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Fibonacci Identities via Fibonacci Functions

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Abstract

We present a differential-calculus-based method which allows one to derive more identities from a given Fibonacci-Lucas identity containing a finite number of terms and having at least one free index. The method has two *independent* components. The first component allows new identities to be obtained directly from an existing identity while the second yields a generalization of the existing identity. The strength of the first component is that no additional information is required about the given original identity. We illustrate the method by providing new generalizations of some well-known identities such as d'Ocagne's identity, Candido's identity, the Gelin-Cesàro identity, and Catalan's identity. The method readily extends to a generalized Fibonacci sequence.

1 Introduction

Let F_j and L_j be the jth Fibonacci and Lucas numbers, defined for all integers by

$$F_j = \frac{\alpha^j - \beta^j}{\alpha - \beta}, \quad L_j = \alpha^j + \beta^j, \tag{1}$$

where $\alpha = (1+\sqrt{5})/2$, the golden ratio, and $\beta = (1-\sqrt{5})/2 = -1/\alpha$. Of course, $\alpha + \beta = 1$, $\alpha\beta = -1$ and $\alpha - \beta = \sqrt{5}$. Let $(G_j)_{j\in\mathbb{Z}}$ be the gibonacci sequence having the same recurrence relation as the Fibonacci sequence but starting with arbitrary initial values; that is, let

$$G_j = G_{j-1} + G_{j-2}, \quad (j \geq 2),$$

with G_0 and G_1 arbitrary numbers (usually integers) not both zero; and

$$G_{-j} = G_{-(j-2)} - G_{-(j-1)}.$$

If, inspired by (1), we introduce infinitely differentiable, complex-valued Fibonacci and Lucas functions, $f(x)$ and $l(x)$, defined by

$$f(x) = \frac{\alpha^x - \beta^x}{\alpha - \beta}, \quad l(x) = \alpha^x + \beta^x, \quad x \in \mathbb{R}; \tag{2}$$

then, clearly,

$$f(x)|_{x=j\in\mathbb{Z}} = F_j, \quad l(x)|_{x=j\in\mathbb{Z}} = L_j;$$

and we will show that (see §4.2 and §5)

$$\Re\left(\frac{d}{dx}f(x)\bigg|_{x=j\in\mathbb{Z}}\right) = \frac{L_j}{\sqrt{5}}\ln\alpha, \quad \Re\left(\frac{d}{dx}l(x)\bigg|_{x=j\in\mathbb{Z}}\right) = F_j\sqrt{5}\ln\alpha, \tag{3}$$

and

$$\Im\left(\frac{d}{dx}f(x)\bigg|_{x=j\in\mathbb{Z}}\right) = -\frac{\pi\beta^j}{\sqrt{5}}, \quad \Im\left(\frac{d}{dx}l(x)\bigg|_{x=j\in\mathbb{Z}}\right) = \pi\beta^j; \tag{4}$$

where, here and throughout this paper, $\Re(X)$ or $\Re X$ denotes the real part of X and $\Im(X)$ or $\Im X$ stands for the imaginary part of X. Many authors have studied various Fibonacci and Lucas functions in the past; we mention Halsey [5], Parker [20], Spickerman [22], Horadam and Shannon [11], and Han et al. [6]. The main difference between the approach in this paper and that in previous work by other authors is that the latter focused on seeking real-valued Fibonacci and Lucas functions. It is, precisely, the complex-valued nature of the Fibonacci and Lucas functions defined in (2) and their derivatives that motivated the method developed in this paper.

Our goal is to present a two-component method, based on (2)–(4) and their extensions, which allows the discovery of more identities from any known Fibonacci-Lucas identity or any gibonacci identity consisting of a finite number of terms and having at least one free index; that is an index that is not being summed over.

To illustrate what we mean, consider the identity

$$\sum_{j=0}^{4n+1} (-1)^{j-1}\binom{4n+1}{j}F_{j+k}^4 = 25^n\left(F_{2n+k+1}^4 - F_{2n+k}^4\right), \tag{5}$$

derived, among other similar results, by Hoggatt and Bicknell [7]. This identity has a free index, k. Working only with the knowledge of (5), our method (first component) allows us to derive the following presumably new identity:

$$\sum_{j=0}^{4n+1} (-1)^{j-1}\binom{4n+1}{j}F_{j+k}^3 L_{j+k} = 25^n\left(F_{2n+k+1}^3 L_{2n+k+1} - F_{2n+k}^3 L_{2n+k}\right); \tag{6}$$

which, in turn, implies the identity

$$\sum_{j=0}^{4n+1}(-1)^{j-1}\binom{4n+1}{j}F_{2j+k}^2=25^nF_{2(4n+k+1)}. \tag{7}$$

We are not done yet, as (7) implies

$$\sum_{j=0}^{4n+1}(-1)^{j-1}\binom{4n+1}{j}F_{4j+2k}=25^nL_{2(4n+k+1)}; \tag{8}$$

which finally implies

$$\sum_{j=0}^{4n+1}(-1)^{j-1}\binom{4n+1}{j}L_{4j+2k}=5^{2n+1}F_{2(4n+k+1)}. \tag{9}$$

Thus, the four identities (6), (7), (8) and (9) all follow from a knowledge of (5).

Our method (second component) provides the following generalization of (5) to (identity (131)):

$$\sum_{j=0}^{4n+1}(-1)^{j-1}\binom{4n+1}{j}F_{j+k}^3G_{j+r}=25^n\left(F_{2n+k+1}^3G_{2n+r+1}-F_{2n+k}^3G_{2n+r}\right).$$

A further generalization is derived in Proposition 32 on page 39.

As another example, consider the following well-known identity (see, for example, Hoggatt and Ruggles [8, Theorem 4])

$$\tan^{-1}\frac{1}{F_{2k+1}}=\tan^{-1}\frac{1}{F_{2k}}-\tan^{-1}\frac{1}{F_{2k+2}}. \tag{10}$$

Our method (first component) shows that (10) implies the following apparently new identity:

$$\frac{L_{2k+1}}{F_{2k+1}^2+1}=\frac{L_{2k}}{F_{2k}^2+1}-\frac{L_{2k+2}}{F_{2k+2}^2+1};$$

and the method (second component) yields a generalization:

$$\frac{G_{2k+r+3}+G_{2k+r+1}}{F_{2k+1}(F_{2k+1}^2+1)}-\frac{G_{r+1}}{F_{2k+1}^2+1}=\frac{G_{r+2}}{F_{2k}^2+1}-\frac{G_r}{F_{2k+2}^2+1}.$$

Yet another example, our method (first component) shows that the following identity of Howard [12, Corollary 3.5]:

$$F_sG_{k+r}+(-1)^{r-1}F_{s-r}G_k=F_rG_{k+s}, \tag{11}$$

having three free indices r, s, and k, implies the following identities:

$$L_sG_{k+r}+(-1)^{r-1}L_{s-r}G_k=F_r(G_{k+s+1}+G_{k+s-1}), \tag{12}$$

$$F_s\left(G_{k+r+1}+G_{k+r-1}\right)+(-1)^rL_{s-r}G_k=L_rG_{k+s}, \tag{13}$$

$$L_s\left(G_{k+r+1}+G_{k+r-1}\right)+(-1)^r5F_{s-r}G_k=L_r\left(G_{k+s+1}+G_{k+s-1}\right). \tag{14}$$

The method (second component) provides the following generalization of (11):

$$H_sG_{k+r}+(-1)^{r-1}H_{s-r}G_k=F_r\left(G_0H_{k+s-1}+G_1H_{k+s}\right); \tag{15}$$

where here, and throughout this paper, $(H_j)_{j\in\mathbb{Z}}$ is a gibonacci sequence with seeds H_0 and H_1. Another generalization of (11) is given in Proposition 36 on page 41.

In Section 5 we will apply the method (second component) to provide a generalization of Candido's identity

$$2\left(F_k^4+F_{k+1}^4+F_{k+2}^4\right)=\left(F_k^2+F_{k+1}^2+F_{k+2}^2\right)^2,$$

to the following (identity (139)):

$$\begin{aligned}&2\left(H_rG_k^3+H_{r+1}G_{k+1}^3+H_{r+2}G_{k+2}^3\right)\\&\quad=\left(G_k^2+G_{k+1}^2+G_{k+2}^2\right)\left(H_rG_k+H_{r+1}G_{k+1}+H_{r+2}G_{k+2}\right),\end{aligned}$$

with a further generalization given in Proposition 38 on page 41; a particular case of which is

$$\begin{aligned}&6\left(F_kF_rF_sF_t+F_{k+1}F_{r+1}F_{s+1}F_{t+1}+F_{k+2}F_{r+2}F_{s+2}F_{t+2}\right)\\&=\left(F_kF_s+F_{k+1}F_{s+1}+F_{k+2}F_{s+2}\right)\left(F_rF_t+F_{r+1}F_{t+1}+F_{r+2}F_{t+2}\right)\\&\quad+\left(F_kF_r+F_{k+1}F_{r+1}+F_{k+2}F_{r+2}\right)\left(F_sF_t+F_{s+1}F_{t+1}+F_{s+2}F_{t+2}\right)\\&\quad+\left(F_kF_t+F_{k+1}F_{t+1}+F_{k+2}F_{t+2}\right)\left(F_rF_s+F_{r+1}F_{s+1}+F_{r+2}F_{s+2}\right).\end{aligned}$$

The method (second component) extends the d'Ocagne identity

$$F_{r+1}F_k-F_rF_{k+1}=(-1)^rF_{k-r},$$

to the gibonacci sequence as

$$G_{r+1}G_k-G_rG_{k+1}=(-1)^r\left(G_1G_{k-r}-G_0G_{k-r+1}\right);$$

and extends the well-known formula for the sum of the squares of two consecutive Fibonacci numbers, namely,

$$F_{k+1}^2+F_k^2=F_{2k+1},$$

to the gibonacci sequence as

$$G_{k+1}^2+G_k^2=G_0G_{2k}+G_1G_{2k+1}.$$

We will also establish the following generalization of Catalan's identity (identity (147)):

$$F_{k+r}G_{k-r+s}+F_{k-r}G_{k+r+s}=2F_kG_{k+s}+(-1)^{k+r+1}F_r^2\left(G_{s+1}+G_{s-1}\right),$$

and the Gelin-Cesàro identity

$$\begin{aligned}&H_{k+r-2}G_{k-1}G_{k+1}G_{k+2}+G_{k-2}H_{k+r-1}G_{k+1}G_{k+2}\\&\quad+G_{k-2}G_{k-1}H_{k+r+1}G_{k+2}+G_{k-2}G_{k-1}G_{k+1}H_{k+r+2}\\&=4H_{k+r}G_k^3-2e_GG_0\left(H_{r+2}+H_r\right)+2e_GG_1\left(H_{r+1}+H_{r-1}\right),\end{aligned}$$

where $e_G=G_0^2-G_1^2+G_0G_1$.

The method (second component) extends the fundamental identity of Fibonacci and Lucas numbers,

$$5F_k^2-L_k^2=(-1)^{k-1}4,$$

to the gibonacci sequence as

$$5G_k^2-\left(G_{k+1}+G_{k-1}\right)^2=(-1)^k4e_G;$$

and offers an extension of the triple-angle formula of Lucas

$$F_{3k}=F_{k+1}^3+F_k^3-F_{k-1}^3,$$

to

$$G_0^2G_{3k-2}+2G_0G_1G_{3k-1}+G_1^2G_{3k}=G_{k+1}^3+G_k^3-G_{k-1}^3.$$

Using the method, the golden ratio power reduction formula

$$\alpha^k=\alpha F_k+F_{k-1},$$

will be shown to imply

$$\begin{aligned}&G_0\left(H_{k+r}+H_{k+r-2}\right)+G_1\left(H_{k+r+1}+H_{k+r-1}\right)\\&\quad=G_r\left(H_k+H_{k-2}\right)+G_{r+1}\left(H_{k+1}+H_{k-1}\right),\end{aligned}$$

which subsumes several Fibonacci-Lucas identities.

Consider a generalized Fibonacci sequence $(W_j)=(W_j(W_0,W_1;P))$ defined, for all integers and arbitrary real numbers W_0, W_1, and $P\neq 0$, by the recurrence relation

$$W_j=PW_{j-1}+W_{j-2},\quad j\geq 2,\tag{16}$$

with $W_{-j}=W_{-j+2}-PW_{-j+1}$.

Two important cases of (W_j) are the special Lucas sequences of the first kind, $(U_j(P))=(W_j(0,1;P))$, and the second kind, $(V_j(P))=(W_j(2,P;P))$; so that

$$U_0=0,\,U_1=1,\quad U_j=PU_{j-1}+U_{j-2},\quad j\geq 2,\tag{17}$$

and

$$V_0 = 2,\, V_1 = P, \quad V_j = PV_{j-1} + V_{j-2}, \quad j \geq 2, \tag{18}$$

with $U_{-j} = U_{-j+2} - PU_{-j+1}$ and $V_{-j} = V_{-j+2} - PV_{-j+1}$.

We will show that the new method also applies to the generalized Fibonacci sequence. For example, the method (first component) shows that the identity [9, Equation (3.14), $Q = -1$]:

$$U_r W_{k+1} + U_{r-1} W_k = W_{k+r}$$

implies

$$V_r W_{k+1} + V_{r-1} W_k = W_{k+r+1} + W_{k+r-1}. \tag{19}$$

The new method presented in this paper complements some previous research (for example the work of Long [16], Dresel [4], and Melham [18]).

The rest of the paper is arranged as follows. In Section 2 we describe the method (first component) and give examples. We cast about for identities to apply the method (first component) in Section 3. Further justification of the method (first component) is addressed in Section 4. A description of the method (second component), with examples including various extensions and generalizations of some known identities, is presented in Section 5. Finally, an extension of the method to the general second order (Horadam) sequence is offered in Section 6.

2 The method, first component

Delaying further justification to Section 4, we present the method (first component) and give examples.

Here then is how to obtain more identities from any given Fibonacci-Lucas identity having a free index:

1. Let k be a free index in the known identity. Replace each Fibonacci number, say $F_{h(k,\ldots)}$, with a certain differentiable function of k, namely, $f(h(k,\ldots))$, with k now considered a variable; and replace each Lucas number, say $L_{h(k,\ldots)}$, with a certain differentiable function $l(h(k,\ldots))$. The subscript h will be considered a function of several variables; that is variable k and other parameters (if any) indicated by ellipsis: "$\ldots$". The explicit form of $f(h(k,\ldots))$ or $l(h(k,\ldots))$ will not enter into consideration.

2. By applying the usual rules of calculus, differentiate, with respect to k, through the identity obtained in step 1.

3. Simplify the equation obtained in step 2 and make the following replacements:

$$f(h(k,\ldots)) \to F_{h(k,\ldots)}, \tag{20}$$

$$l(h(k,\ldots)) \to L_{h(k,\ldots)}. \tag{21}$$

4. Take the real part of the whole expression/equation obtained in step 3, using also the following prescription:

$$\Re\frac{\partial f}{\partial k}(h(k,\ldots)) \to \frac{L_{h(k,\ldots)}}{\sqrt{5}}\ln\alpha, \tag{22}$$

$$\Re\frac{\partial l}{\partial k}(h(k,\ldots)) \to F_{h(k,\ldots)}\sqrt{5}\ln\alpha. \tag{23}$$

Remark 1. Formally, the method (first component) of obtaining new identities from a known Fibonacci-Lucas identity proceeds in two quick steps:

(i) Treat the subscripts of Fibonacci and Lucas numbers as variables and differentiate through the given identity, with respect to the free index of interest, using the rules of differential calculus.

(ii) Make the following replacements:

$$\frac{\partial}{\partial k}F_{h(k,\ldots)} \to \frac{L_{h(k,\ldots)}}{\sqrt{5}}\frac{\partial}{\partial k}h(k,\ldots), \tag{24}$$

$$\frac{\partial}{\partial k}L_{h(k,\ldots)} \to F_{h(k,\ldots)}\sqrt{5}\,\frac{\partial}{\partial k}h(k,\ldots), \tag{25}$$

$$\ln\alpha \to 1, \tag{26}$$

$$i \to 0; \tag{27}$$

where $i = \sqrt{-1}$ is the imaginary unit.

For example, given the double-angle identity

$$F_{2k} = L_k F_k,$$

we have, by step (i),

$$\frac{d}{dk}F_{2k} = \frac{d}{dk}\left(L_k F_k\right) = L_k\frac{d}{dk}F_k + F_k\frac{d}{dk}L_k;$$

so that, by step (ii), using (24) and (25), we get

$$\frac{L_{2k}}{\sqrt{5}}\cdot\frac{d}{dk}(2k) = L_k\cdot\frac{L_k}{\sqrt{5}} + F_k\cdot F_k\sqrt{5};$$

and hence,

$$2L_{2k} = L_k^2 + 5F_k^2. \tag{28}$$

2.1 Examples

We illustrate the method (first component) with some examples from familiar identities.

