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Arithmetic Identities for Some Analogs of the 5-Core Partition Function

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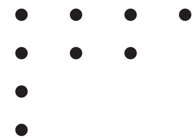
Abstract

Recently, Gireesh, Ray, and Shivashankar studied an analog, $\overline{a}_t(n)$, of the t-core partition function, $c_t(n)$. In this paper, we study the function $\overline{a}_5(n)$ (A053723) in conjunction with $c_5(n)$ (A368490), as well as another analogous function $\overline{b}_5(n)$ (A368495). We also find several arithmetic identities for $\overline{a}_5(n)$ and $\overline{b}_5(n)$.

1 Introduction

A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of a positive integer n is a finite sequence of non-increasing positive integer *parts* λ_i such that $n = \sum_{i=1}^{k} \lambda_i$. The *Ferrers-Young diagram* of the partition λ of n is constructed by placing n nodes in k rows so that the ith row has λ_i nodes. The nodes are marked with the row and column coordinates, similar to how one would mark the position of the elements of a matrix. Let λ'_j denote the number of nodes in column j. The *hook number* $H(i,j)$ for the node at position (i,j) is determined by counting the nodes situated directly below and to the right of it, including the node itself. That is, $H(i,j) = \lambda_i + \lambda'_j - i - j + 1$. If none of the hook numbers of a partition is divisible by t, then it is called a *t-core*.

Example 1. The Ferrers-Young diagram of the partition $\lambda = (4,3,1,1)$ of 9 is given as follows:



The nodes $(1,1)$, $(1,2)$, $(1,3)$, $(1,4)$, $(2,1)$, $(2,2)$, $(2,3)$, $(3,1)$, and $(4,1)$ have hook numbers 7, 4, 3, 1, 5, 2, 1, 2, and 1, respectively. Therefore, λ is a t-core for $t = 6$ and $t \geq 8$.

Granville and Ono [8] proved that for $t \geq 4$, every positive integer n has a t-core. For a recent survey on t-cores, we refer the readers to a paper by Cho, Kim, Nam, and Sohn [5].

Let $c_t(n)$ denote the number of t-cores of n. Then the generating function of $c_t(n)$ is (see Garvan, Kim, and Stanton [6, Eq. (2.1)])

$$\sum_{n=0}^{\infty} c_t(n) q^n = \frac{f_t^t}{f_1}, \tag{1}$$

where for integer $j \geq 1$, $f_j := (q^j; q^j)_\infty$ and throughout the paper, for complex numbers a and q with $|q| < 1$, we define

$$(a;q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k).$$

For $|ab| < 1$, Ramanujan's general theta function $f(a,b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.$$

In this notation, Jacobi's well-known triple product identity [3, p. 35, Entry 19] takes the form

$$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty. \tag{2}$$

Consider the following special cases of $f(a,b)$:

$$\varphi(-q) := f(-q,-q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{f_1^2}{f_2}, \tag{3}$$

$$\psi(-q) := f(-q,-q^3) = \sum_{n=0}^{\infty} (-q)^{n(n+1)/2} = \frac{f_1 f_4}{f_2}, \tag{4}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = f_1, \tag{5}$$

where the q-product representations in the above arise from (2) and manipulation of the q-products.

In the notation of (5), the generating function (1) of $c_t(n)$ may be recast as

$$\sum_{n=0}^{\infty} c_t(n)q^n = \frac{f^t(-q^t)}{f(-q)}. \tag{6}$$

Recently, Gireesh, Ray, and Shivashankar [7, Eq. (1.2)] considered an analog $\overline{a}_t(n)$ of $c_t(n)$ with $f(-q)$ is replaced by $\varphi(-q)$ in (6), namely,

$$\sum_{n=0}^{\infty} \overline{a}_t(n)q^n = \frac{\varphi^t(-q^t)}{\varphi(-q)}.$$

