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On the Reciprocal Sums and Products of mth-order Linear Recursive Sequences

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Abstract

The mth-order linear recursive sequence of $(w_n)_{n\geq 0}$ is defined by the recursion $w_n = a_1 w_{n-1} + a_2 w_{n-2} + \cdots + a_m w_{n-m}$ for $n > m$. In previous discussions of the reciprocal sums and products of $(w_n)_{n\geq 0}$, the condition $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$ was typically imposed. In this paper, we extend previous research and allow the coefficients to be arbitrary positive integers.

1 Introduction

For positive integers a_1, a_2,..., a_m with $a_m \neq 0$, the mth-order linear recursive sequence $(w_n)_{n\geq 0}$ is defined by

$$w_n = a_1 w_{n-1} + a_2 w_{n-2} + \cdots + a_m w_{n-m}, \quad n > m, \tag{1}$$

where the initial values $w_i \in \mathbb{N}$, and at least one is not zero. The characteristic polynomial of the sequence $(w_n)_{n\geq 0}$ is

$$\varphi(x) = x^m - a_1 x^{m-1} - \cdots - a_{m-1}x - a_m = (x-\lambda_1)^{m_1} \cdots (x-\lambda_l)^{m_l}, \tag{2}$$

where the λ_i are called the *roots* of the sequence. If the absolute value of a root of the sequence $(w_n)_{n\geq 0}$ is strictly largest, the root is called the *dominant root*.

If $m = 2$, $a_1 = a_2 = 1$, and $w_0 = 0$, $w_1 = 1$, the resulting sequence $(w_n)_{n\geq 0}$ is the famous Fibonacci sequence $(f_n)_{n\geq 0}$. Ohtsuka and Nakamura [10] considered the reciprocal sums of the Fibonacci sequence and obtained the following result:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{f_k} \right)^{-1} \right\rfloor = \begin{cases} f_{n-2}, & \text{if } n \text{ is even and } n \geq 2; \\ f_{n-2} - 1, & \text{if } n \text{ is odd and } n \geq 1, \end{cases}$$

where $\lfloor z \rfloor$ denotes the floor function.

Computing the floor function of $(\sum\limits_{k=n}^{\infty} \frac{1}{w_k})^{-1}$ is a difficult problem. Some researchers have studied the nearest integer to $(\sum\limits_{k=n}^{\infty} \frac{1}{w_k})^{-1}$. For example, Wu and Zhang [13] proved that there exists a positive integer n_1 such that

$$\left\| (\sum_{k=n}^{\infty} \frac{1}{w_k})^{-1} \right\| = w_n - w_{n-1}, \quad n \geq n_1,$$

where $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$ and $\|z\|$ denotes the nearest integer function, defined by $\|z\| = \lfloor z + \frac{1}{2} \rfloor$.

If two sequences $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ satisfy the condition that $(u_n/v_n)_{n\geq 0}$ tends to 1 as $n \to \infty$, we call them *asymptotically equivalent*. Some researchers have continued the study of reciprocal sums by finding a sequence that is asymptotically equivalent to $\left((\sum\limits_{k=n}^{\infty} \frac{1}{w_k})^{-1} \right)$. Specifically, Trojovský [12] proved that the sequences

$$\left((\sum_{k=n}^{\infty} \frac{1}{P(w_k)})^{-1} \right)_n \quad \text{and} \quad (P(w_n) - P(w_{n-1}))_n$$

are asymptotically equivalent, where $P(z)$ is a non-constant polynomial with $P(z) \in \mathbb{C}[z]$. For more on reciprocal sums and products, see [3, 6, 15, 2, 1, 4, 9, 14, 16, 8].

In previous research concerning reciprocal sums and products of $(w_n)_{n\geq 0}$, the condition $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$ was typically imposed. In this paper, we allow the coefficients to be arbitrary positive integers, and obtain a series of sequences that are asymptotically equivalent to $\left((\sum\limits_{k=n}^{\infty} \frac{1}{w_k})^{-1} \right)$ and $\left(1 - \prod\limits_{k=n}^{\infty} (1 - \frac{1}{w_k}))^{-1} \right)$. The main results are summarized in the following theorem.

