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Self-Complementary Magic Squares of Singly Even Orders

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Abstract

A *self-complementary magic square of order* n is a magic square M whose entries consist of the first n^2 natural numbers such that when each entry i of M is replaced by $n^2 + 1 - i$, the resulting square is equivalent to M (under rotation or reflection). The purpose of this paper is twofold. For any given singly even natural number n, (i) we extend the method of Strachey to produce many magic squares (instead of just one) of order n, and (ii) by making a variation of this extended method, we construct many self-complementary magic squares of order n.

1 Introduction

A *magic square* M is an $n \times n$ array of integers (which may overlap) so that the sum of the entries in each row, column and the diagonal is a constant. The constant sum is called the *magic sum*. In the case that the entries of M are the consecutive integers from $1, 2, \ldots, n^2$, then M is said to be *of order* n. It is easy to see that the magic sum of an n-order magic square is $\frac{n(n^2+1)}{2}$. A *doubly even order magic square* is a magic square matrix of order n, where n is divisible by 4. A *singly even order magic square* is a magic square matrix of order n, where n is even but not divisible by 4.

Suppose $M = (a_{i,j})$ is a magic square of order n. Then M is said to be *ro-symmetrical* if $a_{i,j} + a_{n+1-i,n+1-j} = n^2 + 1$ for all $1 \leq i, j \leq n$. It is known that the magic square constructed using the De la Loubère method is ro-symmetrical and that ro-symmetrical magic square of singly even order does not exist (see [5, p. 203]).

On the other hand, M is said to be *ref-symmetrical* if $a_{i,j} + a_{n+1-i,j} = n^2 + 1$ for all $1 \leq i, j \leq n$. Alternatively, these two notions maybe expressed in the following way.

Let J_k denote the $k \times k$ matrix where all entries are 1 and let $\sigma(M)$ (respectively $\rho(M)$) denote the 180-degree clockwise rotation on M (respectively reflection on the central horizontal of M). Then M is ro-symmetrical (respectively ref-symmetrical) if $M + \sigma(M) = (n^2 + 1)J_n$ (respectively $M + \rho(M) = (n^2 + 1)J_n$). Note that equivalently, in this definition, we could replace $\rho(M)$ by $\pi(M)$ which denotes the reflection on the central vertical of M, in which case we have $a_{i,j} + a_{i,n+1-j} = n^2 + 1$ for all $1 \leq i, j \leq n$ instead.

The methods of construction for magic squares of singly even order appear to be relatively scarce when compared to magic squares of other orders. The earliest record for singly even order case dates back to a letter of Strachey to Coxeter in 1918. The magic square of odd order constructed by the De la Loubère method is used in the Strachey method. For ease of reference, the De la Loubère and the Strachey methods of construction are given in Section 2. The magic squares which are constructed by these two methods are called the *De la Loubère square* and the *Strachey square* respectively.

Recently, in [3], a method of producing a magic square of order $2(2m + 1)$ with the reflectional symmetrical property was presented. Earlier, in [1], magic squares of singly even orders were constructed which involves a special kind of composition (called the Yang-Hui composition). While the Strachey method produces only one magic square for a given singly even order, we show (in Section 3) that the method can be extended to produce (i) many magic squares of a given singly even order, and (ii) we can use any magic square of odd order in the construction (instead of restricting to the De la Loubère magic square).

Two magic squares M_1 and M_2 are said to be *equivalent* if $\phi(M_1) = M_2$ for some $\phi \in D_4$ where D_4 denotes the dihedral group of order 8. Suppose M is a magic square of order n. The *complement* of M is the magic square (of order n) obtained by replacing each entry i of M by $n^2 + 1 - i$. We say that M is *self-complementary* if M is equivalent to its complement. Self-complementary magic squares of order n have been characterized in [3] where it was shown that (i) if n is odd, then M is self-complementary if and only if M is ro-symmetrical, whereas (ii) if n is even, then M is self-complementary if and only if either M is ref-symmetrical or else M is ro-symmetrical. Note that if M is ro-symmetrical (respectively ref-symmetrical), then $\sigma(M) = \overline{M}$ (respectively $\rho(M) = \overline{M}$), where $\overline{M}$ denotes the complement of M.