They obtained some arithmetic identities and multiplicative formulas for $\overline{a}_3(n)$, $\overline{a}_4(n)$, and $\overline{a}_8(n)$ by using Ramanujan's theta functions. (Note that Theorem 1.1 in their paper [7] holds only for a special case. Also, the induction process in the proof of the theorem is not quite correct.) They also studied the arithmetic density of $\overline{a}_t(n)$ by employing the theory of modular forms and found the following Ramanujan-type congruence for $\overline{a}_5(n)$ [7, Theorem 1.10]: For all $n \geq 0$,

$$\overline{a}_5(20n+6) \equiv 0 \pmod{5}. \tag{7}$$

Note that

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{a}_5(n)q^n &= \frac{\varphi^5(-q^5)}{\varphi(-q)} \\ &= 1 + 2q + 4q^2 + 8q^3 + 14q^4 + 14q^5 + 20q^6 + 24q^7 + \cdots . \end{aligned} \tag{8}$$

In this paper, we revisit the function $\overline{a}_5(n)$ in conjunction with $c_5(n)$ as well as another function $\overline{b}_5(n)$ defined by

$$\sum_{n=0}^{\infty} \overline{b}_5(n)q^n = \frac{\psi^5(-q^5)}{\psi(-q)} = 1 + q + q^2 + 2q^3 + 3q^4 - q^5 + 2q^7 - 2q^9 + 6q^{10} + \cdots , \tag{9}$$

where $\psi(-q)$ is as defined in (4).

The sequences $(c_5(n))$, $(\overline{a}_5(n))$, and $(\overline{b}_5(n))$ are A053723, A368490, and A368495, respectively, in [10].

We state our results in the following theorems and corollaries. In the sequel, we assume that $c_5(n) = \overline{a}_5(n) = \overline{b}_5(n) = 0$ for $n < 0$.

We state a recurrence relation for $\overline{a}_5(n)$ and some relations between $\overline{a}_5(n)$ and $c_5(n)$ in the following theorem.

Theorem 2. *For every nonnegative integer n,*

$$\overline{a}_5(5n+2) = 4c_5(5n+1), \tag{10}$$
$$\overline{a}_5(5n+3) = 4c_5(5n+2), \tag{11}$$
$$\overline{a}_5(10n+1) = 2c_5(10n), \tag{12}$$
$$\overline{a}_5(10n+9) = 2c_5(10n+8), \tag{13}$$
$$\overline{a}_5(20n+6) = 10c_5(10n+2), \tag{14}$$
$$\overline{a}_5(20n+14) = 10c_5(10n+6). \tag{15}$$

Furthermore, for every integer $k \geq 2$,

$$\overline{a}_5(5^k n) = \Big(\frac{5^k-1}{4}\Big)\overline{a}_5(5n) - \Big(\frac{5^k-5}{4}\Big)\overline{a}_5(n). \tag{16}$$

The following corollary is immediate from the above theorem.

Corollary 3. *For every nonnegative integer n and every integer $k \geq 2$,*

$$\overline{a}_5(20n+6) \equiv 0 \pmod{10}, \tag{17}$$
$$\overline{a}_5(20n+14) \equiv 0 \pmod{10}, \tag{18}$$

and

$$4\overline{a}_5(5^k n) \equiv 5\overline{a}_5(n) - \overline{a}_5(5n) \pmod{5^k}.$$

Note that (17) implies (7). However, even stronger results implying (17) and (18) are stated in Corollary 7.

Now we state some recurrence relations for $\overline{b}_5(n)$.

Theorem 4. *For every nonnegative integer n and every integer $k \geq 2$, we have*

$$\overline{b}_5(4n+3) = 2\overline{b}_5(2n) \tag{19}$$

and

$$\overline{b}_5\big(5^k(n+3)-3\big) = \Big(\frac{5^k-1}{4}\Big)\overline{b}_5(5n+12) - \Big(\frac{5^k-5}{4}\Big)\overline{b}_5(n). \tag{20}$$

Next we state some identities connecting $\overline{b}_5(n)$ with $\overline{a}_5(n)$ and $c_5(n)$.