Theorem 1. *Let $(w_n)_{n\geq 0}$ be an mth-order linear recursive sequence defined by (1). The sequences*

$$\left((\sum_{k=n}^{\infty} \frac{1}{w_k})^{-1} \right) \quad \textit{and} \quad (w_n - w_{n-1}) \tag{3}$$

are asymptotically equivalent, and the sequences

$$\left((1-\prod_{k=n}^{\infty}(1-\frac{1}{w_k}))^{-1}\right)_n \qquad \text{and} \qquad (w_n - w_{n-1})_n \tag{4}$$

are asymptotically equivalent.

2 Some lemmas

In this section, we shall give several lemmas that are useful for the proofs of the theorem.

Lemma 2. *(Descartes rule of signs). Let $\phi(x) = a_{n_1}x^{n_1} + \cdots + a_{n_k}x^{n_k}$ be a polynomial where the n_i are integers and $n_1 > n_2 > \cdots > n_k \geq 0$. The number of positive roots of $\phi(x)$ is at most the number of sign changes of adjacent nonzero coefficients.*

Lemma 3. *([5, 7] Eneström-Kakeya theorem). Let $\phi'(x) = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial of order n with real coefficients. If $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n$, then all complex roots z of $\phi'(x)$ satisfy $|z| \leq 1$.*

Lemma 4. *Let $\varphi(x) = x^m - a_1x^{m-1} - \cdots - a_{m-1}x - a_m$ be the characteristic polynomial of $(w_n)_{n\geq 0}$, where the coefficients a_1, a_2, ..., a_m are positive integers. Then the following hold:*

- *$\varphi(x)$ has only one positive root λ_1, called λ, and $\lambda > 1$;*
- *The other $m-1$ roots of $\varphi(x)$ lie in the circle $|z| < \lambda$. Therefore λ is the dominant root of $\varphi(x)$.*

Proof. By Lemma 2, the characteristic polynomial $\varphi(x) = x^m - a_1x^{m-1} - \cdots - a_{m-1}x - a_m$ has at most one positive root λ_1, called λ. In addition, $\lim_{x\to\infty} \varphi(x) = +\infty$, and, by $\varphi(1) < 0$, we obtain that $\lambda > 1$ and λ is the only positive root of $\varphi(x)$.

By using the relation

$$\lambda^m = a_1\lambda^{m-1} + a_2\lambda^{m-2} + \cdots + a_{m-1}\lambda + a_m,$$

we obtain that $\varphi(x) = (x-\lambda)\psi(x)$, where

$$\begin{aligned}\psi(x) &= x^{m-1} + (\lambda - a_1)x^{m-2} + (\lambda^2 - a_1\lambda - a_2)x^{m-3} + \cdots \\ &\quad + (\lambda^{m-2} - a_1\lambda^{m-3} - \cdots - a_{m-2})x + \lambda^{m-1} - a_1\lambda^{m-2} - \cdots - a_{m-1}.\end{aligned}$$

Claim 1. If z is a root of $\psi(x)$, then $|z| \leq \lambda$.

In fact, we prove that all the roots of $\Psi(x) := \psi(\lambda x)$ are in the closed unit ball. The polynomial

$$\begin{aligned}\Psi(x) &= \lambda^{m-1}x^{m-1} + (\lambda - a_1)\lambda^{m-2}x^{m-2} + (\lambda^2 - a_1\lambda - a_2)\lambda^{m-3}x^{m-3} + \cdots \\ &\quad + (\lambda^{m-2} - a_1\lambda^{m-3} - \cdots - a_{m-2})\lambda x + \lambda^{m-1} - a_1\lambda^{m-2} - \cdots - a_{m-1} \\ &= \lambda^{m-1}x^{m-1} + (\lambda^{m-1} - a_1\lambda^{m-2})x^{m-2} + (\lambda^{m-1} - a_1\lambda^{m-2} - a_2\lambda^{m-3})x^{m-3} + \cdots \\ &\quad + (\lambda^{m-1} - a_1\lambda^{m-2} - \cdots - a_{m-2}\lambda)x + \lambda^{m-1} - a_1\lambda^{m-2} - \cdots - a_{m-1}.\end{aligned}$$

Since

$$\lambda^{m-1} > \lambda^{m-1} - a_1\lambda^{m-2} > \lambda^{m-1} - a_1\lambda^{m-2} - a_2\lambda^{m-3} > \cdots$$
$$> \lambda^{m-1} - a_1\lambda^{m-2} - \cdots - a_{m-2}\lambda > \lambda^{m-1} - a_1\lambda^{m-2} - \cdots - a_{m-1} = a_m/\lambda > 0,$$

which λ is positive root of $\varphi(x)$. By Lemma 3, this completes the proof of Claim 2.