Subsequently, in [2], a construction for self-complementary magic squares of doubly even order was given; it converts a ro-symmetrical magic square (constructed using a well-known construction called the Generalized Doubly Even Method, [5, pp. 199–200] and [4, p. 527, Section 34.21]) into a ref-symmetrical magic square. In quite the same spirit as in [2], in Section 4, we provide a new method of constructing self-complementary magic squares of singly even order. This time we make use of a ro-symmetrical magic square (which may be constructed using the De la Loubère method) on a variation of the extended Strachey method to produce a ref-symmetrical square of singly even order.

2 De la Loubère and Strachey squares

Let n be an odd integer. A magic square of order n can be constructed by the *De la Loubère method* in which the sequence of integers $1, 2, \ldots, n^2$ are filled successively into an $n \times n$ square according to the following rule.

L(i) Put 1 in the middle cell of the top row of square; fill the next integer diagonally in the north-east direction.

L(ii) When the top row is reached, the next integer is written in the bottom row as if it came immediately above the top row (for example, the integers 2 and 9 in the magic square below for the case $n = 5$).

L(iii) When the last column is reached, the next integer is written in the first column as if it immediately succeeded the last column (for example, the integers 4 and 10 in the magic square below).

L(iv) When the next cell to be filled is already occupied or when the top-right cell is reached, the move drops directly below it and continue to mount again (for example, the integers $6, 16$ and 21 in the magic square below).

$$\begin{bmatrix} 17 & 24 & 1 & 8 & 15 \\ 23 & 5 & 7 & 14 & 16 \\ 4 & 6 & 13 & 20 & 22 \\ 10 & 12 & 19 & 21 & 3 \\ 11 & 18 & 25 & 2 & 9 \end{bmatrix}$$

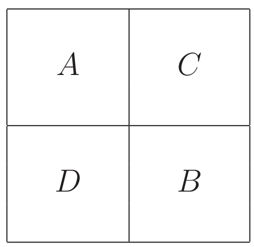
Now consider the case $n = 2(2m + 1)$. A magic square of order n can be constructed by the *Strachey method*. In this case, we begin by partitioning an $n \times n$ square into four $(2m + 1) \times (2m + 1)$ subsquares A, B, C and D as shown below. Then apply the following rule.

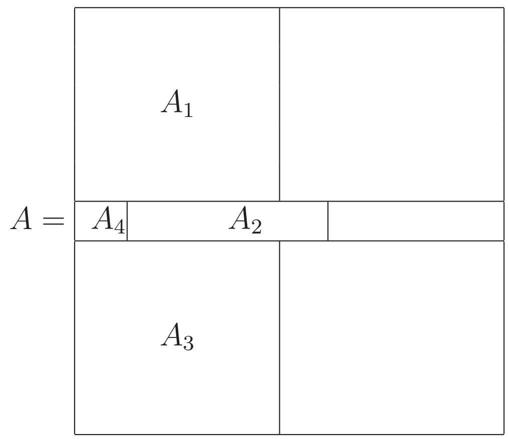
S(i) Let A be the De la Loubère square of order $2m + 1$ and let $B = A + (2m + 1)^2 J_{2m+1}$, $C = A + 2(2m + 1)^2 J_{2m+1}$ and $D = A + 3(2m + 1)^2 J_{2m+1}$.

S(ii) Divide A as shown below where A_1, A_3 are $m \times m$ subsquares which are at the upper left and the bottom left corners of A respectively, A_4 is the first cell of the middle row of A while A_2 is the $1 \times m$ rectangle in the middle row, next to A_4. Divide D in a similar manner and obtain D_1, D_2, D_3 and D_4.

S(iii) Interchange A_i with D_i for $i = 1, 2, 3$.

S(iv) Interchange the last $m - 1$ columns of C with those of B.





For $n = 10$, the Strachey method yields the following magic square of order 10.

$$\begin{bmatrix} 92 & 99 & 1 & 8 & 15 & 67 & 74 & 51 & 58 & 40 \\ 98 & 80 & 7 & 14 & 16 & 73 & 55 & 57 & 64 & 41 \\ 4 & 81 & 88 & 20 & 22 & 54 & 56 & 63 & 70 & 47 \\ 85 & 87 & 19 & 21 & 3 & 60 & 62 & 69 & 71 & 28 \\ 86 & 93 & 25 & 2 & 9 & 61 & 68 & 75 & 52 & 34 \\ 17 & 24 & 76 & 83 & 90 & 42 & 49 & 26 & 33 & 65 \\ 23 & 5 & 82 & 89 & 91 & 48 & 30 & 32 & 39 & 66 \\ 79 & 6 & 13 & 95 & 97 & 29 & 31 & 38 & 45 & 72 \\ 10 & 12 & 94 & 96 & 78 & 35 & 37 & 44 & 46 & 53 \\ 11 & 18 & 100 & 77 & 84 & 36 & 43 & 50 & 27 & 59 \end{bmatrix}.$$