2.2 Example from a connecting formula between Fibonacci and Lucas numbers

In this example we show that

$$L_k = F_{k+1} + F_{k-1} \implies 5F_k = L_{k+1} + L_{k-1}.$$

Following step 1 we write

$$l(k) = f(k+1) + f(k-1)$$

and (step 2) differentiate with respect to k, obtaining

$$\frac{d}{dk} l(k) = \frac{d}{dk} f(k+1) + \frac{d}{dk} f(k-1).$$

Steps 3 and 4 now give

$$\Re \frac{d}{dk} l(k) = \Re \frac{d}{dk} f(k+1) + \Re \frac{d}{dk} f(k-1);$$

and by (22) and (23),

$$F_k \sqrt{5} \ln \alpha = \frac{L_{k+1}}{\sqrt{5}} \ln \alpha + \frac{L_{k-1}}{\sqrt{5}} \ln \alpha;$$

that is

$$5F_k = L_{k+1} + L_{k-1}.$$

2.2.1 Example from the double-angle identity of Fibonacci and Lucas numbers

In this example we demonstrate that:

$$F_{2k} = L_k F_k \implies 2L_{2k} = L_k^2 + 5F_k^2. \tag{29}$$

For the identity $F_{2k} = L_k F_k$, step 1 is

$$f(2k) = l(k) f(k);$$

where k is now considered a variable.

Following step 2, we differentiate with respect to k to obtain

$$2\frac{df}{dk}(2k) = l(k)\frac{df}{dk}(k) + f(k)\frac{dl}{dk}(k).$$

Steps 3 and 4 give

$$2\Re\frac{df}{dk}(2k) = L_k \Re\frac{df}{dk}(k) + F_k \Re\frac{dl}{dk}(k).$$

Thus, using (22) and (23), we have

$$2\frac{L_{2k}}{\sqrt{5}}\ln\alpha = L_k\frac{L_k}{\sqrt{5}}\ln\alpha + F_k\sqrt{5}F_k\ln\alpha;$$

which, dropping $\ln\alpha$ and multiplying through by $\sqrt{5}$, is

$$2L_{2k} = L_k^2 + 5F_k^2.$$

The interested reader may wish to verify the converse of (29), that is

$$2L_{2k} = L_k^2 + 5F_k^2 \implies F_{2k} = L_kF_k.$$

2.2.2 Example from the multiplication formula of Fibonacci and Lucas numbers

Here we show that the multiplication formula

$$F_{k+m} + (-1)^m F_{k-m} = L_m F_k$$

implies

$$L_{k+m} + (-1)^m L_{k-m} = L_m L_k \tag{30}$$

and

$$L_{k+m} - (-1)^m L_{k-m} = 5F_m F_k. \tag{31}$$

We write

$$f(k+m) + (-1)^m f(k-m) = l(m)f(k); \tag{32}$$

so that, treating k as the free index of interest gives

$$\frac{\partial f}{\partial k}(k+m) + (-1)^m\frac{\partial f}{\partial k}(k-m) = l(m)\frac{\partial}{\partial k}f(k).$$

Thus, by steps 3 and 4, we have

$$\Re\frac{\partial f}{\partial k}(k+m) + (-1)^m\Re\frac{\partial f}{\partial k}(k-m) = L_m\Re\frac{\partial f}{\partial k}(k);$$

and hence, using (22) and (23), we obtain

$$\frac{L_{k+m}}{\sqrt{5}}\ln\alpha + (-1)^m\frac{L_{k-m}}{\sqrt{5}}\ln\alpha = L_m\frac{L_k}{\sqrt{5}}\ln\alpha;$$

from which we get (30).

Taking m as the index of interest and differentiating (32) with respect to m yields

$$\frac{\partial f}{\partial m}(k+m) - (-1)^m\frac{\partial f}{\partial m}(k-m) + (-1)^m i\pi f(k-m) = f(k)\frac{\partial}{\partial m}l(m),$$

so that

$$\Re\frac{\partial f}{\partial m}(k+m)-(-1)^m\Re\frac{\partial f}{\partial m}(k-m)=F_k\Re\frac{\partial}{\partial m}l(m);$$

and hence

$$\frac{L_{k+m}}{\sqrt{5}}\ln\alpha-(-1)^m\frac{L_{k-m}}{\sqrt{5}}\ln\alpha=F_kF_m\sqrt{5}\ln\alpha;$$

from which (31) follows.

The reader may verify that the remaining multiplication formula can be discovered by differentiating (30) with respect to m.

2.2.3 Example from an inverse tangent Fibonacci number identity

Consider the following identity:

$$\tan^{-1}\frac{F_{2m}}{F_{2k+2m-1}}=\tan^{-1}\frac{L_m}{L_{2k+m-1}}-\tan^{-1}\frac{L_m}{L_{2k+3m-1}},\quad m\text{ even},\tag{33}$$

which can be derived using the inverse tangent addition formula and basic Fibonacci-Lucas identities.

We now demonstrate that (33) implies

$$\frac{1}{5}\frac{F_{2m}L_{2k+2m-1}}{F_{2k+2m-1}^2+F_{2m}^2}=\frac{L_mF_{2k+m-1}}{L_{2k+m-1}^2+L_m^2}-\frac{L_mF_{2k+3m-1}}{L_{2k+3m-1}^2+L_m^2},\quad m\text{ even}.\tag{34}$$

We treat k as the free index of interest. Step 1 gives the Fibonacci-Lucas function form of (33) as

$$\tan^{-1}\frac{f(2m)}{f(2k+2m-1)}=\tan^{-1}\frac{l(m)}{l(2k+m-1)}-\tan^{-1}\frac{l(m)}{l(2k+3m-1)};$$

so that step 2 yields

$$\begin{aligned}&\frac{2f(2m)}{f(2k+2m-1)^2+f(2m)^2}\frac{\partial f}{\partial k}(2k+2m-1)\\&\quad=\frac{2l(m)}{l(2k+m-1)^2+l(m)^2}\frac{\partial l}{\partial k}(2k+m-1)\\&\qquad-\frac{2l(m)}{l(2k+3m-1)^2+l(m)^2}\frac{\partial l}{\partial k}(2k+3m-1),\end{aligned}$$

which, by step 3, results in

$$\begin{aligned}&\frac{F_{2m}}{F_{2k+2m-1}^2+F_{2m}^2}\frac{\partial f}{\partial k}(2k+2m-1)\\&\quad=\frac{L_m}{L_{2k+m-1}^2+L_m^2}\frac{\partial l}{\partial k}(2k+m-1)-\frac{L_m}{L_{2k+3m-1}^2+L_m^2}\frac{\partial l}{\partial k}(2k+3m-1),\end{aligned}$$

whence taking the real part and replacing the derivatives using (22) and (23) gives (34).

By treating m as the free index, the interested reader can verify, using our method, that (33) also implies

$$\frac{2}{5}\frac{F_{2k-1}}{F_{2k+2m-1}^2+F_{2m}^2}=-\frac{F_{2k-1}}{L_{2k+m-1}^2+L_m^2}+\frac{F_{2k+3m-1}L_m+F_{2k+2m-1}}{L_{2k+3m-1}^2+L_m^2},\quad m \text{ even.}$$

2.3 Extension to a generalized Fibonacci sequence

We now describe how the method (first component) for obtaining new identities from existing ones works for the generalized Fibonacci sequence $(W_j(W_0,W_1;P))$ whose terms are given in (16). The scheme is the following.

1. Let k be a free index in the known identity. Replace each generalized Fibonacci number, say $W_{h(k,\ldots)}$, with a certain differentiable function of k, namely, $w(h(k,\ldots))$, with k now considered a variable.

2. By applying the usual rules of calculus, differentiate, with respect to k, through the identity obtained in step 1.

3. Simplify the equation obtained in step 2 and make the following replacement:
$$w(h(k,\ldots))\to W_{h(k,\ldots)}. \tag{35}$$

4. Take the real part of the equation/expression obtained in step 3, using also the following prescription:
$$\Re\frac{\partial w}{\partial k}(h(k,\ldots))\to\frac{W_{h(k+1,\cdots)}+W_{h(k-1,\cdots)}}{\delta}\ln\sigma; \tag{36}$$
where $\sigma=(P+\delta)/2$ and $\delta=\sqrt{P^2+4}$.

Note that, on account of (70) and (72), for the special Lucas sequences, (35) and (36) reduce to

$$u(h(k,\ldots))\to U_{h(k,\ldots)}, \tag{37}$$
$$\Re\frac{\partial u}{\partial k}(h(k,\ldots))\to\frac{V_{h(k,\ldots)}}{\delta}\ln\sigma \tag{38}$$

and

$$v(h(k,\ldots))\to V_{h(k,\ldots)}, \tag{39}$$
$$\Re\frac{\partial v}{\partial k}(h(k,\ldots))\to U_{h(k,\ldots)}\delta\ln\sigma; \tag{40}$$

of which the Fibonacci and Lucas relations (20)–(23) are particular cases.

For the gibonacci sequence, (35) and (36) reduce to

$$g(h(k,\ldots))\to G_{h(k,\ldots)}, \tag{41}$$
$$\Re\frac{\partial g}{\partial k}(h(k,\ldots))\to\frac{G_{h(k+1,\cdots)}+G_{h(k-1,\cdots)}}{\sqrt{5}}\ln\alpha. \tag{42}$$

2.4 More examples

We give further examples involving the gibonacci sequence and the generalized Fibonacci sequence.

2.4.1 Examples from an identity of Howard

Consider the following identity, derived by Howard [12, Corollary 3.5]:

$$F_sG_{k+r} + (-1)^{r-1}F_{s-r}G_k = F_rG_{k+s},$$

Identity (11) has three free indices r, s, and k.

We write

$$f(s)g(k+r) + (-1)^{r-1}f(s-r)g(k) = f(r)g(k+s). \tag{43}$$

Treating s as the index of interest and differentiating (43) with respect to s gives

$$g(k+r)\frac{d}{ds}f(s) + (-1)^{r-1}g(k)\frac{\partial f}{\partial s}(s-r) = f(r)\frac{\partial g}{\partial s}(k+s); \tag{44}$$

so that, using (20) and (41) we get

$$G_{k+r}\Re\frac{d}{ds}f(s) + (-1)^{r-1}G_k\Re\frac{\partial f}{\partial s}(s-r) = F_r\Re\frac{\partial g}{\partial s}(k+s).$$

We now use (22) to replace the derivatives on the left hand side and (42) to replace the derivative on the right hand side, obtaining

$$L_sG_{k+r} + (-1)^{r-1}L_{s-r}G_k = F_r(G_{k+s+1} + G_{k+s-1}).$$

On the other hand, treating r as the index of interest and differentiating (43) with respect to r yields

$$\begin{aligned} f(s)\frac{\partial g}{\partial r}(k+r) &+ (-1)^{r-1}i\pi f(s-r)g(k) - (-1)^{r-1}g(k)\frac{\partial f}{\partial r}(s-r) \\ &= g(k+s)\frac{d}{dr}f(r); \end{aligned} \tag{45}$$

so that, taking the real part,

$$F_s\Re\frac{\partial g}{\partial r}(k+r) - (-1)^{r-1}G_k\Re\frac{\partial f}{\partial r}(s-r) = G_{k+s}\Re\frac{d}{dr}f(r).$$

Use of (42) and (22) finally gives (identity 13):

$$F_s\left(G_{k+r+1} + G_{k+r-1}\right) + (-1)^rL_{s-r}G_k = L_rG_{k+s}.$$

The interested reader is invited to discover, by differentiating with respect to s, that (13) implies

$$L_s\left(G_{k+r+1} + G_{k+r-1}\right) + (-1)^r5F_{s-r}G_k = L_r\left(G_{k+s+1} + G_{k+s-1}\right);$$

and that differentiating (11) with respect to k does not produce a new result.

2.4.2 Example from a general recurrence relation

Consider the following identity of Horadam [9, Equation (3.14), $Q=-1$]:

$$U_r W_{k+1} + U_{r-1} W_k = W_{k+r}.$$

We write

$$u(r)w(k+1) + u(r-1)w(k) = w(k+r);$$

and differentiate with respect to r, obtaining

$$\frac{d}{dr}u(r)\cdot w(k+1) + \frac{d}{dr}u(r-1)\cdot w(k) = \frac{\partial w}{\partial r}(k+r);$$

so that, taking the real part, we find

$$\Re\frac{d}{dr}u(r)\cdot W_{k+1} + \Re\frac{d}{dr}u(r-1)\cdot W_k = \Re\frac{\partial w}{\partial r}(k+r);$$

and hence, upon using (38) and (36) to replace the derivatives, we derive identity (19):

$$V_r W_{k+1} + V_{r-1} W_k = W_{k+r+1} + W_{k+r-1}.$$

In particular,

$$V_r U_{k+1} + V_{r-1} U_k = V_{k+r},$$
$$V_r V_{k+1} + V_{r-1} V_k = (P^2+4) U_{k+r}.$$

2.4.3 Example from a multiplication formula

Here we will demonstrate that the identity [9, Equation (3.16), $Q=-1$]:

$$W_{k+r} + (-1)^r W_{k-r} = V_r W_k$$

implies the identity

$$(W_{k+r+1} + W_{k+r-1}) - (-1)^r (W_{k-r+1} + W_{k-r-1}) = U_r W_k \delta^2. \tag{46}$$

We write

$$w(k+r) + (-1)^r w(k-r) = v(r)w(k)$$

and differentiate through with respect to r to obtain

$$\frac{\partial w}{\partial r}(k+r) + (-1)^r \pi i w(k-r) - (-1)^r \frac{\partial w}{\partial r}(k-r) = w(k)\frac{d}{dr}v(r);$$

so that

$$\Re\frac{\partial w}{\partial r}(k+r) - (-1)^r \Re\frac{\partial w}{\partial r}(k-r) = w(k)\Re\frac{d}{dr}v(r).$$

Using (36) and (40), we get

$$\frac{W_{k+r+1}+W_{k+r-1}}{\delta}-(-1)^r\frac{(W_{k-r+1}+W_{k-r-1})}{\delta}=W_kU_r\delta;$$

and hence (46).

Identities

$$V_{k+r}-(-1)^rV_{k-r}=U_kU_r\delta^2$$

and

$$U_{k+r}-(-1)^rU_{k-r}=U_rV_k$$

are special cases of (46).

3 Applications

In this section, we pick various known results from the literature and apply the method (first component) to discover new identities.

3.1 New identities from an identity of Long

Long [17, Equation (44)] showed that, for a non-negative integer n and integers k and r,

$$\sum_{j=0}^{n}\binom{n}{j}F_{r+2kj}=L_k^nF_{r+nk},\quad \text{if } k \text{ is even.} \tag{47}$$

Based on the knowledge of (47) alone, we will derive the results stated in the proposition.

Proposition 2. *If n is a non-negative integer, k is an even integer and r is an integer, then*

$$2\sum_{j=0}^{n}j\binom{n}{j}L_{r+2kj}=5nL_k^{n-1}F_{r+nk}F_k+nL_k^nL_{r+nk}, \tag{48}$$

$$2\sum_{j=0}^{n}j\binom{n}{j}F_{r+2kj}=nL_k^{n-1}L_{r+nk}F_k+nL_k^nF_{r+nk}. \tag{49}$$

Identity (47) contains two free indices r and k. Treating r as the index of interest immediately gives the Lucas version of (47), namely,

$$\sum_{j=0}^{n}\binom{n}{j}L_{r+2kj}=L_k^nL_{r+nk},\quad \text{if } k \text{ is even;}$$

coming from

$$\sum_{j=0}^{n}\binom{n}{j}\Re\frac{\partial f}{\partial r}(r+2kj)=l(k)^n\Re\frac{\partial f}{\partial r}(r+nk)$$

and prescription (22).

To derive (48), write (47) as

$$\sum_{j=0}^{n}\binom{n}{j}f(r+2kj)=l(k)^nf(r+nk);$$

treat k as the index of interest and differentiate with respect to k (step 2) to obtain

$$\sum_{j=0}^{n}2j\binom{n}{j}\frac{\partial f}{\partial k}(r+2kj)=nl(k)^{n-1}f(r+nk)\frac{\partial}{\partial k}l(k)+nl(k)^n\frac{\partial f}{\partial k}(r+nk),$$

and, taking the real part,

$$\sum_{j=0}^{n}2j\binom{n}{j}\Re\frac{\partial f}{\partial k}(r+2kj)=nL_k^{n-1}F_{r+nk}\Re\frac{\partial}{\partial k}l(k)+nL_k^n\Re\frac{\partial f}{\partial k}(r+nk). \tag{50}$$

Thus (48) follows from step 4 of Section 2, after using (22) and (23) to replace the derivatives in (50).

To derive (49) treat r as the free index of interest in (48) and write

$$2\sum_{j=0}^{n}j\binom{n}{j}\frac{\partial l}{\partial r}(r+2kj)=5nL_k^{n-1}\frac{\partial f}{\partial r}(r+nk)f(k)+nL_k^n\frac{\partial l}{\partial r}(r+nk).$$

3.2 New identities arising from an identity of Hoggatt and Bicknell

Based on Hoggatt and Bicknell's result [7, Identity 2′]:

$$\sum_{j=0}^{4n+1}(-1)^{j-1}\binom{4n+1}{j}F_{j+k}^4=25^n\left(F_{2n+k+1}^4-F_{2n+k}^4\right),$$

we wish to derive the four identities (6), (7), (8) and (9) stated in the Introduction section.