Claim 2. On the circle $|z| = \lambda$, the polynomial $\varphi(x)$ has the unique root λ.

If $\varphi(z) = 0$, then

$$z^m = a_1 z^{m-1} + a_2 z^{m-2} + \cdots + a_{m-1} z + a_m.$$

The triangle inequality is satisfied:

$$|z|^m \leq a_1 |z|^{m-1} + a_2 |z|^{m-2} + \cdots + a_{m-1} |z| + a_m. \tag{5}$$

If $z = \lambda$, then $\varphi(z) = 0$. So (5) must be an equality. Therefore, $a_1 z^{m-1}$, $a_2 z^{m-2}$, $\ldots$, $a_{m-1}z$, a_m all on the same ray leaving the origin. Since a_1, a_2, $\ldots$, a_m are all the elements of $\mathbb{R}^+$, z^{m-1}, z^{m-2}, $\ldots$, z must be elements of $\mathbb{R}^+$. Therefore we obtain $\varphi(z) \in \mathbb{R}^+$. On the circle $|z| = \lambda$, we obtain $z = \lambda$. This completes the proof of Claim 2.

□

We define $f(x) = \mathcal{O}(g(x))$ to mean that the quotient $f(x)/g(x)$ is bounded for $x \geq a$.

Lemma 5. *Let $(w_n)_{n\geq 0}$ be mth-order linear recursive sequence defined by (1). Then the asymptotic formula of $(w_n)_{n\geq 0}$ is as follows:*

$$w_n = c\lambda^n + \mathcal{O}((\lambda\mu)^{\frac{n}{2}}) \tag{6}$$

where c is a constant. The λ defined by Lemma 4 is the dominant root of $\varphi(x)$, and μ is the largest absolute value of the remaining $m-1$ roots of $\varphi(x)$.

Proof. By [11], there exist unique nonzero polynomials $\ell_1, \ldots, \ell_l \in \mathbb{Q}(\{\lambda_i\}_{i=1}^{l})[x]$, with $\deg \ell_i \leq m_i - 1$ (where m_i is the multiplicity of λ_i as a root of the characteristic polynomial $\varphi(x)$) for $1 \leq i \leq l$, such that

$$w_n = \ell_1(n)\lambda_1^n + \cdots + \ell_l(n)\lambda_l^n, \quad \text{for all } n.$$

By Lemma 4, we get

$$w_n = c\lambda^n + \sum_{i=2}^{l} \ell_i(n)\lambda_i^n, \quad \ell_i(n) \in \mathbb{R}[n],$$

where λ is the dominant root of $\varphi(x)$, c is a nonzero constant, and

$$\deg \ell_i(n) \le m_i - 1, \quad \text{for} \quad i = 2, \ldots, l, \quad m_2 + \cdots + m_l = m - 1,$$

Therefore, we obtain that

$$w_n = c\lambda^n + \mathcal{O}(n^m \mu^n) = c\lambda^n + \mathcal{O}((\lambda\mu)^{\frac{n}{2}}),$$

where $n^m < \left|\frac{\lambda}{\mu}\right|^{\frac{n}{2}}$ for all sufficiently large n. □

Remark 6. For discussing the reciprocal product of mth-order linear recursive sequences, we assume that $\mu > 1$. When $\mu < 1$, as in [3], we write $w_n = c\lambda^n + \mathcal{O}(c^{-n})$, for some $c > 1$).

Lemma 7. *Let $\lambda > |\mu| > 1$. We get*

$$\prod_{k=n}^{\infty}(1 - \frac{1}{c\lambda^k} + \mathcal{O}((\frac{\mu}{\lambda^3})^{\frac{k}{2}})) = 1 - \sum_{k=n}^{\infty} \frac{1}{c\lambda^k} + \mathcal{O}((\frac{\mu}{\lambda^3})^{\frac{n}{2}}), \tag{7}$$

where c is a constant.

