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Integer Sequences from Configurations in the Hausdorff Metric Geometry via Edge Covers of Bipartite Graphs

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Abstract

The Hausdorff metric provides a way to measure the distance between nonempty compact sets in $\mathbb{R}^N$, from which we can build a geometry of sets. This geometry is very different than the standard Euclidean geometry and provides many interesting results. In this paper we focus on line segments in this geometry, where pairs of disjoint sets A and B satisfying certain distance conditions have the property that there are exactly m different sets on the line segment $\overline{AB}$ at every distance from A, where m can assume many values different than one. We provide new families of sets that generate previously unrecorded integer sequences via these values of m by connecting the values of m to the number of edge coverings of a graph corresponding to the sets A and B.

1 Introduction

The Hausdorff metric h imposes a geometry on the hyperspace $\mathcal{H}(\mathbb{R}^N)$ of all nonempty compact subsets of $\mathbb{R}^N$. One notion of betweenness in this geometry is an extension of betweenness in Euclidean geometry. For certain positive integers m, there exists a pair of disjoint finite sets A and B for which there are $m = \#([A, B])$ different sets on the line segment defined by these sets at every distance from one of the sets. There are many fascinating and interesting properties of these numbers $m = \#([A, B])$. For example, for each integer m between 1 and 18 there exist sets A and B such that $\#([A, B]) = m$, but no such sets exist for $m = 19$. Further discussion of this can be found in Section 3.

The number $\#([A, B])$ is related to edge covers of bipartite graphs, which is explained in more detail in Section 4. For each fixed value of m, varying the size of one of the partite sets yields different numbers of edge covers, generating integer sequences. These sequences help us understand more about line segments in $\mathcal{H}(\mathbb{R}^N)$ and edge covers of graphs.

2 The Hausdorff metric

Felix Hausdorff introduced the Hausdorff metric h in the early 20th century as a way to measure the distance between compact sets. The Hausdorff metric imposes a geometry on the space $\mathcal{H}(\mathbb{R}^N)$, which is the subject of our study. Throughout, we let d_E denote the standard Euclidean metric in $\mathbb{R}^N$.

Definition 1. The *Hausdorff distance* h between sets A and B in $\mathcal{H}(\mathbb{R}^N)$ is

$$h(A, B) = \max\{d(A, B), d(B, A)\},$$

where

$$d(A, B) = \max_{a \in A}\{d(a, B)\}$$

and

$$d(x, A) = \min_{a \in A}\{d_E(x, a)\}$$

for every $x \in A$.

In other words, to find the distance $d(A, B)$ from set A to set B, we measure the Euclidean distance from a point x in A farthest from B to a point in B closest to x. The Hausdorff distance between A and B is then the larger of $d(A, B)$ and $d(B, A)$. The mapping h in Definition 1 is called the *Hausdorff metric*. The proof that h is a metric can be found in many topology texts, see Barnsley or Edgar [1, 6], for example.

Example 2. Let $a_1 = (-2, 0)$, $a_2 = (-1, 0)$, $b_1 = (1, -1)$, and $b_2 = (1, 1)$ in $\mathbb{R}^2$, and define segments A and B in $\mathbb{R}^2$ as $A = \overline{a_1 a_2}$ and $B = \overline{b_1 b_2}$ as shown at left in Figure 1. In this example, $d(A, B) = d_E(a_1, (1, 0)) = 3$ and $d(B, A) = d_E(b_1, a_2) = \sqrt{5}$. So $h(A, B) = 3$. (Note that d itself is not a metric since d is not symmetric.)

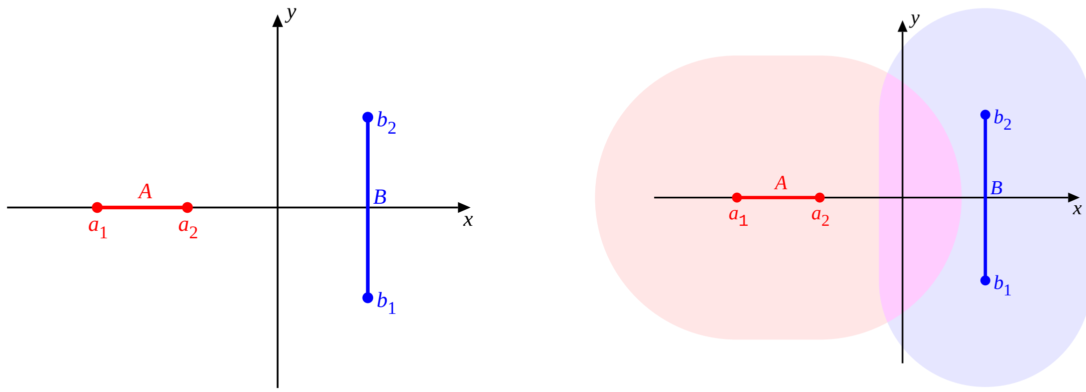


Figure 1: Left: Segments A and B. Right: Dilations of the segments.

3 Betweenness and finite configurations

We can use the Hausdorff metric to extend the notion of Euclidean betweenness to $\mathcal{H}(\mathbb{R}^N)$.

Definition 3. Let A, B, and C be in $\mathcal{H}(\mathbb{R}^N)$. The set C is *between* A and B in $\mathcal{H}(\mathbb{R}^N)$ if

$$h(A, B) = h(A, C) + h(C, B).$$

We adopt the notation ACB from Blumenthal [3] when C is between A and B. So, as in Euclidean geometry, the line segment determined by $A \neq B$ in $\mathcal{H}(\mathbb{R}^N)$ is the set of all sets C satisfying ACB.

