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Counting Partitions by Genus: a Compendium of Results

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Abstract

We study the enumeration of set partitions, according to their length, number of parts, cyclic type, and genus. We introduce genus-dependent Bell, Stirling numbers, and Faà di Bruno coefficients. Besides attempting to summarize what is already known on the subject, we obtain new generic results (in particular for partitions into two parts, for arbitrary genus), and present computer generated new data extending the number of terms known for sequences or families of such coefficients; this also leads to new conjectures.

1 Introduction

This is the second paper in a series devoted to the combinatorics of set partitions and their enumeration according to their genus. In a previous paper [22], functional equations were written between generating functions (G.F.) of partitions, enabling one to count partitions in genus 0, 1 and 2. In the present paper, which is completely independent, our goal is different. We wish to collect as much data as possible on that combinatorics. Accordingly, our paper gathers classical and known results as well as new data, obtained by computer "brute force" calculations and a few exact new results. In many cases, these data suggest conjectures and extrapolations, that we mark with the sign $\stackrel{?}{=}$.

This endeavor has benefited in a tremendous way from the existence of the *On-Line Encyclopedia of Integer Sequences* (OEIS) [17]. Several unexpected connections and identifications have been made possible thanks to this irreplaceable and unique source.

Our paper is organized as follows. In Section 2, we recall some basic definitions: total numbers of partitions are given by Bell numbers, and by Stirling numbers when the number of parts is fixed. The key notion of genus is also recalled. In Section 3, the Bell numbers are refined by fixing the genus of partitions, and by including partitions with or without singletons. Explicit expressions for their G.F. are given in genus 0 to 2, conjectured in genus 3, and the general form is discussed in higher genus. The same steps are repeated in Section 4 for the Stirling numbers, with again exact or conjectured results for their counting and G.F. The rest of the paper is devoted to the counting of partitions of given cyclic type. In Section 5, we review three families of partitions for which this counting is generically known: the famous non-crossing partitions (i.e., of genus 0); the partitions of genus 1 and 2, [22]; the partitions into pairs, i.e., of type $[2^k]$, and arbitrary genus; and the partitions into two parts, for which we obtain a result in arbitrary genus, which is new, to the best of our knowledge. Section 6 gathers data on various types of partitions for which we have only partial results and conjectures: types $[p^k]$ for varying p or k and three-part partitions. Finally, the tables in the appendix contain the number of partitions of the set $\{1, \ldots, n\}$ of arbitrary genus up to $n = 15$.

To put the present work in perspective, let us recall some background. For a long time, the census of partitions according to their genus has been confined to two particular cases.

On the one hand, non-crossing partitions, i.e., of genus 0 in the current approach, and of arbitrary type, have been enumerated by Kreweras [15]. His result reappeared in the context of large size matrix integrals and their "planar" limit [1]: there, connected correlation functions (also known as cumulants) were in one-to-one correspondence with non-crossing partitions. The generating function of the latter was shown to satisfy a remarkably simple functional equation, from which Kreweras result followed by use of Lagrange inversion formula. That functional equation in turn received a simple diagrammatic interpretation by Cvitanovic [7]. Non-crossing cumulants associated with non-crossing partitions then appeared in the framework of free probability in the work of Speicher [18].

On the other hand, and in a totally independent vein, the special class of partitions of a set of even cardinality into pairs and of arbitrary genus was treated by Walsh and Lehman [19], and then by Harer-Zagier. See more references in Section 5.3. It is only recently that general partitions of higher genus were reconsidered by Cori and Hetyei [5, 6], and results on the counting of partitions with a given number of parts and genus 1 and 2 were obtained. Their method relies on a reduction of diagrams to simpler "irreducible" ones, with the same genus, which they proved to be in finite number for given genus. They were able to list these irreducible diagrams for genus 1 and 2. This enabled them to determine the G.F. of partitions with a given number of parts, for genus 1 and 2. See (49)-(51). This method was pushed one step further in [22], using the same reduction to irreducible diagrams, followed by a reconstruction of all partitions from those irreducible diagrams by use of functional identities generalizing Cvitanovic's one in higher genus. That method is limited in genus by the increasing complexity in first listing all irreducible diagrams in genus greater or equal to 3 and then in "dressing" them.

As this short review shows, there is ample room for further progress and we hope the present work will stimulate the curiosity and imagination of some readers.

2 Bell, Stirling, and Faà di Bruno numbers

2.1 Equivalence relations on a set with n elements

Any equivalence relation on a set is specified by a partition α of this set (or set-partition, for short), and conversely, an equivalence relation defines a partition.

The number of equivalence relations on a set with n elements is given by the Bell numbers, which obey the recurrence:

$$B_{n+1} = \sum_{p=0}^{n} \binom{n}{p} B_p, \text{ with } B_1 = 1, \tag{1}$$

hence

$$B_n = \frac{1}{e} \sum_{\ell=0}^{\infty} \frac{\ell^n}{\ell!}, \qquad \text{OEIS sequence A000110}. \tag{2}$$

The exponential generating function of the Bell numbers is

$$\mathcal{B}(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = e^{e^x - 1}. \tag{3}$$

2.2 Equivalence relations on a set with n elements, with k equivalence classes

One may impose that the equivalence relations have a given number, say k, of equivalence classes, i.e., that the partition has k parts.

The number of such relations is given by the Stirling number of the second kind, $S_{n,k}$, which obeys the recurrence relation:

$$S_{n,k} = k\, S_{n-1,k} + S_{n-1,k-1}, \ n > 1, \text{ with } S_{1,k} = 0, \ k > 1, \text{ and } S_{1,1} = 1. \tag{4}$$

An explicit form is

$$S_{n,k} = \frac{1}{k!}\sum_{s=0}^{k}(-1)^{k-s}\binom{k}{s}s^n, \qquad \text{OEIS sequence A008277.} \tag{5}$$

The exponential generating function of the Stirling numbers $S_{n,k}$ is

$$\mathcal{S}(x,y) = e^{y\,(e^x-1)} \tag{6}$$

with the following "sum rule" expressing their sum over k as a Bell number,

$$B_n = \sum_{k=0}^{n} S_{n,k}. \tag{7}$$

2.3 Set with n elements, with k equivalence classes of specified cardinalities

One may further impose that the chosen equivalences classes have specified cardinalities.

The set of cardinalities of the classes of the equivalence relation defined by the set-partition α determines a partition $[\alpha] \vdash n$ of the *integer* n, called the *type* of the partition. It is usual to denote this integer partition as follows: $[\alpha] = [1^{\alpha_1}, \ldots, n^{\alpha_n}]$. It can be represented as a Ferrers diagram or as a Young diagram. The number of equivalence relations on a set with n elements having equivalent classes with cardinalities specified by $[\alpha]$ will be denoted by $C_{n,[\alpha]}$. These numbers are sometimes called the Faà di Bruno coefficients.

$$C_{n,[\alpha]} = \frac{n!}{\prod_{\ell=1}^{n}\alpha_\ell!(\ell!)^{\alpha_\ell}}. \tag{8}$$

Sum rule: calling $|\alpha| = \sum_\ell \alpha_\ell$ the number of parts of the integer partition $[\alpha]$, we have obviously

$$\sum_{\substack{[\alpha]\\|\alpha|=k}} C_{n,[\alpha]} = S_{n,k}. \tag{9}$$

2.4 Genus of partitions on a cyclically ordered set

If the underlying set of n elements is totally ordered (for definiteness one may take it as $\{1,2,3,\ldots,n\}$), or if it is cyclically ordered, one may introduce a new structure, finer than the ones already considered, by determining the genus of set partitions (a non-negative integer). With α a partition of $\{1,2,3,\ldots,n\}$, we associate a *permutation* τ of $\mathcal{S}_n$: its cycles are the parts of α, *with the important constraint that their elements are in increasing order*. We also consider the cyclic permutation $\sigma := (1,2,\ldots,n)$. Then following [14, 19, 20], the genus $g(\alpha)$ is defined by

$$n + 2 - 2g = \#\text{cy}(\tau) + \#\text{cy}(\sigma) + \#\text{cy}(\sigma\circ\tau^{-1}) \tag{10}$$

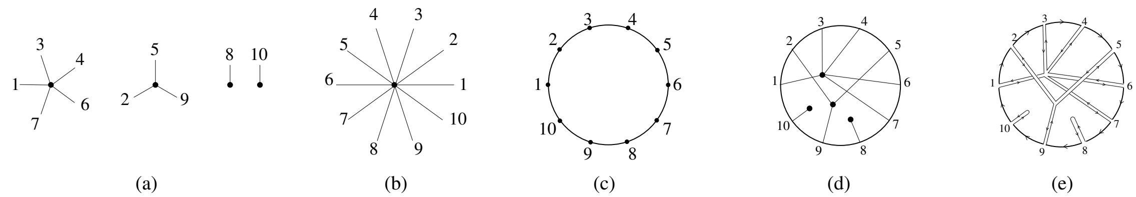

Figure 1: The partition $(\{1,3,4,6,7\},\{2,5,9\},\{8\},\{10\})$ of $\{1,\ldots,10\}$. (a) the four ℓ-vertices; (b) and (c): two equivalent representations of the special 10-vertex; (d) a contribution to $C^{(g)}_{10,[1^2\,3\,5]}$; (e) the double line (fat graph) version of (d), with three faces and thus genus $g=2$.

or in the present case,

$$-2g = |\alpha| - 1 - n + \#\mathrm{cy}(\sigma\circ\tau^{-1}) \tag{11}$$

since here $\#\mathrm{cy}(\sigma)=1$ and $\#\mathrm{cy}(\tau)=\sum\alpha_\ell=|\alpha|$.

For the reader's convenience, we recall briefly the diagrammatic representation of partitions, that makes the topological interpretation of the definition (11) more transparent.

To a given partition, we may also attach a map: it has α_ℓ ℓ-valent vertices, in short ℓ-*vertices* whose edges are numbered clockwise by the elements of the partition, (see Fig. 1a), and a special n-vertex, with its n edges numbered *anti*-clockwise from 1 to n. See. 1b. Edges are connected pairwise by matching their indices. Two maps are regarded as topologically equivalent if they encode the same partition. In fact it is topologically equivalent and more handy to attach n points *clockwise* on a circle, and to connect them pairwise by arcs of the circle. See Fig. 1b. Now the permutation σ describes the connectivity of the n points on the circle, while τ describes how these points are connected through the ℓ-vertices (drawn inside the disk). It is readily seen that the permutation $\sigma\circ\tau^{-1}$ describes the circuits bounding clockwise the faces of the map. This is even more clearly seen if one adopts a double line notation for each edge [12], thus transforming the map into a "fat graph". See Fig. 1e. Consecutive entries of the cycles of $\sigma\circ\tau^{-1}$ label the tails of those oriented edges that are drawn going inward: in this way one can read the entries of $(\{1,8,9,6,5,3,2,10\},\{4\},\{7\})$ by following the three circuits of Fig. 1e. Thus the number of cycles of $\sigma\circ\tau^{-1}$ is the number f of faces of the map. Since each face is homeomorphic to a disk, gluing a disk to each face transforms the map into a closed Riemann surface, to which we may apply Euler's formula

$$2-2g = \#(\text{vertices}) - \#(\text{edges}) + \#(\text{faces}) = 1 + \sum_\ell \alpha_\ell - n + f \tag{12}$$

with $f=\#\mathrm{cy}(\sigma\circ\tau^{-1})$, and we have reproduced (11). In other words, g is the minimal genus of the surface on which the map induced by the partition may be drawn without crossings.

Since f is a positive integer, and letting $k=|\alpha|$ denote the number of parts of the integer partition $[\alpha]$, we have

$$g = \frac{n-k+1-f}{2} \le \frac{(n-k)}{2}. \tag{13}$$

Also note that one-part partitions ($k = 1$) necessarily have genus 0. Hence for $g > 0$, $k \geq 2$ and

$$n \geq 2g + k \geq 2g + 2. \tag{14}$$

Remark 1. Assuming the existence of an order on the underlying set is not really a restriction, since one can always choose one. Each family of set-partitions (or equivalence relations) previously considered will be itself decomposed according to the genus, and we shall introduce notation $B_n^{(g)}$, $S_{n,k}^{(g)}$ and $C_{n,[\alpha]}^{(g)}$, with

$$\sum_g B_n^{(g)} = B_n, \ \sum_g S_{n,k}^{(g)} = S_{n,k}, \ \text{and} \ \sum_g C_{n,[\alpha]}^{(g)} = C_{n,[\alpha]}. \tag{15}$$

Remark 2. Let us emphasize again that the maps associated with set partitions are subject to the important constraint that their vertices (= parts) are ordered. Accordingly, the edges connecting each ℓ-vertex to the n-vertex cannot cross one another, thus respecting their original cyclicity and ordering. Only crossings of edges originating from distinct vertices are allowed. It is that constraint that makes the census of partitions difficult.