Proof. First we prove the following equality:

$$\prod_{k=n}^{n+m}(1 - \frac{1}{c\lambda^k} + \mathcal{O}((\frac{\mu}{\lambda^3})^{\frac{k}{2}})) = 1 - \sum_{k=n}^{n+m} \frac{1}{c\lambda^k} + \mathcal{O}((\frac{\mu}{\lambda^3})^{\frac{n}{2}}), \tag{8}$$

We prove (8) by mathematical induction. When $m = 1$, we have

$$\begin{aligned}
&\prod_{k=n}^{n+1}(1 - \frac{1}{c\lambda^k} + \mathcal{O}((\frac{\mu}{\lambda^3})^{\frac{k}{2}})) \\
&= (1 - \frac{1}{c\lambda^n} + \mathcal{O}((\frac{\mu}{\lambda^3})^{\frac{n}{2}})) \times (1 - \frac{1}{c\lambda^{n+1}} + \mathcal{O}((\frac{\mu}{\lambda^3})^{\frac{n+1}{2}})) \\
&= 1 - \frac{1}{c\lambda^n} - \frac{1}{c\lambda^{n+1}} + \frac{1}{c^2\lambda^{2n+1}} + \mathcal{O}((\frac{\mu}{\lambda^3})^{\frac{n}{2}}) \\
&+ \mathcal{O}((\frac{\mu}{\lambda^3})^{\frac{n+1}{2}}) + \mathcal{O}((\frac{\mu}{\lambda^5})^{\frac{n}{2}}) + \mathcal{O}((\frac{\mu}{\lambda^3})^{\frac{2n+1}{2}}) \\
&= 1 - \frac{1}{c\lambda^n} - \frac{1}{c\lambda^{n+1}} + \mathcal{O}((\frac{\mu}{\lambda^3})^{\frac{n}{2}}) \\
&= 1 - \sum_{k=n}^{n+1} \frac{1}{c\lambda^k} + \mathcal{O}((\frac{\mu}{\lambda^3})^{\frac{n}{2}}).
\end{aligned}$$

That is, (8) holds when $m = 1$. Now suppose that for every integer m, we have

$$\prod_{k=n}^{n+m}(1 - \frac{1}{c\lambda^k} + \mathcal{O}((\frac{\mu}{\lambda^3})^{\frac{k}{2}})) = 1 - \sum_{k=n}^{n+m} \frac{1}{c\lambda^k} + \mathcal{O}((\frac{\mu}{\lambda^3})^{\frac{n}{2}}).$$

Then for $m+1$, we have

$$\begin{aligned}
&\prod_{k=n}^{n+m+1}(1-\frac{1}{c\lambda^k}+\mathcal{O}((\frac{\mu}{\lambda^3})^{\frac{k}{2}}))\\
&=(1-\sum_{k=n}^{n+m}\frac{1}{c\lambda^k}+\mathcal{O}((\frac{\mu}{\lambda^3})^{\frac{n}{2}}))\times(1-\frac{1}{c\lambda^{n+m+1}}+\mathcal{O}((\frac{\mu}{\lambda^3})^{\frac{n+m+1}{2}}))\\
&=1-\sum_{k=n}^{n+m}\frac{1}{c\lambda^k}-\frac{1}{c\lambda^{n+m+1}}+\frac{1}{c\lambda^{n+m+1}}(\sum_{k=n}^{n+m}\frac{1}{c\lambda^k})+\mathcal{O}((\frac{\mu}{\lambda^3})^{\frac{n}{2}})\\
&\quad+\mathcal{O}((\frac{\mu}{\lambda^3})^{\frac{n+m+1}{2}})+\mathcal{O}((\frac{\mu}{\lambda^5})^{\frac{n}{2}})+\mathcal{O}((\frac{\mu}{\lambda^3})^{\frac{2n+m+1}{2}})\\
&=1-\sum_{k=n}^{n+m+1}\frac{1}{c\lambda^k}+\mathcal{O}((\frac{\mu}{\lambda^3})^{\frac{n}{2}}).
\end{aligned} \tag{9}$$

Now (9) follows from (9) and mathematical induction.