Theorem 5. *For every nonnegative integer n, we have*

$$\begin{aligned}
\overline{b}_5(4n+1) &= c_5(n) - 2\overline{b}_5(2n-1), && (21)\\
\overline{b}_5(10n) &= \frac{1}{2}c_5(10n+2), && (22)\\
\overline{b}_5(10n+1) &= c_5(5n+1), && (23)\\
\overline{b}_5(10n+2) &= \frac{1}{4}\overline{a}_5(2n+1) + \frac{1}{2}c_5(2n), && (24)\\
\overline{b}_5(10n+3) &= c_5(5n+2), && (25)\\
\overline{b}_5(10n+4) &= \frac{1}{2}c_5(10n+6), && (26)\\
\overline{b}_5(10n+6) &= 0, && (27)\\
\overline{b}_5(10n+8) &= 0, && (28)\\
\overline{b}_5(20n+5) &= -c_5(5n+1), && (29)\\
\overline{b}_5(20n+7) &= \frac{1}{2}\overline{a}_5(2n+1) + c_5(2n), && (30)\\
\overline{b}_5(20n+9) &= -c_5(5n+2), && (31)\\
\overline{b}_5(20n+15) &= 0, && (32)\\
\overline{b}_5(20n+19) &= 0. && (33)
\end{aligned}$$

Corollary 6. *For positive integers n, $\overline{b}_5(n)$ is 0 for at least 30%, greater than 0 for at least 52%, and less than 0 for at least 10%.*

Proof. Identities (27), (28), (32), and (33) readily imply the observed frequency of zeroes. Similarly, (29) and (31) imply the frequency of negatives. From the identities of (10), (11), (12), and (13), we observe that the sequence $(\overline{a}_5(2n+1))$ is positive in at least 4 out of 5 cases. Together with (22)–(26) and (30), this implies that the frequency of positives is at least equal to

$$\frac{2+2+2\cdot(4/5)+2+2+1\cdot(4/5)}{20},$$

that is, 52%. □

From (14), (15), (22), and (26), we arrive at the following corollary, implying the congruence of (7) by Gireesh, Ray, and Shivashankar [7, Thm. 1.10].

Corollary 7. *For every nonnegative integer n,*

$$\overline{a}_5(20n+6) = 20\overline{b}_5(10n),$$
$$\overline{a}_5(20n+14) = 20\overline{b}_5(10n+4).$$

We state some infinite families of congruences in the following corollary.

Corollary 8. *For every nonnegative integer n and every integer $k \geq 2$,*

$$4\overline{b}_5\Big(5^k(n+3)-3\Big) \equiv 5\overline{b}_5(n) - \overline{b}_5(5n+12) \pmod{5^k},$$
$$\overline{b}_5(5^k(20n+18)-3) \equiv 0 \pmod{\frac{5^k-1}{4}},$$

and

$$\overline{b}_5(5^k(20n+22)-3) \equiv 0 \pmod{\frac{5^k-1}{4}}.$$

Proof. The first congruence readily follows from (20). Again, from (32), (33), and (20) it follows that, for every nonnegative integer n and every integer $k \geq 2$,

$$\overline{b}_5(5^k(20n+18)-3) = \Big(\frac{5^k-1}{4}\Big)\overline{b}_5(100n+87),$$
$$\overline{b}_5(5^k(20n+22)-3) = \Big(\frac{5^k-1}{4}\Big)\overline{b}_5(100n+107),$$

which implies the last two congruences in the corollary. □

We arrange the rest of the paper as follows. In Section 2, we provide some preliminary lemmas. We prove Theorem 2, Theorem 4, and Theorem 5 in Section 3, Section 4, and Section 5, respectively.

2 Preliminary Lemmas

In the following lemma, we state some known theta function identities.