Write (5) as

$$\sum_{j=0}^{4n+1}(-1)^{j-1}\binom{4n+1}{j}f(j+k)^4=25^n\left(f(2n+k+1)^4-f(2n+k)^4\right);$$

and differentiate through, with respect to k, to obtain

$$\begin{aligned}&\sum_{j=0}^{4n+1}(-1)^{j-1}\binom{4n+1}{j}4f(j+k)^3\frac{\partial f}{\partial k}(j+k)\\&\quad=25^n\left(4f(2n+k+1)^3\frac{\partial f}{\partial k}(2n+k+1)-4f(2n+k)^3\frac{\partial f}{\partial k}(2n+k)\right);\end{aligned} \tag{51}$$

and taking the real part:

$$\sum_{j=0}^{4n+1}(-1)^{j-1}\binom{4n+1}{j}4F_{j+k}^3\Re\frac{\partial f}{\partial k}(j+k)$$
$$=25^n\left(4F_{2n+k+1}^3\Re\frac{\partial f}{\partial k}(2n+k+1)-4F_{2n+k}^3\Re\frac{\partial f}{\partial k}(2n+k)\right).$$

Thus,

$$\sum_{j=0}^{4n+1}(-1)^{j-1}\binom{4n+1}{j}F_{j+k}^3\frac{L_{j+k}}{\sqrt{5}}$$
$$=25^n\left(F_{2n+k+1}^3\frac{L_{2n+k+1}}{\sqrt{5}}-F_{2n+k}^3\frac{L_{2n+k}}{\sqrt{5}}\right);$$

and hence (6). Identities (7), (8) and (9) are derived in the same manner; (7) is obtained from (6), etc.

3.3 New identities from an inverse tangent identity

Proposition 3. *If k is an integer, then*

$$\frac{L_{2k+1}}{F_{2k+1}^2+1}=\frac{L_{2k}}{F_{2k}^2+1}-\frac{L_{2k+2}}{F_{2k+2}^2+1}, \tag{52}$$

$$\frac{L_{2k+1}}{L_{2k}L_{2k+2}}\frac{(F_{2k}^2+1)(F_{2k+2}^2+1)}{(F_{2k+1}^2+1)}=\frac{(F_{2k+2}^2+1)}{L_{2k+2}}-\frac{(F_{2k}^2+1)}{L_{2k}}. \tag{53}$$

Recall identity (10):

$$\tan^{-1}\frac{1}{F_{2k+1}}=\tan^{-1}\frac{1}{F_{2k}}-\tan^{-1}\frac{1}{F_{2k+2}}.$$

To derive (52), write (10) as

$$\tan^{-1}\frac{1}{f(2k+1)}=\tan^{-1}\frac{1}{f(2k)}-\tan^{-1}\frac{1}{f(2k+2)}, \tag{54}$$

and differentiate with respect to k to obtain

$$\frac{1}{f(2k+1)^2+1}\frac{df}{dk}(2k+1)$$
$$=\frac{1}{f(2k)^2+1}\frac{df}{dk}(2k)-\frac{1}{f(2k+2)^2+1}\frac{df}{dk}(2k+2),$$

so that, by (20),

$$\frac{1}{F_{2k+1}^2+1}\,\Re\frac{df}{dk}(2k+1)$$
$$=\frac{1}{F_{2k}^2+1}\,\Re\frac{df}{dk}(2k)-\frac{1}{F_{2k+2}^2+1}\,\Re\frac{df}{dk}(2k+2),$$

and hence (52), upon using (22). Identity (53) is a rearrangement of (52).

Simple telescoping of (52) and (53) produces the results stated in the next proposition.

Proposition 4. *If n is an integer, then*

$$\sum_{k=1}^{n}\frac{L_{2k+1}}{F_{2k+1}^2+1}=\frac{3}{2}-\frac{L_{2(n+1)}}{F_{2(n+1)}^2+1},$$
$$\sum_{k=1}^{n}\frac{L_{2k+1}}{L_{2k}L_{2k+2}}\frac{\left(F_{2k}^2+1\right)\left(F_{2k+2}^2+1\right)}{\left(F_{2k+1}^2+1\right)}=\frac{F_{2(n+1)}^2+1}{L_{2n+2}}-\frac{2}{3};$$

with the limiting case:

$$\sum_{k=1}^{\infty}\frac{L_{2k+1}}{F_{2k+1}^2+1}=\frac{3}{2}.$$

3.4 New identities from an identity of Jennings

Jennings [13, Theorem 2] showed, among results of a similar nature, that

$$F_k\sum_{j=0}^{n}(-1)^{(k+1)(n+j)}\binom{n+j}{2j}L_k^{2j}=F_{(2n+1)k}.$$

Writing

$$\sum_{j=0}^{n}(-1)^{(k+1)(n+j)}\binom{n+j}{2j}l(k)^{2j}=\frac{f((2n+1)k)}{f(k)}$$

and differentiating with respect to k gives

$$\sum_{j=0}^{n}(-1)^{(k+1)(n+j)}(n+j)\pi i\binom{n+j}{2j}l(k)^{2j}+\sum_{j=0}^{n}(-1)^{(k+1)(n+j)}2j\binom{n+j}{2j}l(k)^{2j-1}\frac{dl}{dk}(k)$$
$$=\frac{2n+1}{f(k)}\frac{df}{dk}((2n+1)k)-\frac{f((2n+1)k)}{f(k)^2}\frac{df}{dk}(k),$$

and taking the real part,

$$\sum_{j=0}^{n}(-1)^{(k+1)(n+j)}2j\binom{n+j}{2j}L_k^{2j-1}\Re\frac{dl}{dk}(k)$$
$$=\frac{2n+1}{F_k}\Re\frac{df}{dk}((2n+1)k)-\frac{F_{(2n+1)k}}{F_k^2}\Re\frac{df}{dk}(k),$$

which, by (22) and (23) gives

$$\sum_{j=0}^{n}(-1)^{(k+1)(n+j)}2j\binom{n+j}{2j}L_k^{2j-1}F_k\sqrt{5}=\frac{2n+1}{F_k}\frac{L_{(2n+1)k}}{\sqrt{5}}-\frac{F_{(2n+1)k}}{F_k^2}\frac{L_k}{\sqrt{5}};$$

and hence the result stated in the next proposition.

Proposition 5. *For non-negative integers k and n, we have*

$$F_k^3\sum_{j=0}^{n}(-1)^{(k+1)(n+j)}j\binom{n+j}{2j}L_k^{2j}=\frac{1}{10}\left((2n+1)F_{2k}L_{(2n+1)k}-F_{(2n+1)k}L_k^2\right).$$

We also have the following divisibility property.

Proposition 6. *If n and k are non-negative integers, then*

$$10F_k^3 \text{ divides } (2n+1)F_{2k}L_{(2n+1)k}-F_{(2n+1)k}L_k^2.$$

3.5 New identities from Candido's identity

Setting $x=G_k$, $y=G_{k+1}$ in the algebraic identity

$$2\left(x^4+y^4+(x+y)^4\right)=\left(x^2+y^2+(x+y)^2\right)^2,$$

gives the following generalization of Candido's identity:

$$2\left(G_k^4+G_{k+1}^4+G_{k+2}^4\right)=\left(G_k^2+G_{k+1}^2+G_{k+2}^2\right)^2.$$

Writing

$$2(g(k)^4+g(k+1)^4+g(k+2)^4)=\left(g(k)^2+g(k+1)^2+g(k+2)^2\right)^2,$$

and differentiating with respect to k gives

$$\begin{aligned}&2\left(g(k)^3\frac{dg}{dk}(k)+g(k+1)^3\frac{dg}{dk}(k+1)+g(k+2)^3\frac{dg}{dk}(k+2)\right)\\&\quad=\left(g(k)^2+g(k+1)^2+g(k+2)^2\right)\cdot\\&\qquad\left(g(k)\frac{dg}{dk}(k)+g(k+1)\frac{dg}{dk}(k+1)+g(k+2)\frac{dg}{dk}(k+2)\right);\end{aligned}\tag{55}$$

so that applying the prescription (41) and (42) yields

$$\begin{aligned}&2\left(G_k^3(G_{k+1}+G_{k-1})+G_{k+1}^3(G_{k+2}+G_k)+G_{k+2}^3(G_{k+3}+G_{k+1})\right)\\&\quad=(G_k^2+G_{k+1}^2+G_{k+2}^2)\left(G_k(G_{k+1}+G_{k-1})+\right.\\&\qquad\left.G_{k+1}(G_{k+2}+G_k)+G_{k+2}(G_{k+3}+G_{k+1})\right),\end{aligned}$$

which can be arranged as stated in the next proposition.

Proposition 7. *For every integer k, we have*

$$\begin{aligned}&G_k^2\left(G_{k+1}(G_{k+2}+G_k)+G_{k+2}(G_{k+3}+G_{k+1})-G_k(G_{k+1}+G_{k-1})\right)\\&\quad+G_{k+1}^2\left(G_k(G_{k+1}+G_{k-1})+G_{k+2}(G_{k+3}+G_{k+1})-G_{k+1}(G_{k+2}+G_k)\right)\\&\quad+G_{k+2}^2\left(G_k(G_{k+1}+G_{k-1})+G_{k+1}(G_{k+2}+G_k)-G_{k+2}(G_{k+3}+G_{k+1})\right)\\&\quad=0.\end{aligned}$$

In particular,

$$F_k^2F_{2k+3}+F_{k+1}^2F_{2k+2}=F_{k+2}^2F_{2k+1}, \tag{56}$$
$$L_k^2F_{2k+3}+L_{k+1}^2F_{2k+2}=L_{k+2}^2F_{2k+1}. \tag{57}$$

Subtraction of (56) from (57) gives

$$F_{k-1}F_{k+1}F_{2k+3}+F_kF_{k+2}F_{2k+2}=F_{k+1}F_{k+3}F_{2k+1},$$

while their addition yields

$$(F_{k+1}^2+F_{k-1}^2)F_{2k+3}+(F_{k+2}^2+F_k^2)F_{2k+2}=(F_{k+3}^2+F_{k+1}^2)F_{2k+1}.$$

Before closing this section, we bring forth a Candido-type identity of Melham and discover new identities from it. Melham [19, Theorem 1] has shown that

$$6\left(\sum_{j=0}^{2n-1}G_{k+j}^2\right)^2=F_{2n}^2\left(G_{k+n-2}^4+4G_{k+n-1}^4+4G_{k+n}^4+G_{k+n+1}^4\right); \tag{58}$$

from which, writing $f(2n)$ for F_{2n}, $g(k+n-2)$ for G_{k+n-2}, etc. , and differentiating with respect to k, we have

$$\begin{aligned}&\left(12\sum_{j=0}^{2n-1}g(k+j)^2\right)\sum_{j=0}^{2n-1}2g(k+j)\frac{\partial g}{\partial k}(k+j)\\&\quad=f(2n)^2\left(4g(k+n-2)^3\frac{\partial g}{\partial k}(k+n-2)+16g(k+n-1)^3\frac{\partial g}{\partial k}(k+n-1)\right.\\&\quad\left.+16g(k+n)^3\frac{\partial g}{\partial k}(k+n)+4g(k+n+1)^3\frac{\partial g}{\partial k}(k+n+1)\right).\end{aligned}$$

Taking the real part according to the prescription of steps 2, 3, and 4 of Section 2.3, using (20), (41) and (42) to replace the Fibonacci and gibonacci functions and derivatives, we obtain the result stated in the next proposition.

Proposition 8. *If n is a non-negative integer and k is an integer, then*

$$\begin{aligned}&6\sum_{j=0}^{2n-1}G_{j+k}^2\sum_{j=0}^{2n-1}G_{j+k}(G_{j+k+1}+G_{j+k-1})\\&\quad=F_{2n}^2\left(G_{k+n-2}^3(G_{k+n-1}+G_{k+n-3})+4G_{k+n-1}^3(G_{k+n}+G_{k+n-2})\right.\\&\quad\left.+4G_{k+n}^3(G_{k+n+1}+G_{k+n-1})+G_{k+n+1}^3(G_{k+n+2}+G_{k+n})\right).\end{aligned}$$

In particular,

$$6\sum_{j=0}^{2n-1}F_{j+k}^2\sum_{j=0}^{2n-1}F_{2j+2k}=F_{2n}^2\left(F_{k+n-2}^2F_{2(k+n-2)}+4F_{k+n-1}^2F_{2(k+n-1)}\right.$$
$$\left.+4F_{k+n}^2F_{2(k+n)}+F_{k+n+1}^2F_{2(k+n+1)}\right) \tag{59}$$

and

$$6\sum_{j=0}^{2n-1}L_{j+k}^2\sum_{j=0}^{2n-1}F_{2j+2k}=F_{2n}^2\left(L_{k+n-2}^2F_{2(k+n-2)}+4L_{k+n-1}^2F_{2(k+n-1)}\right.$$
$$\left.+4L_{k+n}^2F_{2(k+n)}+L_{k+n+1}^2F_{2(k+n+1)}\right). \tag{60}$$

Subtraction of (59) from (60) gives

$$6\sum_{j=0}^{2n-1}F_{j+k+1}F_{j+k-1}\sum_{j=0}^{2n-1}F_{2j+2k}$$
$$=F_{2n}^2\left(F_{k+n-1}F_{k+n-3}F_{2(k+n-2)}+4F_{k+n}F_{k+n-2}F_{2(k+n-1)}\right.$$
$$\left.+4F_{k+n+1}F_{k+n-1}F_{2(k+n)}+F_{k+n+2}F_{k+n}F_{2(k+n+1)}\right).$$

3.6 New identities from the Gelin-Cesàro identity

The Gelin-Cesàro identity

$$F_{k-2}F_{k-1}F_{k+1}F_{k+2}=F_k^4-1$$

has the following generalization (Horadam and Shannon [10, Identity (2.5), $Q=-1$]):

$$W_{k-2}W_{k-1}W_{k+1}W_{k+2}=W_k^4+(-1)^k\gamma_W W_k^2-h_W^2;$$

where $\gamma_W=e_W(P^2-1)$, $h_W=e_WP$, and $e_W=PW_0W_1+W_0^2-W_1^2$.

For the sequence of Lucas numbers, we have $\gamma_L=0$ and $e_L=5=h_L$, so that

$$L_{k-2}L_{k-1}L_{k+1}L_{k+2}=L_k^4-25;$$

while for the gibonacci sequence, $\gamma_G=0$, $h_G=e_G=G_0G_1+G_0^2-G_1^2$ and

$$G_{k-2}G_{k-1}G_{k+1}G_{k+2}=G_k^4-e_G^2.$$

Writing

$$w(k-2)w(k-1)w(k+1)w(k+2)=w(k)^4+(-1)^k\gamma w(k)^2-h_W^2$$

and differentiating with respect to k and making use of (35) and (36) from Section 2.3 yields the result stated in the next proposition.

Proposition 9. *For every integer k,*

$$\begin{aligned}&(W_{k-1}+W_{k-3})W_{k-1}W_{k+1}W_{k+2}+W_{k-2}(W_k+W_{k-2})W_{k+1}W_{k+2}\\&\quad+W_{k-2}W_{k-1}(W_k+W_{k+2})W_{k+2}+W_{k-2}W_{k-1}W_{k+1}(W_{k+3}+W_{k+1})\\&=2W_k(W_{k+1}+W_{k-1})(2W_k^2+(-1)^k\gamma_W).\end{aligned}\tag{61}$$

In particular,

$$\begin{aligned}&(G_{k-1}+G_{k-3})G_{k-1}G_{k+1}G_{k+2}+G_{k-2}(G_k+G_{k-2})G_{k+1}G_{k+2}\\&\quad+G_{k-2}G_{k-1}(G_k+G_{k+2})G_{k+2}+G_{k-2}G_{k-1}G_{k+1}(G_{k+3}+G_{k+1})\\&=4G_k^3(G_{k+1}+G_{k-1});\end{aligned}$$

with the special cases

$$F_{k+1}F_{k+2}F_{2k-3}+F_{k-1}F_{k-2}F_{2k+3}=2F_k^3L_k=2F_k^2F_{2k}\tag{62}$$

and

$$L_{k+1}L_{k+2}F_{2k-3}+L_{k-1}L_{k-2}F_{2k+3}=2L_k^3F_k=2L_k^2F_{2k};\tag{63}$$

where, to arrive at (62) and (63), we used

$$F_{k+1}+F_{k-1}=L_k,\quad L_{k+1}+L_{k-1}=5F_k,$$

and [23, Identity (16a)]

$$L_mF_n+L_nF_m=2F_{m+n}.$$

Substituting $k+2$ for k and arranging (62) and (63) as

$$\frac{F_{2k+1}}{F_kF_{k+1}}+\frac{F_{2k+7}}{F_{k+3}F_{k+4}}=\frac{2F_{k+2}^2F_{2k+4}}{F_{k+2}^4-1}$$

and

$$\frac{F_{2k+1}}{L_kL_{k+1}}+\frac{F_{2k+7}}{L_{k+3}L_{k+4}}=\frac{2L_{k+2}^2F_{2k+4}}{L_{k+2}^4-25};$$

and the use of the telescoping summation formula

$$\sum_{k=1}^n(-1)^{k-1}\left(f_k+(-1)^{m-1}f_{k+m}\right)=\sum_{k=1}^m(-1)^{k-1}f_k+(-1)^{n-1}\sum_{k=1}^m(-1)^{k-1}f_{k+n}$$

yields the summation identities stated in the next proposition.