The Hausdorff metric is a complicated one, and it is generally not clear how to determine which sets lie on a given line segment. One construction that helps in this regard is the dilation of a set.

Definition 4. Let A be in $\mathcal{H}(\mathbb{R}^N)$ and let s be a positive real number. The *dilation* of A by s is the set

$$(A)_s = \{x \in \mathbb{R}^N \mid d_E(x, a) \leq s \text{ for some } a \in A\}.$$

Example 5. Consider again the segments A and B presented in Example 2. In this case, we have $r = h(A, B) = 3$. Let s be a real number between 0 and 3 and let $t = r - s$. See Figure 1 at right for illustrations of the dilations $(A)_s$ and $(B)_t$ for a fixed value of s.

The dilation of a set A by s has two important properties. The first is that $h(A, (A)_s) = s$. The second is that if C satisfies ACB, then $C \subseteq (A)_s \iff d(C, A) \leq s$ [5, Theorem 4]. It follows that C satisfies ACB with $h(A, C) = s$, then C is a subset of $(A)_s \cap (B)_{h(A,B)-s}$. An example of this intersection is shown at right in Figure 1. In fact, Bogdewicz [4] shows that the set $C_s = (A)_s \cap (B)_{h(A,B)-s}$ always satisfies AC_sB with $h(A, C_s) = s$.

Given sets A and B in $\mathcal{H}(\mathbb{R}^N)$, there are two possibilities for sets C that satisfy ACB. Blackburn et al. [2] show that if there is a point $a \in A$ or a point $b \in B$ such that $d(a, B) \neq$

$h(A, B)$ or $d(b, A) \neq h(A, B)$, then there are infinitely many different sets C that satisfy ACB and $d(A, C) = s$ for any $0 < s < h(A, B)$. The other possibility is the one of interest in this paper and we call such a pair of sets a *configuration*.

Definition 6. A *configuration* is a pair A, B of sets in $\mathcal{H}(\mathbb{R}^N)$ such that $d(a, B) = d(b, A) = h(A, B)$ for every $a \in A$ and $b \in B$.

We will denote the configuration pair A, B as $[A, B]$. Blackburn et al. [2] demonstrate that if A and B are finite sets such that $[A, B]$ is a finite configuration, then there is a finite number k such that there are exactly k sets C satisfying ACB with $h(A, C) = s$ for every $0 < s < h(A, B)$. We denote this number k of sets C satisfying ACB for a finite configuration $[A, B]$ – that is, the number of different sets which exist at the same location on the line segment between A and B in $\mathcal{H}(\mathbb{R}^N)$ – by $\#([A, B])$. Moreover, the number of sets C with ACB and $h(A, C) = s$ is the same for every value of s. Lund et al. [8] show that there are infinitely many different integers that appear as $\#([A, B])$ – all of the Fibonacci numbers and even indexed Lucas numbers, for example. Every integer between 1 and 18 is $\#([A, B])$ for some $[A, B]$, and every number between 20 and 36 is $\#([A, B])$ for some $[A, B]$. However, Blackburn et al. [2] show that there is, surprisingly, no configuration $[A, B]$ with $\#([A, B]) = 19$, and Honigs [7] proves that there is no configuration $[A, B]$ with $\#([A, B]) = 37$. Ovsyannikov [9] extends these results to show that there exist no configurations $[A, B]$ with $\#([A, B])$ equal to 41, 59, or 67.

4 Bipartite graphs

We are interested in determining which integers can occur as $\#([A, B])$ for some finite configuration $[A, B]$. As we will see, there is a correspondence between finite configurations and bipartite graphs, and $\#([A, B])$ is equal to the number of edge covers of the corresponding graph.

We use the standard terminology and notation from graph theory. We consider simple graphs $G = (V, E)$ where V is the vertex set and E is the edge set. We say that two vertices are adjacent if they have an edge in common, two edges are incident if they share a vertex, and an edge and a vertex are incident if the edge contains the vertex. A bipartite graph is a graph $G = (V, E)$ such that there exist disjoint sets V_1 and V_2 (called parts) such that $V = V_1 \cup V_2$ and there are no adjacent vertices within a part.

Now we make the connection between finite configurations and bipartite graphs. Given A and B in $\mathcal{H}(\mathbb{R}^N)$, any set C that lies on the line segment $\overline{AB}$ must satisfy $h(A, C) = s$ and $h(C, B) = h(A, B) - s$ for some $0 < s < h(A, B)$. This implies that $C \subseteq (A)_s \cap (B)_{h(a,b)-s}$. If $[A, B]$ is a finite configuration, then $(A)_s \cap (B)_{h(A,B)-s}$ is a finite set as illustrated at left in Figure 2, where $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, and $C = \{c_1, c_2, c_3, c_4\}$.

