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Analytical Study and Efficient Evaluation of the Josephus Function

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Abstract

In this paper, we present a novel approach to analyzing intrinsic properties of the Josephus function, J_k. The linear structure between extremal points of J_k is fully revealed, leading to the design of an efficient algorithm for evaluating J_k. We also derive algebraic expressions that describe how to recursively compute extremal points, including fixed points. The existence of consecutive extremal and also fixed points of J_k, for all $k \geq 2$, is proven as a consequence, which generalizes Knuth's result for $k = 2$. Moreover, we conduct an extensive comparative numerical experiment to illustrate the performance of the proposed algorithm for evaluating the Josephus function compared to established algorithms. The results show that the proposed scheme is highly effective in computing $J_k(n)$ for large inputs.

1 Introduction

The Josephus problem is a game of elimination that has been studied for nearly two millennia. The earliest known formulation of the problem appears in the historical text [12] written by historian Flavius Josephus. Josephus described a method of serial elimination by casting

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lots, which he and 40 of his soldiers applied while trapped in a cave by the Roman army and facing imminent capture and inevitable massacre.

The general formulation of the problem is as follows: a certain number of people n is arranged in a circle, and an execution method is established. An initial counting point and a direction of rotation are fixed, and after $k-1$ number of people is counted, the next person is executed and removed from the circle. This procedure is repeated with the remaining people until only one person remains, who is released. The objective of this problem is to determine the position of the survivor $J_k(n) \in \{1, 2, \ldots, n\}$ in the initial circle.

The Josephus problem received little attention throughout the first millennium, but new formulations were proposed in the second millennium, such as the rhyming games traced by Oring [17]. In the following centuries, more specific versions of the problem emerged, and it was integrated into a certain category of puzzles or riddles. One notable example is Bachet's presentation [2], where was established a formal procedure to solve the case $k = 3$ and $n = 41$ with total accuracy. In 1776, Euler heuristically found a recursive formula that connects the survivor's position for a given number of condemned with the position of the corresponding survivor when one more condemned is added; see [7]. This was a significant development in the mathematical treatment of the problem, as it required logical argumentation, appropriate symbolic notation, and exploration. However, the computational aspect of the problem has received more attention in recent years, as the advancement of technology allows for the efficient calculation of the survivor's position $J_k(n)$ for a large number of people n and a fixed k. In [23], Tait presents some discussion and figures in this direction, highlighting the importance of fast computational methods for solving the problem. Over time, the problem has attracted the attention of mathematicians from various areas, and the study of the problem was recently extended to the field of permutations. Mathematicians such as Herstein, Kaplansky, Ball, and others studied the problem in group theory and algorithmically; see, for instance, [6, 3, 11, 21, 1, 25]. The Josephus problem also has various modern applications such as computer algorithms, card games, data structures, and image encryption; see, for instance, [26, 4, 5, 14, 27]. When the number of jumps, $k-1$, is equal to one, Knuth derived a closed-form expression for $J_2(n)$ as $2n - 2^{\lfloor \log_2 n \rfloor + 1} + 1$ and developed an efficient algorithm for evaluating J_k for arbitrary values of k; see [13]. This algorithm avoids the recursive nature of the Josephus function and is presented in [8]. We invite the reader interested in delving deeper into the historical origin of this problem and applications to review references such as [16, 10, 20, 19, 22, 18, 15] and [9, Appendix].

This paper introduces a new approach for analyzing the intrinsic properties of the discrete Josephus function, J_k. We formulate algebraic expressions that describe and characterize all extremal points of J_k and include recurrence formulas to compute low and high extremal points. Additionally, we prove the existence of consecutive extremal and fixed points for all $k \geq 3$, generalizing Knuth's result for $k = 2$; see [8]. Moreover, by revealing the discrete and piecewise linear structure of J_k between extremal points, we design an efficient algorithm for evaluating $J_k(n)$ for large n and a given reduction constant k. A comparative computational study is conducted at the last section of the paper to evaluate the performance of the proposed algorithm against established methods, such as Euler/Woodhouse [7, 26], Knuth [8], and Uchiyama [24]. The results of the numerical comparison indicate that the proposed scheme is highly effective for computing $J_k(n)$ for large inputs k and n.

1.1 Notation and definitions

The mathematical formulation of the classical Josephus problem can be stated as follows: Let n be the number of people arranged in a circle which closes up its ranks as individuals are picked out. Starting anywhere (person 1st spot), go sequentially around clockwise, picking out each kth person (this number k is called the *reduction constant*) until but one person is left (this person is called the *survivor*). The position of the survivor is denoted by $J_k(n)$, which belongs to the natural numbers $\mathbb{N}$. This procedure is called the *elimination process*, and it naturally generates a discrete function $J_k : \mathbb{N} \to \mathbb{N}$ for each $k \geq 2$ that we call the Josephus function. Given $n \geq 1$ and $k \geq 2$ integers, we say that the Josephus problem has been solved once we have determined the value of J_k at n. For any two integers ℓ and m such that $\ell \leq m$, let $[[\ell, m]]$ denote the set $\{\ell, \ldots, m\}$. Notice further that $J_k(n) \in [[1, n]]$ for every n.

Definition 1 (Fixed and extremal points)**.** A *fixed point* of J_k is a value n_p such that $n_p = J_k(n_p)$. Additionally, an *extremal point* n_e is defined as a point that satisfies either $J_k(n_e) \in [[1, k-1]]$ or $J_k(n_e) \in [[n_e - k + 2, n_e]]$. If $J_k(n_e) \in [[1, k-1]]$, we refer to n_e as a *low extremal point*. On the other hand, if $J_k(n_e) \in [[n_e - k + 2, n_e]]$, we refer to n_e as a *high extremal point*.

Note that, a fixed point n_p is also a high extremal point because $J_k(n_p) = n_p$. However, there are high extremal points that are not fixed points; see Figure 1.

2 Properties of the Josephus function

In this section, we begin by recalling a recursive formula, which first appeared in Euler's paper [7, pp. 130–131] and establishes a way of determining $J_k(n+1)$ in terms of $J_k(n)$.

Theorem 2 (Euler's formula)**.** *Let $k \geq 2$ and $n \geq 1$, and define $p := J_k(n)$. Then*

$$J_k(n+1) = p + k - \ell(n+1),$$

if $p + k \in [[\ell(n+1) + 1, (\ell+1)(n+1)]]$ for some non-negative integer ℓ.

