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Finite Sums Involving Fibonacci and Lucas Numbers

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Abstract

In this paper, we introduce several identities related to Fibonacci and Lucas numbers, extending the results established by Byrd in 1975. Moreover, we derive some identities involving Fibonacci, Lucas, Bernoulli, Euler, Genocchi, and Stirling numbers. Our main tools are linear operators and their properties.

1 Introduction and preliminaries

Fascination with special numbers and polynomials, such as Bernoulli, Euler, and Genocchi numbers and polynomials, has persisted since the post-Renaissance period due to their wide-ranging applications in various fields of mathematics, computer algorithms, engineering, and beyond. The exploration of these numbers and polynomials has evolved over time, progressing from elementary number theory techniques to more advanced approaches such as real analysis, complex analysis, and operator theory. Notable references on the subject can be found in [1, 5, 12, 13], illustrating the ongoing development of explicit formulas, identities, and properties associated with these special numbers. The Bernoulli numbers B_n, Euler numbers E_n (see A122045 in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [16]), and Genocchi numbers G_n (A036968) can be defined by the exponential generating functions:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \tag{1}$$

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \tag{2}$$

$$\frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}. \tag{3}$$

It is straightforward to show that $G_0 = 0$ and $G_n = 2(1 - 2^n)B_n = nE_{n-1}$ for $n \geq 1$. Thus, the properties of the Genocchi numbers can be deduced from those of the Euler numbers. It is also an old result of Genocchi [7] that the G_n's are all integers.

The Pell and Lucas polynomial sequences are, respectively, defined by the following recurrence relations:

$$P_n(t) = 2tP_{n-1}(t) + P_{n-2}(t),$$
$$W_n(t) = 2tW_{n-1}(t) + W_{n-2}(t);$$

however, they have distinct initial conditions:

$$P_0(t) = 0 \text{ and } P_1(t) = 1,$$
$$W_0(t) = 2 \text{ and } W_1(t) = 2t.$$

From these polynomials we can extract the following numbers: For $t = \frac{1}{2}$, $P_n(\frac{1}{2})$ is the nth Fibonacci number: $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$, $F_1 = 1$ and $n \geq 2$. For $t = \frac{1}{2}$, $W_n(\frac{1}{2})$ is the nth Lucas number: $L_n = L_{n-1} + L_{n-2}$ with $L_0 = 2$, $L_1 = 1$ and $n \geq 2$. The recurrent linear sequences $(F_n)_{n\geq 0}$ and $(L_n)_{n\geq 0}$ share the same characteristic polynomial, $x^2 - x - 1$. The roots of this polynomial are: $\alpha = (1 + \sqrt{5})/2$ (the golden ratio) and $\beta = (1 - \sqrt{5})/2$ (the silver ratio).

The Fibonacci numbers F_n (A000045) and Lucas numbers L_n (A000032) can also be defined by the exponential generating function [11, p. 232]:

$$\sum_{n=0}^{\infty} F_n \frac{t^n}{n!} = \frac{e^{\alpha t} - e^{\beta t}}{\sqrt{5}}, \tag{4}$$

$$\sum_{n=0}^{\infty} L_n \frac{t^n}{n!} = e^{\alpha t} + e^{\beta t}. \tag{5}$$

In 1975, Byrd [2, 3] proved that the following two identity hold for every nonnegative integer n:

$$\sum_{k=0}^{n} \left(\sqrt{5}\right)^k \binom{n}{k} \frac{F_{n-k+1}}{n-k+1} B_k = \beta^n, \tag{6}$$

and

$$\sum_{k=0}^{n} \left(\frac{\sqrt{5}}{2}\right)^k \binom{n}{k} E_k L_{n-k} = 2^{1-n}. \tag{7}$$