Remark 3. Tables and conjectures follow from computer calculations, using Mathematica. From a computational point of view, tables giving the number of set partitions for a set of n elements with given genus, possibly obeying some constraints, can be obtained from the following simple algorithm:

(1) Generate all partitions of this set.

(2) Compute their genus, using formula (10).

(3) Select those partitions that obey the chosen particular constraints (given number of parts, specific cyclic type, absence of singletons, ...).

This method is simple enough to implement, and we could follow the above steps using set partitions generated by Mathematica, for "small" values of n; we also used a predefined command that can generate lists containing all partitions with given number of blocks (parts).

Unfortunately, the large number of generated partitions makes the method intractable when n increases.

A first simplification consists in focusing on partitions without singletons, since the others (or their number) can be simply obtained from the former.

However, when n is too large, typically $n > 12$, the amount of RAM required to hold all these partitions exhausts the possibilities of a typical laptop, even if one restricts his attention to partitions without singletons; moreover, using virtual memory techniques slows down considerably the calculations. In order to handle higher values of n we had to write programs (in Mathematica) that build partitions sequentially, i.e., one at a time rather than in long lists, sometimes generating only those of a given cyclic type, then saving those partitions, or only their number, only if they obey some chosen criteria (according to their genus for instance).

Another method that we used for such higher values of n was to save large lists of partitions into external files and subsequently use stream processing techniques (manipulating pointers) to process each record one at a time.

It is important to notice that, unfortunately, we could not devise an efficient algorithm that would have allowed us to generate directly all the partitions that have a specific genus.

In the following, we shall be using both representations, by pairs of permutations or by fat graphs, in turn or in parallel.

2.5 Partitions with no singletons

If a set partition has no singleton, its associated equivalence relation is such that no element is isolated. Equivalently, each part of the partition contains at least 2 elements. By adding the constraint that the families of partitions considered previously should have no singleton one can define "associated"[1] Bell numbers $\widehat{B}_n$, and "associated" Stirling numbers (of the second kind) $\widehat{S}_{n,k}$. Obviously,

$$\sum_{k=1}^{n} \widehat{S}_{n,k} = \widehat{B}_n. \tag{16}$$

Since the notation $C_{n,[\alpha]}$ already incorporates the partition type, there is no "hat" version of the Faà di Bruno coefficients: either $[\alpha]$ contains singletons, or it does not.

2.5.1 Associated Bell number $\widehat{B}_n$

See OEIS sequence A000296.

Their exponential generating function is

$$\widehat{\mathcal{B}}(x) = e^{e^x - x - 1}. \tag{17}$$

Let us mention the following identities:

$$B_n = \widehat{B}_n + \widehat{B}_{n+1} \tag{18}$$

$$\widehat{B}_n = \sum_{j=0}^{n-2} (-1)^j \, B_{n-1-j}. \tag{19}$$

Using generating functions, the first relation is a consequence of the following equality : $\frac{d}{dz} \exp(e^z - z - 1) = \exp(e^z - 1) - \exp(e^z - z - 1)$. Both relations are mentioned in the OEIS in sequence A000296.

[1]More generally, one could introduce s-associated Bell or Stirling numbers by imposing that each part contains at least s elements but in the present paper we consider only the case $s = 2$. See [2].

2.5.2 Associated Stirling numbers of the second kind $\widehat{S}_{n,k}$

See OEIS sequence A008299 (also see A134991 where they are called Ward numbers).

$$\widehat{S}_{n,k} = \sum_{\ell=0}^{k} (-1)^{\ell} \binom{n}{\ell} S_{n-\ell,k-\ell} \tag{20}$$

Conversely,

$$S_{n,k} = \sum_{\ell=0}^{k-1} \binom{n}{\ell} \widehat{S}_{n-\ell,k-\ell} \tag{21}$$

Their exponential generating function is

$$\widehat{\mathcal{S}}(x,y) = e^{y\,(e^x - x - 1)}. \tag{22}$$

They can be expressed (see OEIS sequence A008299) in terms of the second-order Eulerian numbers $E^{(2)}$, as defined in [9, p. 256], by

$$\widehat{S}_{n,k} = \sum_{\ell=0}^{n-k} \binom{\ell}{n-2k} E^{(2)}_{n-k,n-k-\ell} \tag{23}$$

The $E^{(2)}$'s can themselves be expressed in terms of Stirling numbers of the second kind (see OEIS sequence A340556) by

$$E^{(2)}_{n,k} = \sum_{j=0}^{k} (-1)^{k-j} \binom{2n+1}{k-j} S_{n+j,j} \tag{24}$$

From (23) and (24) one can recover (20).

Notice that

$$C_{n,[1^r,\,\alpha']} = \binom{n}{r} C_{n-r,[\alpha']}, \tag{25}$$

where α' has no singleton.

One can also impose a genus restriction on the partitions without singletons, and as singletons do not affect the genus, one is therefore led to consider numbers $\widehat{B}^{(g)}_n$ and $\widehat{S}^{(g)}_{n,k}$ with, of course $\sum_g \widehat{B}^{(g)}_n = \widehat{B}_n$ and $\sum_g \widehat{S}^{(g)}_{n,k} = \widehat{S}_{n,k}$. Moreover

$$C^{(g)}_{n,[1^r,\,\alpha']} = \binom{n}{r} C^{(g)}_{n-r,[\alpha']}. \tag{26}$$

We shall return to these sequences in the next section.

3 Genus-dependent Bell numbers $B_n^{(g)}$

3.1 Unconstrained partitions: basic numbers $B_n^{(g)}$

3.1.1 Genus 0

Known as Catalan numbers. See OEIS sequence A000108.

$$B_n^{(0)} = \mathcal{C}_n := \frac{1}{(n+1)!} \frac{(2n)!}{n!} \qquad (27)$$
$$= \{1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, \ldots\}$$

The ordinary G.F. is

$$B^{(0)}(x) = \frac{1 - \sqrt{1-4x}}{2x}. \qquad (28)$$

3.1.2 Genus 1

See OEIS sequence A002802. We have

$$B_n^{(1)} = \frac{1}{2^4\, 3} \frac{1}{(2n-3)(2n-1)} \frac{1}{(n-4)!} \frac{(2n)!}{n!}, \qquad (29)$$

The ordinary G.F. is

$$B^{(1)}(x) = \frac{x^4}{(1-4x)^{5/2}}. \qquad \text{See [6].} \qquad (30)$$

3.1.3 Genus 2

The first few terms are listed in the OEIS as sequence A297179. The next formula seems to be new; it is obtained by summing $S_{n,k}^{(2)}$, see (51) given below, over k, the number of parts.

$$B_n^{(2)} = \frac{1}{2^9\, 3^2\, 5} \frac{(5n^3 - 39n^2 + 88n - 84)}{(2n-7)(2n-5)(2n-3)(2n-1)} \frac{1}{(n-6)!} \frac{(2n)!}{n!}, \qquad (31)$$

The ordinary G.F. is

$$B^{(2)}(x) = \frac{x^6\,(1 + 6x - 19x^2 + 21x^3)}{(1-4x)^{11/2}}. \qquad \text{See [6].} \qquad (32)$$

3.1.4 Genus 3

We conjecture that

$$B_n^{(3)} \stackrel{?}{=} \frac{1}{2^{13}\, 3^4\, 5\, 7} \frac{\left(35n^6 - 819n^5 + 7589n^4 - 36009n^3 + 93464n^2 - 129060n + 95040\right)}{(2n-11)(2n-9)(2n-7)(2n-5)(2n-3)(2n-1)} \frac{1}{(n-8)!} \frac{(2n)!}{n!}, \qquad (33)$$

The ordinary G.F. is

$$B^{(3)}(x) \stackrel{?}{=} \frac{x^8\left(1+60x-66x^2-130x^3+1065x^4-2262x^5+1738x^6\right)}{(1-4x)^{17/2}}, \tag{34}$$

This suggests for any $g > 0$ the following Ansatz for the ordinary G.F.

$$B^{(g)}(x) \stackrel{?}{=} \frac{x^{2g+2}P^{(g)}(x)}{(1-4x)^{(6g-1)/2}}, \tag{35}$$

with an overall power of x dictated by (14) and a polynomial $P^{(g)}$ of degree $3(g-1)$.

We shall see in the sequel a repeated appearance of formulae of that type, in particular with the universal "critical exponent" $(6g-1)/2$ in the denominator.

For genus $g \geq 4$, we have incomplete results that corroborate this Ansatz.

3.1.5 Genus 4

The formula below is conjectured, and one should compute $B_n^{(4)}$ for $n = 16, 17, 18, 19$ to determine all the coefficients a_i

$$B_n^{(4)} = \{1, 352, 19261, 541541, 10571561, 162718556\} \text{ for } n = 10, \ldots 15. \tag{36}$$

The ordinary G.F. should be

$$B^{(4)}(x) \stackrel{?}{=} \frac{x^{10}\left(1+306x+4035x^2-16669x^3+63735x^4-136164x^5+a_6x^6+a_7x^7+a_8x^8+a_9x^9\right)}{(1-4x)^{23/2}}.$$

Similarly, we propose for $g = 5, 6$ that

$$B^{(5)}(x) \stackrel{?}{=} \frac{x^{12}(1 + 1320x + 75068x^2 + 218300x^3 + \cdots)}{(1-4x)^{29/2}}$$

$$B^{(6)}(x) \stackrel{?}{=} \frac{x^{14}(1 + 5406x + \cdots)}{(1-4x)^{35/2}}. \tag{37}$$

The first nontrivial coefficient in the numerator of $B^{(g)}(x)$, $g > 0$, appears to be always divisible by 6: $6\times\{1, 10, 51, 220, 901, \ldots\}$, for $g = 2, 3, \ldots$, and we conjecture that this sequence is given by

$$\frac{(d(g) + 8g + 2)(6g-2)!}{C(g)(3g-1)!} - 2(6g-1)$$

in terms of $C(g)$: $C = 12, 30240, 518918400, 28158588057600, 3497296636753920000, \ldots$, and $d(g)$: $d = 0, 10, 68, 318, 1336, 5426, \ldots$, given by

$$C(g) = 3 \times 2^{2g-1}\frac{(2g)!}{g!}\frac{(6g-5)!!}{(2g-3)!!} = 12(2g-1)\frac{(6g-5)!}{(3g-3)!} \tag{38}$$

$$d(g) = \frac{1}{3}(4^{1+g} - 1 - 3(6g-1)). \tag{39}$$

Warning: the sequence $B_n^{(0)}$, shifted in such a way that it starts with 1 for $n = 1$, (resp., $B_n^{(1)}$, shifted in such a way that it starts with 1 for $n = 3$) gives also the number of rooted bi-colored unicellular maps of genus 0 (resp., of genus 1) on n edges. However, the counting of partitions differs from that of maps, as already discussed, and we observe that the above coincidence fails at genus 2 and above. Rooted bi-colored unicellular maps are studied by Goupil and Schaeffer [8].

3.2 Partitions with no singletons: associated numbers $\widehat{B}_n^{(g)}$

For all g one has the recurrence

$$\widehat{B}_n^{(g)} = B_n^{(g)} - \sum_{s=1}^{n} \binom{n}{s} \widehat{B}_{n-s}^{(g)} \text{ with } \widehat{B}_n^{(g)} = 0 \text{ for } n < 2g+2, \text{ and } \widehat{B}_{2g+2}^{(g)} = 1. \quad (40)$$

3.2.1 Genus 0

See OEIS sequence A005043 (Riordan numbers).

$$\widehat{B}_n^{(0)} = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{j}{\lfloor j/2 \rfloor}. \quad (41)$$

The ordinary G.F. is

$$\frac{2}{1 + x + \sqrt{(1-3x)(1+x)}} = \frac{1 - \sqrt{\frac{1-3x}{1+x}}}{2x}, \quad (42)$$

with coefficients

$$0, 1, 1, 3, 6, 15, 36, 91, 232, 603, 1585, 4213, 11298, 30537, 83097, 227475, \ldots.$$

For genus $g > 0$, we again have a general Ansatz for the ordinary G.F. of $\widehat{B}_n^{(g)}$

$$\widehat{B}^{(g)}(x) \stackrel{?}{=} \frac{x^{2(g+1)}(1+x)^{g-1}\, \widehat{P}^{(g)}(x)}{\Delta(x)^{\frac{(6g-1)}{2}}} \quad (43)$$

where $\Delta(x) = (1-3x)(1+x)$ is the discriminant of the algebraic equation satisfied by $\widehat{B}^{(0)}(x)$, namely $\widehat{B}^{(0)}(x) = 1 + (x\widehat{B}^{(0)}(x))^2/(1 - x\widehat{B}^{(0)}(x))$ (see [22]), and $\widehat{P}^{(g)}$ is a polynomial of degree $3(g-1)$. See below.