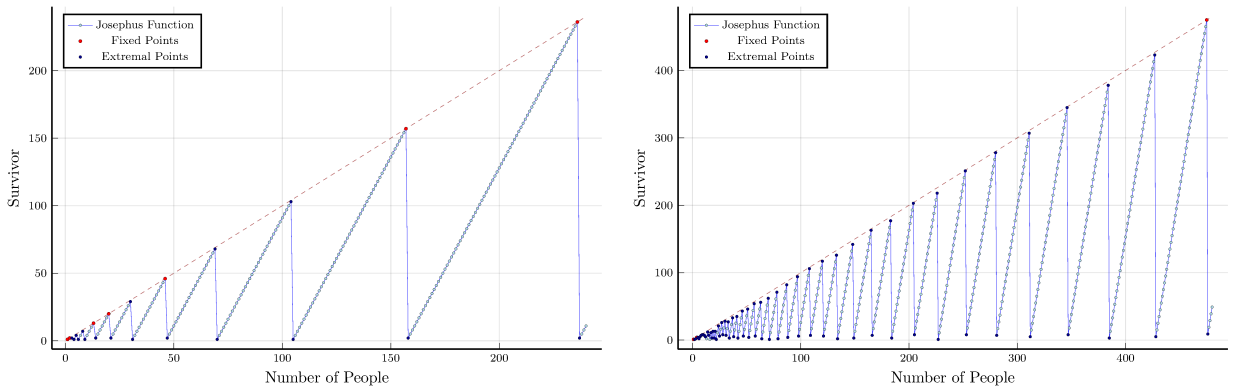
Remark 3. For some special values of the reduction constant k, we have the following observations:

(a) If $k \leq n+1$, then the formula given in Theorem 2 can be simplified as follows:

$$J_k(n+1) = \begin{cases} p + k, & \text{if } p + k \leq n + 1; \\ p + k - (n+1), & \text{if } p + k > n + 1. \end{cases} \tag{1}$$

(b) When $k = 3$, (1) holds for every n. In fact, we just need to check (1) for $n = 1$. In this case, we have $J_3(n+1) = J_3(2) = 2 = J_3(1) + 3 - (1+1) = J_3(n) + k - (n+1)$.

(c) When $k = 2$, no recursive formula is necessary hereafter since Knuth [8] has deduced the following explicit formula: $J_2(n) = 2n - 2^{\lfloor \log_2 n \rfloor + 1} + 1$, for all n.



(a) Graph of $J_3(n)$ for $n \leq 240$.

(b) Graph of $J_{10}(n)$ for $n \leq 480$.

Figure 1: Graphs of the Josephus functions J_3 and J_{10}.

(d) It is possible for two consecutive values of n to have the same image under J_k. For example, when $k = 8$, $J_8(3) = J_8(4) = 3$.

(e) If $n \leq k-1$, then $J_k(n) \leq n \leq k-1$, which implies that n is a low, and also high, extremal point. Moreover, if $n \geq 2k-3$, then there is an alternating sequence of pure low and high extremal points for the Josephus function J_k for every $k \geq 2$.

In Figure 1, we present the graphs of J_3 and J_{10} for various values of n. The extremal points of J_k, n_e, i.e., $J_k(n_e) \in [[1, k-1]] \cup [[n_e - k + 2, n_e]]$, which are studied in the next subsection, are indicated in blue. Moreover, the fixed points n_p of J_k, i.e., $n_p = J_k(n_p)$, that are also extremal points, are depicted in red.

Note that for n_e to be both a low and high extremal point,

$$J_k(n_e) \in [[1, k-1]] \cap [[n_e - k + 2, n_e]],$$

which always happens if $n_e \leq k-1$. In addition, if $n_e \geq 2k-3$ is a high extremal point then $n_e + 1$ is a low extremal point, which reveals a sequence of alternated extremal points. The two graphs depicted in Figure 1 exhibit the intrinsic piecewise linear structure of J_k between its low and high extremal points. Moreover, in general the fixed points for $k \geq 3$ exhibit a chaotic behavior in contrast with the case $k = 2$; see Figure 1 for the cases $k = 3$ and $k = 10$, and also Table 1 with $k = 2, 3, \ldots, 12, 15$. Note that the fixed points of J_2 can be described by the formula: $n_p^{(i)} = 2^i - 1$, where i is a natural number. Note further that $J_2(2^i) = 1$, which makes $n_e = 2^i$ be a low extremal point, for every i; see [13, pp. 184].

In Table 1, we illustrate the first fixed points of the Josephus functions J_k for some values of k.

Now based on Theorem 2, we present a formula for the Josephus function for the value of $n = n_p + 1$, i.e., just after a fixed point $n_p = J_k(n_p)$ is attained.

Proposition 4 (Values of J_k after reaching a fixed point)**.** *Let $k \geq 2$ and n_p be a fixed point.*

Fixed points \ Functions	J_2	J_3	J_4	J_5	J_6	J_7	J_8	J_9	J_{10}	J_{11}	J_{12}	J_{15}
1st	1	1	1	1	1	1	1	1	1	1	1	1
2nd	3	2	21	2	20	2	3	2	4	2	10	2
3rd	7	13	38	46	51	3	13	7	475	4	11	52
4th	15	20	51	542	794	12	15	8	8177	5	19	388
5th	31	46	122	2587	953	68	26	15	11217	49	55	1899
6th	63	157	163	3234	17629	274	1276	17	28954	54	111	30003
7th	127	236	689	6317	21155	593	1905	375	126567	188	290	136887

Table 1: First seventh fixed points of J_k for $k = 2, 3, \ldots, 12, 15$.

Then

$$J_k(n_p+1) = \begin{cases} k-1, & \text{if } k \le n_p+1; \\ n_p+1, & \text{if } k = s_\ell + 1 \text{ for some } \ell \ge 1; \\ k_1 - 1, & \text{if } k = s_\ell + k_1 \text{ for some } \ell \ge 1 \text{ and } 2 \le k_1 \le n_p; \\ n_p, & \text{if } k = s_{\ell+1} \text{ for some } \ell \ge 1, \end{cases}$$

where $s_\ell = \ell(n_p+1)$. *Moreover,* $J_k(n_p) = n_p$ *if and only if* $J_k(n_p+1) = k-1$ *whenever* $n_p \ge k-1$.

