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A Minimal Excludant over Overpartitions

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Abstract

Define the minimal excludant of an overpartition π, denoted $\overline{\text{mex}}(\pi)$, to be the smallest positive integer that is not a part of the non-overlined parts of π. For a positive integer n, the function $\sigma\overline{\text{mex}}(\pi)$ is the sum of the minimal excludants over all overpartitions of n. In this paper, we prove that the $\sigma\overline{\text{mex}}(\pi)$ equals the number of partitions of n into distinct parts using three colors. We also provide an asymptotic formula for $\sigma\overline{\text{mex}}(\pi)$ and show that $\sigma\overline{\text{mex}}(\pi)$ is almost always even and is odd exactly when n is a triangular number. Moreover, we generalize $\overline{\text{mex}}(\pi)$ using the least r-gaps, denoted $\overline{\text{mex}}_r(\pi)$, defined as the smallest part of the non-overlined parts of the overpartition π appearing less than r times. Similarly, for a positive integer n, the function $\sigma_r\overline{\text{mex}}(\pi)$ is the sum of the least r-gaps over all overpartitions of n. We derive a generating function and an asymptotic formula for $\sigma_r\overline{\text{mex}}(\pi)$. Lastly, we study the arithmetic density of $\sigma_r\overline{\text{mex}}(\pi)$ modulo 2^k, where $r = 2^m \cdot 3^n, m, n \in \mathbb{Z}_{\geq 0}$.

1 Introduction

The minimal excludant (mex) of a subset S of a well-ordered set U is the smallest value in U that is not in S. In particular, the minimal excludant of a set S of positive integers, denoted $\text{mex}(S)$, is the least positive integer not in S, i.e., $\text{mex}(S) = \min(\mathbb{Z}^+ \setminus S)$. The history of the minimal excludant goes way back in the 1930s when it was first used in combinatorial game theory by Sprague and Grundy [8, 12].

In 2019, Andrews and Newman [2] studied the minimal excludant of an integer partition π, denoted $\operatorname{mex}(\pi)$, which is defined as the smallest positive integer that is not a part of π. Moreover, they also introduced the arithmetic function

$$\sigma \operatorname{mex}(n) := \sum_{\pi \in \mathcal{P}(n)} \operatorname{mex}(\pi),$$

where $\mathcal{P}(n)$ is the set of all partitions of n.

In their paper, Andrews and Newman proved the following interesting relationship between $\sigma \operatorname{mex}(n)$ and $D_2(n)$ which is the number of partitions of n into distinct parts using two colors:

$$\sigma \operatorname{mex}(n) = D_2(n).$$

Moreover, they showed that $\sigma \operatorname{mex}(n)$ is almost always even; in particular, they showed that $\sigma \operatorname{mex}(n)$ is odd exactly when $n = j(3j \pm 1)$ for some $j \in \mathbb{Z}^+$.

Recall that an overpartition of a positive integer n is a non-increasing sequence of natural numbers whose sum is n in which the first occurrence of a number may be overlined. We denote by $\overline{p}(n)$ the number of overpartitions of n. For example, $\overline{p}(3) = 8$ since there are 8 overpartitions of 3 which are:

$$3, \overline{3}, 2+1, \overline{2}+1, 2+\overline{1}, \overline{2}+\overline{1}, 1+1+1, \overline{1}+1+1.$$

The goal of this paper is to extend the notion of minimal excludant of partitions to overpartitions. There are several ways to obtain such a generalization. (For example, see Section 4 of [7] for one such definition of minimal excludant of overpartitions and its relation to the Ramanujan function $R(q)$.) We propose the following definition below (Definition 1). We justify using this definition through the results we obtain, which are overpartition analogues of results concerning the classical partition function (see Proposition 10). We also note that our definition coincides with the one given in a recent paper of Yang and Zhou on identities involving mex-related partitions [14].

Definition 1. The *minimal excludant of an overpartition* π, denoted $\overline{\operatorname{mex}}(\pi)$, is the smallest positive integer that is not a part of the non-overlined parts of π. For a positive integer n, denote the sum of $\overline{\operatorname{mex}}(\pi)$ over all overpartitions π of n as $\sigma\overline{\operatorname{mex}}(n)$:

$$\sigma\overline{\operatorname{mex}}(n) = \sum_{\pi \in \overline{\mathcal{P}}(n)} \overline{\operatorname{mex}}(\pi),$$

where $\overline{\mathcal{P}}(n)$ is the set of all overpartitions of n. We set $\sigma\overline{\operatorname{mex}}(0) = 1$.

For example, consider $n = 3$. The table below shows all overpartitions of 3 and their corresponding minimal excludant.

π	$\overline{\text{mex}}(\pi)$
3	1
$\overline{3}$	1
$2+1$	3
$\overline{2}+1$	2
$2+\overline{1}$	1
$\overline{2}+\overline{1}$	1
1+1+1	2
$\overline{1}+1+1$	2

Table 1: Minimal excludants of overpartitions of 3.

Thus, $\sigma\overline{\text{mex}}(3) = 13$. The table below shows the first ten values of $\sigma\overline{\text{mex}}(n)$.

n	0	1	2	3	4	5	6	7	8	9
$\sigma\overline{\text{mex}}(n)$	1	3	6	13	24	42	73	120	192	302

Table 2: First ten values of $\sigma\overline{\text{mex}}(n)$.

We observe that these are also the first ten values of the sequence A022568 in OEIS which is $(D_3(n))$, the sequence of number of partitions of n into distinct parts using three colors. In Section 2, we derive the generating function of $\sigma\overline{\text{mex}}(n)$ and prove the aforementioned observation relating $\sigma\overline{\text{mex}}(n)$ and $D_3(n)$. This result generalizes the results of Andrews and Newman, which relates $\sigma\,\text{mex}(n)$ and $D_2(n)$.

Theorem 2. *For all positive integers n, we have*

$$\sigma\overline{\text{mex}}(n) = D_3(n).$$

We also derive an asymptotic formula for $\sigma\overline{\text{mex}}(n)$ and prove a theorem regarding the parity of $\sigma\overline{\text{mex}}(n)$.

Theorem 3. *We have*

$$\sigma\overline{\text{mex}}(n) \sim \frac{e^{\pi\sqrt{n}}}{8n^{3/4}}$$

as $n \to \infty$.