3.2.2 Genus 1

See OEIS sequence A245551.

$$\widehat{B}_n^{(1)} = \sum_{\ell=0}^{n-4} \frac{(-1)^{n-\ell}\, 3^{\ell-2}}{2^{n-4}} \frac{(2\ell+3)!!(2n-2\ell-5)!!}{\ell!(-\ell+n-4)!}, \quad \text{G.F. } \widehat{B}^{(1)}(x) \text{ with } \widehat{P}^{(1)}(x) = 1, \quad (44)$$

$$0, 0, 0, 1, 5, 25, 105, 420, 1596, 5880, 21120, 74415, 258115, 883883, 2994355, 10051860, \ldots.$$

3.2.3 Genus 2

$\widehat{B}_n^{(2)} = 0, 0, 0, 0, 0, 1, 21, 203, 1512, 9513, 53592, 278355, 1359072, 6318312, 28227199, 122005884, \ldots$

$$\text{G.F. } \widehat{B}^{(2)}(x) \text{ with } \widehat{P}^{(2)}(x) = (1 + 9x - 4x^2 + 9x^3). \qquad \text{See [22].} \tag{45}$$

3.2.4 Genus 3

$\widehat{B}_n^{(3)} = 0, 0, 0, 0, 0, 0, 0, 1, 85, 1725, 21615, 208230, 1685112, 12028588, 78029380, 469278810, \ldots$

$$\text{G.F. } \widehat{B}^{(3)}(x) \text{ with } \widehat{P}^{(3)}(x) \overset{?}{=} (1 + 66x + 249x^2 + 226x^3 + 894x^4 - 480x^5 + 406x^6). \tag{46}$$

3.2.5 Genus 4

$$\widehat{B}_n^{(4)} = 0, 0, 0, 0, 0, 0, 0, 1, 341, 15103, 318318, 4615611\ , 52720668\ , \ldots$$

$$\text{G.F. } \widehat{B}^{(4)}(x) \text{ with } P^{(4)}(x) \overset{?}{=} 1 + 315x + 6519x^2 + 20228x^3 + 65718x^4 + 95247x^5 + \cdots$$

with 4 terms missing.

4 Genus-dependent Stirling numbers $S_{n,k}^{(g)}$

4.1 Partitions with k parts: the numbers $S_{n,k}^{(g)}$

4.1.1 Genus 0

Known as the Narayana numbers. See OEIS sequence A001263.

$$S_{n,k}^{(0)} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1} = \frac{1}{k}\binom{n-1}{k-1}\binom{n}{k-1}. \tag{47}$$

Their two-variable G.F. is

$$S^{(0)}(x,y) = \frac{1 + x - xy - \sqrt{(1 + x - xy)^2 - 4x}}{2x}. \tag{48}$$

4.1.2 Genus 1

Conjectured by Yip [21], proved by Cori and Hetyei [5].

$$S_{n,k}^{(1)} = \frac{1}{6}\binom{n}{2}\binom{n-2}{k-2}\binom{n-2}{k} = \frac{1}{6}\binom{k}{2}\binom{n}{k}\binom{n-2}{k}, \tag{49}$$

$$\text{G.F.} \qquad S^{(1)}(x,y) = \frac{x^4y^2}{((1 + x - xy)^2 - 4x)^{5/2}}. \tag{50}$$

4.1.3 Genus 2

Obtained by Cori and Hetyei. See OEIS sequence A297178.

$$
\begin{aligned}
S^{(2)}_{n,k} =& 8\gamma[n-10,k-6] - 4\gamma[n-10,k-5] - 15\gamma[n-10,k-4] + 10\gamma[n-10,k-3] \\
&+ \gamma[n-10,k-2] - 4\gamma[n-9,k-5] + 39\gamma[n-9,k-4] - 10\gamma[n-9,k-3] \\
&- 4\gamma[n-9,k-2] - 15\gamma[n-8,k-4] - 10\gamma[n-8,k-3] + 6\gamma[n-8,k-2] \\
&- 4\gamma[n-7,k-2] + 10\gamma[n-7,k-3] + \gamma[n-6,k-2]
\end{aligned}
\tag{51}
$$

where

$$
\gamma[n,k] = \frac{\binom{n+10}{5}\binom{n+5}{k}\binom{n+5}{n-k}}{\binom{10}{5}}. \tag{52}
$$

We now introduce some new notation:

Definition 4. Given two 2-indexed sequences a_{st} and b_{st}, let

$$
a *_p b = \sum_{0\le s\le t\le p} a_{st} b_{st}. \tag{53}
$$

Using this notation, the result for $S^{(2)}_{n,k}$ may be simplified in the form

$$
S^{(2)}_{n,k} = \frac{1}{30240}\,\chi^{(2)} *_4 \Big(\frac{\Gamma(n-t)\Gamma(n-t+5)}{\Gamma(k-s-1)\Gamma(k-s+4)\Gamma(n-k+s-t-3)\Gamma(n-k+s-t+2)}\Big) \tag{54}
$$

with a triangular array of constants $\chi^{(2)}$ given by

$$
\chi^{(2)}(t,s) = \begin{array}{ccccc}
1 & & & & \\
-4 & 10 & & & \\
6 & -10 & -15 & & \\
-4 & -10 & 39 & -4 & \\
1 & 10 & -15 & -4 & 8
\end{array}. \tag{55}
$$

Then the G.F. is

$$
S^{(2)}(x,y) = \frac{x^6 y^2\, p^{(2)}(x,y)}{((1+x-xy)^2-4x)^{11/2}} \tag{56}
$$

$$
\begin{aligned}
p^{(2)}(x,y) &= \sum_{\substack{0\le t\le 4\\ 0\le s\le t}} \chi^{(2)}(t,s) x^t y^s \\
&= 1 - x(4-10y) + x^2(6-10y-15y^2) - x^3(4+10y-39y^2+4y^3) \\
&\quad + x^4(1+10y-15y^2-4y^3+8y^4)
\end{aligned}
\tag{57}
$$

as first derived by Cori and Hetyei [6].

4.1.4 Observations and conjectures

For $k = 2$, we have an expression that follows from an exact formula for the two-part partitions of arbitrary genus. See below Section 5.4:

$$S_{n,2}^{(g)} = \binom{n}{2g+2} = \frac{1}{\binom{2g+2}{g+1}} \binom{n}{g+1} \binom{n-g-1}{g+1}. \tag{58}$$

Observation 5. All known data for $k = 3$ and $n \leq 15$, $g \leq 6$ are consistent with

$$S_{n,3}^{(g)} \overset{?}{=} \frac{4^{g+1}-1}{3} \frac{(n-g-1)}{g+2} \binom{n}{2g+3} = \frac{4^{g+1}-1}{3\binom{2g+3}{g+1}} \binom{n}{g+2} \binom{n-g-1}{g+2}. \tag{59}$$

In particular

$$S_{n,3}^{(2)} \overset{?}{=} \frac{21}{4} \binom{n}{7} (n-3) = \frac{3}{5} \binom{n}{4} \binom{n-3}{4} \quad \text{and} \quad S_{n,3}^{(3)} \overset{?}{=} 17(n-4) \binom{n}{9}. \tag{60}$$

Also at given g, for the lowest $n = 2g+3$, $S_{2g+3,3}^{(g)} = \frac{4^{g+1}-1}{3}$, OEIS sequence A002450.

Observation 6. The result obtained in (54) for $S_{n,k}^{(2)}$ can be generalized in terms of an expression that encodes all (presently) known results for $S_{n,k}^{(g)}$, $g \geq 1$. This expression is as follows

$$S_{n,k}^{(g)} \overset{?}{=} \frac{1}{C(g)} \chi^{(g)} *_{4g-4} \left(\frac{\Gamma(n-t+g-2)\Gamma(n-t+4g-3)}{\Gamma(k-s-1)\Gamma(k-s+3g-2)\Gamma(n-k+s-t-2g+1)\Gamma(n-k+s-t+g)} \right) \tag{61}$$

$$= \frac{1}{C(g)} \chi^{(g)} *_{4g-4} \left((3g-1)! \binom{n-k+s-t+g-1}{n-k+s-t-2g} \binom{n-t+g-3}{k-s-2} \binom{n-t+4g-4}{n-k+s-t+g-1} \right) \tag{62}$$

with the integer constant $C(g)$ given in (38) and coefficients $\chi^{(g)}$ given for $g = 3, 4$ by

$$\chi^{(3)} = \begin{array}{ccccccccc} 1 \\ -8 & 68 \\ 28 & -340 & 246 \\ -56 & 612 & 294 & -980 \\ 70 & -340 & -3390 & 4480 & 245 \\ -56 & -340 & 5700 & -3500 & -5530 & 1464 \\ 28 & 612 & -3390 & -3500 & 11020 & -1824 & -1208 \\ -8 & -340 & 294 & 4480 & -5530 & -1824 & 2944 & -16 \\ 1 & 68 & 246 & -980 & 245 & 1464 & -1208 & -16 & 180 \end{array}, \quad \chi^{(4)} = \begin{array}{cccccc} 1 \\ -12 & 318 \\ 66 & -2862 & 6831 \\ -220 & 11130 & -33651 & 6072 \\ 495 & -23850 & 30123 & 156660 & -99693 \\ -792 & 28620 & \cdots \\ 924 & -13356 \\ -792 & -13356 \\ 495 & 28620 \\ -220 & -23850 & \cdots \\ 66 & 11130 & 30123 \\ -12 & -2862 & -33651 & 156660 \\ 1 & 318 & 6831 & 6072 & -99693 & \cdots \end{array}$$

where the entries of $\chi^{(3)}$ have been determined from a subset of the existing data, but some of those of $\chi^{(4)}$ are still undetermined at this stage.

Observation 7.

- Each column of the arrays $\chi^{(g)}$ is symmetric.

- The entries of the first column of $\chi^{(g)}$, which is of length $4g-3$, are binomial coefficients with alternated signs $(-1)^t\binom{4(g-1)}{t}$.
- The second column is the product of the line $4(g-1)$ of the triangular array OEIS sequence A144431 (a "sub-Pascal array") by the coefficient $d(g)$ given above in (39).
- This second column can also be obtained as $d(g)$ times an appropriate line of a matrix defined as the inverse of the matrix of partial sums of the signed Pascal triangle (see OEIS sequence A059260).
- The last $(g-1)$ lines of the array $\chi^{(g)}$ have a vanishing sum (a justification is given below).
- The last line of the array $\chi^{(g)}$ is conjectured to be given by (64) (details are given below).

At genus g, the first non-zero coefficients $S^{(g)}_{n,k}$ appear for $(n,k) = (2g+2,2)$, and their "experimental" values up to $n = 4(2g-1)$, $k = 2(2g-1)$ can be used to determine the constants $\chi^{(g)}$ but one can lower these two integers by making use of the previous observations.

As already mentioned, (62), evaluated at $g = 1,2$ gives back $S^{(1)}_{n,k}$ and $S^{(2)}_{n,k}$; moreover its evaluation at $g \geq 3$ is compatible with all presently known "experimental" results, with the Ansatz (59), and the sum over k of $S^{(3)}_{n,k}$ is indeed equal to $B^{(3)}_n$.

This justifies the following conjectures:

Conjecture 8 (Genus $g = 3$ conjecture (weak form))**.** The expression (62), with $g = 3$, gives $S^{(3)}_{n,k}$ for all n and k.

Conjecture 9 (Genus g conjecture (strong form))**.** The expression (62), together with an appropriate triangular array of constants $\chi^{(g)}$ gives $S^{(g)}_{n,k}$ for all $n, k, g > 0$.

The corresponding Ansatz on the G.F. is

$$\text{G.F.} \qquad S^{(g)}(x,y) \overset{?}{=} \frac{x^{2g+2}y^2\, p^{(g)}(x,y)}{((1+x-xy)^2-4x)^{(6g-1)/2}} \tag{63}$$

with $p^{(g)}(x,y) = \sum_{\substack{0\leq t\leq 4(g-1)\\ 0\leq s\leq t}} \chi^{(g)}(t,s)x^t y^s$.

We have $S^{(g)}(x,1) = B^{(g)}(x)$, $p^{(g)}(x,1) = P^{(g)}(x)$. The latter polynomial being (conjectured) of degree $3(g-1)$ in x, this tells us that the last $(g-1)$ lines of the array $\chi^{(g)}$ have a vanishing sum.