Recall that the set $\mathrm{End}(\mathbb{C}[x])$ of linear endomorphisms of $\mathbb{C}[x]$ is both a vector space over $\mathbb{C}$ for the addition and multiplication of an endomorphism by a complex scalar and a non-commutative ring for the addition and composition of endomorphisms. It is clear that defining a linear operator of $\mathbb{C}[x]$ is equivalent to giving the images under this operator of the vectors of any basis of $\mathbb{C}[x]$. Note also that for every scalar $\alpha \in \mathbb{C}$ and $u, v \in \mathrm{End}(\mathbb{C}[x])$, we have $\alpha(u \circ v) = (\alpha u) \circ v = u \circ (\alpha v)$, which endows $\mathrm{End}(\mathbb{C}[x])$ with a structure of algebra over $\mathbb{C}$. Among the known linear operators, we cite the translation operator τ_r defined for every complex number $r \neq 0$ in the canonical basis by [9, 10, 12, 13, 14, 15]:

$$\tau_r(x^n) = (x + r)^n, \quad n \in \mathbb{N},$$

the derivation operator denoted $D = \mathrm{d}/\mathrm{d}x$ defined by:

$$D(x^0) = 0 \text{ and } D(x^n) = nx^{n-1}, \text{ for all } n \geq 1.$$

Recall also that τ_r can be expressed as follows [12, p. 209]:

$$\tau_r = e^{rD} = \sum_{k=0}^{\infty} r^k \frac{D^k}{k!}, \tag{8}$$

and Δ_r is the difference operator defined by $\Delta_r = \tau_r - 1$, for $r \neq 0$.

The umbral calculus provides solid tools for establishing new identities, generalizing old ones and finding well-known ones. These tools by the mean of linear operators not only simplify the proofs of certain formulas, particularly explicit formulas for Euler and Bernoulli polynomials and numbers, but also facilitate the exploration of new explicit formulas and the study of properties verified by other remarkable sequences of polynomials and numbers. In this paper, we utilize these operators and their properties to obtain new identities concerning Bernoulli, Euler, Fibonacci, Lucas, and Genocchi numbers. Interesting results on this subject can be found in [6, 17].

2 Identities concerning Fibonacci and Lucas numbers

In this section, we extend the identities (6) and (7) using elementary properties of Bernoulli, Euler, Fibonacci, and Lucas numbers, as well as operators. We show also some relationships between Genocchi numbers and Lucas numbers.

Theorem 1. *For all integers n, m such that $m \geq n \geq 0$ and every real x, we have*

$$(x+\beta)^m = \sum_{n=0}^{m} \sum_{k=0}^{n} \left(\sqrt{5}\right)^k \binom{n}{k} \binom{m}{n} B_k \frac{F_{n-k+1}}{n-k+1} x^{m-n}. \tag{9}$$

Proof. We have

$$\begin{aligned}
\tau_\beta &= e^{\beta D} \\
&= \frac{e^{\beta D}}{e^{\sqrt{5}D}-1} \left(e^{\sqrt{5}D}-1\right) \\
&= \left(\frac{\sqrt{5}D}{e^{\sqrt{5}D}-1}\right) \left(\frac{e^{\alpha D}-e^{\beta D}}{\sqrt{5}D}\right) \text{(since } \alpha-\beta=\sqrt{5}\text{)} \\
&= \left(\sum_{n=0}^{\infty} \left(\sqrt{5}\right)^n B_n \frac{D^n}{n!}\right) \left(\sum_{n=1}^{\infty} F_n \frac{D^{n-1}}{n!}\right) \text{(according to (1) and (4)).}
\end{aligned} \tag{10}$$

Note that the classical product of two formal series $\sum_{n=0}^{\infty} u_n \frac{t^n}{n!}$ and $\sum_{n=0}^{\infty} v_n \frac{t^n}{n!}$, is defined by

$$\left(\sum_{n=0}^{\infty} u_n t^n\right) \left(\sum_{n=0}^{\infty} v_n t^n\right) = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}, \tag{11}$$

where $c_n = \sum_{k=0}^{n} \binom{n}{k} u_k v_{n-k}$. Note that also

$$D^n(x^m) = \begin{cases} n!\binom{m}{n} x^{m-n}, & \text{if } m \geq n \geq 0; \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

Now, by successively applying relations (11), (12), and evaluating the image of x^m under τ_β and D^n, we obtain both sides of Equality (9). This completes the proof of Theorem 1. □

We can deduce several identities involving Fibonacci and Bernoulli numbers as special cases of Theorem 1. The following corollaries present these results.