3 Beyond Strachey

If every entry of an n-th order magic square is reduced by 1, the resulting square is called a *reduced magic square.*

Definition 1. Suppose B_1 and B_2 are reduced magic squares of order $2m+1$. Let B denote the $2(2m+1) \times 2(2m+1)$ magic square defined by

$$B = \begin{bmatrix} B_1 & B_2 \\ B_1 & B_2 \end{bmatrix}.$$

Example 1. Let B_1 be the reduced magic square of the De la Loubère magic square of order 5, and let B_2 be the square obtained from B_1 by taking a 90-degree clockwise rotation. Then

$$B = \begin{bmatrix} 16 & 23 & 0 & 7 & 14 & 10 & 9 & 3 & 22 & 16 \\ 22 & 4 & 6 & 13 & 15 & 17 & 11 & 5 & 4 & 23 \\ 3 & 5 & 12 & 19 & 21 & 24 & 18 & 12 & 6 & 0 \\ 9 & 11 & 18 & 20 & 2 & 1 & 20 & 19 & 13 & 7 \\ 10 & 17 & 24 & 1 & 8 & 8 & 2 & 21 & 15 & 14 \\ 16 & 23 & 0 & 7 & 14 & 10 & 9 & 3 & 22 & 16 \\ 22 & 4 & 6 & 13 & 15 & 17 & 11 & 5 & 4 & 23 \\ 3 & 5 & 12 & 19 & 21 & 24 & 18 & 12 & 6 & 0 \\ 9 & 11 & 18 & 20 & 2 & 1 & 20 & 19 & 13 & 7 \\ 10 & 17 & 24 & 1 & 8 & 8 & 2 & 21 & 15 & 14 \end{bmatrix}.$$

Definition 2. Let A denote the $2(2m+1) \times 2(2m+1)$ square defined by

$$A = \begin{bmatrix} A_0 & A_2 \\ A_3 & A_1 \end{bmatrix}$$

where

(i) A_0 is a $(2m+1) \times (2m+1)$ matrix whose entries consist of only 0 and 3 such that each row has only $(m+1)$ 0's while each diagonal has only $(m+1)$ 3's,

(ii) A_1 is a $(2m+1) \times (2m+1)$ matrix whose entries consist of only 1 and 2 such that each row and each diagonal has only $(m+2)$ 1's, and

(iii) $A_2 = 3J_{2m+1} - A_1$ and $A_3 = 3J_{2m+1} - A_0$.

It is easy to construct the matrices A_0 and A_1. Figure 1 depicts 3 examples of the matrix A_0 with $m = 2$ while Figure 2 depicts 3 examples of the matrix A_1. More general constructions of A_0 and A_1 will be discussed in Section 5.

$$\begin{bmatrix} 3 & 3 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 \\ 0 & 3 & 3 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 3 & 0 & 0 & 3 \\ 0 & 3 & 0 & 0 & 3 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 3 & 0 & 0 & 3 \\ 0 & 3 & 0 & 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 3 & 3 \end{bmatrix}$$

(i) (ii) (iii)

Figure 1: A_0 with $m = 2$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(i) (ii) (iii)

Figure 2: A_1 with $m = 2$.

Example 2. Suppose A_0 is the matrix in Figure 1 (iii) and A_1 is the matrix in Figure 2 (iii). Then

$$A = \begin{bmatrix} 0 & 0 & 0 & 3 & 3 & 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 & 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 & 0 & 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 & 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 & 1 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 0 & 0 & 2 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 0 & 0 & 2 & 1 & 1 & 1 & 1 \\ 3 & 3 & 0 & 0 & 3 & 2 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 0 & 0 & 2 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 0 & 0 & 2 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

With the matrices A and B as given in the above definitions, we obtain a reduced magic square of order $2(2m+1)$ as follows.

Definition 3. Suppose A and B are as defined in the Definitions 2 and 1 respectively. Let $[A; B]$ denote the $2(2m+1) \times 2(2m+1)$ matrix obtained by superimposing A and B so that the entries of A are the radix digits while those of B are the unit digits.

