell+1)+\ell-1}{\ell} \frac{1}{2^{(k+1)\ell}}.$$

Proof. Let k and n be two fixed positive integers with $k \geq 2$. By applying the first part of Theorem 1 with (12) filling the role of (13); that is with $w = \varepsilon_k$, $f(z) = (2-z)^{-k}$, $\alpha = 1$, $z_0 = 0$, and $g(z) = (2-z)^n$, we get

$$(2 - \varepsilon_k)^n = 2^n + \sum_{\ell=1}^{+\infty} \frac{1}{\ell!} \left(\left(\frac{d}{dz}\right)^{\ell-1} \left((2-z)^{-k\ell}(-n(2-z)^{n-1})\right)\right)(0)$$

$$= 2^n - n \sum_{\ell=1}^{+\infty} \frac{1}{\ell!} \left(\left(\frac{d}{dz}\right)^{\ell-1} (2-z)^{-k\ell+n-1}\right)(0)$$

$$= 2^n - n \sum_{\ell=1}^{+\infty} \frac{1}{\ell} \left[z^{\ell-1}\right] (2-z)^{-k\ell+n-1}.$$

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But according to the generalized binomial formula and (3), we have

$$\left[z^{\ell-1}\right](2-z)^{-k\ell+n-1} = \binom{-k\ell+n-1}{\ell-1}2^{-k\ell+n-\ell}(-1)^{\ell-1} = \binom{k\ell-n+\ell-1}{\ell-1}\frac{2^n}{2^{(k+1)\ell}}$$

for all $\ell \in \mathbb{N}$. Thus

$$(2-\varepsilon_k)^n = 2^n - n\sum_{\ell=1}^{+\infty}\frac{1}{\ell}\binom{k\ell-n+\ell-1}{\ell-1}\frac{2^n}{2^{(k+1)\ell}}$$

$$= 2^n - n\sum_{\ell=0}^{+\infty}\frac{1}{\ell+1}\binom{k(\ell+1)+\ell-n}{\ell}\frac{2^n}{2^{(k+1)(\ell+1)}};$$

that is

$$\rho_k^n = 2^n - n2^{n-k-1}\sum_{\ell=0}^{+\infty}\frac{1}{\ell+1}\binom{k(\ell+1)+\ell-n}{\ell}\frac{1}{2^{(k+1)\ell}},$$

as required. $\qquad\square$

Remark 4.

1. In the proof of Theorem 3, we have omitted the verification of the convergence of the series $\sum_{\ell=0}^{+\infty}\frac{1}{\ell+1}\binom{k(\ell+1)+\ell-n}{\ell}\frac{1}{2^{(k+1)\ell}}$. Note that this convergence can be easily confirmed by using, for example, the elementary upper bounds $\binom{m}{\ell} \leq \frac{2^m}{\sqrt{m+2}}$ (for $1 \leq \ell \leq m$).

2. Actually, the formula of ρ_k in the second part of Theorem 3 was already discovered by Wolfram [14], who obtained it in a more complicated way.

3. As its proof shows, Theorem 3 is actually valid even for $n \in \mathbb{Z}$. The particular cases $n = -1$ and $n = k$ provide the results of Hare et al. [5, Theorem 1.1] (notice that $\rho_k^k = \frac{1}{2-\rho_k}$).

4. Theorem 3 can be also obtained by using *the generalized binomial series* and *the generalized binomial formula*, investigated in [4, §5.4]. Indeed, according to [4], the generalized binomial series $\mathscr{B}_t$ $(t \in \mathbb{R})$ is defined by

$$\mathscr{B}_t(z) := \sum_{\ell=0}^{+\infty}(t\ell)^{\ell-1}\frac{z^\ell}{\ell!}$$

and satisfies the functional equation:

$$\mathscr{B}_t(z)^{1-t} - \mathscr{B}_t(z)^{-t} = z. \tag{15}$$

For all $r \in \mathbb{R}$, we have in addition the generalized binomial formula:

$$\mathscr{B}_t(z)^r = \sum_{\ell=0}^{+\infty}\binom{t\ell+r}{\ell}\frac{r}{t\ell+r}z^\ell \tag{16}$$

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(setting $t = 0$, we obtain the usual binomial formula). Now, giving an integer $k \geq 2$, by transforming the equality $\rho_k^{k+1} - 2\rho_k^k + 1 = 0$ into $\left(\frac{1}{2}\rho_k\right)^{1+k} - \left(\frac{1}{2}\rho_k\right)^k = -\frac{1}{2^{k+1}}$ and comparing this with (15), we derive that $\rho_k = 2\mathscr{B}_{-k}\left(-\frac{1}{2^{k+1}}\right)$. The formula of Theorem 3 then follows by combining this last formula with the generalized binomial formula (16).