We can represent this configuration as a bipartite graph. If $[A, B]$ is a finite configuration, let $V = A \cup B$, and let E be the set $\{(a, b) : a \in A \text{ and } b \in B \text{ such that } d_E(a, b) = h(A, B)\}$. Then $G_{[A,B]} = (V, E)$ forms a bipartite graph with parts A and B and edge set E. Notice

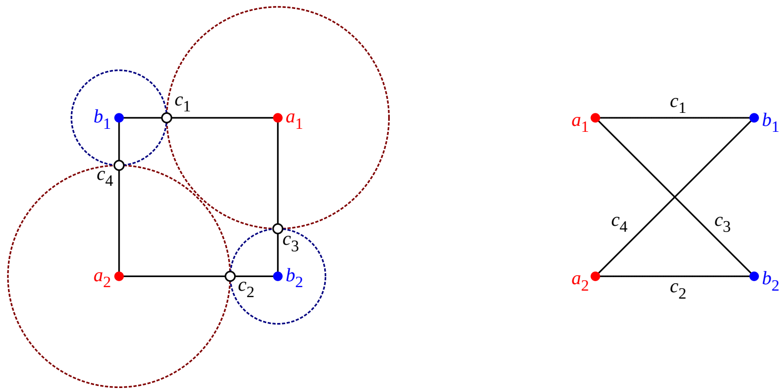


Figure 2: Left: Sets on the line segment $\overline{AB}$ in $\mathcal{H}(\mathbb{R}^N)$. Right: A corresponding bipartite graph.

that if C satisfies ACB with $h(A, C) = s$, then each point $c \in C$ satisfies $a_c c b_c$ for some $a_c \in A$ and $b_c \in B$, with $d_E(a_c, c) = s$. This point c can be identified with the edge (a_c, b_c) in $G_{[A,B]}$ as illustrated at right in Figure 2. The set $E_C = \{(a_c, b_c) : c \in C\}$ is a subset of E. The subsets C of $(A)_s \cap (B)_{h(a,b)-s}$ that satisfy ACB are those sets C such that every $a \in A$ is a distance s from some $c \in C$, and every $b \in B$ is a distance $h(a, b) - s$ from some $c' \in C$. In graph theory terms, E_C will not isolate any vertices in V. Such a set is called an *edge cover*.

Definition 7. An *edge cover* of a graph $G = (V, E)$ is a subset E' of the set E such that every vertex in V is incident to at least one edge in E'.

So E_C is an edge cover of $G_{[A,B]}$. Thus, $\#([A, B])$ is equal to the number of edge covers of $G_{[A,B]}$. Similarly, every bipartite graph can be viewed as a finite configuration [2, Configuration Construction Theorem]. We also use the notation $\#(G)$ for the number of edge coverings of a graph G.

5 Edge covers of complete bipartite graphs

To understand which integers can appear as $\#([A, B])$ for a finite configuration $[A, B]$, we investigate edge covers of bipartite graphs. In an attempt to be systematic, we begin with the complete bipartite graphs, then study what happens when we begin to remove edges. We let $K_{m,n}$ denote the complete bipartite graph with parts V_1 and V_2, where $|V_1| = m$ and $|V_2| = n$. Throughout, we assume $m \leq n$.

We can also calculate the number of edge covers of a bipartite graph using biadjacency matrices whose entries are all 0 or 1 (a $\{0, 1\}$ matrix). The biadjacency matrix of a bipartite graph with parts V_1 and V_2 of size m and n is an $m \times n$ matrix whose ijth entry is 1 if the ith vertex in V_1 is adjacent to the jth vertex in V_2 and is 0 otherwise. Since an edge cover cannot isolate a vertex, the number of edge covers of such a bipartite graph is equal to the number of $\{0,1\}$ $m \times n$ matrices with no zero rows or columns. This number is well-known and can be calculated through a standard inclusion-exclusion principle argument. The next theorem provides the result. (Entry A048291 in the Online Encyclopedia of Integer Sequences (OEIS) gives this formula for $n \times n$ matrices.)

Theorem 8. *Let $E(m,n)$ be the number of edge covers of a $K_{m,n}$ complete bipartite graph. Then*

$$E(m,n) = \sum_{j=0}^{m} \binom{m}{j} (-1)^j (2^{m-j}-1)^n.$$

With this result in hand, we proceed to determine the number of edge covers of the graphs that result from $K_{m,n}$ by removing one to three edges.

6 Edge covers of $K_{m,n}$ minus one edge

To calculate the number of edge covers of a complete bipartite graph that is missing edges, we use the following proposition from Honigs [7].

Proposition 9. *Let $G=\{V,E\}$ be a graph. Let $v_1, v_2 \in V$, and suppose $(v_1,v_2) \notin E$. Let $G' = \{V,E'\}$ where $E'=E \cup \{(v_1,v_2)\}$. Then*

$$\#(G') = 2\#(G) + \#(G-v_1) + \#(G-v_2) + \#(G-v_1-v_2).$$

We use Proposition 9 to determine the number of edge covers of graphs obtained from complete graphs by removing one, two, or three edges. We begin with removing one edge.

Theorem 10. *Let G be a graph obtained from a complete bipartite graph $K_{m,n}$ after removing one edge. Then the number of edge covers of G is $E_1(m,n)$ where*

$$E_1(m,n) = \#(G) = \frac{1}{2}(E(m,n) - E(m-1,n) - E(m,n-1) - E(m-1,n-1)). \tag{1}$$

(The function E in Theorem 10 is defined in Theorem 8.)