Proof. Let us consider the four cases for the values of k with respect to n_p as follows:

Case 1: Assume $k \le n_p + 1$. Note that $s_1 + 1 = n_p + 2 \le n_p + k \le 2n_p + 1 < s_2$. It follows from Theorem 2 that $J_k(n_p+1) = n_p + k - s_1 = n_p + k - (n_p+1) = k-1$.

Case 2: Assume $k = s_\ell + 1$. So, $n_p + k = s_{\ell+1}$. Then by Theorem 2,

$$J_k(n_p+1) = n_p + k - s_\ell = n_p + 1.$$

Case 3: Assume $k = s_\ell + k_1$ $(2 \le k_1 \le n_p)$. Note that

$$s_{\ell+1} + 1 \le n_p + k \le 2n_p + s_\ell = s_{\ell+1} + n_p - 1 < s_{\ell+2}.$$

Then by Theorem 2, $J_k(n_p+1) = n_p + k - s_{\ell+1} = k_1 - 1$.

Case 4: Assume $k = s_{\ell+1}$. Then $s_{\ell+1} + 1 \le n_p + k < s_{\ell+2}$. Theorem 2 implies

$$J_k(n_p+1) = n_p + k - s_{\ell+1} = n_p.$$

On the other hand, assume that $k \le n_p + 1$ and $J_k(n_p+1) = k-1$. By using Euler's formula backwards, we can determine the value of $p = J_k(n_p)$. In particular, p must satisfy the equation $k-1 = p+k-s_\ell$ for some non-negative integer ℓ. This is equivalent to $p = s_\ell - 1$. If $\ell = 0$, then $p = -1$, which is not possible. If $\ell \ge 2$, then $p \ge s_2 - 1 > s_1 - 1 = n_p$, which is also not possible. Hence $\ell = 1$ and $p = n_p$, which implies that n_p is a fixed point of J_k. □

Remark 5. If n_p is a fixed point, then the following interesting facts can be derived from Proposition 4:

(a) If $n_p \ge k-1$, then $J_k(n_p+1) = k-1$. In particular, $J_3(n_p+1) = 2$ for any n_p.

(b) It is possible for two consecutive fixed points to exist for a given value of k. For example, if $k = 9$, then the values for $n = 7$ and $n = 8$ are both fixed points of J_9.

(c) Note that, if a fixed point $n_p \leq k - 1$, then n_p is also a low extremal point.

We now provide a formal presentation of a partially generalized version of Euler's formula, which was previously stated without proof in [20, pp. 47].

Lemma 6 (Generalization of Euler's formula)**.** *Let $n \geq 1$ and $k \geq 2$ be positive integers and define $p := J_k(n)$. Then*

(a) *If $p + km \leq n + m$, for any $m \geq 0$, then $J_k(n + m) = p + km$.*

(b) *If $m \geq 0$ is the smallest value for which $p + km > n + m$, then*

$$J_k(n + m) = p + km - (n + m).$$

Proof.

(a) Let m be any natural number. First of all, $p+km \leq n+m$ if and only if $m \leq (n-p)/(k-1)$. Thus $p + k\bar{m} \leq n + \bar{m}$ for every $0 \leq \bar{m} \leq m$. Let us define the finite sequences: $r_{\bar{m}} := n + \bar{m}$, and $t_{\bar{m}} := J_k(r_{\bar{m}})$ for every integer $\bar{m} = 0, \ldots, m$. Since, $p + k \leq n + 1$, by Theorem 2, we get $t_1 = p + k$. Now, since $t_1 + k = p + k(2) \leq n + 2 = r_1 + 1$, again by Theorem 2, we get $t_2 = t_1 + k$. By applying this procedure m times, we get that $t_{\bar{m}} = t_{\bar{m}-1} + k$ for $\bar{m} \in [[1, m]]$. Then

$$J_k(n + m) = J_k(r_m) = t_m = t_{m-1} + k = \cdots = p + km,$$

proving item (a).

(b) Let us write $p' := p + k(m - 1)$ and $n' := n + (m - 1)$. Note that $p' \leq n'$ by hypothesis. Thus by part (a), $J_k(n') = p'$. Moreover $p'+k > n'+1$. We claim that $p'+k \leq 2(n'+1)$. Otherwise, $2(n+m) = 2(n'+1) < p'+k \leq n'+k = (n+m)+k-1$, which implies that $n+m < k-1$. Then $k < p+k(m-1) \leq (n+m)-1 < k-2$, which is a contradiction. Hence it follows from equation (1) that

$$J_k(n + m) = J_k(n' + 1) = p' + k - (n' + 1) = p + k - (n + m),$$

which proves item (b).

□

In the following subsection, we use Lemma 6 to derive several general conclusions regarding the behavior of the Josephus function J_k.

2.1 Characterization of extremal points

We start by characterizing some initial extremal points for the Josephus function.

Corollary 7 (The first known high extremal point)**.** *The point $2k-2$ is always an extremal point for J_k. In particular,*

$$n_e := \begin{cases} 2k-3, & \text{if } J_k(2k-2) \leq k-1; \\ 2k-2, & \text{if } J_k(2k-2) > k-1, \end{cases} \tag{2}$$

is a high extremal point and $n_e + 1$ is a low extremal point.

Proof. When $J_k(2k-2) \leq k-1$, by the definition of low extremal point, we have that $2k-2$ is a low extremal point and hence $n_e = 2k-3$ is a high extremal point proving the first part of equation (2).

On the other hand, when $J_k(2k-2) > k-1$ we have

$$J_k(2k-2) \geq k = (2k-2) - k + 2 \in [[(2k-2) - k + 2, 2k-2]].$$

Hence $n_e = 2k-2$ is a high extremal point, as desired. The rest of the statement of the corollary follows from the definition of low extremal points. □

In the following, we present our main results. From now on, let us assume that $k \geq 2$ and $n \geq k-1$, which allows an alternating sequence of low and high extremal points for the Josephus function J_k where $k \geq 2$. We next reveal a recurrence procedure to find successive extremal points of the function J_k. In particular, we prove that there is an infinite number of fixed points for J_k.