Taking $m\to\infty$, we have

$$\prod_{k=n}^{\infty}(1-\frac{1}{c\lambda^k}+\mathcal{O}(\frac{\mu}{\lambda^3})^{\frac{k}{2}})=1-\sum_{k=n}^{\infty}\frac{1}{c\lambda^k}+\mathcal{O}((\frac{\mu}{\lambda^3})^{\frac{n}{2}}),$$

which completes the proof. □

3 Proof of the theorem

Proof of Theorem 1. Using the geometric series, we get

$$\frac{1}{1\pm\epsilon}=1+\mathcal{O}(\epsilon), \tag{10}$$

where $\epsilon\to 0$. Using Lemma 5, we obtain

$$\begin{aligned}
\frac{1}{w_k}&=\frac{1}{c\lambda^k+\mathcal{O}((\lambda\mu)^{\frac{k}{2}})}=\frac{1}{c\lambda^k(1+\mathcal{O}((\frac{\mu}{\lambda})^{\frac{k}{2}}))}\\
&=\frac{1}{c\lambda^k}(1+\mathcal{O}(\frac{\mu}{\lambda})^{\frac{k}{2}})=\frac{1}{c\lambda^k}+\mathcal{O}(\frac{\mu}{\lambda^3})^{\frac{k}{2}}.
\end{aligned} \tag{11}$$

Thus

$$\sum_{k=n}^{\infty}\frac{1}{w_k}=\frac{1}{c}\sum_{k=n}^{\infty}\frac{1}{\lambda^k}+\mathcal{O}(\sum_{k=n}^{\infty}(\frac{\mu}{\lambda^3})^{\frac{k}{2}})=\frac{\lambda}{c(\lambda-1)\lambda^n}+\mathcal{O}((\frac{\mu}{\lambda^3})^{\frac{n}{2}}).$$

Taking the reciprocal, we get

$$\begin{aligned}
(\sum_{k=n}^{\infty}\frac{1}{w_k})^{-1}&=\frac{1}{\frac{\lambda}{c(\lambda-1)\lambda^n}+\mathcal{O}((\frac{\mu}{\lambda^3})^{\frac{n}{2}})}=\frac{1}{\frac{\lambda}{c(\lambda-1)\lambda^n}(1+\mathcal{O}((\frac{\mu}{\lambda})^{\frac{n}{2}}))}\\
&=\frac{c(\lambda-1)\lambda^n}{\lambda}(1+\mathcal{O}((\frac{\mu}{\lambda})^{\frac{n}{2}}))=(c\lambda^n-c\lambda^{n-1})(1+\mathcal{O}((\frac{\mu}{\lambda})^{\frac{n}{2}})),
\end{aligned}$$

which yields that

$$\frac{(\sum\limits_{k=n}^{\infty}\frac{1}{w_k})^{-1}}{w_n-w_{n-1}} = \frac{(c\lambda^n - c\lambda^{n-1})(1+\mathcal{O}((\frac{\mu}{\lambda})^{\frac{n}{2}}))}{(c\lambda^n+\mathcal{O}((\lambda\mu)^{\frac{n}{2}}))-(c\lambda^{n-1}+\mathcal{O}((\lambda\mu)^{\frac{n-1}{2}}))}$$
$$= \frac{(c\lambda^n - c\lambda^{n-1})(1+\mathcal{O}((\frac{\mu}{\lambda})^{\frac{n}{2}}))}{(c\lambda^n-c\lambda^{n-1})(1+\mathcal{O}(\frac{(\lambda\mu)^{\frac{n}{2}}}{\lambda^n-\lambda^{n-1}})+\mathcal{O}(\frac{(\lambda\mu)^{\frac{n-1}{2}}}{\lambda^n-\lambda^{n-1}}))} \tag{12}$$
$$= \frac{(c\lambda^n - c\lambda^{n-1})(1+\mathcal{O}((\frac{\mu}{\lambda})^{\frac{n}{2}}))}{(c\lambda^n-c\lambda^{n-1})(1+\mathcal{O}((\frac{\mu}{\lambda})^{\frac{n}{2}})+\mathcal{O}((\frac{\mu}{\lambda})^{\frac{n-1}{2}}))}.$$

We obtain that

$$\frac{(\sum\limits_{k=n}^{\infty}\frac{1}{w_k})^{-1}}{w_n-w_{n-1}} \quad \text{tends to } 1, \text{as } n\to\infty.$$