Proof. Let the graph G' be a complete bipartite graph and the graph G be G' missing one edge, (v_1, w_1). Since we are only removing one edge, the completeness of $K_{m,n}$ implies that it does not matter which edge. Solving for $\#(G)$ in Proposition 9 gives

$$\#(G) = \frac{1}{2}(\#(G') - \#(G-v_1) - \#(G-w_1) - \#(G-v_1-w_1)).$$

As depicted in Figure 3, removing vertex v_1 from G produces a $K_{m-1,n}$ graph. Removing vertex w_1 from G produces a $K_{m,n-1}$ graph. Removing vertices v_1 and w_1 from G produces

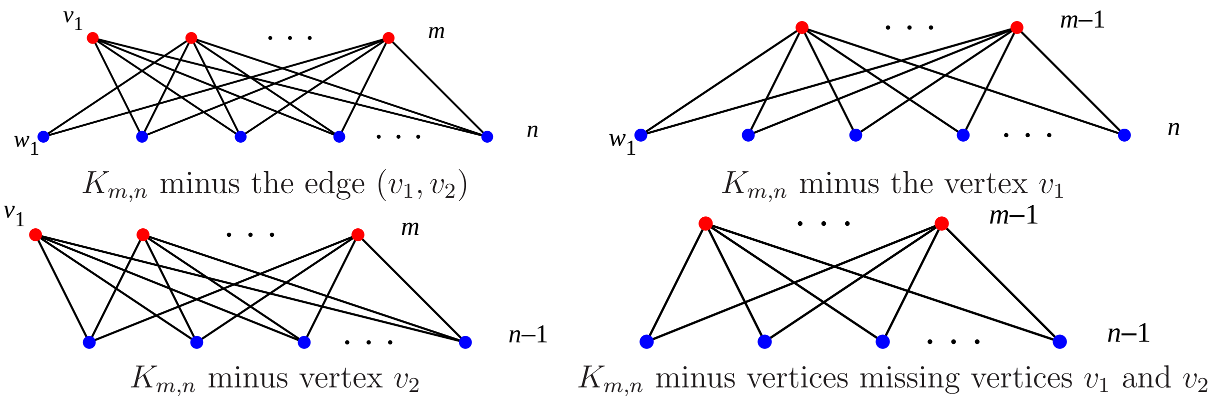


Figure 3: Calculating $\#(K_{m,n}$ minus an edge).

a $K_{m-1,n-1}$ graph. Therefore

$$\begin{aligned}\#(G) &= \frac{1}{2}(\#(K_{m,n}) - \#(K_{m-1,n}) - \#(K_{m,n-1}) - \#(K_{m-1,n-1})) \\ &= \frac{1}{2}(E(m,n) - E(m-1,n) - E(m,n-1) - E(m-1,n-1)).\end{aligned}$$

□

As a result of Theorem 10, we obtain integer sequences by fixing the value of m in calculations of $E_1(m,n)$ and letting n change. As examples, we have the sequences given in Table 1 (sequences simplified with a computer algebra system). The first entry appears as sequences A024023 and A103453 in the OEIS. This graph theoretic approach to these sequences provides a new perspective from which to view these sequences

$E_1(2,n)$	$3^{n-1} - 1$
$E_1(3,n)$ (A335608)	$3 \cdot 7^{n-1} - 5 \cdot 3^{n-1} + 2$
$E_1(4,n)$ (A335609)	$7 \cdot 15^{n-1} - 16 \cdot 7^{n-1} + 4 \cdot 3^{n} - 3$
$E_1(5,n)$ (A335610)	$15 \cdot 31^{n-1} - 43 \cdot 15^{n-1} + 46 \cdot 7^{n-1} - 22 \cdot 3^{n-1} + 4$
$E_1(6,n)$ (A335611)	$31 \cdot 63^{n-1} - 106 \cdot 31^{n-1} + 145 \cdot 15^{n-1} - 100 \cdot 7^{n-1} + 35 \cdot 3^{n-1} - 5$

Table 1: Sequences $(E_1(m,n))_{n \geq m}$.

It is not difficult to see from the sum (1) and Theorem 8 that the highest power exponential term in $E_1(m,n)$ comes from the summands $E(m,n)$ and $E(m,n-1)$. This term is

$$\frac{1}{2}\big((2^m-1)^n - (2^m-1)^{n-1}\big) = (2^{m-1}-1)(2^m-1)^{n-1}.$$

So the sequence $(E_1(m,n))_{n\geq m}$ behaves asymptotically as $(2^m-1)^{n-1}$ as n increases.

To conceptualize how edge covers of complete bipartite graphs missing one edge relate to the Hausdorff metric geometry, we consider the graph G that is $K_{6,2}$ missing one edge. Since $E_1(6,2) = 242$ we know that G has 242 edge covers. The corresponding configuration $[A,B]$ with $A = \{a_1,a_2,a_3,a_4,a_5,a_6\}$ and $B = \{b_1,b_2\}$ is illustrated in Figure 4. Because $\#([A,B]) = \#(G)$, the number of distinct sets C that lie on the segment $\overline{AB}$ in $\mathcal{H}(\mathbb{R}^N)$ at any fixed distance from A is also 242.

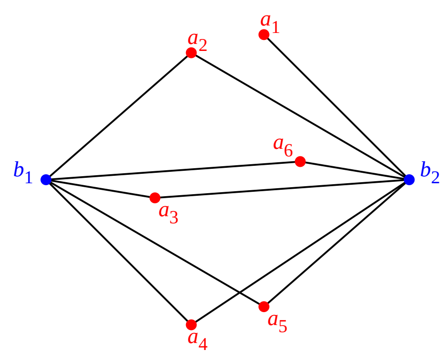


Figure 4: A configuration $[A,B]$.