First, we introduce the following sequences:

$$(\delta_k(r,j))_{r=0,j=1}^{k-2,k-1} \quad \text{with} \qquad \delta_k(r,j) := \begin{cases} 1, & \text{if } r < j-1; \\ 0, & \text{if } r \geq j-1, \end{cases} \tag{3}$$

$$(a_k(n,r))_{n=k-1,r=0}^{\infty,k-2} \quad \text{with} \qquad a_k(n,r) := \frac{k(n+1)-(r+1)}{k-1}, \tag{4}$$

$$(M_k(n,r,j))_{n=k-1,r=0,j=1}^{\infty,k-2,k-1} \quad \text{with} \quad M_k(n,r,j) := \frac{n-r}{k-1} - \delta_k(r,j). \tag{5}$$

The meaning of each sequence term can be summarized as follows: $\delta_k(r,j)$ serves as a variation of Kronecker's delta, while $a_k(n,r)$ is associated with the extremal points of the Josephus function. Lastly, $M_k(n,r,j)$ defines the upper bound of an interval where the Josephus function exhibits linear behavior. We outline some important properties of these sequences in the following lemma.

Lemma 8 (Properties of the Josephus function)**.** *Define $j := J_k(n+1)$. Then*

(a) *If $j \in [[1, k-1]]$, i.e., $n+1$ is a low extremal point, then $m_0 := \left\lfloor \dfrac{n-(j-1)}{k-1} \right\rfloor$ is the largest nonnegative integer m that satisfies the inequality*

$$j + km \leq n + 1 + m. \tag{6}$$

and $[[0, m_0]]$ is the solution set of (6). Moreover, $m_0 \geq 1$ if $n \geq k-2+j$.

(b) *If $j \in [[1, k-1]]$, i.e., $n+1$ is a low extremal point, and $n \equiv r \pmod{k-1}$, then $m_0 = M_k(n, r, j)$ and $n + 1 + m_0 = a_k(n, r) - \delta_k(r, j)$.*

(c) *$a_k(n, r)$ is an integer if and only if $n \equiv r \pmod{k-1}$.*

Proof.

(a) First note that (6) is equivalent to $m \leq (n-(j-1))/(k-1)$. Then since $n \geq k-1$, we have $m_0 = \lfloor (n-(j-1))/(k-1) \rfloor$ as the largest non-negative integer satisfying (6). Therefore, any non-negative integer m satisfying $0 \leq m \leq m_0$ clearly satisfies the inequality (6), since 0 is a trivial solution. Now, note that $n \geq k-2+j$ happens automatically when $n \geq 2k-3$ for $j \leq k-1$. Moreover, since

$$\frac{n-(j-1)}{k-1} \geq \frac{2k-3-(k-2)}{k-1} = 1,$$

we have $m_0 \geq 1$.

On the other hand, if $n \in [[k-1, 2k-4]]$ then from a direct inspection to the value of $(n-(j-1))/(k-1)$, we get that

$$m_0 = \begin{cases} 0, & \text{if } n < k-2+j; \\ 1, & \text{if } n \geq k-2+j. \end{cases} \tag{7}$$

So, in this case, it follows from (7) that $m_0 = 1$ is the largest integer satisfying (6) if $n \geq k-2+j$, proving this item.

(b) Suppose that $n \equiv r \pmod{k-1}$. Note that

$$\frac{(j-1)-r}{k-1} \in \left[\frac{1}{k-1}, \frac{j-1}{k-1}\right] \subset (0, 1)$$

when $r < j-1$, and

$$\frac{r-(j-1)}{k-1} \in \left[\frac{1}{k-1}, 1 - \frac{j}{k-1}\right] \subset (0, 1)$$

when $r > j-1$. Thus using the definition of $M_k(n, r, j)$ given in (5), we get

$$m_0 = \left\{ \begin{array}{ll} \left\lfloor \dfrac{n-r}{k-1} - \dfrac{(j-1)-r}{k-1} \right\rfloor = \dfrac{n-r}{k-1} - 1, & \text{if } r < j-1; \\ \dfrac{n-r}{k-1}, & \text{if } r = j-1; \\ \left\lfloor \dfrac{n-r}{k-1} + \dfrac{r-(j-1)}{k-1} \right\rfloor = \dfrac{n-r}{k-1}, & \text{if } r > j-1; \end{array} \right\} = M_k(n, r, j).$$

The equality $n + 1 + m_0 = a_k(n, r) - \delta_k(r, j)$ follows directly from previous equation and by the definition of $a_k(n, r)$ and $\delta_k(r, j)$ in (3) and (4), respectively.

(c) If $a_k(n,r) = \dfrac{k(n+1)-(r+1)}{k-1}$ is an integer, clearly $k(n+1) \equiv r+1 \pmod{k-1}$. In this case,

$$n \equiv kn-(k-1)n \equiv kn \equiv k(n+1)-k \equiv r+1-k \equiv r-(k-1) \equiv r \pmod{k-1}.$$

Now if $n \equiv r \pmod{k-1}$ then

$$k(n+1) \equiv n+1+(k-1)(n+1) \equiv n+1 \equiv r+1 \pmod{k-1},$$

which implies that $a_k(n,r)$ is an integer.

□

We now present formulas characterizing extremal points of the Josephus function J_k and their images. These expressions are useful in determining recurrence formulas between extremal points and a formula for the image of J_k at any arbitrary n in terms of the high and low extremal points.

Theorem 9 (Extremal point formulas)**.** *Let $n_e \geq 2k-3$ be a high extremal point and define $j := J_k(n_e+1)$. Then*

(a) *$J_k(n_e+1+m) = j+km$, for all $m \in [[0, \lfloor (n_e-(j-1))/(k-1) \rfloor]]$.*

(b) *If $n_e \equiv r \pmod{k-1}$, then the next high extremal point of J_k is*

$$n_e^+ := \frac{k(n_e+1)-(r+1)}{k-1} - \delta_k(r,j). \tag{8}$$

Moreover,

$$J_k(n_e^+) = n_e^+ - \delta_k(r,j)(k-1) - r + (j-1), \tag{9}$$

$$J_k(n_e^+ + 1) = (1-\delta_k(r,j))(k-1) - r + (j-1), \tag{10}$$

and

$$J_k(n_e^+ + 1) = (k-1) - [n_e^+ - J_k(n_e^+)]. \tag{11}$$

(c) *n_e^+ is a fixed point of J_k if and only if $r-(j-1)=0$ and $n_e^+ = \dfrac{k(n_e+1)-j}{k-1}$.*

(d) *J_k is a linear function on $[[n_e+1, n_e^+]]$.*

Proof.