A further conjecture, in accordance with the existing data, is that the terms of highest degree in x, viz $4(g-1)$, of $p^{(g)}(x,y)$ are of the form

$$[p^{(g)}(x,y)]_{x^{4(g-1)}} = (1-y)^{2(g-1)}\left[(1-y)^{4g+1}y^{-2g-3}\sum_{j=0}^{2g-2}\frac{2\; s_{2g+2+j,\,j+1}}{(2g+j+2)(2g+j+1)}y^{-j}\right]_+ \tag{64}$$

where $[\cdot]_+$ is the polynomial part in y of the expression and $s_{p,q}$ are the Stirling numbers of the first kind. See OEIS sequence A185259 where these polynomials are tabulated. If true, this conjecture determines the last line $\chi^{(4)}(13,s)$ of $\chi^{(4)}$ to be

$$\{1, 318, 6831, 6072, -99693, 103950, 196581, -413820, 155628, 146168, -117876, 7776, 8064\}.$$

4.1.5 Particular cases: $S^{(3)}_{n,k}$ and $S^{(4)}_{n,k}$ for small k

The above general conjecture for $S^{(g)}_{n,k}$ leads, when $g = 3, 4$, and small values of $k = 2, 3, 4$, to simple enough formulae that are displayed below. For $k = 2$, they follow from (58). One can check that they are compatible with the known (experimental) values of $S^{(3)}_{n,k}$, up to $n = 15$, (see tables in the appendix).

Genus 3.

$S^{(3)}_{n,2} = \binom{n}{8} = (0, 0, 0, 0, 0, 0, 0, 1, 9, 45, 165, 495, 1287, 3003, 6435, \dots)$ See OEIS sequence A000581.

$S^{(3)}_{n,3} \stackrel{?}{=} 17(n-4)\binom{n}{9} = (0, 0, 0, 0, 0, 0, 0, 0, 85, 1020, 6545, 29920, 109395, 340340, 935935, \dots)$

$S^{(3)}_{n,4} \stackrel{?}{=} \frac{5}{3}(32n^2 - 288n + 613)\binom{n}{10} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 1555, 24145, 194150, 1085370, 4759755, 17482465, \dots)$

Genus 4.

$S^{(4)}_{n,2} = \binom{n}{10} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 11, 66, 286, 1001, 3003, \dots)$ See OEIS sequence A001287.

$S^{(4)}_{n,3} \stackrel{?}{=} \frac{341}{6}(n-5)\binom{n}{11} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 341, 4774, 35464, 186186, 775775, \dots)$

$S^{(4)}_{n,4} \stackrel{?}{=} \frac{11}{2}(65\,n\,(n-11) + 1842)\binom{n}{12} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 14421, 252538, 2288286, 14369355, \dots)$

4.2 Partitions with no singletons: associated numbers $\widehat{S}^{(g)}_{n,k}$

4.2.1 Genus 0

See OEIS sequence A108263.

$$\widehat{S}^{(0)}_{n,k} = \frac{1}{(n-k+1)}\binom{n-k-1}{n-2k}\binom{n}{k}. \tag{65}$$

$$\text{The ordinary G.F. is } \frac{1 + x - \sqrt{(1-x)^2 - 4x^2y}}{2x(xy+1)}. \tag{66}$$

4.2.2 Genus 1

$$\widehat{S}^{(1)}_{n,k} = \frac{1}{6}\binom{k}{2}\binom{n}{k}\binom{n-k}{k}. \tag{67}$$

$$\text{The ordinary G.F. is } \frac{x^4y^2}{((1-x)^2 - 4x^2y)^{5/2}}. \tag{68}$$

4.2.3 Genus 2 and above

It is conjectured that for generic genus $g > 0$, the G.F. has the form

$$\widehat{S}^{(g)}(x,y) \stackrel{?}{=} \frac{x^{2g+2}y^2\hat{p}^{(g)}(x,y)}{((1-x)^2-4x^2y)^{(6g-1)/2}} \tag{69}$$

with $\hat{p}^{(g)}(x,y)$ a polynomial of degree $4(g-1)$ in x. For instance

$$\begin{aligned}\hat{p}^{(2)}(x,y) &= 1+2x(-2+7y)+x^2(6-22y+21y^2)\\ &\quad +x^3(-4+2y+7y^2)+x^4(1+6y-19y^2+21y^3),\end{aligned} \tag{70}$$

as derived in [22].

Remark 10. Note that $p^{(2)}(x,0) = \hat{p}^{(2)}(x,0) = (1-x)^4$ so that the term of order y^2 in $S^{(2)}(x,y)$ or $\widehat{S}^{(2)}(x,y)$, i.e., the G.F. of genus 2 partitions into two parts with or without singleton, is $x^6/(1-x)^7 = \frac{1}{2}\sum_{n=6}^{\infty}\frac{n}{3}x^n\sum_{p=1}^{n-1}\binom{p-1}{2}\binom{n-p-1}{2}$, in agreement with formula (92) below.

By the same token, we may assert that $p^{(g)}(x,0) = \hat{p}^{(g)}(x,0) = (1-x)^{4(g-1)}$ so that the term of order y^2 in $S^{(g)}(x,y)$ or $\widehat{S}^{(g)}(x,y)$ is

$$x^{2g+2}/(1-x)^{2g+3} = \frac{1}{2}\sum_{n=2g}^{\infty}\frac{n}{g+1}x^n\sum_{p=1}^{n-1}\binom{p-1}{g}\binom{n-p-1}{g}.$$

This implies that

$$\widehat{S}^{(0)}_{n,2} = S^{(0)}_{n,2} - n = \frac{1}{2}n(n-3) \quad \text{and} \quad \widehat{S}^{(g)}_{n,2} = S^{(g)}_{n,2} = \binom{n}{2g+2} \text{ for } g>0 \tag{71}$$

in agreement with the result (58). These numbers can be recognized as the elements of the array in OEIS A275514.

4.2.4 Particular case $n = 2k$

Since we assume in this section that there are no singletons and since k denotes the number of parts, the equality $n = 2k$ implies that each part is of length 2, so that the type is determined, $[\alpha] = [2^k]$, and $\widehat{S}^{(g)}_{2k,k} = C^{(g)}_{2k,[2^k]}$. Because of this coincidence, we postpone the study of this particular case to the next section, which is devoted to the study of coefficients $C^{(g)}_{n,[\alpha]}$.

5 Genus-dependent Faà di Bruno coefficients $C^{(g)}_{n,[\alpha]}$. Part I. Fully solved cases

The genus-dependent Faà di Bruno coefficients $C^{(g)}_{n,[\alpha]}$ are explicitly known in many specific cases, for particular types $[\alpha] = [\cdots \ell^{\alpha_\ell} \cdots]$ and/or for particular values of the genus g, most of them discussed and summarized in Section 6. However, to the best of our knowledge, they are generically known in only three families of cases, two of them are classic—the cases of genus 0, for all types, and the partitions of type $[2^p]$, for all g—and the third one is new, the partitions into two parts, i.e., of type $[p, n-p]$, for all g. We review these three cases in this section. In addition, the G.F. of all types of partitions are explicitly known for genus 1 and 2 (see [22]), although the extraction of explicit formulae for the Faà di Bruno coefficients is arduous.

5.1 The particular case $g = 0$

$C^{(0)}_{n,[\alpha]}$ is the number of *noncrossing* partitions (also called planar partitions) of type $[\alpha]$.

$$C^{(0)}_{n,[\alpha]} = \frac{n!}{(n+1-\sum \alpha_\ell)!\ \prod_\ell \alpha_\ell!} = \frac{1}{n+1}\binom{n+1}{\alpha_1, \dots, \alpha_n, n+1-\sum \alpha_j} \tag{72}$$

where the symbol $\binom{\cdots}{\cdots}$ denotes a multinomial coefficient. It was first derived by Kreweras [15], and reappeared later in the context of large random matrices [1] and of free probabilities and their free (or noncrossing) cumulants [18]. One may also collect these expressions into a G.F.

$$Z^{(0)}(x) = 1 + \sum_{n\geq 1} x^n \sum_{[\alpha]\vdash n} C^{(0)}_{n,[\alpha]} \prod_{\ell=1}^{n} \kappa_\ell^{\alpha_\ell} \tag{73}$$

where the κ_ℓ are new indeterminates, from which we may also construct the function

$$W(x) = \sum_{\ell\geq 1} \kappa_\ell x^\ell. \tag{74}$$

Then, it was shown in [1] that (72) is equivalent to the following functional relation between $Z^{(0)}$ and W

$$Z^{(0)}(x) = 1 + W(x\, Z^{(0)}(x)). \tag{75}$$

Also see [7] for a nice graphical interpretation of that identity.

5.2 Genus 1 and 2

As recalled above, the genus 1 and 2 G.F. have been constructed in [22]. We shall use them in the following to substantiate some remarks and conjectures. For illustration, we recall here the expression of the G.F. in genus 1. Let

$$V(x) = xW'(x)\,, \quad X_2(x) = xW'(x) - W(x)\,, \quad Y_2(x) = \frac{1}{2}x^2 W''(x)\,, \quad \tilde{x} = xZ^{(0)}(x). \tag{76}$$

Then

$$Z^{(1)}(x) = \frac{X_2(\tilde{x})Y_2(\tilde{x})}{(1-X_2(\tilde{x}))^4(1-V(x))}. \tag{77}$$

The more cumbersome expression of $Z^{(2)}(x)$ will not be recalled here.

Note that even though we have explicit expressions of their G.F., the determination of Faà di Bruno coefficients for $g = 1$ or 2 is still implicit, contrary to formulae (72).

5.3 Type $[\alpha] = [2^k]$

So $n = 2k$ (k parts of length 2) and $g \le \frac{k}{2}$.

If we focus on the terms with $[\alpha] = [2^k]$, it suffices to specialize the indeterminates κ to $\kappa_\ell = \kappa_2 \delta_{\ell,2}$. By a small abuse of notation, we still use $Z^{(g)}(x)$ and $W(x)$ for these specialized G.F. As already mentioned and explained at the end of Section 4.2, $C^{(g)}_{2k,[2^k]}$ is known for all g and coincides with $\widehat{S}^{(g)}_{2k,k}$. This famous case was first solved by Walsh and Lehman [19, 20] by combinatorial methods; then by Harer and Zagier [10], in the context of the virtual Euler characteristics of the moduli space of curves, by means of matrix integrals; and by Jackson by a character theoretic approach [13]. It has been the object of an abundant literature since then, a good review of which is given in [16]. Also see [3]. The reason for which this case can be solved for arbitrary genus is that the crucial constraint of monotonicity of the cycles of τ is here irrelevant, and we are just dealing with ordinary maps. One finds that

$$\widehat{S}^{(g)}_{2k,k} = C^{(g)}_{2k,[2^k]} = \frac{(2k)!}{(k+1)!(k-2g)!} \left[\left(\frac{u/2}{\tanh u/2} \right)^{k+1} \right]_{u^{2g}} \tag{78}$$

where the notation $[Y]_{u^k}$ means the coefficient of u^k in expression Y. The first few terms are given in Table 1.

g	0	1	2	3	4
$k=1$	1				
$k=2$	2	1			
$k=3$	5	10			
$k=4$	14	70	21		
$k=5$	42	420	483		
$k=6$	132	2310	6468	1485	
$k=7$	429	12012	66066	56628	
$k=8$	1430	60060	570570	1169740	225225

Table 1: Values of $\widehat{S}^{(g)}_{2k,k}$.