Theorem 4. *For a positive integer n, we have*

$$\sigma\overline{\text{mex}}(n) \equiv \begin{cases} 1 \pmod 2, & \text{if } n = \frac{j(j+1)}{2} \text{ for some } j \in \mathbb{N}; \\ 0 \pmod 2, & \text{otherwise.} \end{cases}$$

Ballantine and Merca [3] explored the least r-gap of a partition π, denoted $g_r(\pi)$, which is the smallest part of π appearing less than r times. In particular, $g_1(\pi)$ is the minimal excludant of π. They defined the arithmetic function

$$\sigma_r \operatorname{mex}(n) = \sum_{\pi \in \mathcal{P}(n)} g_r(\pi)$$

which is the sum of the least r-gaps in all partitions of n. They also derived the following generating function for $\sigma_r \operatorname{mex}(n)$:

$$\sum_{n=0}^{\infty} \sigma_r \operatorname{mex}(n) q^n = \frac{(q^{2r}; q^{2r})_\infty}{(q;q)_\infty (q^r; q^{2r})_\infty}$$

where $(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k)$.

In this paper, we generalize $\overline{\operatorname{mex}}(\pi)$ into r-gaps.

Definition 5. The *least r-gap of an overpartition* π, denoted $\overline{\operatorname{mex}}_r(\pi)$ is the smallest part of the non-overlined parts of π appearing less than r times. Moreover, the function

$$\sigma_r \overline{\operatorname{mex}}(n) = \sum_{\pi \in \overline{\mathcal{P}}(n)} \overline{\operatorname{mex}}_r(\pi)$$

is the sum of the least r-gaps over all overpartitions of n. Moreover, we set $\sigma_r \overline{\operatorname{mex}}(0) = 1$.

For example, let $r = 2$ and $n = 3$. The table below shows all overpartitions of 3 and their corresponding least 2-gap.

π	$\overline{\operatorname{mex}}(\pi)$
3	1
$\overline{3}$	1
$2+1$	1
$\overline{2}+1$	1
$2+\overline{1}$	1
$\overline{2}+\overline{1}$	1
1+1+1	2
$\overline{1}+1+1$	2

Table 3: Least 2-gaps of overpartitions of 3.

Thus, $\sigma_2 \overline{\operatorname{mex}}(2) = 10$. The first ten values of $\sigma_2 \overline{\operatorname{mex}}(n)$ are given in the following table below.

n	0	1	2	3	4	5	6	7	8	9
$\sigma_2 \overline{\operatorname{mex}}(n)$	1	2	5	10	18	32	55	90	144	226

Table 4: First ten values of $\sigma_2 \overline{\operatorname{mex}}(n)$.

We observe that these are also the first ten values of the sequence A001936 in OEIS which is the sequence of coefficients of q^n in the expansion of

$$\frac{(-q;q)_\infty(q^4;q^4)_\infty}{(q;q)_\infty(q^2;q^4)_\infty}.$$

In Section 3, we derive the generating function and asymptotic formula for $\sigma_r\overline{\text{mex}}(n)$.

Theorem 6. *For all positive integers r, we have*

$$\sum_{n=0}^{\infty}\sigma_r\overline{\text{mex}}(n)q^n=\frac{(-q;q)_\infty(q^{2r};q^{2r})_\infty}{(q;q)_\infty(q^r;q^{2r})_\infty}.$$

Theorem 7. *For all positive integers r, we have*

$$\sigma_r\overline{\text{mex}}(n)\sim\frac{e^{\pi\sqrt{n}}}{8\sqrt{r}n^{3/4}}$$

as $n\to\infty$.

Chakraborty and Ray [6] studied the arithmetic density of $\sigma_2\operatorname{mex}(n)$ and $\sigma_3\operatorname{mex}(n)$ modulo 2^k for a positive integer k and proved that for almost every nonnegative integer n lying in an arithmetic progression, the integer $\sigma_r\operatorname{mex}(n)$ is a multiple of 2^k where $r\in\{2,3\}$.

We also study the arithmetic density of $\sigma_r\overline{\text{mex}}(n)$ when $r=2^m\cdot 3^n$, where $m,n\in\mathbb{Z}_{\geq 0}$. In Section 4, we prove the following result.

Theorem 8. *Let $r=2^m\cdot 3^n$ where $m,n\in\mathbb{Z}_{\geq 0}$ and $k\geq 1$ be a positive integer. Then*

$$\lim_{X\to+\infty}\frac{\#\{n\leq X:\sigma_r\overline{\text{mex}}(n)\equiv 0 \pmod{2^k}\}}{X}=1.$$

Equivalently, for almost every nonnegative integer n lying in an arithmetic progression, the integer $\sigma_r\overline{\text{mex}}(n)$ is a multiple of 2^k when $r=2^m\cdot 3^n$, $m,n\in\mathbb{Z}_{\geq 0}$.

2 Minimal excludant of an overpartition

2.1 Generating function of $\sigma\overline{\text{mex}}(n)$

Proof of Theorem 2. Let $p^{\overline{\text{mex}}}(m,n)$ be the number of overpartitions π of n with $\overline{\text{mex}}(\pi)=m$. Then we have the following double series $M(z,q)$ in which the coefficient of z^mq^n is

$p^{\overline{\text{mex}}}(m,n)$:

$$M(z,q) := \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} p^{\overline{\text{mex}}}(m,n) z^m q^n = \sum_{m=1}^{\infty} z^m q^1 \cdot q^2 \cdot \cdots \cdot q^{m-1} \cdot \frac{\prod_{n=1}^{\infty}(1+q^n)}{\prod_{\substack{n=1\\n\neq m}}^{\infty}(1-q^n)}$$
$$= \sum_{m=1}^{\infty} z^m q^{\binom{m}{2}} \cdot \frac{(-q;q)_\infty}{(q;q)_\infty} \cdot (1-q^m)$$
$$= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{m=1}^{\infty} z^m q^{\binom{m}{2}} \cdot (1-q^m).$$