Example 3. Suppose A and B are the 10×10 matrices in Examples 2 and 1 respectively. Then

$$[A; B] = \begin{bmatrix} \mathbf{0}16 & \mathbf{0}23 & \mathbf{0}\,0 & \mathbf{3}\,7 & \mathbf{3}14 & \mathbf{1}10 & \mathbf{2}\,9 & \mathbf{2}\,3 & \mathbf{2}22 & \mathbf{2}16 \\ \mathbf{0}22 & \mathbf{0}\,4 & \mathbf{0}\,6 & \mathbf{3}13 & \mathbf{3}15 & \mathbf{1}17 & \mathbf{2}11 & \mathbf{2}\,5 & \mathbf{2}\,4 & \mathbf{2}23 \\ \mathbf{0}\,3 & \mathbf{0}\,5 & \mathbf{3}12 & \mathbf{3}19 & \mathbf{0}21 & \mathbf{1}24 & \mathbf{2}18 & \mathbf{2}12 & \mathbf{2}\,6 & \mathbf{2}\,0 \\ \mathbf{0}\,9 & \mathbf{0}11 & \mathbf{0}18 & \mathbf{3}20 & \mathbf{3}\,2 & \mathbf{1}\,1 & \mathbf{2}20 & \mathbf{2}19 & \mathbf{2}13 & \mathbf{2}\,7 \\ \mathbf{0}10 & \mathbf{0}17 & \mathbf{0}24 & \mathbf{3}\,1 & \mathbf{3}\,8 & \mathbf{1}\,8 & \mathbf{2}\,2 & \mathbf{2}21 & \mathbf{2}15 & \mathbf{2}14 \\ \mathbf{3}16 & \mathbf{3}23 & \mathbf{3}\,0 & \mathbf{0}\,7 & \mathbf{0}14 & \mathbf{2}10 & \mathbf{1}\,9 & \mathbf{1}\,3 & \mathbf{1}22 & \mathbf{1}16 \\ \mathbf{3}22 & \mathbf{3}\,4 & \mathbf{3}\,6 & \mathbf{0}13 & \mathbf{0}15 & \mathbf{2}17 & \mathbf{1}11 & \mathbf{1}\,5 & \mathbf{1}\,4 & \mathbf{1}23 \\ \mathbf{3}\,3 & \mathbf{3}\,5 & \mathbf{0}12 & \mathbf{0}19 & \mathbf{3}21 & \mathbf{2}24 & \mathbf{1}18 & \mathbf{1}12 & \mathbf{1}\,6 & \mathbf{1}\,0 \\ \mathbf{3}\,9 & \mathbf{3}11 & \mathbf{3}18 & \mathbf{0}20 & \mathbf{0}\,2 & \mathbf{2}\,1 & \mathbf{1}20 & \mathbf{1}19 & \mathbf{1}13 & \mathbf{1}\,7 \\ \mathbf{3}10 & \mathbf{3}17 & \mathbf{3}24 & \mathbf{0}\,1 & \mathbf{0}\,8 & \mathbf{2}\,8 & \mathbf{1}\,2 & \mathbf{1}21 & \mathbf{1}15 & \mathbf{1}14 \end{bmatrix}.$$

Here, the boldface integers are the radix digits. By taking the entries to be integers represented in base 5^2, we have the following reduced magic square of order 10.

$$\begin{bmatrix} 16 & 23 & 0 & 82 & 89 & 35 & 59 & 53 & 72 & 66 \\ 22 & 4 & 6 & 88 & 90 & 42 & 61 & 55 & 54 & 73 \\ 3 & 5 & 87 & 94 & 21 & 49 & 68 & 62 & 56 & 50 \\ 9 & 11 & 18 & 95 & 77 & 26 & 70 & 69 & 63 & 57 \\ 10 & 17 & 24 & 76 & 83 & 33 & 52 & 71 & 65 & 64 \\ 91 & 98 & 75 & 7 & 14 & 60 & 34 & 28 & 47 & 41 \\ 97 & 79 & 81 & 13 & 15 & 67 & 36 & 30 & 29 & 48 \\ 78 & 80 & 12 & 19 & 96 & 74 & 43 & 37 & 31 & 25 \\ 84 & 86 & 93 & 20 & 2 & 51 & 45 & 44 & 38 & 32 \\ 85 & 92 & 99 & 1 & 8 & 58 & 27 & 46 & 40 & 39 \end{bmatrix}.$$

Theorem 1. *Suppose A and B are as defined in the Definitions 2 and 1 respectively. Then $[A; B]$ yields a reduced magic square of order $2(2m+1)$ when its entries are considered as integers represented in base $(2m+1)^2$.*

Proof. Clearly B is a magic square with magic sum $4m(m+1)(2m+1)$.