7 Edge covers of $K_{m,n}$ minus two edges

The next step in our calculations of numbers of edge covers of subgraphs of complete bipartite graphs is to consider the case of removing two edges. When we remove two edges from a complete bipartite graph, there are three different cases for the number of edge covers of the graph that are based upon the vertices from which the two edges are removed.

Theorem 11. *Let G be a $K_{m,n}$ complete bipartite graph missing two edges. Partition the vertex set V of G into two parts, V_1 and V_2, where $|V_1| = m$ and $|V_2| = n$.*

1. *The number of edge covers of G when the 2 removed edges are incident to the same vertex in V_1 and not incident to the same vertex in V_2 is $E_{2_1}(m,n)$ where*

$$E_{2_1}(m,n) = \#(G) = \frac{1}{2}(E_1(m,n) - E(m-1,n) - E_1(m,n-1) - E(m-1,n-1)).$$

2. *The number of edge covers of G when the 2 removed edges are not incident to the same vertex in V_1 but are incident to the same vertex in V_2 is $E_{2_2}(m,n)$ where*

$$E_{2_2}(m,n) = \#(G) = \frac{1}{2}(E_1(m,n) - E_1(m-1,n) - E(m,n-1) - E(m-1,n-1)).$$

3. *The number of edge covers of G when the 2 removed edges are not incident to the same vertex in V_1 and are not incident to the same vertex in V_2 is $E_{2_3}(m,n)$ where*

$$E_{2_3}(m,n) = \#(G) = \frac{1}{2}(E_1(m,n) - E_1(m-1,n) - E_1(m,n-1) - E_1(m-1,n-1)).$$

Proof. Assume the graph G' is a $K_{m,n}$ complete bipartite graph missing one edge. Let G be a graph obtained from G' by removing an edge. Let the vertex set V of G be partitioned into two parts $V_1 = \{v_1, v_2, \ldots, v_m\}$ and $V_2 = \{w_1, w_2, \ldots, w_n\}$. We consider three different cases based on the incidence of the removed edges to certain vertices. The removed edges can be incident to the same vertex in V_1 but not in V_2, incident to the same vertex in V_2 but not in V_1, or not incident to the same vertex in V_1 or V_2, as illustrated from left to right, respectively, in Figure 5.

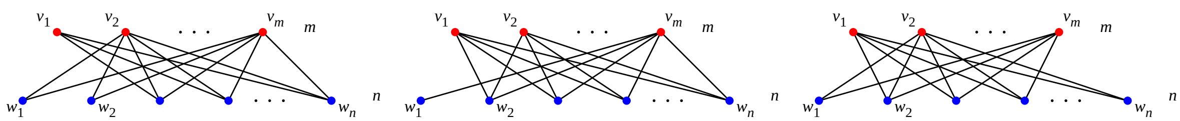
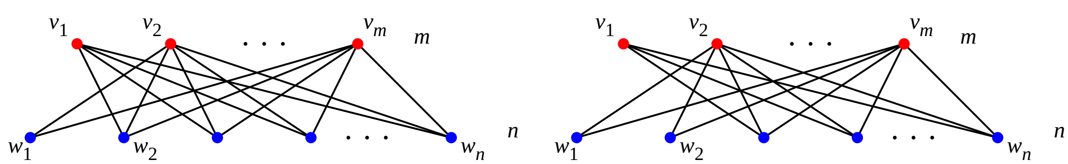


Figure 5: Case 1, Case 2, and Case 3 for removing two edges.

Case 1: Assume that the 2 removed edges in G are incident to the same vertex in V_1 but are not incident to the same vertex in V_2. An illustration of such a graph G can be seen at left in Figure 6. The edges (v_1, w_1) and (v_1, w_2) are missing, and both edges are incident to vertex v_1 in V_1. The graph of G' can be seen at left in Figure 6. Notice that in the graph G' only the edge (v_1, w_1) is missing.



The graph G' The graph G

Figure 6: Case 1: Removing two edges incident to the same vertex.

Since G' is a $K_{m,n}$ complete bipartite graph missing one edge, $\#(G') = E_1(m,n)$.

When we remove the vertex v_1 from G, the result is a $K_{m-1,n}$ complete bipartite graph, as seen at top left in Figure 7. So $\#(G - v_1) = E(m-1,n)$.

When we remove the vertex w_2 from G, the result is a $K_{m,n-1}$ complete bipartite graph missing one edge, as seen at top right in Figure 7. So $\#(G - w_2) = E_1(m,n-1)$.

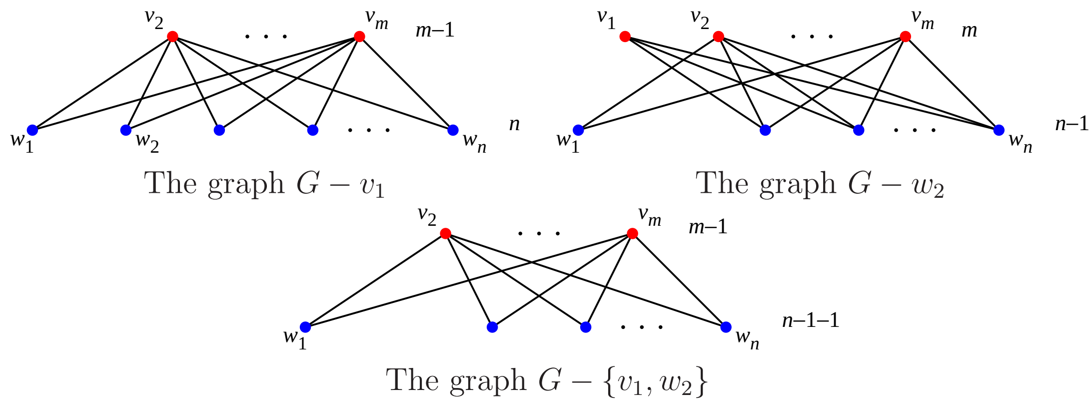


Figure 7: Case 1: Removing vertices from G.