Thus we have

$$\begin{aligned}
\sum_{n\geq 0} \sigma\overline{\text{mex}}(n) q^n &= \frac{\partial}{\partial z}\Big|_{z=1} M(z,q) \\
&= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{m=0}^{\infty} m q^{\binom{m}{2}}(1-q^m) \\
&= \frac{(-q;q)_\infty}{(q;q)_\infty} \left(\sum_{m=1}^{\infty} m q^{\binom{m}{2}} - \sum_{m=1}^{\infty} m q^{\binom{m}{2}} \cdot q^m \right) \\
&= \frac{(-q;q)_\infty}{(q;q)_\infty} \left(\sum_{m=1}^{\infty} m q^{\binom{m}{2}} - \sum_{m=1}^{\infty} (m-1) q^{\binom{m}{2}} \right) \\
&= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{m=0}^{\infty} q^{\binom{m+1}{2}} \\
&= \frac{(-q;q)_\infty}{(q;q)_\infty} \cdot \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \\
&= (-q;q)_\infty \cdot \frac{(q^2;q^2)_\infty}{(q;q)_\infty (q;q^2)_\infty}, \quad \text{(by [1], Eq. (2.2.13))} \\
&= (-q;q)_\infty \cdot (-q;q)_\infty^2 \\
&= (-q;q)_\infty^3 \\
&= \sum_{n\geq 0} D_3(n) q^n.
\end{aligned}$$

□

As an illustration, observe that the thirteen 3-colored partitions of 3 are: $3_1, 3_2, 3_3, 2_1 + 1_1, 2_1 + 1_2, 2_1 + 1_3, 2_2 + 1_1, 2_2 + 1_2, 2_2 + 1_3, 2_3 + 1_1, 2_3 + 1_2, 2_3 + 1_3, 1_1 + 1_2 + 1_3,$ Indeed, $D_3(3) = 13 = \sigma\overline{\text{mex}}(3)$.

2.2 Asymptotic formula for $\sigma\overline{\mathrm{mex}}(n)$

To derive an asymptotic formula for $\sigma\overline{\mathrm{mex}}(n)$, we use the following asymptotic result by Ingham [5] about the coefficients of a power series.

Proposition 9. *Let $A(q) = \sum_{n=0}^{\infty} a(n)q^n$ be a power series with radius of convergence equal to 1. Assume that $(a(n))$ is a weakly increasing sequence of nonnegative real numbers. If there are constants $\alpha, \beta \in \mathbb{R}$, and $C > 0$ such that*

$$A(e^{-t}) \sim \alpha t^{\beta} e^{\frac{C}{t}}, \text{ as } t \to 0^+, \qquad A(e^{-z}) \ll |z|^{\beta} e^{\frac{C}{|z|}} \text{ as } z \to 0,$$

with $z = x + iy$ $(x > 0, y \in \mathbb{R})$ in each region of the form $|y| \leq \Delta x$ for $\Delta > 0$. Then

$$a(n) \sim \frac{\alpha}{2\sqrt{\pi}} \frac{C^{\frac{2\beta+1}{4}}}{n^{\frac{2\beta+3}{4}}} e^{2\sqrt{Cn}}, \text{ as } n \to \infty.$$

Proof of Theorem 3. Note that $\sigma\overline{\mathrm{mex}}(n) = D_3(n)$ and $(D_3(n))$ is an increasing sequence of nonnegative real numbers, thus $\sigma\overline{\mathrm{mex}}(n)$ is also an increasing sequence of nonnegative real numbers. Let $A(q) = (-q;q)_{\infty}^3$, where $a(n) = \sigma\overline{\mathrm{mex}}(n)$ as in Proposition 9. From [4], we have

$$\frac{1}{(e^{-t};e^{-t})_{\infty}} \sim \sqrt{\frac{t}{2\pi}} e^{\frac{\pi^2}{6t}} \text{ as } t \to 0^+. \tag{1}$$

Moreover, we use the following identity:

$$(-q;q)_{\infty} = \frac{1}{(q;q^2)_{\infty}} = \frac{(q^2;q^2)_{\infty}}{(q;q)_{\infty}}. \tag{2}$$

By (1) and (2), as $t \to 0^+$, we obtain

$$(-e^{-t};e^{-t})_{\infty} = \frac{(e^{-2t};e^{-2t})_{\infty}}{(e^{-t};e^{-t})_{\infty}} \sim \frac{\sqrt{\frac{t}{2\pi}} e^{\frac{\pi^2}{6t}}}{\sqrt{\frac{2t}{2\pi}} e^{\frac{\pi^2}{12t}}} = \frac{1}{\sqrt{2}} e^{\frac{\pi^2}{12t}}.$$

Hence, as $t \to 0^+$, we get

$$A(e^{-t}) = (-e^{-t};e^{-t})_{\infty}^3 \sim \left(\frac{1}{\sqrt{2}} e^{\frac{\pi^2}{12t}}\right)^3 = \frac{1}{2\sqrt{2}} e^{\frac{\pi^2}{4t}}. \tag{3}$$

Moreover, from [5], if $z = x + iy$ $(x > 0)$ with $|y| \leq \Delta x$, then

$$\frac{1}{(e^{-z};e^{-z})_{\infty}} \sim \sqrt{\frac{z}{2\pi}} e^{\frac{\pi^2}{6z}}, \quad \text{as } z \to 0. \tag{4}$$

Similarly, we have

$$A(e^{-z}) = (-e^{-z}; e^{-z})_\infty^3 = \frac{(e^{-2z}; e^{-2z})_\infty^3}{(e^{-z}; e^{-z})_\infty^3} \sim \frac{1}{2\sqrt{2}} e^{\frac{\pi^2}{4z}}$$

as $z \to 0$, in these regions. From Remark 2 in [5], this implies that

$$A(e^{-z}) \ll |z|^0 e^{\frac{\pi^2}{4|z|}}, \quad \text{as } z \to 0$$

in each region of the form $|y| \leq \Delta x$ for $\Delta > 0$.

Take $\alpha = \frac{1}{2\sqrt{2}}, \beta = 0$ and $C = \frac{\pi^2}{4}$. By Proposition 9, we obtain

$$\sigma\overline{\text{mex}}(n) \sim \frac{\frac{1}{2\sqrt{2}}}{2\sqrt{\pi}} \frac{\left(\frac{\pi^2}{4}\right)^{1/4}}{n^{3/4}} e^{2\sqrt{\frac{\pi^2}{4}n}} = \frac{e^{\pi\sqrt{n}}}{8n^{3/4}}$$

as $n \to \infty$. □

2.3 Parity of $\sigma\overline{\text{mex}}(n)$

Proof of Theorem 4. We have

$$\begin{aligned}
\sum_{n \geq 0} \sigma\overline{\text{mex}}(n) q^n &= (-q; q)_\infty^3 \\
&= \prod_{n=1}^{\infty} (1 + q^n)^3 \\
&\equiv \prod_{n=1}^{\infty} (1 - q^n)^3 \pmod 2 \\
&= (q; q)_\infty^3 \\
&= \sum_{j=0}^{\infty} (-1)^j (2j+1) q^{\frac{j(j+1)}{2}}, \text{ by Jacobi's identity [10].}
\end{aligned}$$

□

Comparing coefficients, we have that $\sigma\overline{\text{mex}}(n) \equiv 0 \pmod 2$ for $n \neq \frac{j(j+1)}{2}$ for every $j \in \mathbb{N}$ and $\sigma\overline{\text{mex}}(n) \equiv 1 \pmod 2$ otherwise. This shows that $\sigma\overline{\text{mex}}(n)$ is almost always even and is odd exactly when n is a triangular number.