Clearly the column sum of A is $3(2m+1)$.

Note that the row sum of A_0 and likewise of A_1 is $3m$. This implies that the row sum of A_3 and likewise of A_2 is $3(m+1)$. Hence the row sum of A is $3(2m+1)$.

Clearly the diagonal sum of A_0 is $3(m+1)$ and that of A_1 is $3m$. This implies that the diagonal sum of A_3 is $3m$ (since the diagonal has only m 3's), and that the diagonal sum of A_2 is $3(m+1)$ (since the diagonal has $(m-1)$ 1's and $(m+2)$ 2's). Hence the diagonal sum of A is $3(2m+1)$.

Since A and B are both magic square, it follows that $[A; B]$ is also a magic square. Moreover, since the entries of A and B are the radix and unit digits of $[A; B]$ respectively, the magic sum of $[A; B]$ is the magic sum of A multiply with $(2m+1)^2$ plus the magic sum of B yielding $(2m+1)(4(2m+1)^2-1)$.

It remains to show that the entries in $[A; B]$ consists of distinct ordered pairs of entries (x, y) where $x \in A$ and $y \in B$.

Since the entries in $A_0 \cup A_3$ consist of only integers from $\{0, 3\}$ and that the entries of $A_1 \cup A_2$ consist of only integers from $\{1, 2\}$, $(x_1, y_1) \neq (x_2, y_2)$ for any integers $x_1 \in A_0 \cup A_3$, $x_2 \in A_1 \cup A_2$ where $y_1 \in B_1$ and $y_2 \in B_2$.

Moreover $(x_1, y) \neq (x_2, y)$ for any integers $x_1 \in A_0$ and $x_2 \in A_3$ where $y \in B_1$ because $x_1 + x_2 = 3$ by Definition 2(iii).

$(x_1, y) \neq (x_2, y)$ for any integers $x_1 \in A_1$ and $x_2 \in A_2$ where $y \in B_2$ because $x_1 + x_2 = 3$ by Definition 2(iii).

This completes the proof. □

4 Ref-symmetric magic squares

Recall that, if N is a matrix, then $\pi(N)$ (respectively $\rho(N)$) denote the matrix obtained by taking a central vertical (respectively horizontal) reflection on N.

Definition 4. Let M be a reduced ro-symmetric magic squares of order $2m+1$. Write

$$M = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix}$$

where M_1 and M_3 are $m \times (2m+1)$ matrices and M_2 is a $1 \times (2m+1)$ matrix. Let B_1 be obtained from M by replacing M_3 by $\pi(M_3)$. Also, let B_1^* be obtained from B_1 by replacing M_2 by $\pi(M_2)$. Now, let B denote the $2(2m+1) \times 2(2m+1)$ matrix defined by

$$B = \begin{bmatrix} B_1 & \pi(B_1) \\ B_1^* & \pi(B_1^*) \end{bmatrix}.$$

Example 4. Let M be the reduced magic square of the De la Loubère magic square of order 5. Then

$$B = \begin{bmatrix} 16 & 23 & 0 & 7 & 14 & 14 & 7 & 0 & 23 & 16 \\ 22 & 4 & 6 & 13 & 15 & 15 & 13 & 6 & 4 & 22 \\ 3 & 5 & 12 & 19 & 21 & 21 & 19 & 12 & 5 & 3 \\ 2 & 20 & 18 & 11 & 9 & 9 & 11 & 18 & 20 & 2 \\ 8 & 1 & 24 & 17 & 10 & 10 & 17 & 24 & 1 & 8 \\ 16 & 23 & 0 & 7 & 14 & 14 & 7 & 0 & 23 & 16 \\ 22 & 4 & 6 & 13 & 15 & 15 & 13 & 6 & 4 & 22 \\ 21 & 19 & 12 & 5 & 3 & 3 & 5 & 12 & 19 & 21 \\ 2 & 20 & 18 & 11 & 9 & 9 & 11 & 18 & 20 & 2 \\ 8 & 1 & 24 & 17 & 10 & 10 & 17 & 24 & 1 & 8 \end{bmatrix}.$$

It is routine to verify that B is a magic square. Moreover, it has the ref-symmetrical property (horizontally).