Lemma 9. *If $\varphi(-q)$, $\psi(-q)$, and $f(-q)$ are as defined in (3)–(5) and $\chi(-q) := (q;q^2)_\infty$, then*

$$\frac{\varphi^5(q^5)}{\varphi(q)} + 4q\frac{f^5(q^5)}{f(q)} = \varphi(q)\varphi^3(q^5), \tag{34}$$
$$\varphi^2(q) - \varphi^2(q^5) = 4q\chi(q)f_5f_{20}, \tag{35}$$
$$\frac{\psi^5(-q^5)}{\psi(-q)} - \frac{\psi^5(q^5)}{\psi(q)} = 4q^3\frac{\psi^5(q^{10})}{\psi(q^2)} + 2q\frac{f_{20}^5}{f_4}, \tag{36}$$
$$\psi^2(q) - q\psi^2(q^5) = \frac{f(-q^5)\varphi(-q^5)}{\chi(-q)} = f(q,q^4)f(q^2,q^3), \tag{37}$$
$$\frac{f_5^5}{f_1} - 4q^3\frac{f_{20}^5}{f_4} = \frac{f^5(q^5)}{f(q)} + 2q\frac{f_{10}^5}{f_2}, \tag{38}$$
$$\frac{f_2^2}{f_1^4} = \frac{f_{10}^2}{f_5^4} + 4q\frac{f_2f_{10}^5}{f_1^3f_5^5}. \tag{39}$$

Proof. Identities (34) and (35) are identical to Entry 9(ii) and Entry 9(iii) of [3, Chap. 19]. For the proofs of (36) and (37), we refer to Entry 15 and Entry 18 of [4, Chap. 36]. The identity (38) can be found in [2, Eq. (4.7)]. The identity (39) is simply [1, Eq. (2.6)]. □

In the following lemma, we recall some 5-dissection formulas from [9, p. 89, Eq. (8.4.4)] and [3, p. 49, Corollary].

Lemma 10. *Let $R(q)$ be defined as*

$$R(q) := \frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

We have

$$\begin{aligned}\frac{1}{f_1} = \frac{f_{25}^5}{f_5^6}\Big(&\frac{1}{R(q^5)^4} + \frac{q}{R(q^5)^3} + \frac{2q^2}{R(q^5)^2} + \frac{3q^3}{R(q^5)} + 5q^4 \\ &- 3q^5 R(q^5) + 2q^6 R(q^5)^2 - q^7 R(q^5)^3 + q^8 R(q^5)^4\Big), \end{aligned} \tag{40}$$

$$\varphi(q) = \varphi(q^{25}) + 2qf(q^{15},q^{35}) + 2q^4 f(q^5,q^{45}), \tag{41}$$

$$\psi(q) = f(q^{10},q^{15}) + qf(q^5,q^{20}) + q^3\psi(q^{25}). \tag{42}$$

In the following lemma, we present two useful identities on $c_5(n)$.

Lemma 11. *For every nonnegative integer n,*

$$c_5(4n+1) = c_5(2n), \tag{43}$$

$$c_5(5n+4) = 5c_5(n). \tag{44}$$

Proof. See [2, Eq. (4.8)] and [6, Eq. (5.1)]. □

3 Proof of Theorem 2

Proofs of (10) *and* (11). Replacing q by $-q$ in (34), we have

$$\frac{\varphi^5(-q^5)}{\varphi(-q)} = 4q\frac{f_5^5}{f_1} + \varphi(-q)\varphi^3(-q^5), \tag{45}$$

which, by (1) and (8), may be recast as

$$\sum_{n=0}^{\infty} \overline{a}_5(n) q^n = 4\sum_{n=0}^{\infty} c_5(n) q^{n+1} + \varphi(-q)\varphi^3(-q^5).$$

Replacing q by $-q$ in (41) and then using the resulting identity in the above, we have

$$\sum_{n=0}^{\infty} \overline{a}_5(n)q^n = 4\sum_{n=0}^{\infty} c_5(n)q^{n+1} + \varphi^3(-q^5)\Big(\varphi(-q^{25}) - 2qf(-q^{15}, -q^{35}) + 2q^4 f(-q^5, -q^{45})\Big).$$

Equating the coefficients of q^{5n+2} and q^{5n+3} from both sides of the above identity, we arrive at (10) and (11), respectively.