When we remove vertices v_1 and w_2 from G, the result is a $K_{m-1,n-1}$ complete bipartite graph, as seen at bottom in Figure 7. So $\#(G - v_1 - w_2) = E(m-1, n-1)$.

Proposition 9 then gives us

$$\#(G) = \frac{1}{2}(E_1(m,n) - E(m-1,n) - E_1(m,n-1) - E(m-1,n-1)).$$

Case 2: In this case, the 2 removed edges in G are not incident to the same vertex in V_1 but are incident to the same vertex in V_2. Note that this is the same case as case 1 with m and n interchanged.

Case 3: In this case, the 2 removed edges in G are not incident to the same vertex in V_1 and are not incident to the same vertex in V_2. An example of such a graph G can be seen at right in Figure 8. The edges (v_1, w_1) and (v_m, v_n) are missing. Edge (v_1, w_1) is incident to a different vertex in V_1 and a different vertex in V_2 than edge (v_m, w_n). The graph of G' can be seen at left in Figure 8.

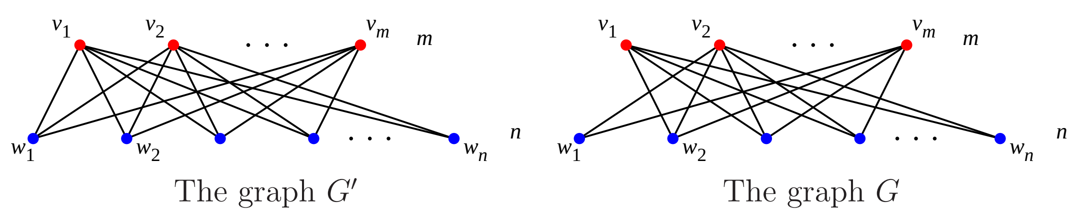
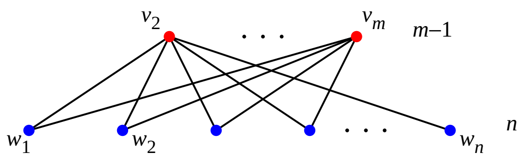


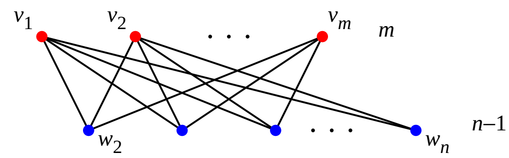
Figure 8: Case 3: Removing two edges incident to the same vertex.

Since G' is a $K_{m,n}$ complete bipartite graph missing one edge, $\#(G') = E_1(m,n)$.

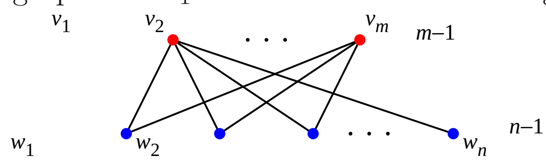
When we remove the vertex v_1 from G, the result is a $K_{m-1,n}$ complete bipartite graph missing one edge, as seen top left in Figure 9, so $\#(G - v_1) = E_1(m-1, n)$.



The graph $G - v_1$



The graph $G - w_1$



The graph $G - \{v_1, w_1\}$

Figure 9: Case 3: Removing vertices from G.

When we remove the vertex w_1 from G, the result is a $K_{m,n-1}$ complete bipartite graph missing one edge, as seen top right in Figure 9, so $\#(G - w_1) = E_1(m, n-1)$.

When we remove vertices v_1 and w_1 from G, the result is a $K_{m-1,n-1}$ complete bipartite graph missing one edge, as seen at bottom in Figure 9, so $\#(G - v_1 - w_1) = E_1(m - 1, n-1)$.

Proposition 9 then gives us

$$\#(G) = \frac{1}{2}(E_1(m,n) - E_1(m-1,n) - E_1(m,n-1) - E_1(m-1,n-1)).$$

□

As a result of Theorem 11, there are three families of integer sequences given by $E_{2_i}(m,n)$ with m fixed and n varying. Some of these sequences are presented in Table 2. The first entry is given in sequences A024023 and A103453 in the OEIS. Once again, this graph theoretic approach to these sequences provides a new perspective from which to view these sequences.