By setting k to 2 in Theorem 3, we derive important formulas for the golden ratio ϕ and its powers. More precisely, we obtain the following corollary:

Corollary 5. *For every positive integer n, we have*

$$\phi^n = 2^n - n2^{n-3} \sum_{\ell=0}^{+\infty} \frac{1}{\ell+1} \binom{3\ell+2-n}{\ell} \frac{1}{2^{3\ell}}.$$

In particular, we have

$$\phi = 2 - \sum_{\ell=0}^{+\infty} \frac{1}{\ell+1} \binom{3\ell+1}{\ell} \frac{1}{2^{3\ell+2}}. \qquad \square$$

More interestingly, we are going to find representations as series (analogous to those of Theorem 3) for the asymptotic equivalent $\mathrm{As}_n^{(k)}$ of $F_n^{(k)}$. To achieve this, we rely on the following key theorem:

Theorem 6. *For all integers k and a, with $k \geq 2$, we have*

$$\sum_{\ell=0}^{+\infty} \binom{(k+1)\ell+a}{\ell} 2^{-(k+1)\ell} = 2^{a+1} \cdot \frac{\rho_k^{-a}}{(k+1)\rho_k - 2k}.$$

Proof. Let k and a be fixed integers, with $k \geq 2$. We apply Corollary 2 for the functions $\varphi(z) = (2-z)^{-k}$ and $g(z) = (2-z)^{-a-1}$, which are both analytic at a neighborhood of 0 and we have $\varphi(0) = 2^{-k} \neq 0$. In this situation, the compositional inverse function ψ of the function $z \mapsto \frac{z}{\varphi(z)} = z(2-z)^k$ (at a neighborhood of 0) satisfies (according to (12)): $\psi(1) = \varepsilon_k$. So by specializing in formula of Corollary 2 the variable z to 1, we get

$$\frac{1 \cdot (g \circ \psi)(1)}{\psi(1) \cdot \left((z(2-z)^k)' \circ \psi\right)(1)} = \sum_{\ell=0}^{+\infty} 1^\ell [z^\ell](2-z)^{-k\ell-a-1}.$$

But since $\psi(1) = \varepsilon_k$ (as clarified above), $\left(z(2-z)^k\right)' = (2-z)^{k-1}\left(2 - (k+1)z\right)$, and for all $\ell \in \mathbb{N}_0$: $[z^\ell](2-z)^{-k\ell-a-1} = \binom{-k\ell-a-1}{\ell}2^{-k\ell-a-1-\ell}(-1)^\ell = \binom{(k+1)\ell+a}{\ell}2^{-(k+1)\ell-a-1}$ (by using the generalized binomial formula and then (3)), it follows that

$$\frac{(2-\varepsilon_k)^{-a-1}}{\varepsilon_k(2-\varepsilon_k)^{k-1}\left(2 - (k+1)\varepsilon_k\right)} = \sum_{\ell=0}^{+\infty} \binom{(k+1)\ell+a}{\ell} 2^{-(k+1)\ell-a-1}.$$

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Finally, since $\varepsilon_k(2 - \varepsilon_k)^k = 1$ (according to (12)), $2 - \varepsilon_k = \rho_k$, and $2 - (k+1)\varepsilon_k = 2 - (k+1)(2 - \rho_k) = (k+1)\rho_k - 2k$, we conclude to

$$\frac{\rho_k^{-a}}{(k+1)\rho_k - 2k} = \sum_{\ell=0}^{+\infty} \binom{(k+1)\ell + a}{\ell} 2^{-(k+1)\ell - a - 1},$$

which immediately gives the required formula of the theorem. $\qquad\square$

Remark 7. The convergence of the series in the formula of Theorem 6 can be verified by using, for example, the Stirling formula $n! \sim_{+\infty} n^n e^{-n}\sqrt{2\pi n}$.