3 Least r-gaps

3.1 Generating function of $\sigma_r\overline{\text{mex}}(n)$

Ballantine and Merca [3] proved that for $n \geq 0$ and $r \geq 1$,

$$\sum_{k=0}^{\infty} p(n - rT_k) = \sigma_r \operatorname{mex}(n).$$

We extend this result to overpartitions and present an analogous proof for the following proposition.

Proposition 10. *For $n \geq 0$ and $r \geq 1$,*

$$\sum_{k=0}^{\infty} \overline{p}(n - rT_k) = \sigma_r\overline{\text{mex}}(n).$$

Proof. Fix $r \geq 1$. For each $k \geq 0$, consider the staircase partition

$$\delta_r(k) = (1^r, 2^r, \ldots, (k-1)^r, k^r)$$

where each part from 1 to k is repeated r times. We create an injection from the set of overpartitions of $n - rT_k$ into the set of overpartitions of n with the following mapping:

$$\phi_{r,n,k} : \overline{\mathcal{P}}(n - rT_k) \hookrightarrow \overline{\mathcal{P}}(n)$$

where for an overpartition π of $n - rT_k$, $\phi_{r,n,k}(\pi)$ is the overpartition obtained by inserting the non-overlined staircase partition $\delta_r(k)$.

For example, if $\pi = 4 + \overline{3} + 2 + \overline{1} + 1 = 11$, we have $\phi_{2,23,3} = 4 + \overline{3} + 2 + \overline{1} + 1 + 3 + 3 + 2 + 2 + 1 + 1 = 23$.

Let $\mathcal{A}_{r,n,k}$ be the image of the overpartitions of $n - rT_k$ under $\phi_{r,n,k}$. We have $\overline{p}(n - rT_k) = |\mathcal{A}_{r,n,k}|$ and $\mathcal{A}_{r,n,k}$ consists of the partitions of n satisfying $\overline{\text{mex}}_r(\pi) > k$.

Now, suppose π is an overpartition of n with $\overline{\text{mex}}_r(\pi) = k$. Then $\pi \in \mathcal{A}_{r,n,i}$, for $i = 0, 1, \ldots, k-1$ and $\pi \notin A_{r,n,j}$ with $j \geq k$. Thus each overpartition of n with $\overline{\text{mex}}_r(\pi) = k$ is counted by the summation $\sum_{k=0}^{\infty} \overline{p}(n - rT_k)$ exactly k times. □

Proof of Theorem 6. We have

$$\sum_{n=0}^{\infty} \sigma_r\overline{\text{mex}}(n) q^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \overline{p}(n - rT_k) \right) q^n, \text{ by Proposition 10}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \overline{p}(n - rT_k) q^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \overline{p}(n) q^{n+rT_k} \\
&= \left(\sum_{n=0}^{\infty} \overline{p}(n) q^n \right) \left(\sum_{k=0}^{\infty} q^{rT_k} \right).
\end{aligned}
$$

Note that the generating function for $\overline{p}(n)$ is

$$
\sum_{n=0}^{\infty} \overline{p}(n) q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}.
$$

Moreover, from [3], we have

$$
\sum_{k=0}^{\infty} q^{rT_k} = \frac{(q^{2r};q^{2r})_\infty}{(q^r;q^{2r})_\infty}.
$$

Thus we have

$$
\sum_{n=0}^{\infty} \sigma_r \overline{\text{mex}}(n) q^n = \left(\sum_{n=0}^{\infty} \overline{p}(n) q^n \right) \left(\sum_{k=0}^{\infty} q^{rT_k} \right) = \frac{(-q;q)_\infty (q^{2r};q^{2r})_\infty}{(q;q)_\infty (q^r;q^{2r})_\infty}.
$$

□

3.2 Asymptotic formula for $\sigma_r \overline{\text{mex}}(n)$

Here, we generalize our asymptotic result in Theorem 3 for the least r-gaps.

Proof of Theorem 7. Note that $\overline{p}(n) < \overline{p}(n+1)$ for $n \in \mathbb{N}$, since for every overpartition of n, say $n = a_1 + a_2 + \cdots + a_l$, we correspondingly have $n + 1 = a_1 + a_2 + \cdots + a_l + 1$ as an overpartition of $n + 1$. Since $\sigma_r \overline{\text{mex}}(n)$ is the sum of the least r-gaps taken over all overpartitions of n, then we can conclude that $\sigma_r \overline{\text{mex}}(n)$ is a weakly increasing sequence.

Let $A(q) = \dfrac{(-q;q)_\infty (q^{2r};q^{2r})_\infty}{(q;q)_\infty (q^r;q^{2r})_\infty}$, where $a(n) = \sigma_r \overline{\text{mex}}(n)$ as in Proposition 9. First,

$$
A(q) = \frac{(-q;q)_\infty (q^{2r};q^{2r})_\infty}{(q;q)_\infty (q^r;q^{2r})_\infty} = \frac{(-q;q)_\infty (-q^r;q^r)_\infty^2 (q^r;q^r)_\infty}{(q;q)_\infty}.
$$

Hence, using (3), as $t \to 0^+$, we get

$$\begin{aligned}
A(e^{-t}) &= \frac{(-e^{-t};e^{-t})_\infty(-e^{-rt};e^{-rt})_\infty^2(e^{-rt};e^{-rt})_\infty}{(e^{-t};e^{-t})_\infty} \\
&\sim \frac{\frac{1}{\sqrt{2}}e^{\frac{\pi^2}{12t}}\left(\frac{1}{\sqrt{2}}e^{\frac{\pi^2}{12rt}}\right)^2\sqrt{\frac{t}{2\pi}}e^{\frac{\pi^2}{6t}}}{\sqrt{\frac{rt}{2\pi}}e^{\frac{\pi^2}{6rt}}} \\
&= \frac{1}{2\sqrt{2r}}e^{\frac{\pi^2}{4t}}.
\end{aligned}$$

Moreover, if $z = x + iy$ $(x > 0)$ with $|y| \le \Delta x$, then from (4), we have

$$A(e^{-z}) = \frac{(-e^{-z};e^{-z})_\infty(-e^{-rz};e^{-rz})_\infty^2(e^{-rz};e^{-rz})_\infty}{(e^{-z};e^{-z})_\infty} \sim \frac{1}{2\sqrt{2r}}e^{\frac{\pi^2}{4z}}, \quad \text{as } z \to 0.$$

From Remark 2 in [5], this implies that $A(e^{-z}) \ll |z|^0 e^{\frac{\pi^2}{4|z|}}$ as $z \to 0$ in each region of the form $|y| \le \Delta x$ for $\Delta > 0$.