Definition 5. Let A^* denote the $2(2m+1) \times 2(2m+1)$ square defined by

$$A^* = \begin{bmatrix} A_0 & A_2 \\ A_3 & A_1 \end{bmatrix}$$

where A_0, A_1, A_2, A_3 as defined in Definition 2 with the following additional conditions.

(i) $\rho(A_0) = A_0$ and $\rho(A_1) = A_1$, and

(ii) the $(m+1)$-th row of A_0 (and of A_1) is symmetrical about the central cell.

Figure 3 depicts an example each for the squares A_0 and A_1.

$$\begin{bmatrix} 3 & 0 & 0 & 0 & 3 \\ 3 & 3 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 3 \\ 3 & 3 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \end{bmatrix}$$

$$A_0 \qquad\qquad A_1$$

Figure 3: A_0 and A_1 with $m = 2$.

Example 5. Suppose A_0 and A_1 are the matrices in Figure 3. Then

$$A^* = \begin{bmatrix} 3 & 0 & 0 & 0 & 3 & 2 & 1 & 2 & 2 & 2 \\ 3 & 3 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 \\ 3 & 0 & 0 & 0 & 3 & 2 & 2 & 1 & 2 & 2 \\ 3 & 3 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 \\ 3 & 0 & 0 & 0 & 3 & 2 & 1 & 2 & 2 & 2 \\ 0 & 3 & 3 & 3 & 0 & 1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 3 & 3 & 3 & 2 & 1 & 1 & 1 & 1 \\ 0 & 3 & 3 & 3 & 0 & 1 & 1 & 2 & 1 & 1 \\ 0 & 0 & 3 & 3 & 3 & 2 & 1 & 1 & 1 & 1 \\ 0 & 3 & 3 & 3 & 0 & 1 & 2 & 1 & 1 & 1 \end{bmatrix}.$$

Definition 6. Suppose A^* and B are as defined in the Definitions 5 and 4 respectively. Let $[A^*; B]$ denote the $2(2m+1) \times 2(2m+1)$ matrix obtained by superimposing A^* and B so that the entries of A are the radix digits while those of B are the unit digits.

Example 6. Suppose A^* and B are the 10×10 matrices in Examples 5 and 4 respectively. Then $[A^*; B]$ is a 10×10 matrix which yields a reduced magic square of order 10 when its entries are considered as integers represented in base 5^2. Moreover it is a ref-symmetrical magic square (with respect to the central horizontal line) which is as depicted below.

$$\begin{bmatrix} 91 & 23 & 0 & 7 & 89 & 64 & 32 & 50 & 73 & 66 \\ 97 & 79 & 6 & 13 & 15 & 40 & 63 & 56 & 54 & 72 \\ 78 & 5 & 12 & 19 & 96 & 71 & 69 & 37 & 55 & 53 \\ 77 & 95 & 18 & 11 & 9 & 34 & 61 & 68 & 70 & 52 \\ 83 & 1 & 24 & 17 & 85 & 60 & 42 & 74 & 51 & 58 \\ 16 & 98 & 75 & 82 & 14 & 39 & 57 & 25 & 48 & 41 \\ 22 & 4 & 81 & 88 & 90 & 65 & 38 & 31 & 29 & 47 \\ 21 & 94 & 87 & 80 & 3 & 28 & 30 & 62 & 44 & 46 \\ 2 & 20 & 93 & 86 & 84 & 59 & 36 & 43 & 45 & 27 \\ 8 & 76 & 99 & 92 & 10 & 35 & 67 & 49 & 26 & 33 \end{bmatrix}.$$

Theorem 2. *Suppose A^* and B are as defined in the Definitions 5 and 4 respectively. Then $[A^*; B]$ yields a reduced ref-symmetric magic square of order $2(2m+1)$ when its entries are considered as integers represented in base $(2m+1)^2$.*

Proof. The magicness of A^* follows from that of A (which has already been shown in the proof of Theorem 1).