The $g = 0$ column is, by (72): $C^{(0)}_{2k,[2^k]} = \frac{1}{k+1}\binom{2k}{k} = \mathcal{C}_k$ (Catalan numbers)[2], whose G.F. is

$$Z^{(0)}(u) = 1 + \sum_{k=1}^{\infty} C^{(0)}_{2k,[2^k]} u^k = \frac{1-\sqrt{1-4u}}{2u} \tag{79}$$

[2]It therefore coincides with $B^{(0)}_k$.

which satisfies

$$Z^{(0}(u) = 1 + u(Z^{(0}(u))^2. \tag{80}$$

(This is the equation (75) expressed here for $W(x) = \kappa_2 x^2$ in the variable $u = \kappa_2 x^2$.) The $g = 1$ column is $\frac{(2k-1)!}{6(k-2)!(k-1)!} = \binom{2k-1}{3}\mathcal{C}_{k-2} = \frac{(k+1)k(k-1)}{12}\mathcal{C}_k$. See OEIS sequence A002802. The k-th row's sum is, by (8), given by $(2k-1)!!$, viz $\{1, 1, 3, 15, 105, 945, \ldots\}$. More generally,

$$C^{(g)}_{2k,[2^k]} = \frac{1}{2^g}\mathcal{C}_k\, R_g(k) \tag{81}$$

with $R_g(k)$ a polynomial of degree $3g$ in k [19, 10] which (for $g > 0$) vanishes for $k = -1, 0, \ldots, 2g-1$, and whose form can be made explicit [8, 3]. There are several expressions for the values that it takes when its argument k is an arbitrary non-negative integer. One of them, in terms of *unsigned* Stirling numbers of the *first* kind $c_{p,q}$[3], are as follows [4]:

$$R_g(k) = \sum_{s=0}^{k}\binom{k}{s}\sum_{j=0}^{k+2-2g}(-1)^{s+1-j}\; c_{k-s+1,k+2-2g-j}\; c_{s+1,j}. \tag{82}$$

As noticed in [10], the numbers $C^{(g)}_{2k,[2^k]}$ (called $\epsilon_g(k)$ there) satisfy a recurrence formula

$$(k+1)C^{(g)}_{2k,[2^k]} = 2(2k-1)C^{(g)}_{2(k-1),[2^{k-1}]} + \frac{1}{2}(2k-1)(2k-2)(2k-3)C^{(g-1)}_{2(k-2),[2^{k-2}]} \tag{83}$$

or in terms of the polynomials R introduced in (81),

$$R_g(k) = R_g(k-1) + \binom{k}{2}R_{g-1}(k-2). \tag{84}$$

In [3], Chapuy gave a combinatorial interpretation of that formula, from which he derived another recurrence equation

$$2gC^{(g)}_{2k,[2^k]} = \sum_{h=1}^{g}\binom{k+2h+1-2g}{2h+1}C^{(g-h)}_{2k,[2^k]}. \tag{85}$$

Proposition 11. *The generating function of the $C^{(g)}_{2k,[2^k]}$ for $g > 0$ is of the form*

$$Z^{(g)}(u) := \sum_k C^{(g)}_{2k,[2^k]}u^k = \frac{u^{2g}\,Q^{(g)}(u)}{(1-4u)^{(6g-1)/2}}, \tag{86}$$

where $Q^{(g)}(u)$ is a polynomial of degree $g-1$ in u satisfying

$$Q^{(g)}(0) = \frac{(4g)!}{2^{2g}(2g+1)!}. \tag{87}$$

[3] Here $c_{p,q}$ is the number of permutations of p elements that have q distinct cycles. They are positive integers such that $s_{p,q} = (-1)^{p-q}\,c_{p,q}$ where the $s_{p,q}$, the Stirling numbers of the first kind, obey $\sum_{p\geq 1} S_{n,p}\, s_{p,q} = \delta_{n,q}$.

Proof. One finds by explicit calculation that $Q^{(1)}(u) = 1$. Equation (83) implies that $Q^{(g)}(u)$ satisfies the following recurrence formula

$$\begin{aligned}
&(1-4u)u\tfrac{d}{du}Q^{(g)}(u)+\Big(2g+1+4u(g-1)\Big)Q^{(g)}(u) = \\
&\quad\Big(3\tbinom{4g-1}{3}+6(-1+4g)(1-14g+8g^2)u+96(g-2)(4g^2-8g-1)u^2+128(g-2)(g-3)(2g-5)u^3\Big)Q^{(g-1)}(u)\\
&\quad+\Big(48g^2-24g+3+24(8g^2-20g-1)u+192(g-3)^2u^2\Big)u(1-4u)\tfrac{d}{du}Q^{(g-1)}(u)\\
&\quad+24\Big(g+u(2g-7)\Big)u^2(1-4u)^2\tfrac{d^2}{du^2}Q^{(g-1)}(u)+4u^3(1-4u)^3\tfrac{d^3}{du^3}Q^{(g-1)}(u)
\end{aligned} \tag{88}$$

that we abbreviate by

$$\mathcal{D}Q^{(g)}(u) = \widehat{\mathcal{D}}Q^{(g-1)}(u)$$

with two linear differential operators in the variable u, $\mathcal{D}$ and $\widehat{\mathcal{D}}$. Equation (88) for $u = 0$ fixes the ratio $Q^{(g)}(0)/Q^{(g-1)}(0) = \frac{(4g-3)(4g-2)(4g-1)}{2(2g+1)}$, which leads to the expression (87) above. The proof of (86) is obtained by induction assuming that $Q^{(g-1)}(u)$ is a polynomial of degree $g-2$. The polynomiality property of $Q^{(g)}(u)$ is then proved as follows: the rhs of (88) might seem to be of degree $g+1$, but in fact its degree is only $g-1$ because of trivial identities

$$\begin{aligned}
\big[\widehat{\mathcal{D}}u^{g-2}\big]_{u^{g+1}}&=128(g-2)(g-3)(2g-5)-4\times192(g-2)(g-3)^2+4^2\times24(2g-7)(g-2)(g-3)+4\times(-4)^3(g-2)(g-3)(g-4)\\
&= 128(g-2)(g-3)\Big((2g-5)-6(g-3)+3(2g-7)-2(g-4)\Big)\equiv0\\
\big[\widehat{\mathcal{D}}u^{g-3}\big]_{u^g}&=128(g-2)(g-3)(2g-5)-4\times192(g-3)^3+4^2\times24(2g-7)(g-3)(g-4)+4\times(-4)^3(g-3)(g-4)(g-5)\equiv0\\
\big[\widehat{\mathcal{D}}u^{g-2}\big]_{u^g}&=96(g-2)\big(4g^2-8g-1\big)+(g-2)\Big(192(g-3)^2-96\big(8g^2-20g-1\big)\Big)+1344(g-3)(g-2)+192(g-4)(g-3)(g-2)\equiv0,
\end{aligned} \tag{89}$$

using the notation introduced in (78). Likewise, in the lhs, $[\mathcal{D}u^{g-1}]_{u^g} \equiv 0$, so that it is consistent to look for a polynomial solution of degree $g-1$ for $Q^{(g)}(u)$. Write $Q^{(g)}(u) = \sum_{r=0}^{g-1} q_r u^r$. Then equation (88) recursively determines the coefficients q_r in terms of those of $Q^{(g-1)}(u)$

$$\big[\mathcal{D}Q^{(g)}(u)\big]_{u^r} = (2g+1+r)q_r + 4(g-r)q_{r-1} = \big[\mathcal{D}Q^{(g-1)}(u)\big]_{u^r} \quad \text{for } 1 \le r \le g-1.$$

Together with the value of $q_0 = Q^{(g)}(0)$ given above, this completely determines all coefficients q_r and completes the proof that $Q^{(g)}(u)$ is a polynomial. □

Note that the expression $(1-4u)$ in the denominator of (86) is—once again—nothing else than the discriminant of the equation (80) satisfied by $Z^{(0)}$.

The first few $Q^{(g)}$ are

$$\begin{gathered}
Q^{(1)}(u) = 1\ ; \quad Q^{(2)}(u) = 21(1+u)\ ; \quad Q^{(3)}(u) = 11(135+558u+158u^2)\ ;\\
Q^{(4)}(u) = 11\times13(1575+13689u+18378u^2+2339u^3)\ ;\\
Q^{(5)}(u) = 3\times13\times17\times19\,(4725+67620u+201348u^2+132356u^3+9478u^4)\ ;\ \cdots
\end{gathered} \tag{90}$$

with $Q^{(g)}(0) = \{1, 21, 1485, 225225, \ldots\} = C^{(g)}_{4g,[2^{2g}]} = \frac{1}{2^g}\,\mathcal{C}_{2g}\,R_g(2g) = \frac{(4g)!}{2^{2g}(2g+1)!}$.
See OEIS sequence A035319.

5.4 Partitions into two parts: type $[\alpha] = [p, n-p]$

In this section, we make use of both the description of partitions by pairs (σ, τ) of permutations and the diagrammatic representation presented in Section 2.4 and apply them to the case of a partition of $\{1, \ldots, n\}$ into two parts of size p and $n-p$. Thus in the diagrammatic representation, n points lie on the circle, numbered and clockwise ordered from 1 to n, while two vertices of valence p and $n-p$, associated by the two increasing cycles of τ are inside the disk, and their edges connect to the points on the circle, *with no crossing of edges originating from the same vertex*. The faces of the map correspond to the cycles of the permutation $\sigma \circ \tau^{-1}$.

Example 12. Figure 2(a) shows a diagrammatic representation of a partition of {1,...,12} into two parts α = ({1,4,5,8,10},{2,3,6,7,9,11,12}). Thus τ = (1,4,5,8,10),(2,3,6,7,9,11,12), σ = (1,2,...,12), and the product $\sigma \circ \tau^{-1}$ = (1,11,10,9,8,6,4,2)(3)(5)(7)(12) has 5 cycles. The map has $f = 5$ faces, which can also be seen by following the circuit indicated by arrows on the figure, and its genus is thus 3 by applying (11).

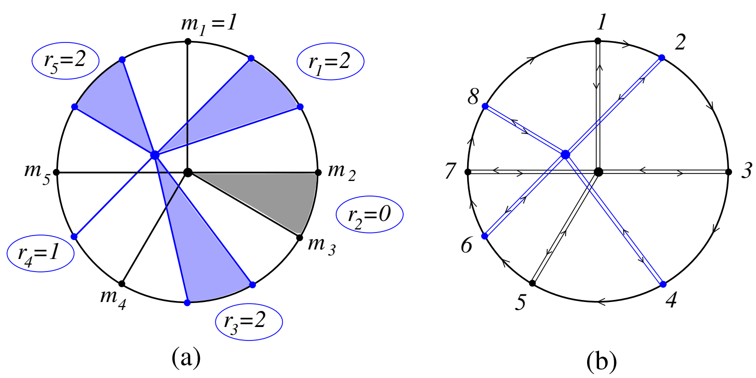

Figure 2: (a) Partition with $n = 12$, $p = 5$, $f = 5$, $s_1 = 1$, $s_2 = 3$, $g = 3$; (b) removing shaded faces, i.e., singletons of $\sigma \circ \tau^{-1}$, relabeling the points and doubling the edges to make the one cycle (or face) $(1, 8, 7, 6, 5, 4, 3, 2)$ more visible, $f' = 1$.

Let C_1 and C_2 denote the two (increasing) cycles of τ, $m_1 < m_2 < \cdots < m_p$ the p elements of C_1, and $p_1 < p_2 < \cdots < p_{n-p}$ those of C_2. (On the figure, we chose $m_1 = 1$ and the corresponding points on the circle are black, those of C_2 are blue.)

Let us consider a singleton x of $\sigma \circ \tau^{-1}$: $x = \sigma(\tau^{-1}(x)) = \tau^{-1}(x) + 1$, i.e., $x = \tau(x-1)$, meaning that $x-1$ and x belong to the same cycle of τ. Let s_1, resp., s_2, be the number of such singletons in cycle C_1, resp., C_2. Diagrammatically, they count the numbers of pairs of edges exiting the same vertex and reaching consecutive points on the circle, or the numbers of grey, resp., blue shaded faces on the figure. Let r_i be the number of integers strictly between m_i and m_{i+1}, $i = 1, 2, \ldots, p$, (with the convention $m_{p+1} \equiv m_1$). Thus $\sum_i r_i = n-p$. For each of the s_1 indices i for which $r_i = 0$, (i.e., $m_{i+1} = m_i + 1$), there is a singleton of

$\sigma \circ \tau^{-1}$ in C_1 (and the associated face on the figure is grey shaded); for each i for which $r_i \geq 1$, there are $r_i - 1$ singletons in C_2 (i.e., blue faces attached to the second vertex). In our example of Fig. 2 there are respectively one and three such faces. The total number of singletons (= shaded faces) is thus

$$\sum_{\substack{i \\ r_i=0}} 1 + \sum_{\substack{i \\ r_i \geq 1}} (r_i - 1) = s_1 + \big((n-p) - (p - s_1)\big) = n - 2p + 2s_1.$$

Interchanging the roles of C_1 and C_2 tells us that it is also equal to $n - 2(n-p) + 2s_2$, hence that

$$s_2 = s_1 + n - 2p.$$

In addition to these singletons, $\sigma \circ \tau^{-1}$ has f' other cycles (= unshaded faces), and a total of $f = s_1 + s_2 + f' = n - 2p + 2s_1 + f'$ faces, whence the genus

$$g = \frac{n-1-f}{2} = p - s_1 - \frac{f'+1}{2} \leq p - 1. \tag{91}$$

or by symmetry between the two vertices, $g \leq n-p-1$, which shows that $g \leq \min(p-1, n-p-1)$. Clearly the minimum value $g = 0$ is reached for $f' = 1$, $s_1 = p - 1$, i.e., with the p-vertex being attached to p consecutive points on the circle. In fact the previous inequality $f' \geq 1$ is an equality. In other words, we have, with the previous notation, the following lemma:

Lemma 13. *For any partition of type $[p, n-p]$, we have $f' = 1$, $g = p - s_1 - 1 = (n-p) - s_2 - 1$.*

Proof. In the spirit of the reduction of diagrams to irreducible ones of the same genus, in Cori–Hetyei procedure [5, 6, 22], we now define a new partition of a new set: the s_1 points $m_{i+1} = m_i + 1$ and s_2 points $p_{j+1} = p_j + 1$ are deleted, n reduced to $n' = n - s_1 - s_2$, p to $p' = p - s_1 = n - p - s_2$, all entries relabeled from 1 to n', but the genus is unchanged $2g' = n' - f' - 1 = 2g$. We are now dealing with a partition of $\{1, \dots, n'\}$ in two parts of equal length p', with associated permutations σ', cyclic on $\{1, \dots, n'\}$, and τ' of type $[p', p']$ with two increasing cycles C'_1 and C'_2. By construction, $\sigma' \circ \tau'^{-1}$ has no singleton. We claim that such a $\sigma' \circ \tau'^{-1}$ has only one cycle, hence is a cyclic permutation.