Proposition 10. *If n is a non-negative integer, then*

$$\sum_{k=1}^n\frac{(-1)^{k-1}F_{k+2}^2F_{2k+4}}{F_{k+2}^4-1}=\frac{5}{6}+\frac{(-1)^{n-1}}{2}\left(\frac{F_{2n+3}}{F_{n+1}F_{n+2}}-\frac{F_{2n+5}}{F_{n+2}F_{n+3}}+\frac{F_{2n+7}}{F_{n+3}F_{n+4}}\right)$$

$$\sum_{k=1}^n\frac{(-1)^{k-1}L_{k+2}^2F_{2k+4}}{L_{k+2}^4-25}=\frac{5}{14}+\frac{(-1)^{n-1}}{2}\left(\frac{F_{2n+3}}{L_{n+1}L_{n+2}}-\frac{F_{2n+5}}{L_{n+2}L_{n+3}}+\frac{F_{2n+7}}{L_{n+3}L_{n+4}}\right);$$

with

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} F_{k+2}^2 F_{2k+4}}{F_{k+2}^4 - 1} = \frac{5}{6},$$
$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} L_{k+2}^2 F_{2k+4}}{L_{k+2}^4 - 25} = \frac{5}{14}.$$

Arranging (61) as

$$\begin{aligned}\frac{W_{j-1} + W_{j-3}}{W_{j-2}} + \frac{W_j + W_{j-2}}{W_{j-1}} + \frac{W_{j+2} + W_j}{W_{j+1}} + \frac{W_{j+3} + W_{j+1}}{W_{j+2}} \\ = \frac{2W_j (W_{j+1} + W_{j-1})(2W_j^3 + (-1)^j \gamma_W)}{W_{j-2} W_{j-1} W_{j+1} W_{j+2}}\end{aligned}$$

and summing produces the next result.

Proposition 11. *If n and k are integers then,*

$$\begin{aligned}\sum_{j=1}^{n} \frac{(-1)^{j-1} 2W_{j+k} (W_{j+k+1} + W_{j+k-1})(2W_{j+k}^2 + (-1)^{j+k} \gamma_W)}{W_{j+k-2} W_{j+k-1} W_{j+k+1} W_{j+k+2}} \\ = (-1)^{n+1} \frac{W_{n+k} + W_{n+k-2}}{W_{n+k-1}} + \frac{W_k + W_{k-2}}{W_{k-1}} \\ + (-1)^{n+1} \frac{W_{n+k+3} + W_{n+k+1}}{W_{n+k+2}} + \frac{W_{k+3} + W_{k+1}}{W_{k+2}};\end{aligned}$$

provided none of the denominators vanishes.

3.7 New identities from a reciprocal series of Fibonacci numbers with subscripts $k2^j$

In this section we apply our method (first component) to discover new results associated with the following identity of Rabinowitz [21, Equation (4)]:

$$\sum_{j=0}^{n} \frac{1}{U_{k2^j}} = \frac{1 + U_{k-1}}{U_k} + \frac{1 - (-1)^k}{U_{2k}} - \frac{U_{k2^n - 1}}{U_{k2^n}}.$$

Writing

$$\sum_{j=0}^{n} \frac{1}{u(k2^j)} = \frac{1 + u(k-1)}{u(k)} + \frac{1 - (-1)^k}{u(2k)} - \frac{u(k2^n - 1)}{u(k2^n)},$$

and differentiating with respect to k gives

$$\begin{aligned}\sum_{j=0}^{n} \frac{-2^j}{u(k2^j)^2}\frac{du}{dk}(k2^j) &= \frac{1}{u(k)}\frac{du}{dk}(k-1) - \frac{(1+u(k-1))}{u(k)^2}\frac{du}{dk}(k)\\ &\quad - \frac{(-1)^k\pi i}{u(2k)} - \frac{2(1-(-1)^k)}{u(2k)^2}\frac{du}{dk}(2k)\\ &\quad - \frac{2^n}{u(k2^n)}\frac{\partial u}{\partial k}(k2^n-1) + \frac{2^n u(k2^n-1)}{u(k2^n)^2}\frac{\partial u}{\partial k}(k2^n).\end{aligned}$$

Taking the real part while using (37)–(40), we have the next result.

Proposition 12. *If n and k are positive integers, then*

$$\sum_{j=0}^{n} \frac{2^j V_{k2^j}}{U_{k2^j}^2} = \frac{(-1)^k 2 + V_k}{U_k^2} + \frac{2(1-(-1)^k)V_{2k}}{U_{2k}^2} - \frac{2^{n+1}}{U_{k2^n}^2},$$

$$\sum_{j=0}^{\infty} \frac{2^j V_{k2^j}}{U_{k2^j}^2} = \frac{(-1)^k 2 + V_k}{U_k^2} + \frac{2(1-(-1)^k)V_{2k}}{U_{2k}^2}.$$

Note that in arriving at the final form of the first expression in Proposition 12, we used

$$U_r V_s - V_r U_s = (-1)^s 2 U_{r-s}.$$

In particular, we have

$$\sum_{j=0}^{n} \frac{2^j V_{2^j}}{U_{2^j}^2} = P + \frac{2\delta^2}{P^2} - \frac{2^{n+1}}{U_{2^n}^2}$$

and

$$\sum_{j=0}^{n} \frac{2^j V_{2^{j+1}}}{U_{2^{j+1}}^2} = \frac{\delta^2}{P^2} - \frac{2^{n+1}}{U_{2^{n+1}}^2};$$

with the special cases

$$\sum_{j=0}^{n} \frac{2^j L_{2^j}}{F_{2^j}^2} = 11 - \frac{2^{n+1}}{F_{2^n}^2}$$

and

$$\sum_{j=0}^{n} \frac{2^j L_{2^{j+1}}}{F_{2^{j+1}}^2} = 5 - \frac{2^{n+1}}{F_{2^{n+1}}^2}.$$

4 Justification of the method

In this section we provide the rationale behind the method that was described in Section 2. To facilitate the discussion, we need the closed formula for the generalized Fibonacci sequence (W_j).

4.1 Closed formula for the generalized Fibonacci sequence

Standard methods for solving difference equations give the closed (Binet) formula of the generalized Fibonacci sequence (W_j) defined by the recurrence relation (16), in the non-degenerate case, $P^2+4>0$, as

$$W_j=\frac{A\sigma^j-B\tau^j}{\sigma-\tau}=\frac{A\sigma^j-B\tau^j}{\delta}, \tag{64}$$

where

$$A=W_1-W_0\tau,\quad B=W_1-W_0\sigma, \tag{65}$$

with

$$\sigma=\frac{P+\sqrt{P^2+4}}{2},\qquad \tau=\frac{P-\sqrt{P^2+4}}{2}; \tag{66}$$

so that

$$\sigma+\tau=P,\quad \sigma-\tau=\sqrt{P^2+4}=\delta,\quad \text{and } \sigma\tau=-1. \tag{67}$$

In particular,

$$U_j=\frac{\sigma^j-\tau^j}{\sigma-\tau},\quad V_j=\sigma^j+\tau^j. \tag{68}$$

Using the Binet formulas, it is readily established that

$$U_{-j}=(-1)^{j-1}U_j,\quad V_{-j}=(-1)^jV_j. \tag{69}$$

It is also straightforward to establish the following:

$$U_{j+1}+U_{j-1}=V_j, \tag{70}$$
$$U_{j+1}-U_{j-1}=PU_j, \tag{71}$$
$$V_{j+1}+V_{j-1}=U_j\delta^2, \tag{72}$$

and

$$V_{j+1}-V_{j-1}=PV_j. \tag{73}$$

As for the gibonacci sequence, we have

$$G_j=\frac{(G_1-G_0\beta)\,\alpha^j-(G_1-G_0\alpha)\,\beta^j}{\sqrt{5}}. \tag{74}$$

Lemma 13. *For an integer j,*

$$A\sigma^j+B\tau^j=W_{j+1}+W_{j-1}, \tag{75}$$

where A and B are as defined in (65).

Proof. Let

$$R_j = A\sigma^j + B\tau^j. \tag{76}$$

Then,

$$\sigma R_j = A\sigma^{j+1} - B\tau^{j-1},$$
$$\tau R_j = -A\sigma^{j-1} + B\tau^{j+1}.$$

Thus,

$$R_j \cdot (\sigma - \tau) = \left(A\sigma^{j+1} - B\tau^{j+1}\right) + \left(A\sigma^{j-1} - B\tau^{j-1}\right);$$

that is

$$R_j\delta = W_{j+1}\delta + W_{j-1}\delta,$$

or

$$R_j = W_{j+1} + W_{j-1}. \tag{77}$$

Identity (75) now follows by equating (76) and (77). □

Identity (75) is at the heart of the justification of the calculus-based method of obtaining Fibonacci identities.

4.2 Justification of the method

We first state a required lemma.

Lemma 14. *If λ is a non-zero real number and x is real, then*

$$\frac{d}{dx}\lambda^x = \begin{cases} \lambda^x \ln \lambda, & \text{if } \lambda > 0; \\ \lambda^x \left(i\pi(2m+1) + \ln(-\lambda)\right), & \text{if } \lambda < 0, \end{cases}$$

where m is an integer.

Thus, if λ is a negative number, then $\dfrac{d}{dx}\lambda^x$ is complex multi-valued with the principal value being

$$\frac{d}{dx}\lambda^x = \lambda^x \left(i\pi + \ln(-\lambda)\right). \tag{78}$$

Proof. If λ is a negative number, then

$$\lambda^x = \exp\left(x \ln \lambda\right) = \exp\left(x\left(\ln(-1) + \ln(-\lambda)\right)\right), \tag{79}$$

where $\ln(-1)$, the complex logarithm of -1, is evaluated as

$$\ln(-1) = \ln\left(\exp(i\pi(2m+1))\right) = i\pi(2m+1), \quad m \in \mathbb{Z};$$

so that (79) can now be written as

$$\lambda^x = (-\lambda)^x \exp\left(i\pi(2m+1)x\right),$$

from which the second result in (14) now follows by differentiation. □

Consider a generalized Fibonacci function $w(x)$ defined by

$$w(x)=\frac{A\sigma^x-B\tau^x}{\sigma-\tau}=\frac{A\sigma^x-B\tau^x}{\delta},\quad x\in\mathbb{R},\tag{80}$$

where A and B are as defined in (65) and σ and τ are as given in (66).

Corresponding to (1), (68) and (74), we have the following special cases of (80):

$$f(x)=\frac{\alpha^x-\beta^x}{\sqrt{5}},\quad l(x)=\alpha^x+\beta^x,$$

$$u(x)=\frac{\sigma^x-\tau^x}{\sigma-\tau},\quad v(x)=\sigma^x+\tau^x,\tag{81}$$

and

$$g(x)=\frac{A_G\alpha^x-B_G\beta^x}{\alpha-\beta}=\frac{A_G\alpha^x-B_G\beta^x}{\sqrt{5}},\tag{82}$$

where

$$A_G=G_1-G_0\beta,\quad B_G=G_1-G_0\alpha.\tag{83}$$

Clearly,

$$w(j)=W_j,\quad j\in\mathbb{Z};\tag{84}$$

that is

$$u(j)=U_j,\quad v(j)=V_j,\quad g(j)=G_j,\quad f(j)=F_j,\quad l(j)=L_j,\quad j\in\mathbb{Z}.$$

Theorem 15. *The following identity holds:*

$$\Re\left(\frac{d}{dx}w(x)\bigg|_{x=j\in\mathbb{Z}}\right)=\frac{W_{j+1}+W_{j-1}}{\delta}\ln\sigma,\tag{85}$$

where, as usual, $\Re(X)$ denotes the real part of X.

Proof. We have

$$\frac{d}{dx}w(x)=\frac{1}{\delta}\left(A\frac{d}{dx}\sigma^x-B\frac{d}{dx}\tau^x\right).$$

From (66), it is clear that $\sigma>0$ and $\tau<0$ for all real numbers P. Thus, employing Lemma 14, we find

$$\begin{aligned}\frac{d}{dx}w(x)&=\frac{1}{\delta}\left(A\sigma^x\ln\sigma-B\tau^x\ln(-\tau)-B\tau^x\pi i(2m+1)\right)\\&=\frac{1}{\delta}\left(A\sigma^x\ln\sigma+B\tau^x\ln\sigma-B\tau^x\ln\sigma-B\tau^x\ln(-\tau)-B\tau^x\pi i(2m+1)\right)\\&=\frac{1}{\delta}\left((A\sigma^x+B\tau^x)\ln\sigma-B\tau^x\ln(-\sigma\tau)-B\tau^x\pi i(2m+1)\right).\end{aligned}$$

Since $\sigma\tau = -1$, we obtain

$$\frac{d}{dx}w(x) = \frac{1}{\delta}\left((A\sigma^x + B\tau^x)\ln\sigma - B\tau^x\pi i(2m+1)\right). \tag{86}$$

Evaluating (86) at $x = j \in \mathbb{Z}$, we have

$$\left.\frac{d}{dx}w(x)\right|_{x=j\in\mathbb{Z}} = \frac{1}{\delta}\left((W_{j+1} + W_{j-1})\ln\sigma - B\tau^j\pi i(2m+1)\right), \text{by (75)}, \tag{87}$$

from which, on taking the real part, (85) follows, since m, τ, δ, B, W_{j+1}, and W_{j-1} are real quantities and σ is a positive number. □

Of course the derivatives given in (3) are particular cases of (85) with $\delta = \sqrt{5}$, $F_{j+1} + F_{j-1} = L_j$, and $L_{j+1} + L_{j-1} = 5F_j$. Similarly, (36), (38), (40) and (42) are all consequences of (85).

Thus, given a (generalized) Fibonacci identity having a free index, on account of (80), (84) and (85), we can replace (generalized) Fibonacci numbers with (generalized) Fibonacci functions, perform differentiation and evaluate at integer values to obtain a new (generalized) Fibonacci identity.

5 The method, second component

The imaginary part of (87) establishes a connection between powers of σ and τ and the (generalized) Fibonacci numbers; through which new (generalized) Fibonacci identities can be obtained. We have

$$\Im\left(\left.\frac{d}{dx}w(x)\right|_{x=j\in\mathbb{Z}}\right) = -\frac{B_W}{\delta}\tau^j\pi(2m+1),$$

where m is some integer and

$$B_W = B = W_1 - W_0\sigma;$$

the principal value being

$$\Im\left(\left.\frac{d}{dx}w(x)\right|_{x=j\in\mathbb{Z}}\right) = -\frac{B_W}{\delta}\tau^j\pi. \tag{88}$$

Specializing to the special Lucas sequences, we have

$$B_U = 1, \quad B_V = P - 2\sigma = \tau - \sigma = -\delta;$$

so that

$$\Im\left(\left.\frac{d}{dx}u(x)\right|_{x=j\in\mathbb{Z}}\right) = -\frac{\pi\tau^j}{\delta}$$

and

$$\Im\left(\frac{d}{dx}v(x)\bigg|_{x=j\in\mathbb{Z}}\right)=\pi\tau^j.$$

For the gibonacci sequence, we have

$$B_G=G_1-G_0\alpha;$$

so that

$$\Im\left(\frac{d}{dx}g(x)\bigg|_{x=j\in\mathbb{Z}}\right)=\frac{G_0\alpha-G_1}{\sqrt{5}}\pi\beta^j.$$

For the Fibonacci and Lucas numbers, we have

$$B_F=1,\quad B_L=-\sqrt{5};$$

so that, in view of (88), the method described in Section 2 can now be applied to a Fibonacci-Lucas identity with the prescription in step 3 and step 4 of Section 2 replaced with the following:

3. Simplify the equation obtained in step 2 and make the following replacements:

$$f(h(k,\ldots))=F_{h(k,\ldots)},\tag{89}$$

$$l(h(k,\ldots))=L_{h(k,\ldots)}.\tag{90}$$

4. Take the imaginary part of the whole expression/equation obtained in step 3, using also the following prescription:

$$\Im\frac{\partial f}{\partial k}(h(k,\ldots))=-\frac{\pi\beta^{h(k,\ldots)}}{\sqrt{5}},\tag{91}$$

$$\Im\frac{\partial l}{\partial k}(h(k,\ldots))=\pi\beta^{h(k,\ldots)}.\tag{92}$$

Remark 16. Formally, the method (second component) of establishing a connection between the powers of β and Fibonacci and Lucas numbers in a given Fibonacci-Lucas identity proceeds in two quick steps:

(i) Treat the subscripts of Fibonacci and Lucas numbers as variables and differentiate through the given identity, with respect to the free index of interest, using the rules of differential calculus.

(ii) Make the following replacements:

$$\frac{\partial}{\partial k}F_{h(k,\ldots)}\to\frac{-\pi\beta^{h(k,\ldots)}}{\sqrt{5}}\frac{\partial}{\partial k}h(k,\ldots),\tag{93}$$

$$\frac{\partial}{\partial k}L_{h(k,\ldots)}\to\pi\beta^{h(k,\ldots)}\frac{\partial}{\partial k}h(k,\ldots),\tag{94}$$

$$\ln\alpha\to 0,\tag{95}$$

$$i\to 1;\tag{96}$$

where $i = \sqrt{-1}$ is the imaginary unit.

For example, given the double-angle formula:

$$F_{2k} = L_k F_k,$$

we have, by step (i),

$$\frac{d}{dk} F_{2k} = \frac{d}{dk} \left(L_k F_k\right) = L_k \frac{d}{dk} F_k + F_k \frac{d}{dk} L_k;$$

so that, by step (ii), using (93) and (94), we get

$$\frac{-\pi\beta^{2k}}{\sqrt{5}} \cdot \frac{d}{dk}(2k) = L_k \cdot \frac{-\pi\beta^k}{\sqrt{5}} + F_k \cdot \pi\beta^k;$$

and hence,

$$2\beta^k = L_k - F_k\sqrt{5}.$$

For the special Lucas sequences, (89)–(92) read

$$u(h(k,\ldots)) = U_{h(k,\ldots)}, \tag{97}$$

$$v(h(k,\ldots)) = V_{h(k,\ldots)}, \tag{98}$$

$$\Im \frac{\partial u}{\partial k}(h(k,\ldots)) = -\frac{\pi\tau^{h(k,\ldots)}}{\delta}, \tag{99}$$

$$\Im \frac{\partial v}{\partial k}(h(k,\ldots)) = \pi\tau^{h(k,\ldots)}; \tag{100}$$

while for the gibonacci sequence, we have

$$g(h(k,\ldots)) = G_{h(k,\ldots)}, \tag{101}$$

$$\Im \frac{\partial g}{\partial k}(h(k,\ldots)) = \frac{(G_0\alpha - G_1)\pi\beta^{h(k,\ldots)}}{\sqrt{5}} = -\frac{B_G}{\sqrt{5}}\pi\beta^{h(k,\ldots)}. \tag{102}$$

5.1 Examples

We now give some examples to illustrate the use of (89)–(92) and (101) and (102) in obtaining new identities from known Fibonacci-Lucas identities.