The fact that $A_2 = 3J_{2m+1} - A_1$ and $A_3 = 3J_{2m+1} - A_0$ together with the condition that $\rho(A_0) = A_0$ and $\rho(A_1) = A_1$ imply that

$$\rho(A^*) + A^* = 3J_{2(2m+1)}. \tag{1}$$

Clearly the rows of B are magic. We assert that the columns of B are also magic.

Since M is ro-symmetrical, we have

$$M_1 + \pi(M_3) = ((2m+1)^2 - 1)J_{m,2m+1} \tag{2}$$

and

$$M_2 + \pi(M_2) = ((2m+1)^2 - 1)J_{1,2m+1} \tag{3}$$

where $J_{k,l}$ is the $k \times l$ matrix whose entries are equal to 1. This proves the assertion.

(2) and (3) imply that

$$B_1 + \rho(B_1^*) = ((2m+1)^2 - 1)J_{2m+1} \tag{4}$$

and this implies that

$$\pi(B_1) + \rho(\pi(B_1^*)) = ((2m+1)^2 - 1)J_{2m+1} \tag{5}$$

(4) and (5) imply that

$$B + \rho(B) = ((2m+1)^2 - 1)J_{2(2m+1)}. \tag{6}$$

To show that both the diagonals are magic, assume that $M = (a_{i,j})$ so that $\{a_{i,i} \mid i = 1, 2, \ldots, 2m+1\}$ and $\{a_{i,2m+2-i} \mid i = 1, 2, \ldots, 2m+1\}$ are the main diagonal and anti-diagonal of M respectively.

Note that B_1 and B_1^* have the same sets of main diagonal and anti-diagonal. The same goes for those of $\pi(B_1)$ and $\pi(B_1^*)$.

Now the main diagonal of B_1 consists of the entries

$$\{a_{i,i} \mid i = 1, 2, \ldots, m+1\} \cup \{a_{i,2m+2-i} \mid i = m+2, m+3, \ldots, 2m+1\}$$

while the main diagonal of $\pi(B_1^*)$ consists of the entries

$$\{a_{i,2m+2-i} \mid i = 1 \ldots, m\} \cup \{a_{i,i} \mid i = m+1, m+2, \ldots, 2m+1\}.$$

The union of these sets turns out to be the set of entries in the diagonal and anti-diagonal of M.

Likewise, the anti-diagonal of B consists of the anti-diagonals of $\pi(B_1)$ and B_1^* whose entries are

$$\{a_{i,i} \mid i = 1, 2, \ldots, m+1\} \cup \{a_{i,2m+2-i} \mid i = m+2, m+3, \ldots, 2m+1\}$$

and

$$\{a_{i,2m+2-i} \mid i = 1 \ldots, m\} \cup \{a_{i,i} \mid i = m+1, m+2, \ldots, 2m+1\}$$

respectively so their union is the set of entries in the diagonal and anti-diagonal of M.

This completes the proof that B is a magic square.

By following the same argument as in the proof of Theorem 1, we see that the entries in $[A^*; B]$ consist of distinct ordered pairs of entries (x, y) where $x \in A^*$ and $y \in B$.

Since A^* and B are magic squares with magic sums $3(2m+1)$ and $(2m+1)((2m+1)^2-1)$ respectively, we conclude (as in the proof of Theorem 1) that $[A^*; B]$ yields a reduced magic square of order $2(2m+1)$. Moreover, by Equations (1) and (6), the resulting reduced magic square is ref-symmetrical. □

5 Some remarks

Some remarks are in order.

1. By taking the $(m+1)$-th row of A_0 to be $(0, 3, \ldots, 3, 0 \ldots, 0)$, and all other rows to be $(3, \ldots, 3, 0, \ldots 0)$ where the numbers of 3 and 0 are m and $m+1$ respectively, and each row in A_1 to be $(1, \ldots, 1, 2, \ldots, 2)$ (where the numbers of 1 and 2 are $m+2$ and $m-1$ respectively), $B_1 = B_2$ to be the reduced De la Loubère magic square of order $2m+1$, the magic square $[A; B]$ is the reduced Strachey square of order $2(2m+1)$.

2. By cyclically permuting the columns of A_1 as constructed in Remark 1, we obtain new matrices which satisfy condition (ii) of Definition 2.

3. By taking the reflection along the central vertical line of the matrix A_0 in Remark 1, we obtain a different matrix satisfying the condition (i) of Definition 2.

4. How many matrices A_0 and A_1 are there that satisfy the conditions of Definition 2 (or Definition 5) for a given natural number m?

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