Since we assume that $\sigma' \circ \tau'^{-1}$ has no singleton, this means that for any $i_1 \in C'_1$, say, $i_2 := \sigma'(\tau'^{-1}(i_1)) = \tau'^{-1}(i_1) + 1$ satisfies $\tau'^{-1}(i_1) < i_2 < i_1$ since each cycle of τ' is increasing and no singleton appears. But again by the monotonicity of each of the cycles of τ', there cannot be between $\tau'^{-1}(i_1)$ and i_1 an element in the same cycle C'_1. This means that i_2 belongs to the other cycle C'_2.

Suppose that $i_2 < i_1 - 1$ and consider the point $i_2 + 1 = \sigma'(i_2)$. Being between $\tau^{-1}(i_1)$ and i_1 it cannot belong to the same cycle C'_1 as i_1. It cannot either belong to cycle C'_2 because it would then equal $\tau'(i_2)$ and thus make a singleton of $\sigma' \circ \tau'^{-1}$: $\sigma'(\tau'^{-1}(i_2 + 1)) = \sigma'(i_2) = i_2 + 1$. Thus the hypothesis $i_2 < i_1 - 1$ must be rejected and we conclude that

$i_2 = \sigma'(\tau'^{-1}(i_1)) = i_1 - 1 = \sigma'^{-1}(i_1)$. Thus $\sigma'^2(\tau'^{-1}(i_1)) = i_1$ and since this applies to arbitrary i_1, we conclude that

$$\sigma'^2 = \tau'$$

i.e., that τ' is the product of two cyclic permutations acting on even, resp., odd integers between 1 and n'; or, in other words, that $\sigma' \circ \tau'^{-1}$ is the cyclic permutation σ'^{-1}. □

The property $f' = 1$, established in the above lemma, may be rephrased as follows: *Let τ be a permutation of $\mathcal{S}_n$ defined as the product of two disjoint increasing cycles, and σ be the circular permutation $(1, 2, ..., n)$, then the cyclic decomposition of the product $\sigma \circ \tau^{-1}$ contains only one non-trivial cycle.*

We could not find this property mentioned in the literature.

Proposition 14. *For general $n \neq 2p, g$, and $p \geq 2$, (otherwise, if $n = 2p$, multiply by $\frac{1}{2}$)*

$$C^{(g)}_{n,[p,n-p]} = \frac{n}{g+1}\binom{p-1}{g}\binom{n-p-1}{g}. \tag{92}$$

Note that this result is symmetric under $p \leftrightarrow n-p$ as it should, and that the bound $g \leq \min(p-1, n-p-1)$ is manifest. An alternative expression is

$$C^{(g)}_{n,[p,n-p]} = \frac{n}{p}\binom{p}{p-1-g}\binom{n-p-1}{g}. \tag{93}$$

Proof. Following the lines of Lemma 13, a partition is completely determined by the choice of the p points m_i on the circle, subject to the condition that there are s_1 pairs of adjacent points. Thus

$$C^{(g)}_{n,[p,n-p]} = \#\{m_1, \dots, m_p \in \{1, n\} \,|\, \#\{i | m_{i+1} = m_i + 1\} = s_1\}, \tag{94}$$

where $g = p - s_1 - 1$. This number may be easily computed by a transfer matrix technique[4]. Suppose the black and blue points of the circle are representing two states of a periodic system on a circle, and assign a weight $1, t, t^2u$ to a transition (i.e., an arc on the circle) between respectively black-black, black-blue and blue-blue points. The matrix

$$M = \begin{pmatrix} 1 & t \\ t & t^2u \end{pmatrix}$$

describes the possible transitions between these states. The number $C^{(g)}_{n,[p,n-p]}$ is then the coefficient of $t^{2p}u^{p-1-g}$ in

$$t_n = \operatorname{tr} M^n. \tag{95}$$

Let $z := t^2$. By virtue of the characteristic equation of M, the numbers t_n satisfy the recurrence relations

$$t_n = t_1 t_{n-1} + z(1-u)t_{n-2} \tag{96}$$

[4]We are quite grateful to Philippe Di Francesco for this suggestion.

with $t_0 = 2$ and $t_1 = 1 + zu$, whence $t_2 = 1 + 2z + z^2u^2$, $t_3 = 1 + 3z + 3z^2u + z^3u^3$. It follows from (96) that

$$[t_n]_{z^pu^{p-1-g}} = [t_{n-1}]_{z^pu^{p-1-g}} + [t_{n-1}]_{z^{p-1}u^{p-2-g}} + [t_{n-2}]_{z^{p-1}u^{p-1-g}} - [t_{n-2}]_{z^{p-1}u^{p-2-g}}, \tag{97}$$

from which all $[t_n]_{z^pu^{p-1-g}}$ may be reconstructed for $p \le n-1, 0 \le g \le p-1$.

Let $D_{n,p,g} := \frac{n}{g+1}\binom{p-1}{g}\binom{n-p-1}{g}$, one may check that the D's satisfy the same relation, namely

$$D_{n,p,g} = D_{n-1,p,g} + D_{n-1,p-1,g} + D_{n-2,p-1,g-1} - D_{n-2,p-1,g}, \tag{98}$$

with $D_{2,1,0} = [t_2]_{zu^0} = 2,\ D_{2,1,1} = [t_2]_{zu^{-1}} = 0,\ D_{3,2,0} = [t_3]_{z^2u} = 3,\ D_{3,2,1} = [t_3]_{z^2u^0} = 0,\ D_{2,1,0} = [t_2]_{zu^0} = 0$. Thus $C^{(g)}_{n,[p,n-p]}$ is equal to $D_{n,p,g}$, which completes the proof of Proposition 14. □

Remark 15. The formula (92), originally proposed for $p \ge 2$, extends trivially to all p: from (26), we learn that $C^{(g)}_{n,[1,n-1]} = nC^{(g)}_{n-1,[n-1]} = n\delta_{g\,0}$, in accordance with the rhs of (92) evaluated at $g = 0$.

Remark 16. One may check that the expression of $C^{(g)}_{n,[p,n-p]}$ is consistent with the Faà di Bruno coefficient (8): $\sum_{g=0}^{p-1} C^{(g)}_{n,[p,n-p]} = \binom{n}{p}$. Once again, this is trivially true for $p = 1$ and, for $2 \le p < n-p$, this is an easy consequence of the celebrated Vandermonde identity, namely:

$$\sum_{k=0}^{m} \binom{n}{k}\binom{\ell}{m-k} = \binom{\ell+n}{m}. \tag{99}$$

Here, using (93) together with (99), one finds:

$$\frac{n}{p}\sum_{g=0}^{p-1} \binom{p}{p-1-g}\binom{n-p-1}{g} = \frac{n}{p}\binom{n-1}{p-1} = \binom{n}{p}. \tag{100}$$

Remark 17. Generating function of the $C^{(g)}_{[p,n-p]}$. One may build a G.F. for the $C^{(g)}_{[p,n-p]}$ adapted to their symmetry under $p \leftrightarrow n-p$:

$$Z^{(g)}(x,v) := \frac{1}{2}\sum_{n=0}^{\infty}\sum_{p=1}^{n-1} C^{(g)}_{[p,n-p]} x^n v^{2p-n} \tag{101}$$

One finds

$$Z^{(g)}(x,v) = \mathcal{Z}^{(g)}(x,1/v) = \frac{(2 - x(v+1/v))x^{2g+2}}{2(1-xv)^{g+2}(1-x/v)^{g+2}}. \tag{102}$$

In particular, for $v = 1$, one recovers the G.F. of the $S^{(g)}_{n,2}$ already encountered in the Remark at the end of Section 4.2, $\sum_n S^{(g)}_{n,2}x^n = \frac{x^{2g+2}}{(1-x)^{2g+3}}$.

Remark 18. Proposition 14 can be checked at low n, for instance $n = 5$, by determining explicitly the partitions contributing to $C^{(g)}_{n,[n-5,5]}$. Their number indeed starts as in Table 2, in agreement with Eqs. (92)–(93).

g	0	1	2	3	4
$n=6$	6				
$n=7$	7	14			
$n=8$	8	32	16		
$n=9$	9	54	54	9	
$n=10$	5	40	60	20	1
$n=11$	11	110	220	110	11
$n=12$	12	144	360	240	36
$n=13$	13	182	546	455	91
$n=14$	14	224	784	784	196
$n=15$	15	270	1080	1260	378

Table 2: Number of partitions contributing to $C^{(g)}_{n,[n-5,5]}$.

5.4.1 Case $[\alpha] = [p^2]$

So $n = 2p$ (2 parts of length p), and $g = \frac{2p-1-f}{2} \leq p-1$. This is of course a particular case of the type $[\alpha] = [p, n-p]$ considered in this section. Here, the Faà di Bruno coefficients are the numbers "of ways to put p identical objects into $g+1$ of altogether p distinguishable boxes". See OEIS sequence A103371. Notice that the last writing below is indeed consistent with (92). One has

$$C^{(g)}_{2p,[p^2]} = \binom{p-1}{p-g-1}\binom{p}{p-g-1} = \binom{p-1}{g}\binom{p}{g+1} = \frac{p}{g+1}\binom{p-1}{g}^2 \tag{103}$$

For small values of p and g these coefficients are gathered with those of the cases $[\alpha] = [p^k]$ studied in Section 3.

The general formula being given above we only notice that the $g = 0$ sequence is just p and that the $g = 1$ sequence defines the pentagonal pyramidal numbers that we shall meet again in Section 3 —see our comments there. Notice also that the sum over g is $\binom{2p-1}{p} = \{1, 3, 10, 35, 126, 462, \ldots\}$,(see OEIS sequence A001700), and that the penultimate term in each row, $\{2, 6, 12, 20, 30, \ldots\}$, is equal to $p(p-1)$. Indeed for $g = p-2$, $C^{(p-2)}_{2p,[p^2]} = \binom{p-1}{p-2}\binom{p}{p-1} = p(p-1)$.

6 Genus-dependent Faà di Bruno coefficients $C^{(g)}_{n,[\alpha]}$. Part II. A compilation of partial results

6.1 About types $[\alpha] = [p^k]$ for given $p = 2, 3, 4, \ldots$ as a function of k

The results for $C^{(g)}_{n,[\alpha]}$ when $[\alpha] = [p^k]$ are gathered in Table 3. For given p and k the values are listed vertically (downward) according to the genus g, for $g = 0, 1, 2, \ldots$ Here $n = kp$, (i.e., k parts of length p); we have $g = (k(p-1)+1-f)/2$, therefore $g \leq k(p-1)/2$. These values have been obtained by an explicit determination of the genus for computer generated set partitions, or obtained from general theorems.

	k=1	k=2	k=3	k=4
p=2	1	2, 1	5, 10	14, 70, 21
p=3	1	3, 6, 1	12, 102, 144, 22	55, 1212, 6046, 7163, 924
p=4	1	4, 18, 12, 1	22, 432, 2007, 2604, 710	140, 7236, 108090, 592824, 1180364, 688270, 50701
p=5	1	5, 40, 60, 20, 1	35, 1240, 12060, 41820, 51565, 18540, 866	285, 26800, 809960, ...
p=6	1	6, 75, 200, 150, 30, 1	51, 2850, 47475, 316700, 905415, 1076238, 462375, 47752	506, 75450, 3837575, ...
p=7	1	7, 126, 525, 700, 315, 42, 1	70, 5670, 144270, ...	819, 177660, 13656006, ...

	k=5	k=6	k=7
p=2	42, 420, 483	132, 2310, 6468, 1485	429, 12012, 66066, 56628
p=3	273, 12330, 149674, 576660, 593303, 69160	1428, 114888, 2771028, ...	7752, 1011486, 42679084, ...
p=4	969, 103680, 3588318, ...	7084, 1359882, 90800208, ...	53820, 16846704, 1929948363, ...
p=5	2530, 495200, 34034480, ...	23751, 8373000, 1097464620, ...	231880, 133685440, 29830376800, ...
p=6	5481, 1707000, 195525750 ...	62832, 35331000, 7670848500, ...	749398, 690413850, 254134018600, ...
p=7	10472, 4755870, 818352528, ...	141778, 116450460, 37838531178, ...	1997688, 2691733464, 1479039054696, ...