By taking $x = -\beta$ in Identity (9), we obtain the following corollary:

Corollary 2. *For all integers n, m such that $m \geq n \geq 0$, we have*

$$\sum_{n=0}^{m} \sum_{k=0}^{n} \left(\sqrt{5}\right)^k \binom{n}{k} \binom{m}{n} B_k \frac{F_{n-k+1}}{n-k+1} \beta^{-n} = 0.$$

By taking $x = -1/2$ in Identity (9), we obtain the following corollary:

Corollary 3. *For all integers n, m such that $m \geq n \geq 0$, we have*

$$\sum_{n=0}^{m}\sum_{k=0}^{n}(-2)^n\left(\sqrt{5}\right)^k\binom{n}{k}\binom{m}{n}B_k\frac{F_{n-k+1}}{n-k+1} = \left(\sqrt{5}\right)^m.$$

By taking $x = 0$ and $m = n$ in (9), we derive a formula due to Byrd that expresses the silver ratio β and its powers in terms of Bernoulli and Fibonacci numbers. More precisely, we obtain:

$$\sum_{k=0}^{n}\binom{n}{k}\left(\sqrt{5}\right)^k\frac{F_{n-k+1}}{n-k+1}B_k = \beta^n, \text{ for all } n \geq 0.$$

The following corollary is a straightforward consequence of Theorem 1.

Corollary 4. *For all nonnegative integers m, n, s with $m - n \geq s$, and every real x, we have*

$$(x+\beta)^{m-s} = \frac{1}{(m)_s}\sum_{n=0}^{m}\sum_{k=0}^{n}\left(\sqrt{5}\right)^k\binom{n}{k}\binom{m}{n}B_k\frac{F_{n-k+1}}{n-k+1}(m-n+1)_s x^{m-n-s},$$

where for $z \in \mathbb{R}$ and $n \in \mathbb{N}$, $(z)_n$ denotes the falling factorial defined as: $(z)_0 = 1$ and $(z)_n = z(z-1)(z-2)\cdots(z-n+1)$ for $n \geq 1$.

Proof. If we differentiate both sides of Relation (9) s times, then we obtain the result. □

Now, we give a double sum involving Euler and Lucas numbers.

Theorem 5. *For all integers m and n with $m \geq n \geq 0$ and every real x, we have*

$$\left(x+\frac{1}{2}\right)^m = \frac{1}{2}\sum_{n=0}^{m}\sum_{k=0}^{n}\left(\frac{\sqrt{5}}{2}\right)^k\binom{n}{k}\binom{m}{n}x^{m-n}E_kL_{n-k}. \tag{13}$$

Proof. Identity (13) can be proved using a similar procedure to that followed in the proof of Theorem 1. Putting $r = \frac{1}{2}$ into Formula (8), we get

$$\begin{aligned}\tau_{\frac{1}{2}} &= e^{\frac{1}{2}D}\\ &= \frac{1}{2}\frac{2}{e^{\frac{\sqrt{5}}{2}D}+e^{-\frac{\sqrt{5}}{2}D}}\times\left(e^{\sigma D}+e^{\beta D}\right)\\ &= \frac{1}{2}\left(\sum_{n=0}^{\infty}\left(\frac{\sqrt{5}}{2}\right)^n E_n\frac{D^n}{n!}\right)\left(\sum_{n=0}^{\infty}L_n\frac{D^n}{n!}\right) \quad \text{(according to (2) and (5)).} \end{aligned} \tag{14}$$

According to the classical product of two formal series in (11) and calculating the image of x^m under $\tau_{\frac{1}{2}}$ and D^n, we get both sides of Equality (13). □

Several identities involving the Lucas and Euler numbers can be deduced as special cases of Identity (13); these are presented in the following corollaries.