In addition, by Lemma 7 and identity (11), we obtain

$$\prod_{k=n}^{\infty}(1-\frac{1}{w_k}) = \prod_{k=n}^{\infty}(1-\frac{1}{c\lambda^k}+\mathcal{O}(\frac{\mu}{\lambda^3})^{\frac{k}{2}})$$
$$= 1-\sum_{k=n}^{\infty}\frac{1}{c\lambda^k}+\mathcal{O}(\frac{\mu}{\lambda^3})^n = 1-\frac{\lambda}{c(\lambda-1)\lambda^n}+\mathcal{O}(\frac{\mu}{\lambda^3})^n.$$

Taking the reciprocal, we get

$$(1-\prod_{k=n}^{\infty}(1-\frac{1}{w_k}))^{-1} = \frac{1}{\frac{\lambda}{c(\lambda-1)\lambda^n}+\mathcal{O}((\frac{\mu}{\lambda^3})^{\frac{n}{2}})} = \frac{1}{\frac{\lambda}{c(\lambda-1)\lambda^n}(1+\mathcal{O}((\frac{\mu}{\lambda})^{\frac{n}{2}}))}$$
$$= \frac{c(\lambda-1)\lambda^n}{\lambda}(1+\mathcal{O}((\frac{\mu}{\lambda})^{\frac{n}{2}})) = (c\lambda^n-c\lambda^{n-1})(1+\mathcal{O}((\frac{\mu}{\lambda})^{\frac{n}{2}})),$$

which yields that

$$\frac{(1-\prod\limits_{k=n}^{\infty}(1-\frac{1}{w_k}))^{-1}}{w_n-w_{n-1}} = \frac{(c\lambda^n - c\lambda^{n-1})(1+\mathcal{O}((\frac{\mu}{\lambda})^{\frac{n}{2}}))}{(c\lambda^n+\mathcal{O}((\lambda\mu)^{\frac{n}{2}}))-(c\lambda^{n-1}+\mathcal{O}((\lambda\mu)^{\frac{n-1}{2}}))}$$
$$= \frac{(c\lambda^n - c\lambda^{n-1})(1+\mathcal{O}((\frac{\mu}{\lambda})^{\frac{n}{2}}))}{(c\lambda^n-c\lambda^{n-1})(1+\mathcal{O}(\frac{(\lambda\mu)^{\frac{n}{2}}}{\lambda^n-\lambda^{n-1}})+\mathcal{O}(\frac{(\lambda\mu)^{\frac{n-1}{2}}}{\lambda^n-\lambda^{n-1}}))} \tag{13}$$
$$= \frac{(c\lambda^n - c\lambda^{n-1})(1+\mathcal{O}((\frac{\mu}{\lambda})^{\frac{n}{2}}))}{(c\lambda^n-c\lambda^{n-1})(1+\mathcal{O}((\frac{\mu}{\lambda})^{\frac{n}{2}})+\mathcal{O}((\frac{\mu}{\lambda})^{\frac{n-1}{2}}))}.$$

We obtain that

$$\frac{(1-\prod\limits_{k=n}^{\infty}(1-\frac{1}{w_k}))^{-1}}{w_n-w_{n-1}} \quad \text{tends to } 1, \text{as } n\to\infty,$$

which completes the proof. □

Remark 8. We now discuss the relative error of the asymptotic behavior of the result in Theorem 1. By identities (12) and (13), we obtain

$$\begin{aligned}\left|\frac{(\sum\limits_{k=n}^{\infty}\frac{1}{w_k})^{-1}}{w_n-w_{n-1}}-1\right| &= \left|\frac{1+\mathcal{O}((\frac{\mu}{\lambda})^{\frac{n}{2}})}{1+\mathcal{O}((\frac{\mu}{\lambda})^{\frac{n}{2}})+\mathcal{O}((\frac{\mu}{\lambda})^{\frac{n-1}{2}})}-1\right| \\ &= \mathcal{O}((\frac{\mu}{\lambda})^{\frac{n-1}{2}})+\mathcal{O}((\frac{\mu}{\lambda})^{\frac{2n-1}{2}})=\mathcal{O}((\frac{\mu}{\lambda})^{\frac{n-1}{2}}),\end{aligned} \tag{14}$$

and

$$\begin{aligned}\left|\frac{(1-\prod\limits_{k=n}^{\infty}(1-\frac{1}{w_k}))^{-1}}{w_n-w_{n-1}}-1\right| &= \left|\frac{1+\mathcal{O}((\frac{\mu}{\lambda})^{\frac{n}{2}})}{1+\mathcal{O}((\frac{\mu}{\lambda})^{\frac{n}{2}})+\mathcal{O}((\frac{\mu}{\lambda})^{\frac{n-1}{2}})}-1\right| \\ &= \mathcal{O}((\frac{\mu}{\lambda})^{\frac{n-1}{2}})+\mathcal{O}((\frac{\mu}{\lambda})^{\frac{2n-1}{2}})=\mathcal{O}((\frac{\mu}{\lambda})^{\frac{n-1}{2}}).\end{aligned} \tag{15}$$