Table 3: Table of coefficients $C^{(g)}_{n,[\alpha]}$ for $n = k\,p$, $[\alpha] = [p^k]$.

In this section we describe some generic features of the sequences that are obtained for increasing values of p, for various choices of $[\alpha] = [p^k]$. The situation where $[\alpha] = [p^2]$, which is a particular case of the type $[\alpha] = [p, n-p]$ considered in Section 5.4—a "solved case"—(see (92) or (93)), was already discussed at the end of 5.4.

We recall from (8) that:

- the k-th row's sum (over g) in the Table of $[p^k]$ is $(pk)!/(k!(p!)^k)$. See OEIS sequences A025035, A025036, A025037, A025038, A025039, for $p = 3, \dots, 7$.
- and from (72) that $C^{(0)}_{p.k,[p^k]} = \frac{1}{pk+1}\binom{pk+1}{k} = \frac{1}{(p-1)k+1}\binom{pk}{k}$.

6.1.1 Genus $g = 1$: Observations and conjectures

For $k = 2$, it follows from Proposition 11 that

$$C^{(1)}_{2p,[p^2]} = \frac{p(p-1)^2}{2} \tag{104}$$

which are the "pyramidal pentagonal numbers", $\{0, 1, 6, 18, 40, 75, 126, 196, 288, 405, \dots\}$. See OEIS sequences A002411. Furthermore, we observe that these pyramidal pentagonal numbers factorize the coefficients $C^{(1)}_{pk,[p^k]}$

$$C^{(1)}_{pk,[p^k]} \stackrel{?}{=} C^{(1)}_{2p,[p^2]}\phi(p,k) \tag{105}$$

$$\phi(p,k) = \begin{array}{cccccc} 0 & 1 & 10 & 70 & 420 & \cdots \\ 0 & 1 & 17 & 202 & 2055 & \cdots \\ 0 & 1 & 24 & 402 & 5760 & \cdots \\ 0 & 1 & 31 & 670 & 12380 & \cdots \\ 0 & 1 & 38 & 1006 & 22760 & \cdots \\ 0 & 1 & 45 & 1410 & 37745 & \cdots \\ \vdots \end{array} \qquad 2 \leq p \leq 7 \tag{106}$$

in which the third column is an arithmetic series $7p-4$, the fourth $34p^2 - 38 + 10$, etc.

In other words, the above tables are compatible with the following expressions[5]:

$$\begin{aligned} C^{(1)}_{3p,[p^3]} &= \frac{p(p-1)^2}{2}(7p-4), \\ C^{(1)}_{4p,[p^4]} &\overset{?}{=} \frac{p(p-1)^2}{2}(34p^2 - 38p + 10), \\ C^{(1)}_{5p,[p^5]} &\overset{?}{=} \frac{p(p-1)^2}{2}\frac{5}{6}(169p^3 - 279p^2 + 146p - 24), \\ C^{(1)}_{6p,[p^6]} &\overset{?}{=} \frac{p(p-1)^2}{2}(533p^4 - 1160p^3 + \frac{1813}{2}p^2 - \frac{599}{2}p + 35) \\ C^{(1)}_{7p,[p^7]} &\overset{?}{=} \frac{p(p-1)^2}{2}\frac{7}{120}(32621p^5 - 87970p^4 + 91335p^3 - 45410p^2 + 10744p - 960). \end{aligned} \tag{107}$$

The constant terms in the polynomial ϕ appear to be (up to a sign $(-1)^k$) the "tetrahedral (or triangular pyramidal) numbers": $k(k^2-1)/6$, OEIS sequence A000292.

6.1.2 Genus $g = 2$: Observations and conjectures

$$\begin{aligned} C^{(2)}_{3p,[p^3]} &\overset{?}{=} \frac{1}{8}p(p-1)^2(p-2)(27 - 55p + 26p^2), \\ C^{(2)}_{4p,[p^4]} &\overset{?}{=} \frac{1}{6}p(p-1)^2(287 - 1248p + 1908p^2 - 1218p^3 + 274p^4), \\ C^{(2)}_{5p,[p^5]} &\overset{?}{=} \frac{1}{144}p(p-1)^2(-30576 + 194318p - 467213p^2 + 532986p^3 - 288895p^4 + 59500p^5). \end{aligned} \tag{108}$$

In each case, the conjecture has been tested on at least two more values than those used in the extrapolation.

[5] The expression for $k = 3$ is a particular case of a formula that will be discussed in Section 6.2.

6.2 Cases $[\alpha] = [n-p-q, p, q]$. Partition of n into $k = 3$ parts

All the data that have been collected in that case, when the genus is 0 or 1, (see the Tables in the Appendix), are given below.

$$C^{(0)}_{n,[n-p-q,p,q]} = n(n-1) \tag{109}$$

$$C^{(1)}_{n,[n-p-q,p,q]} = \frac{n}{2}(-5(n-1)^2 + 3(p^2+q^2+r^2-1) + 6pqr + (r+p)(r+q)(p+q)), \tag{110}$$

where in the last expression, the symmetry in the exchange of p, q and $r := n-p-q$ is manifest. These expressions have to be multiplied by $\frac{1}{2}$ if two of the three integers $p, q, n-p-q$ are equal, and by $\frac{1}{6}$ if $p = q = n-p-q$.

Equation (109) for genus 0 is an immediate consequence of Kreweras' formula, (72). Equation (110) was presented as a conjecture in a first version of this paper but it has been subsequently proved by Hock. See our comments in the acknowledgments section.

7 Tables

The following tables provide the coefficients $C^{(g)}_{n,[\alpha]}$ for $2 \le n \le 15$ and $\alpha_1 = 0$. Partitions without singletons are ordered by increasing number of parts $|\alpha|$ and then by lexicographic order on $[\alpha]$.

	g / $[\alpha]$	0	1	2	3	4	5
$n=2$	$[2]$	1	0	0	0	0	0
$n=3$	$[3]$	1	0	0	0	0	0
$n=4$	$[4]$	1	0	0	0	0	0
	$[2^2]$	2	1	0	0	0	0
$n=5$	$[5]$	1	0	0	0	0	0
	$[2,3]$	5	5	0	0	0	0
$n=6$	$[6]$	1	0	0	0	0	0
	$[2,4]$	6	9	0	0	0	0
	$[3^2]$	3	6	1	0	0	0
	$[2^3]$	5	10	0	0	0	0
$n=7$	$[7]$	1	0	0	0	0	0
	$[2,5]$	7	14	0	0	0	0
	$[3,4]$	7	21	7	0	0	0
	$[2^2,3]$	21	70	14	0	0	0
$n=8$	$[8]$	1	0	0	0	0	0
	$[2,6]$	8	20	0	0	0	0
	$[3,5]$	8	32	16	0	0	0
	$[4^2]$	4	18	12	1	0	0
	$[2^2,4]$	28	128	54	0	0	0
	$[2,3^2]$	28	152	100	0	0	0
	$[2^4]$	14	70	21	0	0	0
$n=9$	$[9]$	1	0	0	0	0	0
	$[2,7]$	9	27	0	0	0	0
	$[3,6]$	9	45	30	0	0	0
	$[4,5]$	9	54	54	9	0	0
	$[2^2,5]$	36	207	135	0	0	0
	$[2,3,4]$	72	531	603	54	0	0
	$[3^3]$	12	102	144	22	0	0
	$[3,2^3]$	84	630	546	0	0	0

	g / $[\alpha]$	0	1	2	3	4	5
$n=10$	$[10]$	1	0	0	0	0	0
	$[2,8]$	10	35	0	0	0	0
	$[3,7]$	10	60	50	0	0	0
	$[4,6]$	10	75	100	25	0	0
	$[5^2]$	5	40	60	20	1	0
	$[2^2,6]$	45	310	275	0	0	0
	$[2,3,5]$	90	830	1340	260	0	0
	$[2,4^2]$	45	450	840	240	0	0
	$[3^2,4]$	45	510	1115	430	0	0
	$[2^3,4]$	120	1165	1685	180	0	0
	$[2^2,3^2]$	180	1985	3565	570	0	0
	$[2^5]$	42	420	483	0	0	0
$n=11$	$[11]$	1	0	0	0	0	0
	$[2,9]$	11	44	0	0	0	0
	$[3,8]$	11	77	77	0	0	0
	$[4,7]$	11	99	165	55	0	0
	$[5,6]$	11	110	220	110	11	0
	$[2^2,7]$	55	440	495	0	0	0
	$[2,3,6]$	110	1210	2530	770	0	0
	$[2,4,5]$	110	1375	3564	1793	88	0
	$[3^2,5]$	55	770	2277	1452	66	0
	$[3,4^2]$	55	825	2684	2035	176	0
	$[2^3,5]$	165	1936	3905	924	0	0
	$[2^2,3,4]$	495	7007	19085	8063	0	0
	$[2,3^3]$	165	2607	8195	4433	0	0
	$[3,2^4]$	330	4620	10395	1980	0	0

	g / $[\alpha]$	0	1	2	3	4	5
$n=12$	$[12]$	1	0	0	0	0	0
	$[2,10]$	12	54	0	0	0	0
	$[3,9]$	12	96	112	0	0	0
	$[4,8]$	12	126	252	105	0	0
	$[5,7]$	12	144	360	240	36	0
	$[6^2]$	6	75	200	150	30	1
	$[2^2,8]$	66	600	819	0	0	0
	$[2,3,7]$	132	1680	4308	1800	0	0
	$[2,4,6]$	132	1968	6510	4740	510	0
	$[2,5^2]$	66	1032	3672	3072	474	0
	$[3^2,6]$	66	1092	4062	3640	380	0
	$[3,4,5]$	132	2436	10500	12084	2568	0
	$[4^3]$	22	432	2007	2604	710	0
	$[2^3,6]$	220	3000	7730	2910	0	0
	$[2^2,3,5]$	660	11232	40716	28968	1584	0
	$[2^2,4^2]$	330	5988	23877	20097	1683	0
	$[2,3^2,4]$	660	13218	59076	59442	6204	0
	$[3^4]$	55	1212	6046	7163	924	0
	$[2^4,4]$	495	8616	28590	14274	0	0
	$[2^3,3^2]$	990	19104	73050	45456	0	0
	$[2^6]$	132	2310	6468	1485	0	0
$n=13$	$[13]$	1	0	0	0	0	0
	$[2,11]$	13	65	0	0	0	0
	$[3,10]$	13	117	156	0	0	0
	$[4,9]$	13	156	364	182	0	0
	$[5,8]$	13	182	546	455	91	0
	$[6,7]$	13	195	650	650	195	13
	$[2^2,9]$	78	793	1274	0	0	0
	$[2,3,8]$	156	2249	6825	3640	0	0
	$[2,4,7]$	156	2691	10803	10335	1755	0
	$[2,5,6]$	156	2912	13130	15470	4238	130
	$[3^2,7]$	78	1482	6630	7670	1300	0
	$[3,4,6]$	156	3393	18161	28145	9945	260
	$[3,5^2]$	78	1768	9984	16796	7046	364
	$[4^2,5]$	78	1872	11271	20904	10322	598
	$[2^3,7]$	286	4420	13819	7215	0	0
	$[2^2,3,6]$	858	16900	76037	76765	9620	0
	$[2^2,4,5]$	858	18720	97539	125606	27547	0
	$[2,3^2,5]$	858	20488	117806	175916	45292	0
	$[2,3,4^2]$	858	21684	133887	222781	71240	0
	$[3^3,4]$	286	7878	53404	100997	37635	0
	$[2^4,5]$	715	14612	63323	53053	3432	0
	$[2^3,3,4]$	2860	68172	363610	419068	47190	0
	$[2^2,3^3]$	1430	37336	221715	298649	41470	0
	$[3,2^5]$	1287	30030	138138	100815	0	0