Relying on Theorem 6, we are now able to provide a series representation (involving binomial coefficients) for the asymptotic equivalent $\mathrm{As}_n^{(k)}$ of $F_n^{(k)}$. This is the subject of the following theorem:

Theorem 8. *For all nonnegative integers k and n, with $k \geq 2$ and $n \neq 1$, we have*

$$\mathrm{As}_n^{(k)} = 2^{n-2} \sum_{\ell=0}^{+\infty} \left(\binom{(k+1)\ell - n}{\ell} - \binom{(k+1)\ell - n}{\ell - 1} \right) 2^{-(k+1)\ell}.$$

Proof. Let k and n be two fixed nonnegative integers such that $k \geq 2$ and $n \neq 1$. By applying Theorem 6 for $a = -n$, we get

$$\sum_{\ell=0}^{+\infty} \binom{(k+1)\ell - n}{\ell} 2^{-(k+1)\ell} = 2^{-n+1} \cdot \frac{\rho_k^n}{(k+1)\rho_k - 2k}. \qquad (17)$$

Similarly, by applying Theorem 6 for $a = k + 1 - n$, we get

$$\sum_{\ell=0}^{+\infty} \binom{(k+1)\ell + k + 1 - n}{\ell} 2^{-(k+1)\ell} = 2^{k+2-n} \cdot \frac{\rho_k^{n-k-1}}{(k+1)\rho_k - 2k}. \qquad (18)$$

But we notice that

$$\sum_{\ell=0}^{+\infty} \binom{(k+1)\ell + k + 1 - n}{\ell} 2^{-(k+1)\ell} = \sum_{\ell=0}^{+\infty} \binom{(k+1)(\ell+1) - n}{\ell} 2^{-(k+1)\ell}$$

$$= \sum_{\ell=1}^{+\infty} \binom{(k+1)\ell - n}{\ell - 1} 2^{-(k+1)(\ell-1)}$$

$$= 2^{k+1} \sum_{\ell=0}^{+\infty} \binom{(k+1)\ell - n}{\ell - 1} 2^{-(k+1)\ell}$$

(since $\binom{(k+1)\ell - n}{\ell - 1} = 0$ for $\ell = 0$, because $n \neq 1$). Thus Formula (18) is simplified to

$$\sum_{\ell=0}^{+\infty} \binom{(k+1)\ell - n}{\ell - 1} 2^{-(k+1)\ell} = 2^{-n+1} \frac{\rho_k^{n-k-1}}{(k+1)\rho_k - 2k}. \qquad (19)$$

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Next, by subtracting side by side Equation (19) from Equation (17) and then multiplying by 2^{n-2}, we get

$$2^{n-2} \sum_{\ell=0}^{+\infty} \left(\binom{(k+1)\ell - n}{\ell} - \binom{(k+1)\ell - n}{\ell - 1} \right) 2^{-(k+1)\ell} = \frac{1}{2} \cdot \frac{\rho_k^n - \rho_k^{n-k-1}}{(k+1)\rho_k - 2k}$$

$$= \frac{1}{2} \cdot \frac{\rho_k - \rho_k^{-k}}{(k+1)\rho_k - 2k} \cdot \rho_k^{n-1}$$

$$= \frac{1}{2} \cdot \frac{\rho_k - (2 - \rho_k)}{(k+1)\rho_k - 2k} \cdot \rho_k^{n-1} \qquad \text{(in view of (12))}$$

$$= \frac{\rho_k - 1}{(k+1)\rho_k - 2k} \rho_k^{n-1}$$

$$= \mathrm{As}_n^{(k)} \qquad\qquad \text{(according to (11)),}$$

as required. $\qquad\square$

Observing Formula of Theorem 8, and because $\mathrm{As}_n^{(k)}$ is close to $F_n^{(k)}$ for n large, a natural question arises: can replacing the series therein with its truncation at some positive integer $N(n,k)$ (depending on n and k) yield the k-Fibonacci numbers? Exploring this direction, we have established the following result, which is essentially equivalent to that of Howard and Cooper [6, Theorem 2.4], but with some corrections (see below for the details).