Proofs of (12) *and* (13). Multiplying both sides of (38) by $\dfrac{f_5^5 f_2}{f_{10}^5 f_1}$, we have

$$\frac{\varphi^5(-q^5)}{\varphi(-q)} - 4q^3\frac{\psi^5(-q^5)}{\psi(-q)} = \frac{\varphi^5(-q^{10})}{\varphi(-q^2)} + 2q\frac{f_5^5}{f_1},$$

which, by (1), (8), and (9), yields

$$\sum_{n=0}^{\infty} \overline{a}_5(n)q^n - 4\sum_{n=0}^{\infty} \overline{b}_5(n)q^{n+3} = \sum_{n=0}^{\infty} \overline{a}_5(n)q^{2n} + 2\sum_{n=0}^{\infty} c_5(n)q^{n+1}. \tag{46}$$

Comparing the coefficients of q^{2n+1} from both sides, we find that

$$\overline{a}_5(2n+1) - 4\overline{b}_5(2n-2) = 2c_5(2n). \tag{47}$$

Replacing n by $5n$ and $5n+4$, we obtain

$$\overline{a}_5(10n+1) = 4\overline{b}_5(10n-2) + 2c_5(10n)$$

and

$$\overline{a}_5(10n+9) = 4\overline{b}_5(10n+6) + 2c_5(10n+8),$$

respectively. Using (27) and (28) in the above, we arrive at (12) and (13).

Proofs of (14) *and* (15). Equating the coefficients of q^{2n} from both sides of (46), we have

$$\overline{a}_5(2n) - 4\overline{b}_5(2n-3) = \overline{a}_5(n) + 2c_5(2n-1). \tag{48}$$

From (47) and (48), it follows that

$$\overline{a}_5(4n+2) - 4\overline{b}_5(4n-1) = \overline{a}_5(2n+1) + 2c_5(4n+1), . \tag{49}$$

Again, employing (1) and (9), it follows from (36) that

$$\sum_{n=0}^{\infty} \overline{b}_5(n)q^n - \sum_{n=0}^{\infty} \overline{b}_5(n)(-q)^n = 4\sum_{n=0}^{\infty} (-1)^n \overline{b}_5(n)q^{2n+3} + 2\sum_{n=0}^{\infty} c_5(n)q^{4n+1}. \tag{50}$$

Equating the coefficients of q^{4n+3} from both sides of the above, we have

$$\overline{b}_5(4n+3) = 2\overline{b}_5(2n). \tag{51}$$

It follows from (47) and (51) that

$$\overline{a}_5(2n+1) = 2c_5(2n) + 2\overline{b}_5(4n-1).$$

Using (43) and the above identity in (49), we obtain

$$\overline{a}_5(4n+2) = 3\overline{a}_5(2n+1) - 2c_5(2n), \tag{52}$$

which by replacement of n with $5n+1$ yields

$$\overline{a}_5(20n+6) = 3\overline{a}_5(10n+3) - 2c_5(10n+2). \tag{53}$$

Again, replacing n by $2n$ in (11), we have

$$\overline{a}_5(10n+3) = 4c_5(10n+2). \tag{54}$$

It follows from (53) and (54) that

$$\overline{a}_5(20n+6) = 10c_5(10n+2),$$

which is (14).

Next, replacing n by $5n+3$ in (52), we have

$$\overline{a}_5(20n+14) = 3\overline{a}_5(10n+7) - 2c_5\big(10n+6\big). \tag{55}$$

Again, replacing n by $2n+1$ in (10), we have

$$\overline{a}_5(10n+7) = 4c_5(10n+6). \tag{56}$$

It follows from (55) and (56) that

$$\overline{a}_5(20n+14) = 10c_5(10n+6),$$

which is (15).