By setting $x = -1/2$ in (13), then we get the following corollary:

Corollary 6. *For all integers m and n with $m \geq n \geq 0$, we have*

$$\sum_{n=0}^{m}\sum_{k=0}^{n}(-1)^n \frac{(\sqrt{5})^k}{2^{k+m-n}}\binom{n}{k}\binom{m}{n}E_k L_{n-k} = 0.$$

If we set $x = \frac{1}{2}$ in (13), then we get the following conclusion:

Corollary 7. *For all integers m and n with $m \geq n \geq 0$, we have*

$$\sum_{n=0}^{m}\sum_{k=0}^{n}\frac{(\sqrt{5})^k}{2^{k+m-n}}\binom{n}{k}\binom{m}{n}E_k L_{n-k} = 2.$$

By taking $x = \sqrt{5}/2$ in (13), we derive a formula for the golden ratio α and its powers as a finite double sum involving Lucas and Euler numbers. More precisely, we obtain the following corollary:

Corollary 8. *For all integers m and n with $m \geq n \geq 0$, we have*

$$\sum_{n=0}^{m}\sum_{k=0}^{n}\left(\frac{\sqrt{5}}{2}\right)^{k+m-n}\binom{n}{k}\binom{m}{n}E_k L_{n-k} = 2\alpha^m.$$

By taking $x = -\sqrt{5}/2$ in (13), we derive a formula which express the silver ratio β and its powers as a finite double sum in terms of Lucas and Euler numbers. More precisely, we obtain the following corollary:

Corollary 9. *For all integers m and n with $m \geq n \geq 0$, we have*

$$\sum_{n=0}^{m}\sum_{k=0}^{n}(-1)^{m-n}\left(\frac{\sqrt{5}}{2}\right)^{k+m-n}\binom{n}{k}\binom{m}{n}E_k L_{n-k} = 2\beta^m.$$

The following result is due to Byrd [3]:

Corollary 10. *When $x = 0$ and $m = n$, Identity (13) reduces to*

$$\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\sqrt{5}}{2}\right)^{k} E_k L_{n-k} = 2^{1-n}\ , \ \text{for all } n \geq 0.$$

In the following, we present new identities linking Genocchi numbers to Lucas numbers.

Theorem 11. *For all integers m and n with $m \geq n \geq 0$, and every real x, we have*

$$(x+\beta)^m = \frac{1}{2\sqrt{5}} \sum_{n=0}^{m} \sum_{k=0}^{n} \left(\sqrt{5}\right)^k \binom{n}{k} \binom{m}{n} G_k \frac{L_{n-k+1}}{n-k+1} x^{m-n}. \tag{15}$$

Proof. Putting $r = \beta$ in Formula (8), we get

$$\begin{aligned}
\tau_\beta &= e^{\beta D} \\
&= \frac{e^{\beta D}}{e^{\sqrt{5}D}+1} \left(e^{\sqrt{5}D}+1\right) \\
&= \frac{1}{2\sqrt{5}} \left(\frac{2(\sqrt{5}D)}{e^{\sqrt{5}D}+1}\right) \left(\frac{e^{\alpha D}+e^{\beta D}}{D}\right) \\
&= \frac{1}{2\sqrt{5}} \left(\sum_{n=0}^{\infty} \left(\sqrt{5}\right)^n G_n \frac{D^n}{n!}\right) \left(\sum_{n=1}^{\infty} L_n \frac{D^{n-1}}{n!}\right) \quad \text{(according to (3) and (5))}.
\end{aligned} \tag{16}$$

Using Relation (11) and calculating the image of x^m under τ_β and D^n, we get both sides of Equality (15). □

In the following corollaries, we consider some special cases of Identity (15).