Theorem 9. *For all integers $n, k \geq 2$, we have*

$$F_{n+k-2}^{(k)} = 2^{n-2} \sum_{0 \leq \ell \leq \frac{n-1}{k+1}} \left(\binom{(k+1)\ell - n}{\ell} - \binom{(k+1)\ell - n}{\ell - 1} \right) 2^{-(k+1)\ell}.$$

To prepare for the proof of Theorem 9, let us provisionally define (for all integers $n, k \geq 2$)

$$G_n^{(k)} := 2^{n-2} \sum_{0 \leq \ell \leq \frac{n-1}{k+1}} \left(\binom{(k+1)\ell - n}{\ell} - \binom{(k+1)\ell - n}{\ell - 1} \right) 2^{-(k+1)\ell}.$$

We first establish the following two propositions:

Proposition 10. *Let $n, k \geq 2$ be two integers. Then we have*

$$\begin{aligned} G_n^{(k)} &= 2^{n-2} & \text{for } 2 \leq n \leq k+1, \\ G_{k+2}^{(k)} &= 2^k - 1. \end{aligned} \qquad (20)$$

Proof. For $2 \leq n \leq k+1$, the only integer ℓ in the range $[0, \frac{n-1}{k+1}]$ is $\ell = 0$. So the definition of $G_n^{(k)}$ gives

$$G_n^{(k)} = 2^{n-2} \left(\binom{-n}{0} - \binom{-n}{-1} \right) = 2^{n-2}$$

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(since $\binom{-n}{0} = 1$ and $\binom{-n}{-1} = 0$), as required.

Next, for $n = k + 2$, the only integers in the range $[0, \frac{n-1}{k+1}] = [0, 1]$ are 0 and 1. So the definition of $G_{k+2}^{(k)}$ gives

$$G_{k+2}^{(k)} = 2^k \left(\left(\underbrace{\binom{-k-2}{0}}_{=1} - \underbrace{\binom{-k-2}{-1}}_{=0} \right) + \left(\underbrace{\binom{-1}{1}}_{=-1} - \underbrace{\binom{-1}{0}}_{=1} \right) 2^{-k-1} \right)$$

$$= 2^k \left(1 - 2^{-k} \right)$$

$$= 2^k - 1,$$

as required. This completes the proof of the proposition. $\qquad\qquad\qquad\qquad\square$

Proposition 11. *For all integers $n, k \geq 2$, we have*

$$G_{n+k+1}^{(k)} = 2G_{n+k}^{(k)} - G_n^{(k)}.$$

Proof. Let $n, k \geq 2$ be two fixed integers. We have by definition

$$2G_{n+k}^{(k)} = 2^{n+k-1} \sum_{0 \leq \ell \leq \frac{n+k-1}{k+1}} \left(\binom{(k+1)\ell - n - k}{\ell} - \binom{(k+1)\ell - n - k}{\ell - 1} \right) 2^{-(k+1)\ell}. \qquad (21)$$

But summing on ℓ such that $0 \leq \ell \leq \frac{n+k-1}{k+1}$ is equivalent to summing on ℓ such that $0 \leq \ell \leq \frac{n+k}{k+1}$, except perhaps in the case where $\frac{n+k}{k+1}$ is an integer. Suppose that we are in the latter (i.e., $\frac{n+k}{k+1} \in \mathbb{Z}$). Then since $\frac{n+k}{k+1} \geq \frac{k+2}{k+1} > 1$, we have $\frac{n+k}{k+1} \geq 2$. Thus for $\ell = \frac{n+k}{k+1} \geq 2$, we find that $\binom{(k+1)\ell - n - k}{\ell} - \binom{(k+1)\ell - n - k}{\ell - 1} = \binom{0}{\ell} - \binom{0}{\ell - 1} = 0$, implying that the sum for $0 \leq \ell \leq \frac{n+k-1}{k+1}$ in (21) can be replaced by the sum for $0 \leq \ell \leq \frac{n+k}{k+1}$ even in the case where $\frac{n+k}{k+1} \in \mathbb{Z}$. So we have in all cases

$$2G_{n+k}^{(k)} = 2^{n+k-1} \sum_{0 \leq \ell \leq \frac{n+k}{k+1}} \left(\binom{(k+1)\ell - n - k}{\ell} - \binom{(k+1)\ell - n - k}{\ell - 1} \right) 2^{-(k+1)\ell}. \qquad (22)$$