Note that in the definitions in (1), F_j and L_j do not change when α and β are interchanged. Thus, α and β can be interchanged in a Fibonacci-Lucas identity involving Fibonacci numbers, Lucas numbers, α and β and no other irrational numbers. More generally, we have the observation stated in Proposition 17.

Proposition 17. *A (generalized) Fibonacci identity involving (generalized) Fibonacci numbers as well as σ and τ and no other irrational numbers remains valid under the exchange of σ and τ.*

Proof. From (64) and (65), we have

$$W_j(\sigma,\tau)=\frac{W_1-W_0\tau}{\sigma-\tau}\sigma^j-\frac{W_1-W_0\sigma}{\sigma-\tau}\tau^j.$$

It is straightforward to verify that $W_j(\sigma,\tau)=W_j(\tau,\sigma)$; and hence, the proposition. □

The generalizations obtained in this section rest on Proposition 17.

5.1.1 Generalizations of the fundamental identity of Fibonacci and Lucas numbers

Differentiating the Fibonacci-Lucas function form

$$5f(k)^2-l(k)^2=(-1)^{k-1}4,$$

of the fundamental identity

$$5F_k^2-L_k^2=(-1)^{k-1}4,$$

and applying the prescription of (89)–(92) yields

$$5F_k\beta^{k+r}+L_k\beta^{k+r}\sqrt{5}=(-1)^k2\beta^r\sqrt{5}, \tag{103}$$

where r is an arbitrary integer; and also, by Proposition 17,

$$5F_k\alpha^{k+r}-L_k\alpha^{k+r}\sqrt{5}=(-1)^{k-1}2\alpha^r\sqrt{5}. \tag{104}$$

Combining (103) and (104) according to the Binet form (74) leads to the result stated in Proposition 18.

Proposition 18. *If k and r are integers, then*

$$5F_kG_{k+r}-L_k\left(G_{k+r+1}+G_{k+r-1}\right)=(-1)^{k-1}2\left(G_{r+1}+G_{r-1}\right).$$

Writing the function form of the identity of Proposition 18 as

$$\begin{aligned}&5f(k-r)g(k)-l(k-r)\left(g(k+1)+g(k-1)\right)\\&\quad=(-1)^{k-r-1}2\left(g(r+1)+g(r-1)\right),\end{aligned}$$

and differentiating with respect to r, using again the prescription (89)–(92) and (101) and (102), we find

$$\begin{aligned}&5\beta^{k-r}G_k+\beta^{k-r}\sqrt{5}\left(G_{k+1}+G_{k-1}\right)\\&\quad=(-1)^{k-r}2\sqrt{5}\left(G_{r+1}+G_{r-1}\right)+(-1)^{k-r}2\sqrt{5}\left(G_0\beta^{r-1}-G_1\beta^r\right),\end{aligned} \tag{105}$$

and, on account of Proposition 17, also

$$\begin{aligned}&5\alpha^{k-r}G_k-\alpha^{k-r}\sqrt{5}\left(G_{k+1}+G_{k-1}\right)\\&\quad=(-1)^{k-r+1}2\sqrt{5}\left(G_{r+1}+G_{r-1}\right)+(-1)^{k-r+1}2\sqrt{5}\left(G_0\alpha^{r-1}-G_1\alpha^r\right).\end{aligned} \tag{106}$$

Combining (105) and (106) gives the next result.

Proposition 19. *If k, r, and s are integers, then*

$$\begin{aligned}5G_kH_{k+s-r} &- \left(G_{k+1}+G_{k-1}\right)\left(H_{k+s-r+1}+H_{k+s-r-1}\right)\\ &= (-1)^{k-r+1}2\left(G_{r+1}+G_{r-1}\right)\left(H_{s+1}+H_{s-1}\right)\\ &\quad + (-1)^{k-r}2\left(G_0\left(H_{r+s}+H_{r+s-2}\right)+G_1\left(H_{r+s+1}+H_{r+s-1}\right)\right).\end{aligned} \tag{107}$$

Setting $r=0$ in (107) gives

$$\begin{aligned}5G_kH_{k+s} &- \left(G_{k+1}+G_{k-1}\right)\left(H_{k+s+1}+H_{k+s-1}\right)\\ &= (-1)^{k-1}2\left(G_1\left(H_{s+1}+H_{s-1}\right)-G_0\left(H_{s+2}+H_s\right)\right),\end{aligned}$$

which upon using $s=0$ and $(H_j)=(G_j)$ gives

$$5G_k^2-\left(G_{k+1}+G_{k-1}\right)^2=(-1)^k4e_G,$$

where, as usual, $e_G=G_0^2-G_1^2+G_0G_1$.

5.1.2 Generalizations of the formula for the sum of squares of two consecutive Fibonacci numbers

Differentiating the Fibonacci function form

$$f(k+1)^2+f(k)^2=f(2k+1),$$

of the identity

$$F_{k+1}^2+F_k^2=F_{2k+1},$$

we have

$$f(k+1)\frac{df}{dk}(k+1)+f(k)\frac{df}{dk}(k)=\frac{df}{dk}(2k+1),$$

and taking the imaginary part, by (89),

$$F_{k+1}\Im\frac{df}{dk}(k+1)+F_k\Im\frac{df}{dk}(k)=\Im\frac{df}{dk}(2k+1),$$

and by (91),

$$\beta^{k+1}F_{k+1}+\beta^kF_k=\beta^{2k+1}.$$

Thus

$$\beta^{r+k+1}F_{k+1}+\beta^{r+k}F_k=\beta^{2k+r+1},$$

or

$$\beta^{s+1}F_{k+1}+\beta^sF_k=\beta^{k+s+1}, \tag{108}$$

where s is an arbitrary integer, and also

$$\alpha^{s+1}F_{k+1}+\alpha^sF_k=\alpha^{k+s+1}. \tag{109}$$

Combining (108) and (109) according to the Binet formula, we have the next result, equivalent to Vajda [23, Identity (8)].

Proposition 20. *If k and s are integers, then*

$$F_{k+1}G_{s+1} + F_kG_s = G_{k+s+1}. \tag{110}$$

Writing $k-1$ for k and setting $s=0$ in (110) gives the well-known result:

$$F_kG_1 + F_{k-1}G_0 = G_k. \tag{111}$$

Differentiating the Fibonacci function form of (110), that is

$$f(k+1)g(s+1) + f(k)g(s) = g(k+s+1),$$

with respect to k and using (91) and (102) gives

$$-\beta^{k+1}G_{s+1} - \beta^kG_s = \beta^{k+s+1}\left(G_0\alpha - G_1\right),$$

or

$$\beta^{k+1}G_{s+1} + \beta^kG_s = G_0\beta^{k+s} + G_1\beta^{k+s+1} \tag{112}$$

and also

$$\alpha^{k+1}G_{s+1} + \alpha^kG_s = G_0\alpha^{k+s} + G_1\alpha^{k+s+1}. \tag{113}$$

Combining (112) and (113) produces the next result.

Proposition 21. *If k and s are integers, then*

$$H_{k+1}G_{s+1} + H_kG_s = G_0H_{k+s} + G_1H_{k+s+1}.$$

In particular,

$$G_{k+1}^2 + G_k^2 = G_0G_{2k} + G_1G_{2k+1}.$$

5.1.3 Generalizations of the d'Ocagne identity

Differentiating with respect to k, the Fibonacci function form

$$f(r+1)f(k) - f(r)f(k+1) = (-1)^r f(k-r),$$

of the d'Ocagne identity

$$F_{r+1}F_k - F_rF_{k+1} = (-1)^r F_{k-r},$$

gives, upon taking the imaginary part while using (89) and (91),

$$\beta^kF_{r+1} - \beta^{k+1}F_r = (-1)^r\beta^{k-r},$$

and also

$$\alpha^kF_{r+1} - \alpha^{k+1}F_r = (-1)^r\alpha^{k-r},$$

and hence, the result stated in the next proposition.

Proposition 22. *If r and k are integers, then*

$$F_{r+1}G_k - F_rG_{k+1} = (-1)^r G_{k-r}. \tag{114}$$

Differentiating the Fibonacci function form of (114):

$$f(r+1)g(k) - f(r)g(k+1) = (-1)^r g(k-r),$$

with respect to r, we find

$$\begin{aligned} &g(k)\frac{df}{dr}(r+1) - g(k+1)\frac{df}{dr}(r) \\ &\qquad = (-1)^r \pi g(k-r)i - (-1)^r \frac{\partial g}{\partial r}(k-r); \end{aligned}$$

and consequent upon use of (91), (101) and (102),

$$\begin{aligned} &\beta^{r+s}G_{k+1} - \beta^{r+s+1}G_k \\ &\qquad = (-1)^r \beta^s G_{k-r}\sqrt{5} + (-1)^r \left(G_0\beta^{k+s-r-1} + G_1\beta^{k+s-r}\right) \end{aligned}$$

and also

$$\begin{aligned} &\alpha^{r+s}G_{k+1} - \alpha^{r+s+1}G_k \\ &\qquad = (-1)^{r-1} \alpha^s G_{k-r}\sqrt{5} + (-1)^r \left(G_0\alpha^{k+s-r-1} + G_1\alpha^{k+s-r}\right) \end{aligned}$$

and hence, the identity stated in the next proposition.

Proposition 23. *If r, s, and k are integers, then*

$$H_{r+s}G_{k+1} - H_{r+s+1}G_k = (-1)^{r-1}\left(H_{s+1} + H_{s-1}\right)G_{k-r} + (-1)^r\left(G_0H_{k+s-r-1} + G_1H_{k+s-r}\right). \tag{115}$$

Writing $r-s$ for r in (115) and setting $s=0$ gives

$$H_rG_{k+1} - H_{r+1}G_k = (-1)^{r-1}\left(2H_1 - H_0\right)G_{k-r} + (-1)^r\left(G_0H_{k-r-1} + G_1H_{k-r}\right);$$

and, in particular,

$$G_rG_{k+1} - G_{r+1}G_k = (-1)^r\left(G_0G_{k-r+1} - G_1G_{k-r}\right).$$

5.1.4 Generalizations of Fibonacci power formulas

The well-known identity

$$G_{k+1}^2 + G_{k-2}^2 = 2\left(G_k^2 + G_{k-1}^2\right), \tag{116}$$

has the gibonacci function form

$$g(k+1)^2 + g(k-2)^2 = 2\left(g(k)^2 + g(k-1)^2\right),$$

which, upon differentiation, gives

$$g(k+1)\frac{\partial g}{\partial k}(k+1) + g(k-2)\frac{\partial g}{\partial k}(k-2) = 2g(k)\frac{\partial g}{\partial k}(k) + 2g(k-1)\frac{\partial g}{\partial k}(k-1),$$

and by (101) and (102):

$$\beta^{k+1}G_{k+1} + \beta^{k-2}G_{k-2} = 2\left(\beta^k G_k + \beta^{k-1}G_{k-1}\right);$$

and multiplying through by β^{s-k}, s an arbitrary integer:

$$\beta^{s+1}G_{k+1} + \beta^{s-2}G_{k-2} = 2\left(\beta^s G_k + \beta^{s-1}G_{k-1}\right); \tag{117}$$

and also

$$\alpha^{s+1}G_{k+1} + \alpha^{s-2}G_{k-2} = 2\left(\alpha^s G_k + \alpha^{s-1}G_{k-1}\right). \tag{118}$$

Combining (117) and (118) according to the Binet formula yields the following generalization of (116).

Proposition 24. *If s and k are integers, then*

$$G_{k+1}H_{s+1} + G_{k-2}H_{s-2} = 2G_k H_s + 2G_{k-1}H_{s-1}.$$

Long's identities [16, (31)–(35)] are all special cases of the above proposition.

The reader is invited to apply the method (second component) to verify that the identity

$$G_{k+1}^3 = 3G_k^3 + 6G_{k-1}^3 - 3G_{k-2}^3 - G_{k-3}^3, \quad \text{[2, Equation (3)]},$$

has the following generalization.

Proposition 25. *If k, r, and s are integers, then*

$$G_{k+1}H_{r+1}I_{s+1} = 3G_k H_r I_s + 6G_{k-1}H_{r-1}I_{s-1} - 3G_{k-2}H_{r-2}I_{s-2} - G_{k-3}H_{r-3}I_{s-3},$$

where (G_j), (H_j), and (I_j) are gibonacci sequences.

5.1.5 Generalizations of a triple-angle identity of Lucas

Differentiating the Fibonacci function form

$$f(3k) = f(k+1)^3 + f(k)^3 - f(k-1)^3,$$

of the identity (see Vorob'ev [24, p. 16]):

$$F_{3k} = F_{k+1}^3 + F_k^3 - F_{k-1}^3, \tag{119}$$

gives, after using (89) and (91),

$$\beta^{3k} = F_{k+1}^2\beta^{k+1} + F_k^2\beta^k - F_{k-1}^2\beta^{k-1},$$

that is

$$\beta^{2k+s} = F_{k+1}^2\beta^{s+1} + F_k^2\beta^s - F_{k-1}^2\beta^{s-1}, \tag{120}$$

where s is an arbitrary integer; and also

$$\alpha^{2k+s} = F_{k+1}^2\alpha^{s+1} + F_k^2\alpha^s - F_{k-1}^2\alpha^{s-1}. \tag{121}$$

Combining (120) and (121), we have the first generalization of (119).

Proposition 26. *If k and s are integers, then*

$$G_{2k+s} = F_{k+1}^2 G_{s+1} + F_k^2 G_s - F_{k-1}^2 G_{s-1}.$$

The reader is invited to check that application of the method (second component) two more times with respect to k gives the full generalization of (119), stated in Proposition 27.

Proposition 27. *If k, r, and s are integers, then*

$$\begin{aligned} &G_0H_0I_{k+r+s-2} + (G_0H_1 + G_1H_0)\, I_{k+r+s-1} + G_1H_1I_{k+r+s} \\ &= G_{s+1}H_{r+1}I_{k+1} + G_sH_rI_k - G_{s-1}H_{r-1}I_{k-1}, \end{aligned}$$

where (G_j), (H_j), and (I_j) are gibonacci sequences with initial terms G_0, G_1; H_0, H_1 and I_0, I_1.

In particular,

$$G_0^2G_{3k-2} + 2G_0G_1G_{3k-1} + G_1^2G_{3k} = G_{k+1}^3 + G_k^3 - G_{k-1}^3,$$

with the special cases:

$$\begin{aligned} F_{3k} &= F_{k+1}^3 + F_k^3 - F_{k-1}^3, \\ 5L_{3k} &= L_{k+1}^3 + L_k^3 - L_{k-1}^3. \end{aligned}$$

Identities (42)–(45) of Long [16] are special cases of the identity stated in Proposition 27.

5.1.6 Identities from the golden ratio power reduction formula

Differentiating the Fibonacci function form

$$\alpha^k = \alpha f(k) + f(k-1),$$

of the golden ratio power reduction formula

$$\alpha^k = \alpha F_n + F_{n-1};$$

we obtain

$$\alpha^k \ln\alpha = \alpha\frac{\partial}{\partial k}f(k) + \frac{\partial}{\partial k}f(k-1),$$

which, upon use of (22), gives

$$\alpha^k\sqrt{5} = \alpha L_k + L_{k-1},$$

and hence

$$\alpha^{k+r}\sqrt{5} = \alpha^{r+1}L_k + \alpha^r L_{k-1}, \tag{122}$$

where r is an arbitrary integer. Also,

$$-\beta^{k+r}\sqrt{5} = \beta^{r+1}L_k + \beta^r L_{k-1}, \tag{123}$$

Combining (122) and (123) gives the next result.

Proposition 28. *If k and r are integers, then*

$$G_{k+r+1} + G_{k+r-1} = L_k G_{r+1} + L_{k-1}G_r.$$

Differentiating the function form of the identity stated in Proposition 28 with respect to k, making use of (92) and (102) while taking the imaginary part gives

$$-\frac{B_G}{\sqrt{5}}\beta^{k+r+1} - \frac{B_G}{\sqrt{5}}\beta^{k+r-1} = \beta^k G_{r+1} + \beta^{k-1}G_r;$$

that is

$$\left(G_0\alpha - G_1\right)\beta^{k+r+1} + \left(G_0\alpha - G_1\right)\beta^{k+r-1} = \beta^k G_{r+1}\sqrt{5} + \beta^{k-1}G_r\sqrt{5},$$

and hence

$$G_0\beta^{k+r} + G_1\beta^{k+r+1} + G_0\beta^{k+r-2} + G_1\beta^{k+r-1} = -\beta^k G_{r+1}\sqrt{5} - \beta^{k-1}G_r\sqrt{5}. \tag{124}$$

Also,

$$G_0\alpha^{k+r} + G_1\alpha^{k+r+1} + G_0\alpha^{k+r-2} + G_1\alpha^{k+r-1} = \alpha^k G_{r+1}\sqrt{5} + \alpha^{k-1}G_r\sqrt{5}. \tag{125}$$

Combining (124) and (125) using the Binet formula and the lemma gives the result stated in the next proposition.