Taking $x = -\beta$ in (15), we obtain the following corollary:

Corollary 12. *For all integers m and n with $m \geq n \geq 0$, we have*

$$\sum_{n=0}^{m} \sum_{k=0}^{n} \left(\sqrt{5}\right)^k \binom{n}{k} \binom{m}{n} G_k \frac{L_{n-k+1}}{n-k+1} \beta^{m-n} = 0.$$

Taking $x = -\frac{1}{2}$ in (15), we get the following corollary:

Corollary 13. *For all integers m and n with $m \geq n \geq 0$, we have*

$$\sum_{n=0}^{m} \sum_{k=0}^{n} 2^n \left(\sqrt{5}\right)^{k-n} \binom{n}{k} \binom{m}{n} G_k \frac{L_{n-k+1}}{n-k+1} = 2\left(\sqrt{5}\right)^{1-m}.$$

Taking $x = 0$ and $m = n$ in (15), we obtain the following identity:

Corollary 14. *For $n \geq 0$, we have*

$$\sum_{k=0}^{n} \left(\sqrt{5}\right)^{k-1} \binom{n}{k} G_k \frac{L_{n-k+1}}{n-k+1} = 2\beta^n.$$

3 Generalization of three identities

In this section, we generalize identities (9), (13), and (15). The next theorem extends the result of Theorem 1.

Theorem 15. *For all nonnegative integers n, m and $q \geq 1$ with $m \geq n$, and every real x, we have the formula:*

$$\begin{aligned}(x+q\beta)^m &= \sum_{n=0}^{m} \sum_{j_1+j_2+\cdots+j_q+\ell_1+\ell_2+\cdots+\ell_q=n} \left(\sqrt{5}\right)^{\ell_1+\ell_2+\cdots+\ell_q} \binom{n}{j_1, j_2, \ldots, j_q} \binom{m}{n} \\ &\quad \times B_{\ell_1} B_{\ell_2} \cdots B_{\ell_q} \frac{F_{j_1+1}}{j_1+1} \frac{F_{j_2+1}}{j_2+1} \cdots \frac{F_{j_q+1}}{j_q+1} x^{m-n}, \end{aligned} \tag{17}$$

where $\binom{n}{j_1, j_2, \ldots, j_q} = \frac{n!}{j_1! j_2! \cdots j_q!}$ is the multinomial coefficient.

Proof. Upon exponentiating both sides of (10) q times, and using the fact that for every nonzero complex number r and every integer $q \geq 1$ we have $\tau_r^q = \tau_{qr}$, we obtain

$$\begin{aligned}\tau_\beta^q &= \tau_{q\beta} \\ &= \sum_{n=0}^{\infty} \sum_{j_1+j_2+\cdots+j_q+\ell_1+\ell_2+\cdots+\ell_q=n} \left(\sqrt{5}\right)^{\ell_1+\ell_2+\cdots+\ell_q} \binom{n}{j_1, j_2, \ldots, j_q} \binom{m}{n} \\ &\quad \times B_{\ell_1} B_{\ell_2} \cdots B_{\ell_q} \frac{F_{j_1+1}}{j_1+1} \frac{F_{j_2+1}}{j_2+1} \cdots \frac{F_{j_q+1}}{j_q+1} D^{m-n}. \end{aligned} \tag{18}$$

Calculating the image of x^m under $\tau_{q\beta}$ and D^{m-n} in (18), we get both sides of Equality (17). □

Remark 16. If we take $x = 0$, $m = n$, and $q = 1$ in Relation (17), then we obtain Identity (6).