(a) By Lemma 8(a), $j+km \leq n_e+1+m$ for every $m \in [[0, m_0]]$ where

$$m_0 := \lfloor (n_e-(j-1))/(k-1) \rfloor .$$

Additionally, Lemma 6(a) implies that $J_k(n_e+1+m) = j+km$ for every

$$m \in [[0, \lfloor (n_e-(j-1))/(k-1) \rfloor]]$$

as desired.

(b) Equation (8) follows directly from the previous item, and Lemma 8(b) that $m_0 \geq 1$ and

$$\begin{aligned} J_k(n_e^+) &= J_k(n_e + 1 + m_0) = j + km_0 \\ &= \begin{cases} j + k\left(\dfrac{n_e - r}{k-1} - 1\right), & \text{if } r < j-1; \\ j + k\left(\dfrac{n_e - r}{k-1}\right), & \text{if } r \geq j-1; \end{cases} \\ &= \begin{cases} a_k(n_e, r) - 1 - (k-1) - (r - (j-1)), & \text{if } r < j-1; \\ a_k(n_e, r) - (r - (j-1)), & \text{if } r \geq j-1; \end{cases} \\ &= n_e^+ - \delta_k(r,j)(k-1) - r + j - 1, \end{aligned}$$

where $m_0 = \lfloor (n_e - (j-1))/(k-1) \rfloor$, which proves (9). Now using Lemma 6(b) and Lemma 8(b), we have

$$\begin{aligned} J_k(n_e^+ + 1) &= J_k(n_e + 1 + (m_0 + 1)) = j + k(m_0 + 1) - (n_e + 1 + (m_0 + 1)) \\ &= \begin{cases} j + k\left(\dfrac{n_e - r}{k-1}\right) - n_e - 1 - \dfrac{n_e - r}{k-1}, & \text{if } r < j-1; \\ j + k\left(\dfrac{n_e - r}{k-1} + 1\right) - n_e - 2 - \dfrac{n_e - r}{k-1}, & \text{if } r \geq j-1; \end{cases} \\ &= \begin{cases} -(r - (j-1)), & \text{if } r < j-1; \\ (k-1) - (r - (j-1)), & \text{if } r \geq j-1; \end{cases} \\ &= (1 - \delta_k(r,j))(k-1) - r + j - 1, \end{aligned}$$

establishing (10). On the other hand, since

$$(1 - \delta_k(r,j))(k-1) - r + j - 1 \in [[1, k-1]],$$

n_e^+ is certainly the next high extremal point of J_k. Additionally,

$$\begin{aligned} (k-1) - [n_e^+ - J_k(n_e^+)] &= (k-1) - [n_e^+ - (n_e^+ - \delta_k(r,j)(k-1) - r + j - 1)] \\ &= (k-1) - [\delta_k(r,j)(k-1) + r - j + 1] \\ &= (1 - \delta_k(r,j))(k-1) - r + j - 1 \\ &= J_k(n_e^+ + 1), \end{aligned}$$

proving (11).

(c) If n_e^+ is a fixed point of J_k, then (9) implies that $\delta_k(r,j)(k-1) + r - j + 1 = 0$. Therefore, $r - j + 1 = 0$ and $n_e^+ = a_k(n_e, j-1) = (k(n_e+1) - j)/(k-1)$. Conversely, if $r - j + 1 = 0$ and $n_e^+ = (k(n_e+1) - j)/(k-1)$, then (9) implies

$$J_k(n_e^+) = n_e^+ - \delta_k(r,j)(k-1) - r + j - 1 = n_e^+ - 0(k-1) - 0 = n_e^+.$$

Thus n_e^+ is a fixed point of J_k.

(d) Follows from (a).

□

The following results present a recurrence formula that allows us to compute successive high extremal points for the Josephus function J_k. We also provide algebraic expressions for the images of J_k at extremal points and at an arbitrary n. This recurrence relation is of great importance in establishing the existence of fixed points and developing an effective extremal algorithm for evaluating the Josephus function.

Corollary 10 (Recurrence formula for computing high extremal points)**.** *Let $n_e^{(i)} \geq 2k-3$ be a high extremal point of the Josephus function J_k, with corresponding functional value $J_k(n_e^{(i)})$. Given*

$$r_i \equiv n_e^{(i)} \pmod{k-1}, \tag{12}$$

and

$$c_i := \begin{cases} 1, & \text{if } r_i < k-2-n_e^{(i)} + J_k(n_e^{(i)}); \\ 0, & \text{if } r_i \geq k-2-n_e^{(i)} + J_k(n_e^{(i)}). \end{cases} \tag{13}$$

Then we can compute the next high extremal point

$$n_e^{(i+1)} = \frac{k(n_e^{(i)}+1)-(r_i+1)}{k-1} - c_i, \tag{14}$$

and its functional value

$$J_k(n_e^{(i+1)}) = (k-1) - n_e^{(i)} + J_k(n_e^{(i)}) + k\left\lfloor \frac{2n_e^{(i)} - J_k(n_e^{(i)}) - (k-2)}{k-1} \right\rfloor. \tag{15}$$

Moreover, for any $n \in [[n_e^{(i)}+1, n_e^{(i+1)}]]$, we have

$$J_k(n) = k(n - n_e^{(i+1)}) + J_k(n_e^{(i+1)}). \tag{16}$$

Proof. Assume that $n_e^{(i)}$ is a given high extremal point and its corresponding value through J_k is known. Then, by Theorem 9(c), we have $J_k(n_e^{(i)}+1) = (k-1) - [n_e^{(i)} - J_k(n_e^{(i)})]$.

Set $j := (k-1) - [n_e^{(i)} - J_k(n_e^{(i)})]$. Notice that $j - 1 = k - 2 - n_e^{(i)} + J_k(n_e^{(i)})$. Find r_i and c_i by employing equations (12) and (13), respectively.