Proof of (16). With the aid of (8), we recast (45) as

$$\sum_{n=0}^{\infty} \overline{a}_5(n)q^n = 4q\frac{f_5^5}{f_1} + \varphi(-q)\varphi^3(-q^5). \tag{57}$$

Employing the 5-dissections of $1/f_1$ from (40) and that of $\varphi(-q)$ from (41) in the above identity, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{a}_5(n)q^n = 4q\frac{f_{25}^5}{f_5}\Big(&\frac{1}{R(q^5)^4}+\frac{q}{R(q^5)^3}+\frac{2q^2}{R(q^5)^2}+\frac{3q^3}{R(q^5)}+5q^4\\ &-3q^5R(q^5)+2q^6R(q^5)^2-q^7R(q^5)^3+q^8R(q^5)^4\Big)\\ &+\varphi^3(-q^5)\Big(\varphi(-q^{25})-2qf(-q^{15},-q^{35})+2q^4f(-q^5,-q^{45})\Big).\end{aligned}$$

Extracting the terms involving q^{5n} from both sides of the above, and then replacing q^5 by q, we find that

$$\sum_{n=0}^{\infty}\overline{a}_5(5n)q^n = 20q\frac{f_5^5}{f_1}+\varphi^3(-q)\varphi(-q^5). \tag{58}$$

Subtracting (57) from (58),

$$\sum_{n=0}^{\infty}\overline{a}_5(5n)q^n-\sum_{n=0}^{\infty}\overline{a}_5(n)q^n = 16q\frac{f_5^5}{f_1}+\varphi(-q)\varphi(-q^5)\big(\varphi^2(-q)-\varphi^2(-q^5)\big). \tag{59}$$

Employing (40) and (41) in the above and then extracting the terms involving q^{5n}, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\overline{a}_5(25n)q^n-\sum_{n=0}^{\infty}\overline{a}_5(5n)q^n\\ &=80q\frac{f_5^5}{f_1}+\varphi(-q)\big(\varphi^3(-q^5)-24q\varphi(-q^5)f(-q^3,-q^7)f(-q,-q^9)\big)\\ &\quad-\varphi(-q)\varphi(-q^5).\end{aligned}$$

Replacing q by $-q$ in (35) and then employing in the above identity, we find that

$$\sum_{n=0}^{\infty}\overline{a}_5(25n)q^n-\sum_{n=0}^{\infty}\overline{a}_5(5n)q^n = 80q\frac{f_5^5}{f_1}+5\varphi(-q)\varphi(-q^5)\big(\varphi^2(-q)-\varphi^2(-q^5)\big),$$

which, by (59), yields

$$\sum_{n=0}^{\infty}\overline{a}_5(25n)q^n-\sum_{n=0}^{\infty}\overline{a}_5(5n)q^n = 5\sum_{n=0}^{\infty}\overline{a}_5(5n)q^n-5\sum_{n=0}^{\infty}\overline{a}_5(n)q^n.$$

Equating the coefficients of q^n from both sides, we find that, for every nonnegative integer n,

$$\overline{a}_5(25n)=6\overline{a}_5(5n)-5\overline{a}_5(n). \tag{60}$$

Now (16) follows by mathematical induction on $k\geq 2$.

4 Proof of Theorem 4

Note that (19) is identical to (51). Therefore, we proceed to prove only (20).