Corollary 17. *In particular, taking $q = 2$ and $x = \sqrt{5}$ in Relation (17), we get*

$$\sum_{n=0}^{m} \sum_{j_1+j_2+\ell_1+\ell_2=n} \left(\sqrt{5}\right)^{\ell_1+\ell_2+m-n} \binom{n}{j_1, j_2} \binom{m}{n} B_{\ell_1} B_{\ell_2} \frac{F_{j_1+1}}{j_1+1} \frac{F_{j_2+1}}{j_2+1} = 1$$

for $m \geq n \geq 2$.

The following theorem generalizes Identity (13).

Theorem 18. *For all nonnegative integers n, m and $q \geq 1$ with $m \geq n$, and every real x, we have*

$$\begin{aligned} & 2^q \left(x+\frac{q}{2}\right)^m \\ & = \sum_{n=0}^{m} \sum_{j_1+j_2+\cdots+j_q+\ell_1+\ell_2+\cdots+\ell_q=n} \left(\frac{\sqrt{5}}{2}\right)^{\ell_1+\ell_2+\cdots+\ell_q} \binom{n}{j_1, j_2, \ldots, j_q} \binom{m}{n} \\ & \quad \times E_{\ell_1} E_{\ell_2} \cdots E_{\ell_q} L_{j_1} L_{j_2} \cdots L_{j_q} x^{m-n}. \end{aligned} \tag{19}$$

Proof. The proof of Relation (19) can be obtained by exponentiating both sides of Relation (14) q times. □

Remark 19. If we take $x = 0$, $m = n$, and $q = 1$ in Relation (14), then we obtain Identity (7).

The following theorem generalizes Relation (15).

Theorem 20. *For all nonnegative integers n, m and $q \geq 1$ with $m \geq n$, and every real x, we have*

$$\begin{aligned} & \left(2\sqrt{5}\right)^q (x+q\beta)^m \\ & = \sum_{n=0}^{m} \sum_{j_1+j_2+\cdots+j_q+\ell_1+\ell_2+\cdots+\ell_q=n} \left(\sqrt{5}\right)^{\ell_1+\ell_2+\cdots+\ell_q} \binom{n}{j_1, j_2, \ldots, j_q} \binom{m}{n} \\ & \quad \times G_{\ell_1} G_{\ell_2} \cdots G_{\ell_q} \frac{L_{n-j_1+1}}{n-j_1+1} \frac{L_{n-j_2+1}}{n-j_2+1} \cdots \frac{L_{n-j_q+1}}{n-j_q+1} x^{m-n}. \end{aligned} \tag{20}$$

Proof. The proof of Relation (20) is obtained by exponentiating both sides of Relation (16) q times. □

4 Additional identities

In this section, we give some identities involving the Genocchi, Fibonacci, and Stirling numbers. The Stirling numbers of the first kind $s(n,k)$ (A048994) and the second kind $S(n,k)$ (A008277), respectively, can be defined by their exponential generating functions [4]:

$$\frac{\log^k(1+x)}{k!} = \sum_{n=0}^{\infty} s(n,k)\frac{x^n}{n!} \quad \text{and} \quad \frac{(e^x-1)^k}{k!} = \sum_{n=0}^{\infty} S(n,k)\frac{x^n}{n!}. \tag{21}$$

A well-known explicit formula for the Stirling numbers of the second kind $S(n,k)$ is given by [4]

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j^n.$$

Recall that the Genocchi polynomials $G_n(x)$ ($n \in \mathbb{N}$) can be defined by the exponential generating function:

$$\sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} = \frac{2te^{xt}}{e^t+1},$$

and when $x = 0$ we have $G_n(0) = G_n$. The following lemma will be used later.