Take $\alpha = \dfrac{1}{2\sqrt{2r}}$, $\beta = 0$ and $C = \frac{\pi^2}{4}$, by Proposition 9, we obtain

$$\sigma_r\overline{\text{mex}}(n) \sim \frac{\frac{1}{2\sqrt{2r}}}{2\sqrt{\pi}}\frac{\left(\frac{\pi^2}{4}\right)^{1/4}}{n^{3/4}}e^{2\sqrt{\frac{\pi^2}{4}n}} = \frac{e^{\pi\sqrt{n}}}{8\sqrt{r}n^{3/4}}$$

as $n \to \infty$. □

4 Distribution of $\sigma_r\overline{\text{mex}}(n)$

4.1 Preliminaries

We first discuss some preliminaries about modular forms. We define the upper-half complex plane

$$\mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$$

and the modular group

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, ad - bc = 1; a, b, c, d \in \mathbb{Z} \right\}.$$

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, the modular group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ by the following linear fractional transformation:

$$Az = \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}.$$

Moreover, if $N \in \mathbb{Z}^+$, we define the following *congruence subgroups* of $SL_2(\mathbb{Z})$ of level N:

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N} \right\}$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

Note that the following inclusions are true:

$$\Gamma(N) \subseteq \Gamma_1(N) \subseteq \Gamma_0(N) \subseteq \mathrm{SL}_2(\mathbb{Z}).$$

Modular forms are complex functions on $\mathbb{H}$ that transforms nicely under these congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. For this paper, we are interested on modular forms transforming nicely with respect to $\Gamma_0(N)$ having a Nebentypus character χ defined as follows.

Definition 11. Let χ be a Dirichlet character modulo N (a positive integer). Then a modular form $f \in M_k(\Gamma_1(N))$ has *Nebentypus character* χ if

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

for all $z \in \mathbb{H}$ and all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. The space of all such modular forms is denoted $M_k(N, \chi)$.

In particular, we look at modular forms involving the Dedekind eta function which is defined as follows.

Definition 12. The *Dedekind eta function* is the function $\eta(z)$ where $z \in \mathbb{H}$:

$$\eta(z) = e^{\frac{\pi i z}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}).$$

Defining $q := e^{2\pi i z}$, we have:

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).$$

Definition 13. A function $f(z)$ is called an *eta-product* if it is expressible as a finite product of the form

$$f(z) = \prod_{\delta | N} \eta^{r_\delta}(\delta z)$$

where N and each r_δ is an integer.

We use the next two theorems to prove that an eta-product is a holomorphic modular form.

Theorem 14 (Gordon, Hughes, Newman)**.** *If $f(z) = \prod_{\delta|N} \eta^{r_\delta}(\delta z)$ is an eta-product for which*

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24} \tag{5}$$

and

$$\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24} \tag{6}$$

then $f(z)$ satisfies

$$f(Az) = \chi(d)(cz+d)^k f(z)$$

for all $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ where $k = \sum_{\delta|N} r_\delta$. Here the character χ is defined by $\chi(d) = \left(\frac{(-1)^k s}{d}\right)$ and $s = \prod_{\delta|N} \delta^{r_\delta}$.

Theorem 15 (Ligozat)**.** *Let c, d and N be positive integers with $d \mid N$ and $\gcd(c,d) = 1$. With the notation as above, if the eta-product $f(z)$ satisfies (5) and (6), then the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is*

$$\frac{1}{24} \sum_{\delta|N} \frac{N \gcd(d,\delta)^2 r_\delta}{\gcd\left(d, \frac{N}{d}\right) d\delta}.$$

4.2 Proof of main result

Before we prove Theorem 8, we prove two propositions first.

Proposition 16. *Let k be a positive integer. Then*

$$f_{r,k}(z) := \frac{\eta(48z)\eta(24rz)^{2^k-1}}{\eta(24z)^2\eta(48rz)^{2^{k-1}-2}} \equiv \sum_{n=0}^{\infty} \sigma_r \overline{\text{mex}}(n) q^{24n+3r} \pmod{2^k}.$$

Proof. Consider

$$g(z) = \frac{\eta(24rz)^2}{\eta(48rz)} = \frac{(q^{24r};q^{24r})_\infty^2}{(q^{48r};q^{48r})_\infty}.$$

By the binomial theorem, $(q^r;q^r)_\infty^{2^k} \equiv (q^{2r};q^{2r})_\infty^{2^{k-1}} \pmod{2^k}$. Thus we have

$$(q^{24r};q^{24r})_\infty^{2^k} \equiv (q^{48r};q^{48r})_\infty^{2^{k-1}} \pmod{2^k},$$

and so

$$g^{2^{k-1}}(z) = \frac{(q^{24r};q^{24r})_\infty^{2^k}}{(q^{48r};q^{48r})_\infty^{2^{k-1}}} \equiv 1 \pmod{2^k}.$$

Now, consider

$$\begin{aligned}\frac{\eta(48z)\eta(48rz)^2}{\eta(24z)^2\eta(24rz)} \cdot g^{2^{k-1}}(z) &= \frac{\eta(48z)\eta(48rz)^2}{\eta(24z)^2\eta(24rz)} \cdot \frac{\eta(24rz)^{2^k}}{\eta(48rz)^{2^{k-1}}} \\ &= \frac{\eta(48z)\eta(24rz)^{2^k-1}}{\eta(24z)^2\eta(48rz)^{2^{k-1}-2}} \\ &= f_{r,k}(z).\end{aligned}$$