	g / $[\alpha]$	0	1	2	3	4	5	6
$n=14$	$[14]$	1	0	0	0	0	0	0
	$[2,12]$	14	77	0	0	0	0	0
	$[3,11]$	14	140	210	0	0	0	0
	$[4,10]$	14	189	504	294	0	0	0
	$[5,9]$	14	224	784	784	196	0	0
	$[6,8]$	14	245	980	1225	490	49	0
	$[7^2]$	7	126	525	700	315	42	1
	$[2^2,10]$	91	1022	1890	0	0	0	0
	$[2,3,9]$	182	2926	10248	6664	0	0	0
	$[2,4,8]$	182	3556	16758	19894	4655	0	0
	$[2,5,7]$	182	3934	21420	32620	13034	882	0
	$[2,6^2]$	91	2030	11550	18900	8645	826	0
	$[3^2,8]$	91	1946	10157	14406	3430	0	0
	$[3,4,7]$	182	4536	28938	56434	28280	1750	0
	$[3,5,6]$	182	4858	33824	75040	48174	6090	0
	$[4^2,6]$	91	2562	18893	45654	33285	4620	0
	$[4,5^2]$	91	2660	20496	52710	42679	7490	0
	$[2^3,8]$	364	6265	22981	15435	0	0	0
	$[2^2,3,7]$	1092	24290	129948	170380	34650	0	0
	$[2^2,4,6]$	1092	27587	176351	307020	114940	3640	0
	$[2^2,5^2]$	546	14343	96726	182756	80094	3913	0
	$[2,3^2,6]$	1092	29988	209510	415870	178920	5460	0
	$[2,3,4,5]$	2184	65674	513450	1200738	696794	43680	0
	$[2,4^3]$	364	11529	95613	243180	162099	12740	0
	$[3^3,5]$	364	11844	100660	262696	173348	11648	0
	$[3^2,4^2]$	546	18690	168273	475195	356951	31395	0
	$[2^4,6]$	1001	23240	123214	146020	21840	0	0
	$[2^3,3,5]$	4004	111608	754614	1286642	365652	0	0
	$[2^3,4^2]$	2002	58912	427777	809823	278061	0	0
	$[2^2,3^2,4]$	6006	191968	1525454	3268552	1314320	0	0
	$[2,3^4]$	1001	34692	301070	725193	339444	0	0
	$[2^5,4]$	2002	56378	354263	473242	60060	0	0
	$[2^4,3^2]$	5005	153832	1075165	1663893	255255	0	0
	$[2^7]$	429	12012	66066	56628	0	0	0

	g / $[\alpha]$	0	1	2	3	4	5	6
$n=15$	$[15]$	1	0	0	0	0	0	0
	$[2,13]$	15	90	0	0	0	0	0
	$[3,12]$	15	165	275	0	0	0	0
	$[4,11]$	15	225	675	450	0	0	0
	$[5,10]$	15	270	1080	1260	378	0	0
	$[6,9]$	15	300	1400	2100	1050	140	0
	$[7,8]$	15	315	1575	2625	1575	315	15
	$[2^2,11]$	105	1290	2700	0	0	0	0
	$[2,3,10]$	210	3720	14760	11340	0	0	0
	$[2,4,9]$	210	4575	24720	35070	10500	0	0
	$[2,5,8]$	210	5145	32760	61215	32340	3465	0
	$[2,6,7]$	210	5430	37200	78000	50550	8610	180
	$[3^2,9]$	105	2490	14835	24920	7700	0	0
	$[3,4,8]$	210	5880	43470	102165	66675	6825	0
	$[3,5,7]$	210	6420	53010	146040	127290	26940	450
	$[3,6^2]$	105	3300	28200	81500	77075	19380	650
	$[4^2,7]$	105	3375	29385	87510	84975	19575	300
	$[4,5,6]$	210	7185	67260	221310	252090	79575	3000
	$[5^3]$	35	1240	12060	41820	51565	18540	866
	$[2^3,9]$	455	8610	36190	29820	0	0	0
	$[2^2,3,8]$	1365	33705	208320	336210	96075	0	0
	$[2^2,4,7]$	1365	38955	293670	644535	347175	25650	0
	$[2^2,5,6]$	1365	41580	342000	853500	579405	74040	0
	$[2,3^2,7]$	1365	42105	344595	853035	522450	38250	0
	$[2,3,4,6]$	2730	94290	886620	2671710	2296950	354000	0
	$[2,3,5^2]$	1365	48825	479040	1530360	1449285	274905	0
	$[2,4^2,5]$	1365	51240	529425	1816410	1913445	417840	0
	$[3^3,6]$	455	16905	171635	570805	550950	90650	0
	$[3^2,4,5]$	1365	55020	612705	2303715	2692020	641475	0
	$[3,4^3]$	455	19215	224925	902400	1158840	321790	0
	$[2^4,7]$	1365	35250	219660	337050	82350	0	0
	$[2^3,3,6]$	5460	172350	1398740	3159450	1515700	54600	0
	$[2^3,4,5]$	5460	188025	1708035	4524870	2847420	185640	0
	$[2^2,3^2,5]$	8190	304335	2994180	8823360	6327465	461370	0
	$[2^2,3,4^2]$	8190	319605	3329865	10604790	8605395	780780	0
	$[2,3^3,4]$	5460	229365	2581215	9060870	8289600	854490	0
	$[3^5]$	273	12330	149674	576660	593303	69160	0
	$[2^5,5]$	3003	97140	761880	1493520	482292	0	0
	$[2^4 3,4]$	15015	554250	5104260	12450900	5524200	0	0
	$[2^3,3^3]$	10010	399660	4013730	10965465	5632135	0	0
	$[2^6,3]$	5005	180180	1471470	2622620	450450	0	0

The following table can be obtained either directly, by generating for each n all partitions with a fixed number of parts, then calculating their genus, or, indirectly, by summing appropriate rows of the table of the genus-dependent Faà di Bruno coefficients, while taking into account singletons (since the following is a table for $S_{n,k}^{(g)}$, not for $\widehat{S}_{n,k}^{(g)}$). For each value of n the number of parts, k appears vertically, and the genus $g = 0, 1, \ldots$ increases in each row.

1	1

1	1
2	1

1	1
2	3
3	1

1	1
2	6, 1
3	6
4	1

1	1
2	10, 5
3	20, 5
4	10
5	1

1	1
2	15, 15, 1
3	50, 40
4	50, 15
5	15
6	1

1	1
2	21, 35, 7
3	105, 175, 21
4	175, 175
5	105, 35
6	21
7	1

1	1
2	28, 70, 28, 1
3	196, 560, 210
4	490, 1050, 161
5	490, 560
6	196, 70
7	28
8	1

1	1
2	36, 126, 84, 9
3	336, 1470, 1134, 85
4	1176, 4410, 2184
5	1764, 4410, 777
6	1176, 1470
7	336, 126
8	36
9	1

1	1
2	45, 210, 210, 45, 1
3	540, 3360, 4410, 1020
4	2520, 14700, 15330, 1555
5	5292, 23520, 13713
6	5292, 14700, 2835
7	2520, 3360
8	540, 210
9	45
10	1

1	1
2	55, 330, 462, 165, 11
3	825, 6930, 13860, 6545, 341
4	4950, 41580, 75075, 24145
5	13860, 97020, 121275, 14575
6	19404, 97020, 63063
7	13860, 41580, 8547
8	4950, 6930
9	825, 330
10	55
11	1

1	1
2	66, 495, 924, 495, 66, 1
3	1210, 13200, 37422, 29920, 4774
4	9075, 103950, 289905, 194150, 14421
5	32670, 332640, 729960, 284130
6	60984, 485100, 685608, 91960
7	60984, 332640, 233772
8	32670, 103950, 22407
9	9075, 13200
10	1210, 495
11	66
12	1

1	1
2	78, 715, 1716, 1287, 286, 13
3	1716, 23595, 90090, 109395, 35464, 1365
4	15730, 235950, 942942, 1085370, 252538
5	70785, 990990, 3396393, 2797080, 253253
6	169884, 1981980, 4972968, 2196480
7	226512, 1981980, 3063060, 443872
8	169884, 990990, 738738
9	70785, 235950, 52767
10	15730, 23595
11	1716, 715
12	78
13	1

1	1
2	91, 1001, 3003, 3003, 1001, 91, 1
3	2366, 40040, 198198, 340340, 186186, 21840
4	26026, 495495, 2690688, 4759755, 2288286, 131495
5	143143, 2642640, 13096083, 18768750, 5424419
6	429429, 6936930, 27432405, 25965940, 2671669
7	736164, 9513504, 26342316, 12737296
8	736164, 6936930, 11477466, 1761760
9	429429, 2642640, 2063061
10	143143, 495495, 114114
11	26026, 40040
12	2366, 1001
13	91
14	1

1	1
2	105, 1365, 5005, 6435, 3003, 455, 15
3	3185, 65065, 405405, 935935, 775775, 184275, 5461
4	41405, 975975, 6921915, 17482465, 14369355, 2564835
5	273273, 6441435, 43702659, 97222125, 58891833, 4235595
6	1002001, 21471450, 123708585, 205865660, 68645577
7	2147145, 38648610, 169954785, 177957780, 20033013
8	2760615, 38648610, 115450335, 59768280
9	2147145, 21471450, 37492455, 6017440
10	1002001, 6441435, 5219214
11	273273, 975975, 230230
12	41405, 65065
13	3185, 1365
14	105
15	1

Table of $S_{n,k}^{(g)}$, from $n = 1$ to $n = 15$.

The following table can be obtained either directly, by generating, for each n, all partitions without singletons, with a fixed number of parts, then calculating their genus, or, indirectly, by summing appropriate rows of the table of the genus-dependent Faà di Bruno coefficients (without singletons). For each value of n the number of parts, k appears vertically, and the genus $g = 0, 1, \ldots$ increases in each row. Warning: the table starts at $n = 2$.

$n=2$	
1	1

$n=3$	
1	1

$n=4$	
1	1
2	2, 1

$n=5$	
1	1
2	5, 5

$n=6$	
1	1
2	9, 15, 1
3	5, 10

$n=7$	
1	1
2	14, 35, 7
3	21, 70, 14

$n=8$	
1	1
2	20, 70, 28, 1
3	56, 280, 154
4	14, 70, 21

$n=9$	
1	1
2	27, 126, 84, 9
3	120, 840, 882, 76
4	84, 630, 546

$n=10$	
1	1
2	35, 210, 210, 45, 1
3	225, 2100, 3570, 930
4	300, 3150, 5250, 750
5	42, 420, 483

$n=11$	
1	1
2	44, 330, 462, 165, 11
3	385, 4620, 11550, 6050, 330
4	825, 11550, 31185, 13420
5	330, 4620, 10395, 1980

$n=12$	
1	1
2	54, 495, 924, 495, 66, 1
3	616, 9240, 31878, 27940, 4642
4	1925, 34650, 137445, 118580, 10395
5	1485, 27720, 101640, 59730
6	132, 2310, 6468, 1485

$n=13$	
1	1
2	65, 715, 1716, 1287, 286, 13
3	936, 17160, 78078, 102960, 34606, 1352
4	4004, 90090, 492492, 709280, 191334
5	5005, 120120, 648648, 770770, 92092
6	1287, 30030, 138138, 100815

$n=14$	
1	1
2	77, 1001, 3003, 3003, 1001, 91, 1
3	1365, 30030, 174174, 322322, 182182, 21658
4	7644, 210210, 1513512, 3273270, 1797796, 112476
5	14014, 420420, 3132129, 6236230, 2319317
6	7007, 210210, 1429428, 2137135, 315315
7	429, 12012, 66066, 56628

$n=15$	
1	1
2	90, 1365, 5005, 6435, 3003, 455, 15
3	1925, 50050, 360360, 890890, 760760, 182910, 5446
4	13650, 450450, 4129125, 12512500, 11606595, 2238600
5	34398, 1261260, 12381369, 37087050, 28261233, 2406040
6	28028, 1051050, 9879870, 24909885, 11638627
7	5005, 180180, 1471470, 2622620, 450450

Table of $\widehat{S}^{(g)}_{n,k}$, from $n = 2$ to $n = 15$.

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After the present article was made available as a preprint (its version 1 on arXiv) we received several comments from A. Hock suggesting that several partitions functions obtained in the present paper could be re-written by using changes of functions suggested by the formalism of topological recursion. This is in particular so for $g = 1$ in the case of partitions into three parts where expressions (110) presented as conjectures in the first version of our paper could be proved by Hock by using this method; his work has now been published [11].

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