$E_{2_1}(2,n)$	$3^{n-2} - 1$
$E_{2_1}(3,n)$ (A335612)	$9 \cdot 7^{n-2} - 11 \cdot 3^{n-2} + 2$
$E_{2_1}(4,n)$ (A335613)	$49 \cdot 15^{n-2} - 76 \cdot 7^{n-2} + 10 \cdot 3^{n-1} - 3$
$E_{2_1}(5,n)$ (A337416)	$225 \cdot 31^{n-2} - 421 \cdot 15^{n-2} + 250 \cdot 7^{n-2} - 58 \cdot 3^{n-2} + 4$
$E_{2_1}(6,n)$ (A337417)	$961 \cdot 63^{n-2} - 2086 \cdot 31^{n-2} + 1615 \cdot 15^{n-2} - 580 \cdot 7^{n-2} + 95 \cdot 3^{n-2} - 5$

$E_{2_2}(3,n)$ (A337418)	$7^{n-1} - 2 \cdot 3^{n-1} + 1$
$E_{2_2}(4,n)$ (A340173)	$3 \cdot 15^{n-1} - 8 \cdot 7^{n-1} + 7 \cdot 3^{n-1} - 2$
$E_{2_2}(5,n)$ (A340174)	$7 \cdot 31^{n-1} - 23 \cdot 15^{n-1} + 4 \cdot 7^{n} - 5 \cdot 3^{n} + 3$
$E_{2_2}(6,n)$ (A340175)	$15 \cdot 63^{n-1} - 58 \cdot 31^{n-1} + 89 \cdot 15^{n-1} - 68 \cdot 7^{n-1} + 26 \cdot 3^{n-1} - 4$
$E_{2_3}(2,n)$	3^{n-2}
$E_{2_3}(3,n)$ (A340199)	$9 \cdot 7^{n-2} - 7 \cdot 3^{n-2} + 1$
$E_{2_3}(4,n)$ (A340200)	$49 \cdot 15^{n-2} - 60 \cdot 7^{n-2} + 22 \cdot 3^{n-2} - 2$
$E_{2_3}(5,n)$ (A340201)	$225 \cdot 31^{n-2} - 357 \cdot 15^{n-2} + 202 \cdot 7^{n-2} - 46 \cdot 3^{n-2} + 3$
$E_{2_3}(6,n)$ (A340897)	$961 \cdot 63^{n-2} - 1830 \cdot 31^{n-2} + 1359 \cdot 15^{n-2} - 484 \cdot 7^{n-2} + 79 \cdot 3^{n-2} - 4$

Table 2: Sequences $(E_{2_k}(m,n))_{n \geq m}$.

As an example of interpreting edge covers of complete bipartite graphs from a geometric perspective, consider the configurations $[A,B]$ (left) and $[A',B']$ (right) as shown in Figure 10, where $A = A' = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ and $B = B' = \{b_1, b_2\}$. Both configurations have corresponding $K_{2,6}$ graphs missing two edges. We have $\#([A,B]) = E_{2_3}(2,6) = 81$ and $\#([A',B']) = E_{2_1}(2,6) = 80$, so there are 81 distinct sets that lie on the line segment $\overline{AB}$ at each fixed distance from A, and 80 distinct sets that lie on the line segment $\overline{A'B'}$ at each distance from A'. These findings correspond to the sequence 3^{n-2} and the fifth term of A024023 in the OEIS, respectively.

8 Edge covers of $K_{m,n}$ minus three edges

We will take one more step in our calculations of number of edge covers and determine the number of edge covers of a $K_{m,n}$ complete bipartite graph missing 3 edges. In this case, there are 6 different possibilities for the removal of edges as indicated in Theorem 12. The functions E_{2_1}, E_{2_2}, and E_{2_3} are defined in Theorem 11.

Theorem 12. *Let G be a $K_{m,n}$ complete bipartite graph missing three edges. Let the vertex set V of G be partitioned into two parts V_1 and V_2 where $|V_1| = m$ and $|V_2| = n$.*

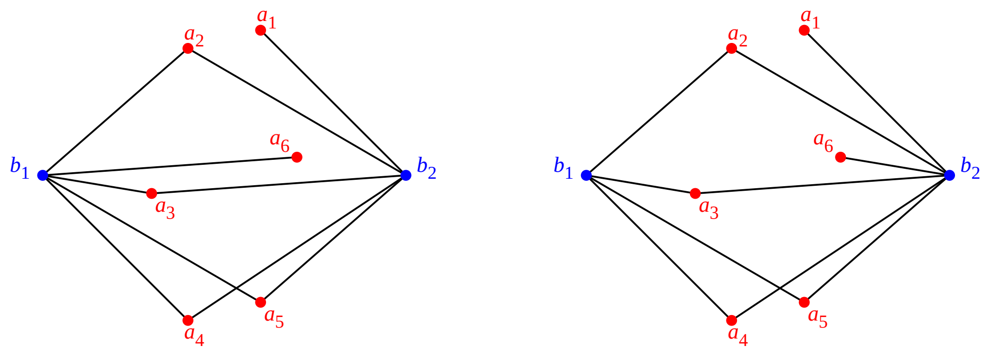


Figure 10: Two complete configurations missing two edges.

1. *The number of edge covers of G when all 3 removed edges are incident to the same vertex in V_1 but are incident to different vertices in V_2 is $E_{3_1}(m,n)$ where*

$$E_{3_1}(m,n) = \#(G) = \frac{1}{2}(E_{2_1}(m,n) - E(m-1,n) - E_{2_1}(m,n-1) - E(m-1,n-1)). \quad (2)$$

2. *The number of edge covers of G when all 3 removed edges are incident to different vertices in V_1 and none of the removed edges are incident to the same vertex in V_2 is $E_{3_2}(m,n)$ where*

$$E_{3_2}(m,n) = \#(G) = \frac{1}{2}(E_{2_3}(m,n) - E_{2_3}(m-1,n) - E_{2_3}(m,n-1) - E_{2_3}(m-1,n-1)). \quad (3)$$