Observe that

$$\begin{aligned}f_{r,k}(z) &= \frac{\eta(48z)\eta(48rz)^2}{\eta(24z)^2\eta(24rz)} \cdot g^{2^k-1}(z) \\ &\equiv \frac{\eta(48z)\eta(48rz)^2}{\eta(24z)^2\eta(24rz)} \pmod{2^k} \\ &= q^{3r}\frac{(q^{48};q^{48})_\infty(q^{48r};q^{48r})_\infty^2}{(q^{24};q^{24})_\infty^2(q^{24r};q^{24r})_\infty}.\end{aligned}$$

Note that

$$\begin{aligned}\sum_{n=0}^{\infty}\sigma_r\overline{\mathrm{mex}}(n)q^n &= \frac{(-q;q)_\infty(q^{2r};q^{2r})_\infty}{(q;q)_\infty(q^r;q^{2r})_\infty} \\ &= \frac{(-q;q)_\infty(q^{2r};q^{2r})_\infty^2}{(q;q)_\infty(q^r;q^r)_\infty} \\ &= \frac{(q^2;q^2)_\infty(q^{2r};q^{2r})_\infty^2}{(q;q)_\infty^2(q^r;q^r)_\infty}.\end{aligned}$$

Thus we have

$$f_{r,k}(z) \equiv q^{3r}\sum_{n=0}^{\infty}\sigma_r\overline{\mathrm{mex}}(n)q^{24n} \pmod{2^k} = \sum_{n=0}^{\infty}\sigma_r\overline{\mathrm{mex}}(n)q^{24n+3r}.$$

□

Proposition 17. *Let $r = 2^m \cdot 3^n$ where $m, n \in \mathbb{Z}_{\geq 0}$ and $k \geq m+2n+1$ be an integer greater than 3. Then $f_{r,k}(z) \in M_{2^{k-2}}(\Gamma_0(N), \chi)$, where*

$$N = \begin{cases} 2^7 \cdot 3^{n+1}, & \text{if } m = 0,1,2; \\ 2^{m+4} \cdot 3^{n+1}, & \text{if } m \geq 3. \end{cases}$$

Proof. Let $r = 2^m \cdot 3^n$ where $m, n \in \mathbb{Z}_{\geq 0}$. First, the weight of $f_{r,k}(z)$ is

$$\ell = \frac{1}{2}\sum_{\delta|N} r_\delta = \frac{1}{2}\left[1 + (2^k - 1) - 2 - (2^{k-1} - 2)\right] = 2^{k-1} - 2^{k-2} = 2^{k-2}.$$

Second, since $f_{r,k}(z) = \dfrac{\eta(48z)\eta(24rz)^{2^k-1}}{\eta(24z)^2\eta(48rz)^{2^{k-1}-2}}$, then $\delta_1 = 48, \delta_2 = 24r, \delta_3 = 24$ and $\delta_4 = 48r$ with $r_{48} = 1, r_{24r} = 2^k - 1, r_{24} = -2$, and $r_{48r} = 2 - 2^{k-1}$. Clearly, $f_{r,k}(z)$ satisfies equation (5) since

$$\sum_{\delta|N} \delta r_\delta = 48 \cdot 1 + 24r \cdot (2^k - 1) + 24 \cdot (-2) + 48r \cdot (2 - 2^{k-1}) \equiv 0 \pmod{24}.$$

Moreover, to satisfy equation (6), we can let $N = 48ru$, where u is the smallest positive integer satisfying

$$\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}.$$

Then we have

$$\begin{aligned}
\sum_{\delta|N} \frac{N}{\delta} r_\delta &= \frac{48ru}{48} + \frac{48ru}{24r}(2^k - 1) - \frac{48ru}{24}(2) - \frac{48ru}{48r}(2^{k-1} - 2) \\
&= ru + 2u(2^k - 1) - 4ru - u(2^{k-1} - 2) \\
&= u(2^{k+1} - 2^{k-1} - 3r) \\
&= u(3 \cdot 2^{k-1} - 3r) \equiv 0 \pmod{24}.
\end{aligned}$$

We have the following:

- If $m = 0$, then $u = 8$, and so $N = 48 \cdot (2^0 \cdot 3^n) \cdot 8 = 2^7 \cdot 3^{n+1}$.
- If $m = 1$, then $u = 4$, and so $N = 48 \cdot (2^1 \cdot 3^n) \cdot 4 = 2^7 \cdot 3^{n+1}$.
- If $m = 2$, then $u = 2$, and so $N = 48 \cdot (2^1 \cdot 3^n) \cdot 4 = 2^7 \cdot 3^{n+1}$.
- If $m \geq 3$, then $u = 1$, and so $N = 48 \cdot (2^m \cdot 3^n) \cdot 1 = 2^{m+4} \cdot 3^{n+1}$.

To prove that $f_{r,k}(z) \in M_{2k-2}(\Gamma_0(N), \chi)$, it suffices to show that $f_{r,k}(z)$ is holomorphic at all cusps of $\Gamma_0(N)$. From Theorem 15, the order of vanishing of $f_{r,k}(z)$ at the cusp $\frac{c}{d}$ where $d|N$ and $\gcd(c, d) = 1$, is

$$\frac{N}{24}\sum_{\delta|N} \frac{\gcd(d, \delta)^2 r_\delta}{\gcd\left(d, \frac{N}{d}\right) d\delta}.$$

Hence $f_{r,k}(z) = \dfrac{\eta(48z)\eta(24rz)^{2^k-1}}{\eta(24z)^2\eta(48rz)^{2^{k-1}-2}}$ is holomorphic at the cusp $\frac{c}{d}$ if and only if

$$\frac{N}{24}\sum_{\delta|N}\frac{\gcd(d,\delta)^2 r_\delta}{\gcd\left(d,\frac{N}{d}\right)d\delta} \geq 0 \iff \sum_{\delta|N}\frac{\gcd(d,\delta)^2 r_\delta}{\delta} \geq 0.$$

That is,

$$\frac{\gcd(d,48)^2}{48} - 2\frac{\gcd(d,24)^2}{24} + (2^k-1)\frac{\gcd(d,24r)^2}{24r} - (2^{k-1}-2)\frac{\gcd(d,48r)^2}{48r} \geq 0.$$

Equivalently,

$$r\gcd(d,48)^2 - 4r\gcd(d,24)^2 + (2^{k+1}-2)\gcd(d,24r)^2 - (2^{k-1}-2)\gcd(d,48r)^2 \geq 0. \quad (7)$$

Now, if $N = 2^7\cdot 3^{n+1}$, then $d = 2^t\cdot 3^s, 0 \leq t \leq 7, 0 \leq s \leq n+1$. Similarly, if $N = 2^{m+4}\cdot 3^{n+1}$, then $d = 2^t \cdot 3^s, 0 \leq t \leq m+4, 0 \leq s \leq n+1$.