On the one hand, by Theorem 9(b), the next high extremal point of J_k, $n_e^{(i+1)}$, is

$$n_e^{(i+1)} = a_k(n_e^{(i)}, r_i) - \delta_k(r_i, j) = \frac{k(n_e^{(i)}+1)-(r_i+1)}{k-1} - c_i,$$

which tell us that (14) holds. Also, it follows from Theorem 9(b) that:

$$\begin{aligned} J_k(n_e^{(i+1)}) &= j + k\left\lfloor \frac{n_e^{(i)} - (j-1)}{k-1} \right\rfloor \\ &= (k-1) - n_e^{(i)} + J_k(n_e^{(i)}) + k\left\lfloor \frac{2n_e^{(i)} - J_k(n_e^{(i)}) - (k-2)}{k-1} \right\rfloor, \end{aligned}$$

which verifies the validity of (15). Now, if $n \in [[n_e^{(i)} + 1, n_e^{(i+1)}]]$, the point $(n, J_k(n))$ can be determined by finding the intersection of the vertical line at (n, n) and the line with slope k that passes through the point $(n_e^{(i+1)}, J_k(n_e^{(i+1)}))$, which gives us (16). □

We now include a similar recurrence formula for the low extremal points.

Corollary 11 (Recurrence formula for computing low extremal points)**.** *Let $\check{n}_e^{(i)} \geq 2k-3$ be a low extremal point of the Josephus function J_k, with functional value $J_k(\check{n}_e^{(i)})$. Given $\check{r}_i \equiv \check{n}_e^{(i)} \pmod{k-1}$ and*

$$\check{c}_i := \begin{cases} 1, & \text{if } \check{r}_i < J_k(\check{n}_e^{(i)}); \\ 0, & \text{if } \check{r}_i \geq J_k(\check{n}_e^{(i)}). \end{cases}$$

Then we can compute the next low extremal point as follows:

$$\check{n}_e^{(i+1)} = \frac{k\check{n}_e^{(i)} - \check{r}_i}{k-1} - \check{c}_i + 1, \tag{17}$$

and its functional value as:

$$J_k(\check{n}_e^{(i+1)}) = k - \check{n}_e^{(i+1)} + J_k(\check{n}_e^{(i)}) + k \left\lfloor \frac{\check{n}_e^{(i)} - J_k(\check{n}_e^{(i)})}{k-1} \right\rfloor. \tag{18}$$

Moreover, for any $n \in [[\check{n}_e^{(i)} + 1, \check{n}_e^{(i+1)}]]$, we have

$$J_k(n) = kn + J_k(\check{n}_e^{(i+1)}) - (k-1)\check{n}_e^{(i+1)}. \tag{19}$$

Proof. The proof of the formulas for $\check{n}_e^{(i+1)}$ and $J_k(\check{n}_e^{(i+1)})$ are similar to Corollary 10. The expression for $J_k(n)$ follows directly from (16) and (11). □

Remark 12 (Knuth's formula is recovered). When $k = 2$, the above corollary can be used to derive Knuth's formula for the low and high extremal (fixed) points, as well as the explicit formula for $J_2(n)$ for every $n \geq 1$. To do this, we substitute $k = 2$ into equations (17) and (18), which yields $\check{r}_i = 0$ and $\check{c}_i = 1$ for all $i \geq 1$. More than that, Corollary 7 can be used to start a straightforward induction argument, to show that the low extremal point $\check{n}_e^{(i)} = 2^i$ and the high extremal point, which in this case coincides with a fixed point, $n_p^{(i)} = \check{n}_e^{(i)} - 1 = 2^i - 1$ for all $i \geq 1$. Moreover, the explicit formula for $J_2(n)$ can be obtained directly from (19) by observing that $J_2(\check{n}_e^{(i)}) = 1$ and $i = \lfloor \log_2 n \rfloor$ whenever $n_e^{(i)}$ is a low extremal point and $n \geq n_e^{(i)}$. Specifically, we have $J_2(n) = 2n - 2^{\lfloor \log_2 n \rfloor + 1} + 1$.

We are now ready to prove the existence of infinitely many fixed points for J_k when $k \geq 3$, which has already been established by Knuth for $k = 2$. The approach we take to prove the existence of fixed points involves applying the recurrence form of Corollary 10 successively. We state this formally in the following theorem.

Theorem 13 (Existence of infinitely many fixed points)**.** *There exist infinitely many fixed points for the Josephus function J_k.*

Proof. When $k = 2$ the result was proved by Knuth [13, pp. 162, 184]. So, let us assume that $k \geq 3$. Then if $n \geq 2k-3$ and $J_k(n)$ is known, and if n is either a low extremal point or a point that is not a high extremal point, then we can reach a high extremal point by using Lemma 6. Without loss of generality, we can assume that n is a high extremal point that is not a fixed point of J_k, and we seek to find a fixed point.

The proof proceeds by contradiction. Assume that the repeated application of Corollary 10 generates only high extremal points, which are no fixed points of J_k. Let n denote $n_e^{(0)}$. So, $n_e^{(0)} + 1$ is a low extremal point of J_k, and by Proposition 4, $j_0 := J_k(n_e^{(0)} + 1) < k - 1$. Let $r_0 \leq k-2$ such that $n_e^{(0)} \equiv r_0 \pmod{k-1}$. Then by applying Corollary 10, the integer $n_e^{(1)}$ is a high extremal point. Based on our general assumption $r_0 \neq j_0 - 1$. Notice that $n_e^{(1)}$ can be written as follows:

$$n_e^{(1)} = \alpha(n_e^{(0)} + 1) - \frac{r_0 + 1}{k-1} - c_0 = \alpha(n+1) + d_0 - 1,$$

where $\alpha := k/(k-1)$ and $d_0 := (1 - c_0) - (r_0 + 1)/(k-1)$. Based on our assumption and on Corollary 10, an infinite sequence $(n_e^{(m)})_{m \geq 0}$ of high extremal points can be generated, where none of its terms is a fixed point such that $j_m := J_k(n_e^{(m)} + 1) < k - 1$ (a low extremal point), $n_e^{(m)} \equiv r_m \pmod{k-1}$ ($r_m \neq j_m - 1$), and for $m \geq 1$:

$$\begin{aligned} n_e^{(m)} &= \alpha^m(n_e^{(0)} + 1) + (\alpha^{m-1}d_0 + \cdots + d_{m-1}) - 1 \\ &= \alpha^m(n+1) + \left(\sum_{i=0}^{m-1} \alpha^{m-1-i} d_i\right) - 1, \end{aligned} \tag{20}$$

where $d_i := (1 - c_i) - (r_i + 1)/(k-1)$. Therefore (20) holds for all $m \geq 1$. Now, observe that it can be rewritten as follows:

$$\begin{aligned} n_e^{(m)} &= \alpha^{m-1}\left(\alpha(n+1) + \sum_{i=0}^{m-1} \frac{d_i}{\alpha^i} - \frac{1}{\alpha^{m-1}}\right) \\ &= \frac{k^{m-1}}{(k-1)^{m-1}}\left(\frac{k}{k-1}(n+1) + \frac{1}{k-1}\sum_{i=0}^{m-1} \frac{(k-1)d_i}{\alpha^i} - \frac{(k-1)^{m-1}}{k^{m-1}}\right) \\ &= \frac{k^{m-1}}{(k-1)^m}\left(k(n+1) + \sum_{i=0}^{m-1}(k-1)d_i\left(\frac{k-1}{k}\right)^i - \frac{(k-1)^m}{k^{m-1}}\right). \end{aligned} \tag{21}$$

Note further that $(k-1)d_i = (k-1)(1-c_i) - r_i - 1$, where $r_i \in [[0, k-2]]$ and $c_i \in \{0, 1\}$. Hence if we define

$$\beta_m := k(n+1) + \sum_{i=0}^{m-1}(k-1)d_i\left(\frac{k-1}{k}\right)^i - \frac{(k-1)^m}{k^{m-1}},$$

then for all $m \geq 1$, β_m satisfies the inequalities

$$1 \leq k(n-k+2) - 1 \leq \beta_m \leq k(n+k-1), \tag{22}$$

where we have used that $-(k-1) \leq (k-1)d_i \leq k-2$, the fact that $n \geq 2k-3$ implies that $n \geq k-1$ because $k \geq 3$, and

$$\sum_{i=0}^{m-1} \left(\frac{k-1}{k}\right)^i = \frac{1-\left(\frac{k-1}{k}\right)^m}{1-\left(\frac{k-1}{k}\right)} \leq k.$$

Using now (21), we get

$$n_e^{(m)} = \frac{k^{m-1}}{(k-1)^m} \cdot \beta_m,$$

and hence, $n_e^{(m)}$ is a positive integer number for all m, if β_m grows as $(k-1)^m$ for m large, or it approaches 0 as $\frac{(k-1)^m}{k^{m-1}}$. These two possibilities do not occur because (22) guarantees that β_m is uniformly far from 0 and bounded. Thus for some $m_0 \geq 1$, r_{m_0} coincides with $j_{m_0} - 1$, and $n_e^{(m_0)}$ is a fixed point by Theorem 9(c), proving the result. □

3 Numerical experiments

In this section, we present a numerical comparison of four algorithms to evaluate the Josephus function. The computational experiments were carried out on an iMac 3.6 GHz 10-Core Intel Core i9 with 32GB of RAM. The algorithms were implemented in the Julia programming language v1.8.

3.1 The algorithms

First, let us describe three well-known methods for solving the Josephus problem (Euler's, Knuth's, and Uchiyama's algorithms); see, for instance, [8, 24, 26]. Moreover, we propose a novel scheme for evaluating the Josephus function called the Extremal algorithm.

3.1.1 Euler's algorithm

Euler's algorithm is a natural interpretation of the recurrence relation proposed first in [7, & 8, pp. 130–131]. In [26, pp. 57], Woodhouse gave the following modern algorithmic description:

$$J_k(n) = ((\cdots((1+k)_{(2)}+k)_{(3)}+\cdots+k)_{(n-1)}+k)_{(n)}$$

where for any positive integers r and s the symbol $(r)_{(s)}$ denotes the integer satisfying: $(r)_{(s)} \equiv r \pmod{s}$ and $1 \leq (r)_{(s)} \leq s$. This scheme clearly has an intrinsic recursive behavior and complexity $O(n)$, i.e., there is necessary about n function evaluations to reach the solution.

To compute the Josephus function J_k at n, Euler's algorithm requires evaluating J_k for all preceding values, that is, $n-1$ evaluations of J_k.

3.1.2 Knuth's algorithm

In equation (3.19) from [8, Chapter 3, Section 3, pp. 81] Knuth presents the following algorithm:

$$\begin{aligned}
&D := 1;\\
&\text{While } D \leq (k-1)n \text{ do } D := \left\lceil \frac{k}{k-1} D \right\rceil;\\
&J_k(n) = kn + 1 - D.
\end{aligned}$$

Knuth's algorithm significantly reduces the number of evaluations required to compute $J_k(n)$ compared to Euler's algorithm. Instead of the recursive computation of Euler's algorithm, the Knuth's algorithm only requires $O(\ln(n))$ evaluations before computing $J_k(n)$. This results in a more efficient computation process than Euler's.

3.1.3 Uchiyama's algorithm

In equation (11) from [24, Section 4, pp. 329], Uchiyama presents the following algorithm:

Let $n_1 = 1$, set

$$c_1^\star = c_1 := J_k(2) = \begin{cases} 1, & \text{if } k \text{ is even;} \\ 2, & \text{if } k \text{ is odd.} \end{cases}$$

For $i \geq 1$ compute:

$$\begin{aligned}
n_{i+1} &= \left\lfloor \frac{k(n_i+1) - c_i}{k-1} \right\rfloor;\\
c_{i+1}^\star &:= c_i + (k-1)(n_{i+1}+1) - k(n_i+1);\\
c_{i+1} &\equiv \begin{cases} c_{i+1}^\star \pmod{n_{i+1}+1}, & \text{if } 1 \leq c_{i+1} \leq n_{i+1}+1; \\ c_{i+1}^\star, & \text{otherwise.} \end{cases}
\end{aligned}$$

If $n_i < n \leq n_{i+1}$, then $J_k(n) = c_i + k(n - n_i - 1)$.

Uchiyama's algorithm, like Knuth's algorithm, improves to $O(\ln(n))$ the number of evaluations required to compute $J_k(n)$ compared to Euler's algorithm.