Next, we have by definition

$$G_n^{(k)} = 2^{n-2} \sum_{0 \leq \ell \leq \frac{n-1}{k+1}} \left(\binom{(k+1)\ell - n}{\ell} - \binom{(k+1)\ell - n}{\ell - 1} \right) 2^{-(k+1)\ell}$$

$$= 2^{n-2} \sum_{1 \leq \ell \leq \frac{n+k}{k+1}} \left(\binom{(k+1)(\ell-1) - n}{\ell - 1} - \binom{(k+1)(\ell-1) - n}{\ell - 2} \right) 2^{-(k+1)(\ell-1)}$$

$$= 2^{n+k-1} \sum_{1 \leq \ell \leq \frac{n+k}{k+1}} \left(\binom{(k+1)\ell - n - k - 1}{\ell - 1} - \binom{(k+1)\ell - n - k - 1}{\ell - 2} \right) 2^{-(k+1)\ell}.$$

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But since $\ell = 0$ gives $\binom{(k+1)\ell - n - k - 1}{\ell - 1} - \binom{(k+1)\ell - n - k - 1}{\ell - 2} = \binom{-n-k-1}{-1} - \binom{-n-k-1}{-2} = 0$ (because $-n - k - 1 \leq -5$), we have even

$$G_n^{(k)} = 2^{n+k-1} \sum_{0 \leq \ell \leq \frac{n+k}{k+1}} \left(\binom{(k+1)\ell - n - k - 1}{\ell - 1} - \binom{(k+1)\ell - n - k - 1}{\ell - 2} \right) 2^{-(k+1)\ell}. \quad (23)$$

By subtracting (side to side) (23) from (22) and using the formulas

$$\binom{(k+1)\ell - n - k}{\ell} - \binom{(k+1)\ell - n - k - 1}{\ell - 1} = \binom{(k+1)\ell - n - k - 1}{\ell}$$

and

$$\binom{(k+1)\ell - n - k}{\ell - 1} - \binom{(k+1)\ell - n - k - 1}{\ell - 2} = \binom{(k+1)\ell - n - k - 1}{\ell - 1}$$

that result from (2), we get

$$2G_{n+k}^{(k)} - G_n^{(k)} = 2^{n+k-1} \sum_{0 \leq \ell \leq \frac{n+k}{k+1}} \left(\binom{(k+1)\ell - n - k - 1}{\ell} - \binom{(k+1)\ell - n - k - 1}{\ell - 1} \right) 2^{-(k+1)\ell}$$

$$= G_{n+k+1}^{(k)},$$

as required. ☐

We are now ready to prove Theorem 9.

Proof of Theorem 9. Let $k \geq 2$ be a fixed integer. We have to show that $F_{n+k-2}^{(k)} = G_n^{(k)}$ for all integer $n \geq 2$. Since the two sequences $(F_{n+k-2}^{(k)})_{n \geq 2}$ and $(G_n^{(k)})_{n \geq 2}$ satisfy the same linear recurrence relation of order $(k+1)$, which is

$$u_{n+k+1} = 2u_{n+k} - u_n$$

(according to (5) and to Proposition 11) then it suffices to show that $F_{n+k-2}^{(k)}$ and $G_n^{(k)}$ coincide at the first $(k+1)$ consecutive values of n; that is

$$F_{n+k-2}^{(k)} = G_n^{(k)} \qquad \text{for } 2 \leq n \leq k + 2. \quad (24)$$

To confirm (24), we distinguish the following two cases:

• *1st case*: (if $2 \leq n \leq k + 1$). In this case, we have $k \leq n + k - 2 \leq 2k - 1$, implying (according to (6)) that $F_{n+k-2}^{(k)} = 2^{n-2}$. Since we have also (according to (20)) $G_n^{(k)} = 2^{n-2}$ then $F_{n+k-2}^{(k)} = G_n^{(k)}$ for this case.