Let $(\star)$ be the left-hand side of inequality (7). We now prove that $(\star) \geq 0$ for $k \geq m+2n+1$. We divide our proof into 6 cases.

Case 1: $d = 1$.
We have $\gcd(d,48) = 1$, $\gcd(d,24) = 1$, $\gcd(d,24r) = 1$, and $\gcd(d,48r) = 1$. Then

$$\begin{aligned}(\star) &= (2^m\cdot 3^n) - 4(2^m\cdot 3^n) + (2^{k+1}-2) - (2^{k-1}-2)\\ &= 2^{k+1} - 2^{k-1} - 3\cdot(2^m\cdot 3^n)\\ &= 3\cdot 2^{k-1} - 2^m\cdot 3^{n+1}.\end{aligned}$$

If we let $k \geq m+2n+1$, then

$$\begin{aligned}3\cdot 2^{k-1} &\geq 3\cdot 2^{m+2n}\\ &= 3\cdot 2^m\cdot 2^{2n}\\ &\geq 3\cdot 2^m\cdot 3^n\\ &= 2^m\cdot 3^{n+1},\end{aligned}$$

proving that $(\star) \geq 0$ for $k \geq m+2n+1$.

Case 2: $d = 3^s, 1 \leq s \leq n+1$.
We have $\gcd(d,48) = 3$, $\gcd(d,24) = 3$, $\gcd(d,24r) = \gcd(d,2^{m+3}\cdot 3^{n+1}) = 3^s$, and $\gcd(d,48r) = \gcd(d,2^{m+4}\cdot 3^{n+1}) = 3^s$. Then

$$\begin{aligned}(\star) &= (2^m\cdot 3^n)\cdot 9 - 4(2^m\cdot 3^n)\cdot 9 + (2^{k+1}-2)\cdot 3^{2s} - (2^{k-1}-2)\cdot 3^{2s}\\ &= 3^{2s}\cdot 3\cdot 2^{k-1} - 3\cdot 9\cdot(2^m\cdot 3^n)\\ &= 2^{k-1}\cdot 3^{2s+1} - 2^m\cdot 3^{n+3}.\end{aligned}$$

If we let $k \geq m + 2n - 4s + 5$, then

$$\begin{aligned} 2^{k-1} \cdot 3^{2s+1} &\geq 2^{m+2n-4s+4} \cdot 3^{2s+1} \\ &= 2^m \cdot 2^{2n-4s+4} \cdot 3^{2s+1} \\ &\geq 2^m \cdot 3^{n-2s+2} \cdot 3^{2s+1} \\ &= 2^{m+4} \cdot 3^{n+3}, \end{aligned}$$

proving that $(\star) \geq 0$ for $k \geq m+2n-4s+5$. Moreover, since $m+2n-4s+5 \leq m+2n+1$, it follows that $(\star) \geq 0$ for $k \geq m + 2n + 1$.

Case 3: $d = 2^t, 0 < t \leq 3$.
We have $\gcd(d, 48) = 2^t$, $\gcd(d, 24) = 2^t$, $\gcd(d, 24r) = 2^t$, and $\gcd(d, 48r) = 2^t$. Then

$$\begin{aligned} (\star) &= (2^m \cdot 3^n) \cdot 2^{2t} - 4(2^m \cdot 3^n) \cdot 2^{2t} + (2^{k+1} - 2) \cdot 2^{2t} - (2^{k-1} - 2) \cdot 2^{2t} \\ &= 2^{2t} \cdot 3 \cdot 2^{k-1} - 3 \cdot 2^{2t} \cdot (2^m \cdot 3^n) \\ &= 2^{k+2t-1} \cdot 3 - 2^{m+2t} \cdot 3^{n+1}. \end{aligned}$$

If we let $k \geq m + 2n + 1$, then

$$\begin{aligned} 2^{k+2t-1} \cdot 3 &\geq 2^{m+2n+2t} \cdot 3 \\ &= 2^{m+2t} \cdot 2^{2n} \cdot 3 \\ &\geq 2^{m+2t} \cdot 3^n \cdot 3 \\ &= 2^{m+2t} \cdot 3^{n+1}, \end{aligned}$$

proving that $(\star) \geq 0$ for $k \geq m + 2n + 1$.

Case 4: $d = 2^t, 3 < t \leq m + 4$.
We have $\gcd(d, 48) = 2^4$, $\gcd(d, 24) = 2^3$, $\gcd(d, 24r) = \gcd(d, 2^{m+3} \cdot 3^{n+1}) = 2^t$, and $\gcd(d, 48r) = \gcd(d, 2^{m+4} \cdot 3^{n+1}) = 2^t$. Then

$$\begin{aligned} (\star) &= (2^m \cdot 3^n) \cdot 2^8 - 4(2^m \cdot 3^n) \cdot 2^6 + (2^{k+1} - 2) \cdot 2^{2t} - (2^{k-1} - 2) \cdot 2^{2t} \\ &= 2^{2t} \cdot 3 \cdot 2^{k-1} \\ &= 2^{k+2t-1} \cdot 3 \geq 0 \text{ for } k \geq 1. \end{aligned}$$

Case 5: $d = 2^t \cdot 3^s, 0 < t \leq 3, 1 \leq s \leq n + 1$.
We have $\gcd(d, 48) = 2^t \cdot 3$, $\gcd(d, 24) = 2^t \cdot 3$, $\gcd(d, 24r) = \gcd(d, 2^{m+3} \cdot 3^{n+1}) = 2^t \cdot 3^s$, and $\gcd(d, 48r) = \gcd(d, 2^{m+4} \cdot 3^{n+1}) = 2^t \cdot 3^s$. Then

$$\begin{aligned} (\star) &= (2^m \cdot 3^n) \cdot (2^{2t} \cdot 3^2) - 4(2^m \cdot 3^n) \cdot (2^{2t} \cdot 3^2) + (2^{k+1} - 2) \cdot (2^{2t} \cdot 3^{2s}) \\ &\quad - (2^{k-1} - 2) \cdot (2^{2t} \cdot 3^{2s}) \\ &= (2^{2t} \cdot 3^{2s}) \cdot 3 \cdot 2^{k-1} - 3 \cdot (2^{2t} \cdot 3^2) \cdot (2^m \cdot 3^n) \\ &= 2^{k+2t-1} \cdot 3^{2s+1} - 2^{m+2t} \cdot 3^{n+3}. \end{aligned}$$