Proposition 29. *If k and r are integers, then*

$$G_0\left(H_{r+k} + H_{r+k-2}\right) + G_1\left(H_{r+k+1} + H_{r+k-1}\right) = G_r\left(H_k + H_{k-2}\right) + G_{r+1}\left(H_{k+1} + H_{k-1}\right).$$

In particular,

$$H_{k+r+1} + H_{k+r-1} = F_r\left(H_k + H_{k-2}\right) + F_{r+1}\left(H_{k+1} + H_{k-1}\right)$$

and

$$5H_{k+r} = L_r\left(H_k + H_{k-2}\right) + L_{r+1}\left(H_{k+1} + H_{k-1}\right);$$

with the special cases

$$\begin{aligned}
L_{r+k} &= F_r L_{k-1} + F_{r+1}L_k,\\
F_{r+k} &= F_r F_{k-1} + F_{r+1}F_k,\\
5F_{r+k} &= L_r L_{k-1} + L_{r+1}L_k,\\
L_{r+k} &= L_r F_{k-1} + L_{r+1}F_k.
\end{aligned}$$

5.1.7 Sum of products of the terms of two gibonacci sequences

Simple telescoping of the identity

$$G_{j+k}^2 = G_{j+k+1}G_{j+k} - G_{j+k}G_{j+k-1},$$

gives

$$\sum_{j=1}^{n} G_{j+k}^2 = G_{n+k}G_{n+k+1} - G_k G_{k+1},$$

whose gibonacci function form is

$$\sum_{j=1}^{n} g(j+k)^2 = g(n+k)g(n+k+1) - g(k)g(k+1).$$

Differentiating:

$$\begin{aligned}2\sum_{j=1}^{n} g(j+k)\frac{\partial g}{\partial k}(j+k) &= g(n+k)\frac{\partial g}{\partial k}(n+k+1) + g(n+k+1)\frac{\partial g}{\partial k}(n+k)\\ &\quad - g(k)\frac{\partial g}{\partial k}(k+1) - g(k+1)\frac{\partial g}{\partial k}(k),\end{aligned}$$

and, upon use of (101) and (102), we have

$$2\sum_{j=1}^{n} \beta^{j+k}G_{j+k} = \beta^{n+k+1}G_{n+k} + \beta^{n+k}G_{n+k+1} - \beta^k G_{k+1} - \beta^{k+1}G_k;$$

which, multiplying through by β^{s-k}, s an arbitrary integer, gives

$$2\sum_{j=1}^{n} \beta^{j+s}G_{j+k} = \beta^{n+s+1}G_{n+k} + \beta^{n+s}G_{n+k+1} - \beta^s G_{k+1} - \beta^{s+1}G_k, \tag{126}$$

and hence also

$$2\sum_{j=1}^{n} \alpha^{j+s}G_{j+k} = \alpha^{n+s+1}G_{n+k} + \alpha^{n+s}G_{n+k+1} - \alpha^s G_{k+1} - \alpha^{s+1}G_k. \tag{127}$$

Combining (126) and (127) according to the Binet formula yields the result stated in the next proposition.

Proposition 30. *If k, s, and n are integers, then*

$$\sum_{j=1}^{n} G_{j+k}H_{j+s} = G_{n+k}H_{n+s+1} + G_{n+k+1}H_{n+s} - G_{k+1}H_s - G_k H_{s+1}. \tag{128}$$

Using generating function techniques, Berzsenyi [1] found alternative expressions for the special case

$$\sum_{j=0}^{2n+s} G_j G_{j+2k+s}, \quad s = 1 \text{ or } 0.$$

Long's identities [16, (4)–(7)] are alternative expressions for special cases of the above proposition. Kronenburg [15, Identity (11.1)] also derived (128).

5.1.8 Generalizations of Hoggatt and Bicknell's identity (5)

Taking the imaginary part of (51) on page 15, using (89), we have

$$\begin{aligned}\sum_{j=0}^{4n+1} (-1)^{j-1} \binom{4n+1}{j} F_{j+k}^3 \Im \frac{\partial f}{\partial k}(j+k) \\ = 25^n \left(F_{2n+k+1}^3 \Im \frac{\partial f}{\partial k}(2n+k+1) - F_{2n+k}^3 \Im \frac{\partial f}{\partial k}(2n+k) \right);\end{aligned}$$

which, applying (91), gives

$$\sum_{j=0}^{4n+1} (-1)^{j-1} \binom{4n+1}{j} F_{j+k}^3 \beta^{j+k} = 25^n \left(F_{2n+k+1}^3 \beta^{2n+k+1} - F_{2n+k}^3 \beta^{2n+k} \right),$$

and hence

$$\sum_{j=0}^{4n+1} (-1)^{j-1} \binom{4n+1}{j} F_{j+k}^3 \beta^{j+r} = 25^n \left(F_{2n+k+1}^3 \beta^{2n+r+1} - F_{2n+k}^3 \beta^{2n+r} \right); \tag{129}$$

where r is an arbitrary integer.

Interchanging α and β in (129), we also have

$$\sum_{j=0}^{4n+1} (-1)^{j-1} \binom{4n+1}{j} F_{j+k}^3 \alpha^{j+r} = 25^n \left(F_{2n+k+1}^3 \alpha^{2n+r+1} - F_{2n+k}^3 \alpha^{2n+r} \right). \tag{130}$$

Combining (129) and (130) according to the Binet formula, we have the result stated in the next proposition.

Proposition 31. *If n is a non-negative integer and k is an integer, then*

$$\sum_{j=0}^{4n+1} (-1)^{j-1} \binom{4n+1}{j} F_{j+k}^3 G_{j+r} = 25^n \left(F_{2n+k+1}^3 G_{2n+r+1} - F_{2n+k}^3 G_{2n+r} \right). \tag{131}$$

Observe that (5) and (6) are particular cases of (131).

By applying the method (second component) three more times, with k as the index of interest, the reader is invited to establish the further generalization presented in the next proposition.

Proposition 32. *If n is a non-negative integer and r, k, and s are integers, then*

$$\begin{aligned}&\sum_{j=0}^{4n+1}(-1)^{j-1}\binom{4n+1}{j}E_{j+k}G_{j+r}H_{j+s}I_{j+t}\\&\qquad=25^n\left(E_{2n+k+1}G_{2n+r+1}H_{2n+s+1}I_{2n+t+1}-E_{2n+k}G_{2n+r}H_{2n+s}I_{2n+t}\right);\end{aligned}$$

where $(E)_{j\in\mathbb{Z}}$, $(G)_{j\in\mathbb{Z}}$, $(H)_{j\in\mathbb{Z}}$, and $(I)_{j\in\mathbb{Z}}$ are gibonacci sequences with seeds E_0, E_1; G_0, G_1; H_0, H_1 and I_0, I_1.

5.1.9 Generalizations of an identity of Melham

In this section, we present a generalization of Melham's identity, (58) on page 19, that is

$$6\left(\sum_{j=0}^{2n-1}G_{k+j}^2\right)^2=F_{2n}^2\left(G_{k+n-2}^4+4G_{k+n-1}^4+4G_{k+n}^4+G_{k+n+1}^4\right);$$

whose Fibonacci-gibonacci function derivative is

$$\begin{aligned}&\left(6\sum_{j=0}^{2n-1}g(k+j)^2\right)\sum_{j=0}^{2n-1}g(k+j)\frac{\partial g}{\partial k}(k+j)\\&\quad=f(2n)^2\left(g(k+n-2)^3\frac{\partial g}{\partial k}(k+n-2)+4g(k+n-1)^3\frac{\partial g}{\partial k}(k+n-1)\right.\\&\quad\left.+4g(k+n)^3\frac{\partial g}{\partial k}(k+n)+g(k+n+1)^3\frac{\partial g}{\partial k}(k+n+1)\right).\end{aligned}$$

Taking the imaginary part and applying (101) and (102) leads to

$$\begin{aligned}&6\sum_{j=0}^{2n-1}G_{k+j}^2\sum_{j=0}^{2n-1}\beta^{k+j}G_{k+j}\\&\qquad=F_{2n}^2\left(\beta^{k+n-2}G_{k+n-2}^3+4\beta^{k+n-1}G_{k+n-1}^3+4\beta^{k+n}G_{k+n}^3+\beta^{k+n+1}G_{k+n+1}^3\right),\end{aligned}$$

which multiplying through by β^{r-k}, r an arbitrary integer, gives

$$\begin{aligned}&6\sum_{j=0}^{2n-1}G_{k+j}^2\sum_{j=0}^{2n-1}\beta^{r+j}G_{k+j}\\&\qquad=F_{2n}^2\left(\beta^{r+n-2}G_{k+n-2}^3+4\beta^{r+n-1}G_{k+n-1}^3+4\beta^{r+n}G_{k+n}^3+\beta^{r+n+1}G_{k+n+1}^3\right),\end{aligned}\tag{132}$$

and also

$$6\sum_{j=0}^{2n-1}G_{k+j}^2\sum_{j=0}^{2n-1}\alpha^{r+j}G_{k+j} \\ = F_{2n}^2\left(\alpha^{r+n-2}G_{k+n-2}^3+4\alpha^{r+n-1}G_{k+n-1}^3+4\alpha^{r+n}G_{k+n}^3+\alpha^{r+n+1}G_{k+n+1}^3\right). \tag{133}$$

Combining (132) and (133) according to the Binet formula yields the next result.

Proposition 33. *If k and r are integers, then*

$$6\sum_{j=0}^{2n-1}G_{k+j}^2\sum_{j=0}^{2n-1}H_{r+j}G_{k+j} \\ = F_{2n}^2\left(H_{r+n-2}G_{k+n-2}^3+4H_{r+n-1}G_{k+n-1}^3+4H_{r+n}G_{k+n}^3+H_{r+n+1}G_{k+n+1}^3\right).$$

Repeated application of the method (second component) two more times to the identity stated in Proposition 33 with k as the index of interest establishes the next result.

Proposition 34. *If k, r, s, and t are integers, then*

$$2\sum_{j=0}^{2n-1}G_{k+j}J_{t+j}\sum_{j=0}^{2n-1}H_{r+j}I_{s+j}+2\sum_{j=0}^{2n-1}I_{s+j}J_{t+j}\sum_{j=0}^{2n-1}G_{k+j}H_{r+j} \\ +2\sum_{j=0}^{2n-1}G_{k+j}I_{s+j}\sum_{j=0}^{2n-1}H_{r+j}J_{t+j} \\ = F_{2n}^2\,(G_{k+n-2}H_{r+n-2}I_{s+n-2}J_{t+n-2}+4\,G_{k+n-1}H_{r+n-1}I_{s+n-1}J_{t+n-1} \\ +4\,G_{k+n}H_{r+n}I_{s+n}J_{t+n}+G_{k+n+1}H_{r+n+1}I_{s+n+1}J_{t+n+1});$$

where (G_j), (H_j), (I_j), and (J_j) are gibonacci sequences.

5.1.10 Generalizations of an identity of Howard

Taking the imaginary part of (44) on page 12 gives

$$G_{k+r}\Im\frac{d}{ds}f(s)+(-1)^{r-1}G_k\Im\frac{\partial f}{\partial s}(s-r)=F_r\Im\frac{\partial g}{\partial s}(k+s); \tag{134}$$

which, on using (91) and (102), yields

$$\beta^sG_{k+r}+(-1)^{r-1}\beta^{s-r}G_k=\beta^{k+s-1}G_0F_r+\beta^{k+s}G_1F_r \tag{135}$$

and hence, also

$$\alpha^sG_{k+r}+(-1)^{r-1}\alpha^{s-r}G_k=\alpha^{k+s-1}G_0F_r+\alpha^{k+s}G_1F_r. \tag{136}$$

Combining (135) and (136) provides the following generalization of Howard's identity (11).

Proposition 35. *If r, s, and k are integers, then*

$$H_sG_{k+r}+(-1)^{r-1}H_{s-r}G_k=F_r\left(G_0H_{k+s-1}+G_1H_{k+s}\right).$$

By taking the imaginary part of (45) on page 12, the reader is invited to establish the result stated in Proposition 36.

Proposition 36. *If k, r, s, and t are integers, then*

$$F_s\left(G_0H_{k+r+t-1}+G_1H_{k+r+t}\right)-(-1)^rF_{s-r}G_k\left(H_{t+1}+H_{t-1}\right)+(-1)^rG_kH_{s-r+t}=G_{k+s}H_{r+t}.$$

5.1.11 Generalizations of Candido's identity

Taking the imaginary part of (55) on page 18 according to the prescription of (101) and (102), we have

$$\begin{aligned}&2\left(\beta^kG_k^3+\beta^{k+1}G_{k+1}^3+\beta^{k+2}G_{k+2}^3\right)\\&\quad=\left(G_k^2+G_{k+1}^2+G_{k+2}^2\right)\left(\beta^kG_k+\beta^{k+1}G_{k+1}+\beta^{k+2}G_{k+2}\right);\end{aligned}$$

which, upon multiplication by β^{r-k} gives

$$\begin{aligned}&2\left(\beta^rG_k^3+\beta^{r+1}G_{k+1}^3+\beta^{r+2}G_{k+2}^3\right)\\&\quad=\left(G_k^2+G_{k+1}^2+G_{k+2}^2\right)\left(\beta^rG_k+\beta^{r+1}G_{k+1}+\beta^{r+2}G_{k+2}\right);\end{aligned}\tag{137}$$

which, on account of Proposition 17, also implies

$$\begin{aligned}&2\left(\alpha^rG_k^3+\alpha^{r+1}G_{k+1}^3+\alpha^{r+2}G_{k+2}^3\right)\\&\quad=\left(G_k^2+G_{k+1}^2+G_{k+2}^2\right)\left(\alpha^rG_k+\alpha^{r+1}G_{k+1}+\alpha^{r+2}G_{k+2}\right).\end{aligned}\tag{138}$$

Combining according to the Binet formula gives the following generalization of Candido's identity.

Proposition 37. *If r and k are integers, then*

$$\begin{aligned}&2\left(H_rG_k^3+H_{r+1}G_{k+1}^3+H_{r+2}G_{k+2}^3\right)\\&\quad=\left(G_k^2+G_{k+1}^2+G_{k+2}^2\right)\left(H_rG_k+H_{r+1}G_{k+1}+H_{r+2}G_{k+2}\right).\end{aligned}\tag{139}$$

By differentiating the gibonacci function form of (139) three more times with respect to k, the reader is invited to demonstrate the further generalization of the Candido identity stated in Proposition 38.

Proposition 38. *If k, r, s, and t are integers, then*

$$\begin{aligned}&6\left(G_kH_rM_sN_t+G_{k+1}H_{r+1}M_{s+1}N_{t+1}+G_{k+2}H_{r+2}M_{s+2}N_{t+2}\right)\\&=\left(G_kM_s+G_{k+1}M_{s+1}+G_{k+2}M_{s+2}\right)\left(H_rN_t+H_{r+1}N_{t+1}+H_{r+2}N_{t+2}\right)\\&\quad+\left(G_kH_r+G_{k+1}H_{r+1}+G_{k+2}H_{r+2}\right)\left(M_sN_t+M_{s+1}N_{t+1}+M_{s+2}N_{t+2}\right)\\&\quad+\left(G_kN_t+G_{k+1}N_{t+1}+G_{k+2}N_{t+2}\right)\left(H_rM_s+H_{r+1}M_{s+1}+H_{r+2}M_{s+2}\right),\end{aligned}$$

where $(M_j)_{j\in\mathbb{Z}}$ and $(N_j)_{j\in\mathbb{Z}}$ are gibonacci sequences with seeds M_0 and M_1 and N_0 and N_1.

In particular, we have

$$
\begin{aligned}
&6\left(F_kF_rF_sF_t + F_{k+1}F_{r+1}F_{s+1}F_{t+1} + F_{k+2}F_{r+2}F_{s+2}F_{t+2}\right)\\
&= \left(F_kF_s + F_{k+1}F_{s+1} + F_{k+2}F_{s+2}\right)\left(F_rF_t + F_{r+1}F_{t+1} + F_{r+2}F_{t+2}\right)\\
&\quad + \left(F_kF_r + F_{k+1}F_{r+1} + F_{k+2}F_{r+2}\right)\left(F_sF_t + F_{s+1}F_{t+1} + F_{s+2}F_{t+2}\right)\\
&\quad + \left(F_kF_t + F_{k+1}F_{t+1} + F_{k+2}F_{t+2}\right)\left(F_rF_s + F_{r+1}F_{s+1} + F_{r+2}F_{s+2}\right).
\end{aligned}
$$

5.2 Generalizations of a Lucas number identity

Differentiating the Lucas function:

$$l(2r) + 2(-1)^r = l(r)^2,$$

of the well-known identity

$$L_{2r} + 2(-1)^r = L_r^2,$$

gives

$$\frac{d}{dr}l(2r) + (-1)^r i\pi = l(r)\frac{d}{dr}l(r),$$

which, employing (90) and (92), yields

$$\beta^{2r} + (-1)^r = \beta^r L_r,$$

or multiplying through by β^{s-r},

$$\beta^{s+r} + (-1)^r\beta^{s-r} = \beta^s L_r, \tag{140}$$

and also

$$\alpha^{s+r} + (-1)^r\alpha^{s-r} = \alpha^s L_r, \tag{141}$$

Combining (140) and (141) according to the Binet formula gives the following multiplication formula (also Vajda [23, Formula (10a)]).