• *2nd case*: (If $n = k + 2$). In this case, we have $F_{n+k-2}^{(k)} = F_{2k}^{(k)} = 2^k - 1$ (according to (6)) and $G_n^{(k)} = G_{k+2}^{(k)} = 2^k - 1$ (according to (20)). Hence $F_{n+k-2}^{(k)} = G_n^{(k)}$ also for this case. This confirms (24) and completes this proof. ☐

By shifting the index n to $(n - k + 2)$ in Theorem 9, we derive the following corollary:

Corollary 12. *For all integers n and k such that $n \geq k \geq 2$, we have*

$$F_n^{(k)} = 2^{n-k} \sum_{0 \leq \ell \leq \frac{n-k+1}{k+1}} \left(\binom{(k+1)\ell - n + k - 2}{\ell} - \binom{(k+1)\ell - n + k - 2}{\ell - 1} \right) 2^{-(k+1)\ell}. \quad \Box$$

By taking $k = 2$ in Corollary 12, we derive in particular an expression of the usual Fibonacci numbers as finite sums involving binomial coefficients and nonpositive powers of 8. This is given by the following corollary:

Corollary 13. *For all integer $n \geq 2$, we have*

$$F_n = 2^{n-2} \sum_{0 \leq \ell \leq \frac{n-1}{3}} \left(\binom{3\ell - n}{\ell} - \binom{3\ell - n}{\ell - 1} \right) 2^{-3\ell}. \quad \Box$$

We end this paper by deriving a result from Corollary 12 that corrects that of Howard and Cooper [6, Theorem 2.4].

Corollary 14. *For all integers n and k such that $n \geq k \geq 2$ and $n \neq 2k - 1$, we have*

$$F_n^{(k)} = 2^{n-k} + \sum_{1 \leq \ell \leq \frac{n-k+1}{k+1}} (-1)^\ell \left(\binom{n - (\ell+1)k + 2}{\ell} - \binom{n - (\ell+1)k}{\ell - 2} \right) 2^{n-(k+1)\ell - k}.$$

Proof. Let $n, k \in \mathbb{Z}$ be fixed with $n \geq k \geq 2$ and $n \neq 2k - 1$. According to Formula (2), we have

$$\binom{a}{\ell} - \binom{a}{\ell - 1} = \binom{a-1}{\ell} - \binom{a-1}{\ell - 2} \qquad (\forall (a, \ell) \in \mathbb{Z}^2 \setminus \{(0, 0), (0, 1)\}).$$

By applying this last formula for each of the couples $(a, \ell) = ((k+1)\ell - n + k - 2, \ell)$ $(0 \leq \ell \leq \frac{n-k+1}{k+1})$, which are all different from $(0, 0)$ and $(0, 1)$, we transform Formula of Corollary 12 to

$$F_n^{(k)} = 2^{n-k} \sum_{0 \leq \ell \leq \frac{n-k+1}{k+1}} \left(\binom{(k+1)\ell + k - n - 3}{\ell} - \binom{(k+1)\ell + k - n - 3}{\ell - 2} \right) 2^{-(k+1)\ell}.$$

Then, by applying Formula (3) for all the binomial coefficients occurring in the last formula, we conclude to

$$F_n^{(k)} = 2^{n-k} \sum_{0 \leq \ell \leq \frac{n-k+1}{k+1}} (-1)^\ell \left(\binom{n - (\ell+1)k + 2}{\ell} - \binom{n - (\ell+1)k}{\ell - 2} \right) 2^{-(k+1)\ell}$$

$$= 2^{n-k} + \sum_{1 \leq \ell \leq \frac{n-k+1}{k+1}} (-1)^\ell \left(\binom{n - (\ell+1)k + 2}{\ell} - \binom{n - (\ell+1)k}{\ell - 2} \right) 2^{n-(k+1)\ell - k},$$

as required. $\qquad \Box$

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Remark 15.

1. Howard and Cooper [6, Theorem 2.4] stated Formula of Corollary 14 with the erroneous condition of summation $1 \leq \ell \leq \frac{n-1}{k+1}$. This condition coincides with ours only for $k = 2$. Therefore, Corollary 14 corrects Theorem 2.4 of [6].

2. Applying (3) for all the binomial coefficients appearing in Formula of Corollary 12, we transform the later to

$$F_n^{(k)} = 2^{n-k} + \sum_{1 \leq \ell \leq \frac{n-k+1}{k+1}} (-1)^\ell \left(\binom{n - (\ell+1)k + 1}{\ell} + \binom{n - (\ell+1)k}{\ell - 1} \right) 2^{n-(k+1)\ell - k}$$

(valid for $n \geq k \geq 2$). Note that the last formula and the formula of Corollary 14 have the advantage that the binomial coefficients which appear in them are ordinary (i.e., have nonnegative entries). Additionally, the two formulas in question are equivalent (we pass from the one to the other via Formula (2)).

4   Acknowledgment

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(Concerned with sequence A000045.)

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