3.1.4 Extremal algorithm

To evaluate the Josephus function J_k, we propose the Extremal algorithm, which employs a strategy of computing recursively high extremal points $n_e^{(i)}$ $(i = 1, 2, \ldots, m)$ until $n_e^{(m)}$ is greater than or equal to n; see Corollary 10. This approach capitalizes on the linear piecewise structure of the Josephus function, enabling the Extremal algorithm to efficiently compute $J_k(n)$ compared to other methods.

The Extremal algorithm for evaluating J_k at $n \geq 2k-3$ can be described as follows. We start by defining $j := J_k(2k-2)$ and the first high extremal point $n_e^{(1)}$ and $J_k(n_e^{(1)})$ are given by:

$$\left(n_e^{(1)}, J_k(n_e^{(1)})\right) := \begin{cases} (2k-3, j+k-2), & \text{if } j \leq k-1; \\ (2k-2, j), & \text{if } j > k-1. \end{cases}$$

For $i \geq 1$ compute:

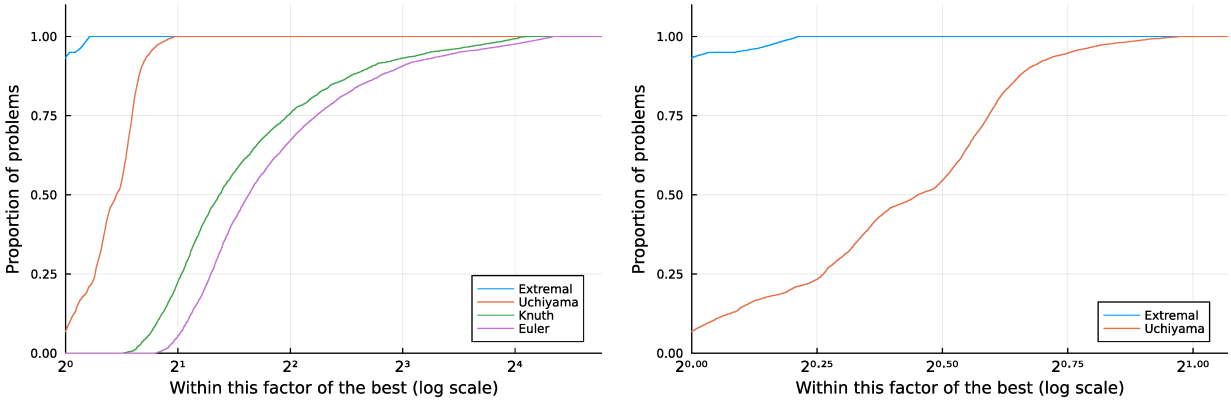
$$\left.\begin{aligned} & r_i \equiv n_e^{(i)} \pmod{k-1}; \\ & c_i := \begin{cases} 1, & \text{if } r_i < k-2-n_e^{(i)} + J_k(n_e^{(i)}); \\ 0, & \text{if } r_i \geq k-2-n_e^{(i)} + J_k(n_e^{(i)}); \end{cases} \\ & J_k(n_e^{(i+1)}) = (k-1) - n_e^{(i)} + J_k(n_e^{(i)}) + k \left\lfloor \frac{2n_e^{(i)} - J_k(n_e^{(i)}) - (k-2)}{k-1} \right\rfloor; \\ & n_e^{(i+1)} = \frac{k(n_e^{(i)}+1) - (r_i+1)}{k-1} - c_i. \end{aligned}\right\}$$

If $n_e^{(i)} < n \leq n_e^{(i+1)}$, then $J_k(n) = J_k(n_e^{(i+1)}) + k(n - n_e^{(i+1)})$.

The Extremal algorithm is also more efficient than Euler's algorithm. Like Knuth's, and Uchiyama's algorithms, it requires $O(\ln(n))$ evaluations before computing $J_k(n)$.

3.2 A comparative computational study

To compare the performance of all algorithms, we conducted extensive numerical experiments by measuring their CPU time. We employed a performance profile testing a total of 452500 problems, uniformly distributed for $n \in [50000 : 100 : 100000]$ and $k \in [50 : 10 : 1000]$. We compared the Extremal algorithm with three other algorithms, namely Euler, Knuth, and Uchiyama, and our results indicate that the Extremal algorithm outperformed the other three in almost all problem instances (see Figure 2), making it an excellent choice for solving the Josephus problem for large inputs.



(a) PP: All Methods.

(b) PP: Extremal vs Uchiyama.

Figure 2: Performance profiles (PP) for $n \in [50000 : 100 : 100000], k \in [50 : 10 : 1000]$.

To provide a more detailed analysis of our results, we present Table 2, which summarizes descriptive statistics of the benchmark of problems, including the minimum (min), mean, maximum (max), and standard deviation (std) of total CPU time. The Extremal algorithm consistently performed better than Euler's, Knuth's, and Uchiyama's algorithms, indicating its superiority for solving the Josephus problem.

Algorithms	min	mean	max	std
Euler	0.000813209	0.00120984	0.00161632	0.000231202
Knuth	0.000659197	0.00101147	0.00137245	0.00019701
Uchiyama	7.234e-5	0.000518487	0.0010305	0.000246989
Extremal	7.6873e-5	0.000367613	0.000633953	0.000144699

Table 2: Statistics of the experiments.

4 Concluding remarks

In this paper, we presented a novel study characterizing the Josephus function structure. We use the function's piecewise linear structure to identify extremal points (including fixed points) of the Josephus function, J_k, via a recurrence formula. We have developed an efficient algorithm for evaluating J_k for large values of n based on the successive computation of the high extremal points of J_k. The effectiveness of the proposed scheme was validated through its comparison to established algorithms. The results of the comparative study demonstrate the remarkable performance of the proposed approach in computing the Josephus function for large inputs. It is noteworthy that we can employ the recurrence formula to calculate low extremal points, and a similar approach can be designed to address the Josephus problem. The analytical study presented in this paper can have substantial practical implications for applications such as scheduling, network optimization, and distributed algorithms. However, finding a recurrence formula for computing fixed points, analogous to the one we obtained for extremal points, remains an open problem. Addressing this problem may lead to further insights and improvements in solving the Josephus problem.

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