Proposition 39. *If r and s are integers, then*

$$G_{s+r} + (-1)^r G_{s-r} = L_r G_s.$$

Differentiating the function

$$g(s+r) + (-1)^r g(s-r) = l(r)g(s),$$

with respect to r gives

$$\frac{\partial g}{\partial r}(s+r) + (-1)^r \pi i g(s-r) - (-1)^r\frac{\partial g}{\partial r}(s-r) = g(s)\frac{d}{dr}l(r),$$

and taking the imaginary part, making use of (92), (101) and (102), we find

$$\begin{aligned} &G_0\left(\beta^{k+r+s-1}-(-1)^r\beta^{k+s-r-1}\right)+G_1\left(\beta^{k+s-1+r}-(-1)^r\beta^{k+s-1-r}\right)\\ &\qquad=(-1)^r\beta^kG_{s-r}\sqrt{5}-\beta^{r+k}G_s\sqrt{5}, \end{aligned}$$

and also

$$\begin{aligned} &G_0\left(\alpha^{k+r+s-1}-(-1)^r\alpha^{k+s-r-1}\right)+G_1\left(\alpha^{k+s-1+r}-(-1)^r\alpha^{k+s-1-r}\right)\\ &\qquad=(-1)^{r-1}\alpha^kG_{s-r}\sqrt{5}+\alpha^{r+k}G_s\sqrt{5}, \end{aligned}$$

and hence the next result.

Proposition 40. *If r, k, and s are integers, then*

$$\begin{aligned} &G_s\left(H_{r+k+1}+H_{r+k-1}\right)+(-1)^{r-1}G_{s-r}\left(H_{k+1}+H_{k-1}\right)\\ &\qquad=G_0F_r\left(H_{k+s}+H_{k+s-2}\right)+G_1F_r\left(H_{k+s+1}+H_{k+s-1}\right). \end{aligned}$$

Note that we used

$$G_{s+r}-(-1)^rG_{s-r}=F_r\left(G_{s+1}+G_{s-1}\right),\quad [23, \text{Formula (10b)}].$$

Note also that $(H_j)=(L_j)$ in the proposition gives the Howard identity (11) while $(H_j)=(F_j)$ produces (12).

Remark 41. It should be noted that in order that the results obtained by applying the method (second component) be valid, it is necessary that the (generalized) Fibonacci function identity obtained from the original (generalized) Fibonacci number identity holds for all real x; not just integers. Also, the derivative of $(-1)^x$ is an imaginary number for real x. In order not to lose this value, therefore, $(-1)^{2k}$ must not be set equal to unity in the original Fibonacci identity when converting to the Fibonacci function form. The method (second component) involves taking the imaginary part. This point is taken into consideration in the examples presented in § 5.2.1 to 5.2.3.

5.2.1 A generalization of the Gelin-Cesàro identity

Since

$$G_{k-1}G_{k+1}=G_k^2-(-1)^ke_G,\quad [23, \text{Identity 28}],$$

where $e_G=G_0^2-G_1^2+G_1G_0$, and

$$\begin{aligned} G_{k-2}G_{k+2}&=\left(G_k-G_{k-1}\right)\left(G_k+G_{k+1}\right)\\ &=G_k^2+\left(G_kG_{k+1}-G_{k-1}G_k\right)-G_{k-1}G_{k+1}\\ &=G_k^2+G_k^2-G_{k-1}G_{k+1}\\ &=G_k^2+G_k^2-\left(G_k^2-(-1)^ke_G\right)\\ &=G_k^2+(-1)^ke_G; \end{aligned}$$

we have the following generalization of the Gelin-Cesàro identity:

$$G_{k-2}G_{k-1}G_{k+1}G_{k+2} = G_k^4 - (-1)^{2k}e_G^2,$$

where we have retained $(-1)^{2k}$ to allow a direct conversion to the gibonacci function form which is required to hold for all real numbers k, namely,

$$g(k-2)g(k-1)g(k+1)g(k+2) = g(k)^4 - (-1)^{2k}e_G^2. \tag{142}$$

Differentiating (142) gives

$$\begin{aligned}
&\frac{d}{dk}g(k-2)g(k-1)g(k+1)g(k+2)\\
&\quad + g(k-2)\frac{d}{dk}g(k-1)g(k+1)g(k+2)\\
&\quad + g(k-2)g(k-1)\frac{d}{dk}g(k+1)g(k+2)\\
&\quad + g(k-2)g(k-1)g(k+1)\frac{d}{dk}g(k+2)\\
&\quad = 4g(k)^3\frac{d}{dk}g(k) - 2i(-1)^{2k}\pi e_G^2;
\end{aligned}$$

so that taking the imaginary part, we have

$$\begin{aligned}
&B_G\beta^{k-2}G_{k-1}G_{k+1}G_{k+2} + G_{k-2}B_G\beta^{k-1}G_{k+1}G_{k+2}\\
&\qquad + G_{k-2}G_{k-1}B_G\beta^{k+1}G_{k+2} + G_{k-2}G_{k-1}G_{k+1}B_G\beta^{k+2}\\
&= 4G_k^3B_G\beta^k + 2(-1)^{2k}e_G^2\sqrt{5};
\end{aligned}$$

and substituting $(G_1 - G_0\alpha)$ for B_G from (102) and multiplying through by β^r yields

$$\begin{aligned}
&\beta^{k+r-2}G_{k-1}G_{k+1}G_{k+2} + G_{k-2}\beta^{k+r-1}G_{k+1}G_{k+2}\\
&\qquad + G_{k-2}G_{k-1}\beta^{k+r+1}G_{k+2} + G_{k-2}G_{k-1}G_{k+1}\beta^{k+r+2}\\
&= 4G_k^3\beta^{k+r} + 2\left(\beta^{r+1}G_0 - \beta^rG_1\right)e_G\sqrt{5},
\end{aligned} \tag{143}$$

which also implies

$$\begin{aligned}
&\alpha^{k+r-2}G_{k-1}G_{k+1}G_{k+2} + G_{k-2}\alpha^{k+r-1}G_{k+1}G_{k+2}\\
&\qquad + G_{k-2}G_{k-1}\alpha^{k+r+1}G_{k+2} + G_{k-2}G_{k-1}G_{k+1}\alpha^{k+r+2}\\
&= 4G_k^3\alpha^{k+r} - 2\left(\alpha^{r+1}G_0 - \alpha^rG_1\right)e_G\sqrt{5}.
\end{aligned} \tag{144}$$

Note that we used

$$\frac{1}{B_G} = \frac{1}{G_1 - G_0\alpha} = \frac{G_0\beta - G_1}{e_G}.$$

Combining (143) and (144), we arrive at the generalization of the Gelin-Cesàro identity stated in Proposition 42.

Proposition 42. *If k and r are integers, then*

$$\begin{aligned} &H_{k+r-2}G_{k-1}G_{k+1}G_{k+2} + G_{k-2}H_{k+r-1}G_{k+1}G_{k+2} \\ &\quad + G_{k-2}G_{k-1}H_{k+r+1}G_{k+2} + G_{k-2}G_{k-1}G_{k+1}H_{k+r+2} \\ &= 4H_{k+r}G_k^3 - 2e_G G_0 \left(H_{r+2} + H_r\right) + 2e_G G_1 \left(H_{r+1} + H_{r-1}\right). \end{aligned}$$

5.2.2 Generalizations of Catalan's identity

Upon differentiating the Fibonacci function form

$$f(k-r)f(k+r) = f(k)^2 + (-1)^{k+r+1} f(r)^2,$$

of Catalan's identity

$$F_{k-r}F_{k+r} = F_k^2 + (-1)^{k+r+1} F_r^2,$$

with respect to k and applying the prescription (89) and (91), we obtain

$$F_{k+r}\beta^{k-r+s} + F_{k-r}\beta^{k+r+s} = 2F_k\beta^{k+s} + (-1)^{k+r}\beta^s F_r^2\sqrt{5}, \tag{145}$$

and hence also

$$F_{k+r}\alpha^{k-r+s} + F_{k-r}\alpha^{k+r+s} = 2F_k\alpha^{k+s} - (-1)^{k+r}\alpha^s F_r^2\sqrt{5}. \tag{146}$$

Combining (145) and (146) according to the Binet formula gives the following generalization of Catalan's identity.

Proposition 43. *If k, r, and s are integers, then*

$$F_{k+r}G_{k-r+s} + F_{k-r}G_{k+r+s} = 2F_kG_{k+s} + (-1)^{k+r+1}F_r^2\left(G_{s+1} + G_{s-1}\right). \tag{147}$$

Writing

$$F_{k-s+r}G_{k-r} + F_{k-s-r}G_{k+r} = 2F_{k-s}G_k + (-1)^{k-s+r+1}F_r^2\left(G_{s+1} + G_{s-1}\right), \tag{148}$$

and setting $k = s$ gives the multiplication formula

$$G_{s+r} - (-1)^r G_{s-r} = F_r\left(G_{s+1} + G_{s-1}\right), \tag{149}$$

derived also by Vajda [23, Formula (10b)].

Applying the method (second component) to (149) with r as the index of interest gives the next result.

Proposition 44. *If r, s, and t are integers, then*

$$L_r\left(G_0H_{s+t-1} + G_1H_{s+t}\right) - (-1)^r G_{s-r}\left(H_{t+1} + H_{t-1}\right) = H_{r+t}\left(G_{s+1} + G_{s-1}\right).$$

In particular, setting $t = s$, $(H_j) = (G_j)$, and using [23, Formula (10a)]:

$$G_{n+m} + (-1)^m G_{n-m} = L_m G_n,$$

we have

$$G_0 G_{2s-1} + G_1 G_{2s} = G_s(G_{s+1} + G_{s-1}).$$

Differentiating the Fibonacci function form of (148), namely,

$$\begin{aligned} &f(k-s+r)g(k-r) + f(k-s-r)g(k+r) \\ &\quad = 2f(k-s)g(k) + (-1)^{k-s+r+1} f(r)^2 \left(g(s+1) + g(s-1)\right), \end{aligned}$$

with respect to s, taking the imaginary part and making use of (89), (91), (101) and (102) yields the identity stated in Proposition (45).

Proposition 45. *If k, r, s, and t are integers, then*

$$\begin{aligned} &G_{k+s-r} H_{k+r+t} + G_{k+s+r} H_{k-r+t} \\ &\quad = 2G_{k+s} H_{k+t} \\ &\quad + (-1)^{k+r} F_r^2 \left(G_0 \left(H_{s+t} + H_{s+t-2}\right) + G_1 \left(H_{s+t+1} + H_{s+t-1}\right)\right) \\ &\quad - (-1)^{k+r} F_r^2 \left(G_{s+1} + G_{s-1}\right)\left(H_{t+1} + H_{t-1}\right). \end{aligned}$$

In particular,

$$G_{k-r} G_{k+r} - G_k^2 = (-1)^{k+r} F_r^2 e_G.$$

5.2.3 Generalization of an identity obtained from an inverse tangent relation

The method (second component) cannot be applied to (10) because (54) on page 16 is valid only for integers k. In order to redeem the situation, we proceed as follows:

$$\begin{aligned} \tan^{-1}\frac{1}{F_{2k}} - \tan^{-1}\frac{1}{F_{2k+2}} &= \tan^{-1}\frac{F_{2k+2} - F_{2k}}{F_{2k}F_{2k+2} + 1} \\ &= \tan^{-1}\frac{F_{2k+1}}{F_{2k+1}^2 + (-1)^{2k+1} + 1}, \end{aligned}$$

where we refrained from setting $(-1)^{2k+1} = -1$ to ensure that the Fibonacci function form

$$\tan^{-1}\frac{1}{f(2k)} - \tan^{-1}\frac{1}{f(2k+2)} = \tan^{-1}\frac{f(2k+1)}{f(2k+1)^2 + (-1)^{2k+1} + 1}, \tag{150}$$

holds for all real numbers k.

Differentiating (150) with respect to k and taking the imaginary part, we find

$$\begin{aligned} &\frac{1}{F_{2k}^2 + 1}\Im\frac{df}{dk}(2k) - \frac{1}{F_{2k+2}^2 + 1}\Im\frac{df}{dk}(2k+2) \\ &\quad = \frac{1}{F_{2k+1}^2 + 1}\Im\frac{df}{dk}(2k+1) - \frac{\pi}{F_{2k+1}\left(F_{2k+1}^2 + 1\right)}, \end{aligned}$$

which, upon use of (89) and (91), gives

$$\frac{\beta^{2k}}{F_{2k}^2+1}-\frac{\beta^{2k+2}}{F_{2k+2}^2+1}=\frac{\beta^{2k+1}}{F_{2k+1}^2+1}+\frac{\sqrt{5}}{F_{2k+1}\left(F_{2k+1}^2+1\right)},$$

that is

$$\frac{\alpha^{r+2}}{F_{2k}^2+1}-\frac{\alpha^{r}}{F_{2k+2}^2+1}=\frac{\alpha^{2k+r+2}\sqrt{5}}{F_{2k+1}\left(F_{2k+1}^2+1\right)}-\frac{\alpha^{r+1}}{F_{2k+1}^2+1},\tag{151}$$

where r is an arbitrary integer and also

$$\frac{\beta^{r+2}}{F_{2k}^2+1}-\frac{\beta^{r}}{F_{2k+2}^2+1}=-\frac{\beta^{2k+r+2}\sqrt{5}}{F_{2k+1}\left(F_{2k+1}^2+1\right)}-\frac{\beta^{r+1}}{F_{2k+1}^2+1}.\tag{152}$$

By combining (151) and (152), we have the next result.

Proposition 46. *If r and k are integers, then*

$$\frac{G_{r+2}}{F_{2k}^2+1}-\frac{G_{r}}{F_{2k+2}^2+1}=\frac{G_{2k+r+3}+G_{2k+r+1}}{F_{2k+1}(F_{2k+1}^2+1)}-\frac{G_{r+1}}{F_{2k+1}^2+1}.$$

6 Extension of the method to the Horadam sequence

In this section we extend the method to a general non-degenerate second order sequence (Horadam sequence).

The Horadam sequence $(W_j)=(W_j(W_0,W_1;P,Q))$ is defined, for all integers and arbitrary real numbers W_0, W_1, $P\neq 0$, and $Q\neq 0$, by the recurrence relation

$$W_j=PW_{j-1}-QW_{j-2},\quad j\geq 2,\tag{153}$$

with $W_{-j}=\left(PW_{-j+1}-W_{-j+2}\right)/Q$.

Associated with (W_j) are the Lucas sequences of the first kind, $(U_j(P,Q))=(W_j(0,1;P,Q))$, and the second kind, $(V_j(P,Q))=(W_j(2,P;P,Q))$; that is

$$U_0=0,\, U_1=1,\quad U_j=PU_{j-1}-QU_{j-2},\quad j\geq 2,\tag{154}$$

and

$$V_0=2,\, V_1=P,\quad V_j=PV_{j-1}-QV_{j-2},\quad j\geq 2,\tag{155}$$

with $U_{-j}=\left(PU_{-j+1}-U_{-j+2}\right)/Q$ and $V_{-j}=\left(PV_{-j+1}-V_{-j+2}\right)/Q$.

Note that, for convenience and since no confusion can arise, we have retained the notation $(W_j)=(W_j(W_0,W_1;P,Q))$ for the Horadam sequence and $(U_j)=(U_j(P,Q))$ and $(V_j)=(V_j(P,Q))$ for the Lucas sequences.

The closed formula for $W_j(W_0,W_1;P,Q)$ in the non-degenerate case, $P^2-4Q>0$, remains

$$W_j=\frac{A\sigma^j-B\tau^j}{\sigma-\tau}=\frac{A\sigma^j-B\tau^j}{\delta},\tag{156}$$

where

$$A = W_1 - W_0\tau, \quad B = W_1 - W_0\sigma, \tag{157}$$

with σ and τ now given by

$$\sigma = \frac{P + \sqrt{P^2 - 4Q}}{2}, \qquad \tau = \frac{P - \sqrt{P^2 - 4Q}}{2}; \tag{158}$$

so that

$$\sigma + \tau = P, \quad \sigma - \tau = \sqrt{P^2 - 4Q} = \delta, \quad \text{and } \sigma\tau = Q. \tag{159}$$

In particular,

$$U_j = \frac{\sigma^j - \tau^j}{\sigma - \tau}, \quad V_j = \sigma^j + \tau^j. \tag{160}$$

The following identities, of which (70) to (73) are particular cases, are easy to derive:

$$U_{j+1} - QU_{j-1} = V_j, \tag{161}$$

$$U_{j+1} + QU_{j-1} = PU_j, \tag{162}$$

$$V_{j+1} - QV_{j-1} = U_j\delta^2, \tag{163}$$

and

$$V_{j+1} + QV_{j-1} = PV_j.$$

Identity (75) of Lemma 13 on page 24 now reads

$$A\sigma^j + B\tau^j = W_{j+1} - QW_{j-1}. \tag{164}$$

We define the Horadam function $w(x)$ by

$$w(x) = \frac{A\sigma^x - B\tau^x}{\sigma - \tau} = \frac{A\sigma^x - B\tau^x}{\delta}, \quad x \in \mathbb{R}, \tag{165}$$

where A and B are as defined in (157) and σ and τ are as given in (158).

We now discuss the extension of the method to the Horadam sequence. We distinguish the following three cases:

1. $Q < 0$;
2. $P > 0$ and $Q > 0$;
3. $P < 0$ and $Q > 0$.