If we let $k \geq m + 2n - 4s + 5$, then

$$\begin{aligned} 2^{k+2t-1} \cdot 3^{2s+1} &\geq 2^{m+2n+2t-4s+4} \cdot 3^{2s+1} \\ &= 2^{m+2t} \cdot 2^{2n-4s+4} \cdot 3^{2s+1} \\ &\geq 2^{m+2t} \cdot 3^{n-2s+2} \cdot 3^{2s+1} \\ &= 2^{m+2t} \cdot 3^{n+3}, \end{aligned}$$

proving that $(\star) \geq 0$ for $k \geq m+2n-4s+5$. Moreover, since $m+2n-4s+5 \leq m+2n+1$, it follows that $(\star) \geq 0$ for $k \geq m + 2n + 1$.

Case 6: $d = 2^t \cdot 3^s, 3 < t \leq m + 4, 1 \leq s \leq n + 1$.
We have $\gcd(d, 48) = 2^4 \cdot 3$, $\gcd(d, 24) = 2^3 \cdot 3$, $\gcd(d, 24r) = \gcd(d, 2^{m+3} \cdot 3^{n+1}) = 2^t \cdot 3^s$, and $\gcd(d, 48r) = \gcd(d, 2^{m+4} \cdot 3^{n+2}) = 2^t \cdot 3^s$. Then

$$\begin{aligned} (\star) &= (2^m \cdot 3^n) \cdot (2^8 \cdot 3^2) - 4(2^m \cdot 3^n) \cdot (2^6 \cdot 3^2) + (2^{k+1} - 2) \cdot (2^{2t} \cdot 3^{2s}) \\ &\quad - (2^{k-1} - 2) \cdot (2^{2t} \cdot 3^{2s}) \\ &= (2^{2t} \cdot 3^{2s}) \cdot 3 \cdot 2^{k-1} \\ &= 2^{k+2t-1} \cdot 3^{2s+1} \geq 0 \text{ for } k \geq 1. \end{aligned}$$

In all possible cases, we have that $(\star) \geq 0$ for $k \geq m+2n+1$ where k is a positive integer greater than 3. Hence by Theorem 15, $f_{r,k}(z)$ is a modular form of weight 2^{k-2}.

□

Lastly, we will use Serre's theorem [13] regarding the coefficients of the Fourier expansion of a holomorphic modular form to prove our final result.

Theorem 18 (Serre)**.** *Let k, m be positive integers. If $f(z) \in M_k(\Gamma_0(N), \chi)$ has Fourier expansion $f(z) = \sum_{n=0}^{\infty} c(n)q^n \in \mathbb{Z}[[q]]$, then there is a constant $\alpha > 0$ such that*

$$\#\{n \leq X : c(n) \not\equiv 0 \pmod{m}\} = \mathcal{O}\left(\frac{X}{\log^\alpha X}\right).$$

Proof of Theorem 8. Let $r = 2^m \cdot 3^n$ where $m, n \in \mathbb{Z}_{\geq 0}$ and $k \geq m + 2n + 1$, $k \geq 3$ be a positive integer. Since $f_{r,k}(z) \in M_{2^{k-2}}(\Gamma_0(N), \chi)$ and the Fourier coefficients of $f_{r,k}(z)$ are integers, then by Serre's theorem, we can find a constant $\alpha > 0$ such that

$$\#\{n \leq X : \sigma_r\overline{\text{mex}}(n) \not\equiv 0 \pmod{2^k}\} = \mathcal{O}\left(\frac{X}{\log^\alpha X}\right),$$

for $k \geq m + 2n + 1$. Then

$$\lim_{X \to +\infty} \frac{\#\{n \leq X : \sigma_r\overline{\text{mex}}(n) \equiv 0 \pmod{2^k}\}}{X} = 1.$$

Equivalently, for almost every nonnegative integer n lying in an arithmetic progression, $\sigma_r\overline{\text{mex}}(n)$ is a multiple of 2^k where $r = 2^m \cdot 3^n$ where $m, n \in \mathbb{Z}_{\geq 0}$ and $k \geq m+2n+1, k \geq 3$. Consequently, $\sigma_r\overline{\text{mex}}(n)$ is a multiple of 2^k, where $k \geq 1$. □

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References

[1] G. E. Andrews, *The Theory of Partitions,* Addison-Wesley Publishing Company, 1976.

[2] G. E. Andrews and D. Newman, Partitions and the minimal excludant, *Ann. Comb.* **23** (2019), 249–254.

[3] C. Ballantine and M. Merca, Bisected theta series, least r-gaps in partitions, and polygonal numbers, *Ramanujan J.* **52** (2020), 433–444.

[4] S. C. Bhoria, P. Eyyunni, P. S. Kaur, and B. Maji, Minimal excludant over partitions into distinct parts, /Serre: Command not found.

[5] K. Bringmann, C. Jennings-Shaffer, and K. Mahlburg, On a Tauberian theorem of Ingham and Euler-Maclaurin summation, *Ramanujan J.* **61** (2023), 55–86.

[6] K. Chakraborty and C. Ray, Distribution of generalized mex-related integer partitions, *Hardy-Ramanujan J.* **43** (2021), 122–128.

[7] A. Dhar, A. Mukhopadhyay, and R. Sarma, Generalization of the extended minimal excludant of Andrews and Newman, *J. Integer Sequences* **26** (2023).

[8] P. Grundy, Mathematics and games, *Eureka* **2** (1939), 6–9.

[9] A. E. Ingham, A Tauberian theorem for partitions, *Ann. of Math (2)* **42** (1941), 1075–1090.

[10] C. Jacobi, *Fundamenta nova theoriae function ellipticarum*, Mathematische Werke **1** (1829), 49–239.

[11] L. Kilford, *Modular Forms: A Classical and Computational Introduction*, Imperial College Press, 2008.

[12] R. Sprague, Über mathematische Kampfspiele, *Tohoku Mathematical J.* **41** (1935), 438–444.

[13] J.-P. Serre, Divisibilité des coefficients des formes modulaires de poids entier, *C.R. Acad. Sci. Paris* **279** (1974), 679–682

[14] J. Yang and L. Zhou, Identities on mex-related partitions, *Ramanujan J.* **63** (2023), 1–25.

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