Replacing q by $-q$ in (37), we have

$$q\psi^2(-q^5) = \frac{f(q^5)\varphi(q^5)}{\chi(q)} - \psi^2(-q).$$

Multiplying both sides of the above identity by $\dfrac{\psi^3(-q^5)}{\psi(-q)}$, we find that

$$q\frac{\psi^5(-q^5)}{\psi(-q)} = \frac{f_{10}^5}{f_2} - \psi(-q)\psi^3(-q^5),$$

which, by (9), can be recast as

$$\sum_{n=0}^{\infty} \overline{b}_5(n)q^{n+1} = \frac{f_{10}^5}{f_2} - \psi(-q)\psi^3(-q^5). \tag{61}$$

Employing the 5-dissection of $1/f_2$ from (40) and that of $\psi(-q)$ from (42) in (61), and then extracting the terms involving q^{5n+3} from both sides of the resulting identity, we obtain

$$\sum_{n=0}^{\infty} \overline{b}_5(5n+2)q^n = 5q\frac{f_{10}^5}{f_2} + \psi^3(-q)\psi(-q^5). \tag{62}$$

Multiplying (61) by q and then subtracting from (62),

$$\begin{aligned}&\sum_{n=0}^{\infty} \overline{b}_5(5n+2)q^n - \sum_{n=0}^{\infty} \overline{b}_5(n)q^{n+2}\\ &= 4q\frac{f_{10}^5}{f_2} + \psi(-q)\psi(-q^5)\big(\psi^2(-q) + q\psi^2(-q^5)\big).\end{aligned} \tag{63}$$

Again, using (40) and (42) in the above identity and then extracting the terms involving q^{5n+4} from both sides, we have

$$\begin{aligned}&\sum_{n=0}^{\infty} \overline{b}_5(25n+22)q^n - \sum_{n=0}^{\infty} \overline{b}_5(5n+2)q^n\\ &= 20q\frac{f_{10}^5}{f_2} + \psi(-q)\big(6\psi(-q^5)f(q^2,-q^3)f(-q,q^4) - q\psi^3(-q^5)\big)\\ &\quad - \psi^3(-q)\psi(-q^5).\end{aligned}$$

Replacing q by $-q$ in (37) and then employing in the above identity, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\overline{b}_5(25n+22)q^n-\sum_{n=0}^{\infty}\overline{b}_5(5n+2)q^n\\&=20q\frac{f_{10}^5}{f_2}+5\psi(-q)\psi(-q^5)\big(\psi^2(-q)+q\psi^2(-q^5)\big).\end{aligned}\tag{64}$$

From (63) and (64), it follows that

$$\sum_{n=0}^{\infty}\overline{b}_5(25n+22)q^n-\sum_{n=0}^{\infty}\overline{b}_5(5n+2)q^n=5\sum_{n=0}^{\infty}\overline{b}_5(5n+2)q^n-5\sum_{n=0}^{\infty}\overline{b}_5(n)q^{n+2}.$$

Comparing the coefficients of q^n from both sides of the above equation, we find that, for any nonnegative integer n,

$$\overline{b}_5(25n+72)=6\overline{b}_5(5n+12)-5\overline{b}_5(n).\tag{65}$$

The general recurrence relation (20) now follows by mathematical induction on $k\geq 2$.

5 Proof of Theorem 5

Proofs of (21), (22), *and* (26). Equating the coefficients of q^{4n+1} from both sides of (50), we have

$$\overline{b}_5(4n+1)=c_5(n)-2\overline{b}_5(2n-1),$$

which is (21).

Replacing n by $n+1$ in (47) and rearranging the terms,

$$4\overline{b}_5(2n)=\overline{a}_5(2n+3)-2c_5(2n+2).\tag{66}$$

Replacing n by $5n$ in the above identity and using (54), we have

$$\begin{aligned}4\overline{b}_5(10n)&=\overline{a}_5(10n+3)-2c_5(10n+2)\\&=4c_5(10n+2)-2c_5(10n+2)\\&=2c_5(10n+2),\end{aligned}$$

which leads to (22).

Next, replacing n by $5n+2$ in (66) and employing (56), we obtain

$$\begin{aligned}4\overline{b}_5(10n+4)&=\overline{a}_5(10n+7)-2c_5(10n+6)\\&=4c_5(10n+6)-2c_5(10n+6)\\&=2c_5(10n+6),\end{aligned}$$

implying (26).