Lemma 21. *For every nonzero complex r and every positive integer k, we have*

$$\Delta_r^k(x^n) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (x+rj)^n, \quad \text{for all } n \geq 0.$$

Proof. It suffices to observe that

$$\Delta_r^k = (\tau_r - 1)^k = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \tau_{rj}.$$

□

Theorem 22. *For $n \geq 1$, we have*

$$\frac{G_n}{n} = \sum_{j=0}^{n-1} \frac{(-1)^j j!}{2^j} S(n-1, j). \tag{22}$$

Proof. We consider the operator

$$\Omega_G = \frac{2D}{e^D+1}.$$

Now Ω_G can be expressed as follows:

$$\Omega_G = \frac{2D}{e^D+1} = \frac{1}{1+\frac{1}{2}\Delta} \circ D.$$

Then we deduce that

$$\Omega_G = \left(\sum_{j=0}^{\infty} \frac{(-1)^j}{2^j} \Delta^j \right) \circ D.$$

Thus, we have for $n \geq 0$

$$\begin{aligned} G_n(x) = \Omega_G(x^n) &= n \sum_{j=0}^{\infty} \frac{(-1)^j}{2^j} \Delta^j(x^{n-1}) \\ &= n \sum_{j=0}^{n-1} \frac{(-1)^j}{2^j} \Delta^j(x^{n-1}). \end{aligned}$$

Putting $x = 0$ in the last equality, we obtain the desired result. □

Note that Identity (22) was proven by Guo and Qi [8] using the formula of higher order derivatives.

Theorem 23. *For all nonnegative integers $m, n,$ and every real x, we have*

$$\begin{aligned}
&\frac{x}{\left(\sqrt{5}\right)^n}\prod_{i=1}^{n-1}(x-\alpha i-(n-i)\beta)\\
&=\sum_{n=0}^{m}\sum_{k=n}^{m}\sum_{j=0}^{k}\prod_{i=0}^{n-1}(-1)^{k-j}\frac{F_n}{k!\left(\alpha\sqrt{5}\right)^n}s(n,k)\binom{k}{j}\\
&\quad\times(x+\alpha j)(x-\alpha ij-(n-1-i)\beta).
\end{aligned} \tag{23}$$

Proof. We consider the finite difference operator

$$\Lambda_F=\frac{e^{\alpha D}-e^{\beta D}}{\sqrt{5}}.$$

The canonical basis associated with the operator $\tau_\alpha-\tau_\beta$ is the sequence of polynomials $\left(U_n^{\tau_\alpha,\tau_\beta}(x)\right)_{n\geq 0}$ such that

$$U_0^{\tau_\alpha,\tau_\beta}(x)=1 \text{ and } U_n^{\tau_\alpha,\tau_\beta}(x)=\frac{x}{\left(\sqrt{5}\right)^n}\prod_{i=1}^{n-1}(x-\alpha i-(n-i)\beta) \text{ for all } n\geq 1.$$

We have

$$\Lambda_F=\frac{e^{\alpha D}-e^{\beta D}}{\sqrt{5}}=\sum_{n=0}^{\infty}F_n\frac{D^n}{n!}. \tag{24}$$

Formula (24) can be written as follows:

$$\Lambda_F=\frac{\tau_\alpha-\tau_\beta}{\sqrt{5}}=\sum_{n=0}^{\infty}\frac{F_n}{\alpha^n}\frac{\ln^n(1+\Delta_\alpha)}{n!}.$$

Then, using the first identity in (21) and Lemma 21, we get

$$\begin{aligned}
\Lambda_F&=\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{F_n}{\alpha^n}s(n,k)\frac{\Delta_\alpha^k}{k!}\\
&=\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\sum_{j=0}^{k}(-1)^{k-j}\frac{F_n}{\alpha^n}s(n,k)\binom{k}{j}\frac{\tau_{\alpha j}}{k!}.
\end{aligned}$$

Finally, calculating the image of $U_n^{\tau_\alpha,\tau_\beta}(x)$ under the operators $\tau_{\alpha j}$ and Λ_F, we obtain (23). □

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