6.1 Case 1: $Q < 0$

If the Horadam parameter, Q, is *negative*, then we see from (158) that σ is positive and τ is negative for all real numbers P; and equation (87) on page 27 in the proof of the theorem now reads

$$\left.\frac{d}{dx}w(x)\right|_{x=j\in\mathbb{Z}} = \frac{1}{\delta}\left((W_{j+1} - QW_{j-1})\ln\sigma - B\tau^j\ln(-Q) - B\tau^j\pi i(2m+1)\right), \text{by (164)}.$$

Taking the real and imaginary parts, we have

$$\Re\left(\left.\frac{d}{dx}w(x)\right|_{x=j\in\mathbb{Z}}\right) = \frac{1}{\delta}\left((W_{j+1} - QW_{j-1})\ln\sigma - B_W\tau^j\ln(-Q)\right) \tag{166}$$

and

$$\Im\left(\left.\frac{d}{dx}w(x)\right|_{x=j\in\mathbb{Z}}\right) = -\frac{B_W}{\delta}\tau^j\pi(2m+1), \tag{167}$$

where m is some integer and

$$B_W = B = W_1 - W_0\sigma.$$

Equation (166) reduces to (85) when $Q = -1$. Equation (167) is the same as (88) on page 27 in Section 5, except that the values of σ and τ now depend on P and Q.

A description of how the method (first component) for obtaining new identities from existing ones works for the general second order (Horadam) sequence $(W_j(W_0, W_1; P, Q))$ with $Q < 0$ now follows.

1. Let k be a free index in the known identity. Replace each Horadam number, say $W_{h(k,\ldots)}$, with a certain differentiable function of k, namely, $w(h(k,\ldots))$, with k now considered a variable.

2. By applying the usual rules of calculus, differentiate, with respect to k, through the identity obtained in step 1.

3. Simplify the equation obtained in step 2 and take the real part, using also the following prescription:

$$w(h(k,\ldots)) \to W_{h(k,\ldots)}, \tag{168}$$

$$\Re\left(\frac{\partial w}{\partial k}(h(k,\ldots))\right) \to \frac{1}{\delta}\left(\left(W_{h(k+1,\ldots)} - QW_{h(k-1,\ldots)}\right)\ln\sigma - B\tau^{h(k,\ldots)}\ln(-Q)\right); \tag{169}$$

where $\sigma = (P+\delta)/2$, $\tau = (P-\delta)/2$, and $\delta = \sqrt{P^2 - 4Q}$.

In particular, for the Lucas sequences, we have

$$u(h(k,\ldots)) \to U_{h(k,\ldots)}, \tag{170}$$

$$\Re\frac{\partial u}{\partial k}(h(k,\ldots)) \to \frac{V_{h(k,\ldots)}}{\delta}\ln\sigma - \frac{\tau^{h(k,\ldots)}}{\delta}\ln(-Q)\,; \tag{171}$$

and

$$v(h(k,\ldots)) \to V_{h(k,\ldots)}, \tag{172}$$

$$\Re\frac{\partial v}{\partial k}(h(k,\ldots)) \to U_{h(k,\ldots)}\delta\ln\sigma + \tau^{h(k,\ldots)}\ln(-Q)\,; \tag{173}$$

of which the generalized Fibonacci relations (35)–(40) on page 11 are particular cases.

Next, we describe how the method (second component) works for the general second order (Horadam) sequence $(W_j(W_0, W_1; P, Q))$ when $Q < 0$. The scheme is the following.

1. Let k be a free index in the known identity. Replace each Horadam number, say $W_{h(k,\ldots)}$, with a certain differentiable function of k, namely, $w(h(k,\ldots))$, with k now considered a variable.

2. By applying the usual rules of calculus, differentiate, with respect to k, through the identity obtained in step 1.

3. Simplify the equation obtained in step 2 and take the imaginary part, using also the following prescription:

$$w(h(k,\ldots)) \to W_{h(k,\ldots)}, \tag{174}$$

$$\Im\frac{\partial w}{\partial k}(h(k,\ldots)) \to -\frac{B_W}{\delta}\pi\tau^{h(k,\ldots)} = \frac{W_0\sigma - W_1}{\delta}\pi\tau^{h(k,\ldots)}; \tag{175}$$

   where $\sigma = (P+\delta)/2$, $\tau = (P-\delta)/2$ and $\delta = \sqrt{P^2 - 4Q}$.

In particular, for the Lucas sequences, we have

$$u(h(k,\ldots)) \to U_{h(k,\ldots)}, \tag{176}$$

$$\Im\frac{\partial u}{\partial k}(h(k,\ldots)) \to -\frac{B_U}{\delta}\pi\tau^{h(k,\ldots)} = -\frac{\pi\tau^{h(k,\ldots)}}{\delta}; \tag{177}$$

and

$$v(h(k,\ldots)) \to V_{h(k,\ldots)}, \tag{178}$$

$$\Im\frac{\partial v}{\partial k}(h(k,\ldots)) \to -\frac{B_V}{\delta}\pi\tau^{h(k,\ldots)} = \pi\tau^{h(k,\ldots)}; \tag{179}$$

of which the Fibonacci and Lucas relations (89)–(92) on page 28 are particular cases.

Note that Proposition 17 on page 29 on the interchangeability of σ and τ, remains valid; and that the method (second component) is applicable provided the expression obtained after substituting Fibonacci, Lucas, gibonacci and Horadam functions $f(x)$, $l(x)$, $u(x)$, $v(x)$, $g(x)$ and $g(x)$ in the given identity holds for all real numbers, as noted in the remark on page 43 .

We give an example; but first we state a needed Lemma.

Lemma 47. *If k is an integer, then*

$$\frac{W_{k+1}-\sigma W_k}{W_1-\sigma W_0}=\tau^k,\quad \frac{W_{k+1}-\tau W_k}{W_1-\tau W_0}=\sigma^k.$$

Proof. We prove, by induction, the first identity for a non-negative integer k; and invoke a theorem of Bruckman and Rabinowitz [3] that if an identity involving generalized Fibonacci numbers is true for all positive subscripts, it is true for all non-positive subscripts as well.

The identity is obviously true for the base case $k=0$. We assume the identity holds for $k=1,2,\ldots n$. We have

$$\begin{aligned}\frac{W_{(n+1)+1}-\sigma W_{n+1}}{W_1-\sigma W_0}&=\frac{PW_{n+1}-QW_n-\sigma W_{n+1}}{W_1-\sigma W_0}\\&=\frac{(\sigma+\tau)W_{n+1}-\sigma\tau W_n-\sigma W_{n+1}}{W_1-\sigma W_0}\\&=\frac{\tau W_{n+1}-\sigma\tau W_n}{W_1-\sigma W_0}=\frac{W_{n+1}-\sigma W_n}{W_1-\sigma W_0}\tau=\tau^n\tau=\tau^{n+1}.\end{aligned}$$

Thus, the identity holds for $n+1$ whenever it holds for n. □

Consider the following identity ([9, Equation (3.14)]):

$$U_rW_{k+1}-QU_{r-1}W_k=W_{k+r}.$$

We write

$$u(r)w(k+1)-Qu(r-1)w(k)=w(k+r);$$

and differentiate with respect to r, obtaining

$$\frac{d}{dr}u(r)\cdot w(k+1)-\frac{d}{dr}u(r-1)\cdot Qw(k)=\frac{\partial w}{\partial r}(k+r).\tag{180}$$

Taking the real part, we get

$$\Re\frac{d}{dr}u(r)\cdot W_{k+1}-\Re\frac{d}{dr}u(r-1)\cdot QW_k=\Re\frac{\partial w}{\partial r}(k+r);$$

which, using (171) and (169), after some rearrangement, gives:

$$\begin{aligned}&((V_rW_{k+1}-QV_{r-1}W_k)-(W_{k+r+1}-QW_{k+r-1}))\ln\sigma\\&\quad=\left((W_{k+1}-\sigma W_k)-(W_1-\sigma W_0)\,\tau^k\right)\tau^r\ln(-Q)\,.\end{aligned}\tag{181}$$

On account of Lemma 47, the right hand side of (181) vanishes and we obtain

$$V_r W_{k+1} - Q V_{r-1} W_k = W_{k+r+1} - Q W_{k+r-1}, \quad Q < 0. \tag{182}$$

If $Q = -1$, then (182) reduces to identity (19) derived in Section 2.4.2.

Next, taking the imaginary part of (180), we find

$$\Im \frac{d}{dr} u(r) \cdot W_{k+1} - \Im \frac{d}{dr} u(r-1) \cdot Q W_k = \Im \frac{\partial w}{\partial r}(k+r);$$

which, using (177) and (175), gives

$$-\tau^r W_{k+1} + Q\tau^{r-1} W_k = (W_0\sigma - W_1)\,\tau^{k+r},$$

that is

$$-\tau^r W_{k+1} + Q\tau^{r-1} W_k = W_0 Q \tau^{k+r-1} - W_1 \tau^{k+r} \tag{183}$$

and also

$$-\sigma^r W_{k+1} + Q\sigma^{r-1} W_k = W_0 Q \sigma^{k+r-1} - W_1 \sigma^{k+r}. \tag{184}$$

Combining (183) and (184), using the Binet formula, we have the result stated in Proposition 48.

Proposition 48. *If r and k are integers, then*

$$Z_r W_{k+1} - Q Z_{r-1} W_k = W_1 Z_{k+r} - Q W_0 Z_{k+r-1}, \quad Q < 0; \tag{185}$$

where $W_j = W_j\,(W_0, W_1; P, Q)$ and $Z_j = Z_j\,(Z_0, Z_1; P, Q)$ are two Horadam sequences.

6.2 Case 2: $P > 0$ and $Q > 0$

If the Horadam parameters P and Q are both *positive*, then it is clear from (158) that σ and τ are both positive. In this case we have

$$\frac{d}{dx} w(x) = \frac{1}{\delta}\left(A \frac{d}{dx}\sigma^x - B \frac{d}{dx}\tau^x\right) = \frac{1}{\delta}\left((A\sigma^x + B\tau^x)\ln\sigma - B\tau^x \ln Q\right);$$

so that

$$\left.\frac{d}{dx} w(x)\right|_{x=j\in\mathbb{Z}} = \frac{1}{\delta}\big((W_{j+1} - Q W_{j-1})\ln\sigma - B\tau^j \ln Q\big).$$

A description of how the method (first component) for obtaining new identities from existing ones works for the general second order (Horadam) sequence $(W_j(W_0, W_1; P, Q))$, $P > 0$, and $Q > 0$ now follows.

1. Let k be a free index in the known identity. Replace each Horadam number, say $W_{h(k,\ldots)}$, with a certain differentiable function of k, namely, $w(h(k,\ldots))$, with k now considered a variable.

2. By applying the usual rules of calculus, differentiate, with respect to k, through the identity obtained in step 1.

3. Simplify the equation obtained in step 2, using also the following prescription:

$$w(h(k,\ldots)) \to W_{h(k,\ldots)}, \tag{186}$$

$$\frac{\partial w}{\partial k}\left(h(k,\ldots)\right) \to \frac{1}{\delta}\left(\left(W_{h(k+1,\ldots)} - QW_{h(k-1,\ldots)}\right)\ln\sigma - B\tau^{h(k,\ldots)}\ln Q\right); \tag{187}$$

where, as usual, $\sigma = (P+\delta)/2$, $\tau = (P-\delta)/2$, and $\delta = \sqrt{P^2-4Q}$.

In particular, for the Lucas sequences, we have

$$u(h(k,\ldots)) \to U_{h(k,\ldots)}, \tag{188}$$

$$\frac{\partial u}{\partial k}(h(k,\ldots)) \to \frac{V_{h(k,\ldots)}}{\delta}\ln\sigma - \frac{\tau^{h(k,\ldots)}}{\delta}\ln Q; \tag{189}$$

and

$$v(h(k,\ldots)) \to V_{h(k,\ldots)}, \tag{190}$$

$$\frac{\partial v}{\partial k}(h(k,\ldots)) \to U_{h(k,\ldots)}\delta\ln\sigma + \tau^{h(k,\ldots)}\ln Q. \tag{191}$$

Returning to the example in the previous section, using (189) and (187) in (180) gives

$$\begin{aligned}&\left((V_rW_{k+1} - QV_{r-1}W_k) - (W_{k+r+1} - QW_{k+r-1})\right)\ln\sigma\\ &\quad = \left((W_{k+1} - \sigma W_k) - (W_1 - \sigma W_0)\,\tau^k\right)\tau^r\ln Q, \quad Q>0,\end{aligned}$$

which is the same as (181) with $\ln(-Q)$ replaced with $\ln Q$; and which in view of Lemma 47 gives

$$V_rW_{k+1} - QV_{r-1}W_k = W_{k+r+1} - QW_{k+r-1}, \quad P>0, \quad Q>0. \tag{192}$$

6.3 Case 3: $P<0$ and $Q>0$

If the parameter P is negative and the parameter Q is positive, then it is obvious from (158) that σ and τ are both negative numbers. In this case we have

$$\begin{aligned}\frac{d}{dx}w(x) &= \frac{1}{\delta}\left(A\frac{d}{dx}\sigma^x - B\frac{d}{dx}\tau^x\right)\\ &= \frac{1}{\delta}\left(A\sigma^x\left(i\pi(2m+1) + \ln(-\sigma)\right) - B\tau^x\left(i\pi(2n+1) + \ln(-\tau)\right)\right)\\ &= \frac{1}{\delta}\left((A\sigma^x + B\tau^x)\ln(-\sigma) - B\tau^x\ln Q\right) + \frac{i\pi}{\delta}\left((2m+1)A\sigma^x - (2n+1)B\tau^x\right),\end{aligned}$$

where m and n are integers; so that

$$\Re\left(\frac{d}{dx}w(x)\bigg|_{x=j\in\mathbb{Z}}\right)=\frac{1}{\delta}\left((W_{j+1}-QW_{j-1})\ln(-\sigma)-B_W\tau^j\ln Q\right) \tag{193}$$

and

$$\Im\left(\frac{d}{dx}w(x)\bigg|_{x=j\in\mathbb{Z}}\right)=\frac{2\pi}{\delta}\left(mA\sigma^j-nB\tau^j\right)+\pi W_j, \tag{194}$$

where m and n are integers.

We now describe how the method (first component) works for the general second order (Horadam) sequence $(W_j(W_0,W_1;P,Q))$ with $P<0$ and $Q>0$.

1. Let k be a free index in the known identity. Replace each Horadam number, say $W_{h(k,\ldots)}$, with a certain differentiable function of k, namely, $w(h(k,\ldots))$, with k now considered a variable.

2. By applying the usual rules of calculus, differentiate, with respect to k, through the identity obtained in step 1.

3. Simplify the equation obtained in step 2 and take the real part, using also the following prescription:

$$w(h(k,\ldots))\to W_{h(k,\ldots)}, \tag{195}$$

$$\Re\left(\frac{\partial w}{\partial k}(h(k,\ldots))\right)\to\frac{1}{\delta}\left(\left(W_{h(k+1,\ldots)}-QW_{h(k-1,\ldots)}\right)\ln(-\sigma)-B\tau^{h(k,\ldots)}\ln Q\right); \tag{196}$$

   where $\sigma=(P+\delta)/2$, $\tau=(P-\delta)/2$, and $\delta=\sqrt{P^2-4Q}$.

In particular, for the Lucas sequences, we have

$$u(h(k,\ldots))\to U_{h(k,\ldots)}, \tag{197}$$

$$\Re\frac{\partial u}{\partial k}(h(k,\ldots))\to\frac{V_{h(k,\ldots)}}{\delta}\ln(-\sigma)-\frac{\tau^{h(k,\ldots)}}{\delta}\ln Q; \tag{198}$$

and

$$v(h(k,\ldots))\to V_{h(k,\ldots)}, \tag{199}$$

$$\Re\frac{\partial v}{\partial k}(h(k,\ldots))\to U_{h(k,\ldots)}\delta\ln(-\sigma)+\tau^{h(k,\ldots)}\ln Q. \tag{200}$$

We see from (194) that the method (second component) will, in general, not generate new identities for the general second order (Horadam) sequence $(W_j(W_0,W_1;P,Q))$ when $P<0$ and $Q>0$. Taking the principal value in (194) gives

$$\Im\frac{\partial w}{\partial k}(h(k,\ldots))\to\pi W_{h(k,\ldots)} \tag{201}$$

for a Horadam function $w(h(k,\ldots))$ and hence does not facilitate a generalization of an identity obtained from the first component.

To illustrate the first component for the case $P < 0$ and $Q > 0$, equation (181) on page 51 now reads

$$\begin{aligned}((V_r W_{k+1} - QV_{r-1}W_k) - (W_{k+r+1} - QW_{k+r-1}))\ln(-\sigma)\\ = \left((W_{k+1} - \sigma W_k) - (W_1 - \sigma W_0)\,\tau^k\right)\tau^r \ln Q, \quad Q > 0,\end{aligned}$$

which in view of Lemma 47 gives

$$V_r W_{k+1} - QV_{r-1}W_k = W_{k+r+1} - QW_{k+r-1}, \quad P < 0, \quad Q > 0. \tag{202}$$

Comparing (182), (192) and (202), we find the result stated in the next proposition.

Proposition 49. *If k and r are integers, then*

$$V_r W_{k+1} - QV_{r-1}W_k = W_{k+r+1} - QW_{k+r-1}, \quad Q \neq 0. \tag{203}$$

In particular, on account of (161) and (163), we have for the Lucas sequences,

$$\begin{aligned}V_r U_{k+1} - QV_{r-1}U_k = V_{k+r},\\ V_r V_{k+1} - QV_{r-1}V_k = \delta^2 U_{k+r}.\end{aligned}$$

We note that identity (203) is a particular case of Howard [12, Theorem 3.1].

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