When $n = 101$, by the identities (14) and (15), we can determine the magnitude of the relative error of the asymptotic behavior for the following two sequences $(w_n)_{n_1\geq 0}$ and $(w_n)_{n_2\geq 0}$:

$$\begin{aligned} w_{n_1} &= 2w_{n-1}+3w_{n-2}+7w_{n-3}+w_{n-4}+w_{n-5}, \\ w_{n_2} &= 3w_{n-1}+5w_{n-2}+w_{n-3}+6w_{n-4}+w_{n-5}. \end{aligned}$$

w_n	λ	μ	$\left\lvert\frac{\mu}{\lambda}\right\rvert^{\frac{n-1}{2}}$
w_{n_1}	≈ 3.2421	≈ -1.1509	$\approx 3.2401\times 10^{-23}$
w_{n_2}	≈ 4.2965	≈ -1.4946	$\approx 1.1762\times 10^{-23}$

Table 1: The higher-order linear recurrences.

The computations are given in Table 1. We used the software `Mathematica`.

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References

[1] G. Choi and Y. Choo, On the reciprocal sums of square of generalized bi-periodic Fibonacci numbers, *Miskolc Math. Notes.* **19** (2018), 201–209.

[2] Y. Choo, On the reciprocal sums of products of two generalized bi-periodic Fibonacci numbers, *Mathematics* **9** (2021), 178.

[3] T. T. Du and Z. G. Wu, On the reciprocal products of generalized Fibonacci sequences, *J. Inequal. Appl.* **2022** (2022), 154.

[4] T. T. Du and Z. G. Wu, On the reciprocal sums of products of mth-order linear recurrence sequences, *Electron Res. Arch.* **31** (2023), 5766–5779.

[5] G. Eneström, Remarque sur un théorème relatif aux racines de l'équation $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$ où tous les coefficientes a sont réels et positifs, *Tohoku Math. J.* **18** (1920), 34–36.

[6] S. Holliday and T. Komatsu, On the sum of reciprocal generalized Fibonacci numbers, *Integers* **11** (2011), 441–455.

[7] S. Kakeya, On the limits of the roots of an algebraic equation with positive coefficients, *Tohoku Math. J.* **2** (1912), 140–142.

[8] E. Kilic and T. Arikan, More on the infinite sum of reciprocal Fibonacci, Pell and higher order recurrences, *Appl. Math. Comput.* **219** (2013), 7783–7788.

[9] T. Komatsu, On the nearest integer of the sum of reciprocal Fibonacci numbers, *Aportaciones Mat., Investig.* **20** (2011), 171–184.

[10] H. Ohtsuka and S. Nakamura, On the sum of reciprocal Fibonacci numbers, *Fibonacci Quart.* **46** (2008), 153–159.

[11] T. N. Shorey and R. Tijdeman, *Exponential Diophantine Equations*, Cambridge Tracts in Mathematics, Vol. 87, Cambridge University Press, 1986.

[12] P. Trojovský, On the sum of reciprocal of polynomial applied to higher order recurrences, *Mathematics* **7** (2019), 638.

[13] Z. G. Wu and H. Zhang, On the reciprocal sums of higher-order sequences, *Adv. Diff. Equ.* **2013** (2013), 189.

[14] Z. G. Wu and J. Zhang, On the higher power sums of reciprocal higher-order sequences, *Sci. World J.* **2014** (2014), Article ID 521358.

[15] W. P. Zhang and T. T. Wang, The infinite sum of reciprocal Pell numbers, *Appl. Math. Comput.* **218** (2012), 6164–6167.

[16] H. Zhang and Z. G. Wu, On the reciprocal sums of the generalized Fibonacci sequences, *Adv. Diff. Equ.* **2013** (2013), Article ID 377.

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