Proofs of (23), (25), (27), *and* (28). Employing (42) in (61), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \overline{b}_5(n) q^{n+1} \\
&= \sum_{n=0}^{\infty} c_5(n) q^{2n} - \psi^3(-q^5)\big(f(q^{10}, -q^{15}) - qf(-q^5, q^{20}) - q^3\psi(-q^{25})\big).
\end{aligned}$$

Comparing the coefficients of the terms involving q^{10n+2}, q^{10n+4}, q^{10n+7}, and q^{10n+9} from both sides of the above identity, we arrive at the desired results of (23), (25), (27), and (28), respectively.

Proofs of (29), (31), (32), *and* (33). Replacing n by $5n+1$ in (21) and then applying (23), we have

$$\begin{aligned}
\overline{b}_5(20n+5) &= c_5(5n+1) - 2\overline{b}_5(10n+1) \\
&= c_5(5n+1) - 2c_5(5n+1) \\
&= -c_5(5n+1),
\end{aligned}$$

which proves (29).

Similarly, replacing n by $5n+2$ in (21) and then using (25), we arrive at (31).

Replacing n by $5n+3$ in (19) and then employing (27), we have

$$\begin{aligned}
\overline{b}_5(20n+15) &= 2\overline{b}_5(10n+6) \\
&= 0,
\end{aligned}$$

which proves (32).

In a similar manner, replacing n by $5n+4$ in (19) and utilizing (28), we obtain (33).

Proofs of (24) *and* (30). From (3) and (39), we see that

$$\begin{aligned}
\varphi^3(-q)\varphi(-q^5) &= \frac{f_1^6 f_5^2}{f_2^3 f_{10}} \\
&= \frac{f_1^2 f_5^6}{f_2 f_{10}^3} - 4q\frac{f_1^3 f_5 f_{10}^2}{f_2^2} \\
&= \varphi(-q)\varphi^3(-q^5) - 4q\frac{f_5^5}{f_1} + 16q^2\frac{f_{10}^5}{f_2}.
\end{aligned}$$

Utilizing (59), we recast the above identity as

$$\sum_{n=0}^{\infty} \overline{a}_5(5n) q^n = \sum_{n=0}^{\infty} \overline{a}_5(n) q^n + 12q\frac{f_5^5}{f_1} + 16q^2\frac{f_{10}^5}{f_2}.$$

Extracting the terms with odd powers of q from both sides, we arrive at

$$\overline{a}_5(10n+5) = \overline{a}_5(2n+1) + 12c_5(2n). \tag{67}$$

Now, replacing n by $10n+5$ in (48),

$$4\bar{b}_5(20n+7) = \bar{a}_5(20n+10) - \bar{a}_5(10n+5) - 2c_5(20n+9).$$

Employing (52) with n replaced by $5n+2$ and (43) with n replaced by $5n+2$ in the above identity, and then using (44), we find that

$$\begin{aligned}4\bar{b}_5(20n+7) &= 3\bar{a}_5(10n+5) - 2c_5(10n+4) - \bar{a}_5(10n+5) - 2c_5(10n+4)\\ &= 2\bar{a}_5(10n+5) - 4c_5(10n+4)\\ &= 2\bar{a}_5(10n+5) - 20c_5(2n).\end{aligned}$$

Applying (67) in the above identity, we obtain

$$2\bar{b}_5(20n+7) = \bar{a}_5(2n+1) + 2c_5(2n),$$

which implies (30).

Finally, replacing n by $5n+1$ in (19) and then applying (30), we have

$$\begin{aligned}\bar{b}_5(10n+2) &= \frac{1}{2}\bar{b}_5(20n+7)\\ &= \frac{1}{4}\bar{a}_5(2n+1) + \frac{1}{2}c_5(2n),\end{aligned}$$

which is (24).

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