3. *The number of edge covers of G when all 3 removed edges are incident to different vertices in V_1 but all 3 removed edges are incident to the same vertex in V_2 (note that this equation returns a value of 0 in the $K_{3,n}$ case) is $E_{3_3}(m,n)$ where*

$$E_{3_3}(m,n) = \#(G) = \frac{1}{2}(E_{2_2}(m,n) - E_{2_2}(m-1,n) - E(m,n-1) - E(m-1,n-1)). \quad (4)$$

4. *The number of edge covers of G when exactly 2 of the removed edges are incident to the same vertex in V_1 but none of the removed edges are incident to the same vertex in V_2 is $E_{3_4}(m,n)$ where*

$$E_{3_4}(m,n) = \#(G) = \frac{1}{2}(E_{2_3}(m,n) - E_1(m-1,n) - E_{2_3}(m,n-1) - E_1(m-1,n-1)). \quad (5)$$

5. *The number of edge covers of G when all 3 removed edges are incident to different vertices in V_1 but exactly 2 removed edges are incident to the same vertex in V_2 is $E_{3_5}(m,n)$ where*

$$E_{3_5}(m,n) = \#(G) = \frac{1}{2}(E_{2_3}(m,n) - E_{2_3}(m-1,n) - E_1(m,n-1) - E_1(m-1,n-1)). \quad (6)$$

6. *The number of edge covers of G when exactly 2 removed edges are incident to the same vertex in V_1 and exactly 2 removed edges are incident to the same vertex in V_2 is $E_{3_6}(m,n)$ where*

$$E_{3_6}(m,n) = \#(G) = \frac{1}{2}(E_{2_3}(m,n) - E_1(m-1,n) - E_1(m,n-1) - E(m-1,n-1)). \quad (7)$$

Proof. Assume the graph G' is a $K_{m,n}$ complete bipartite graph missing two edges. Let the graph G be a graph obtained from G' by removing an edge. Let the vertex set V of G be partitioned into two parts V_1 and V_2 where $|V_1| = m$ and $|V_2| = n$. We consider six different cases based on the incidence of the removed edges to certain vertices. The possible cases are those listed in Theorem 12.

Case 1: In this case, all 3 removed edges are incident to the same vertex in V_1. An example of a graph of G', with edges (v_1, w_2) and (v_1, w_3) missing, and a corresponding graph G missing the additional edge (v_1, w_1) can be seen in Figure 11. All 3 edges are incident to v_1 in V_1 but are incident to different vertices in V_2.

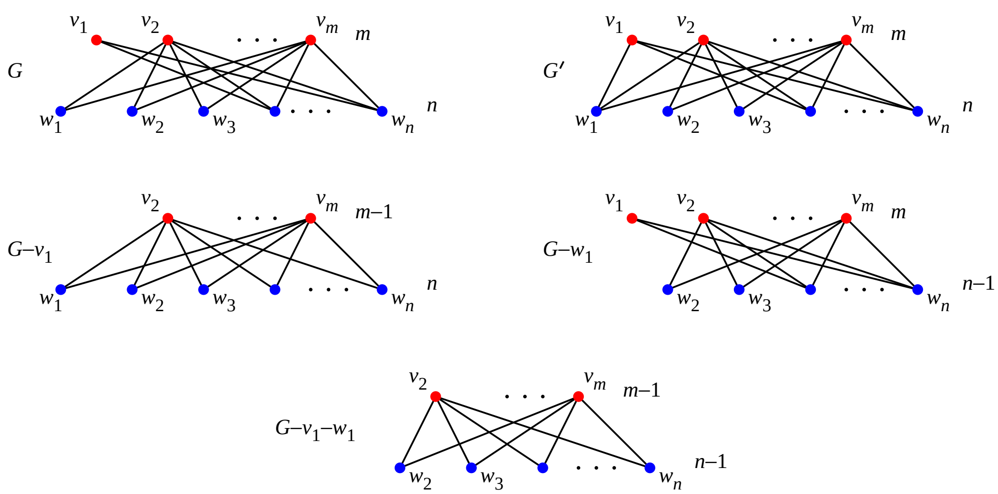


Figure 11: The graphs for case 1.

Since G' is a $K_{m,n}$ complete bipartite graph missing two edges that are incident to the same vertex in V_1 but different vertices in V_2, we have that $\#(G') = E_{2_1}(m,n)$.

When we remove the vertex v_1 from G, the result is a $K_{m-1,n}$ complete bipartite graph, as seen in Figure 11. So $\#(G - v_1) = E(m-1,n)$.

When we remove the vertex w_1 from G, the result is a $K_{m,n-1}$ complete bipartite graph missing two edges that are incident to the same vertex in V_1 but incident to different vertices in V_2, as seen in Figure 11. So $\#(G - w_1) = E_{2_1}(m, n-1)$.

When we remove vertices v_1 and w_1 from G, the result is a $K_{m-1,n-1}$ complete bipartite graph, as seen in Figure 11. This means we can use E to calculate $\#(G - v_1 - w_1)$, so $\#(G - v_1 - w_1) = E(m-1, n-1)$.

Proposition 9 then gives us

$$\#(G) = \frac{1}{2}(E_{2_1}(m,n) - E(m-1,n) - E_{2_1}(m,n-1) - E(m-1,n-1)).$$

Case 2: In this case, all 3 edges are incident to different vertices in V_1 and none of the vertices are incident to the same vertex in V_2. An example of a graph G' missing vertices (v_2, w_2) and (v_m, w_n) and a corresponding G missing the additional edge (v_1, w_1) can be seen in Figure 12. None of the edges are incident to the same vertices in V_1 or in V_2.

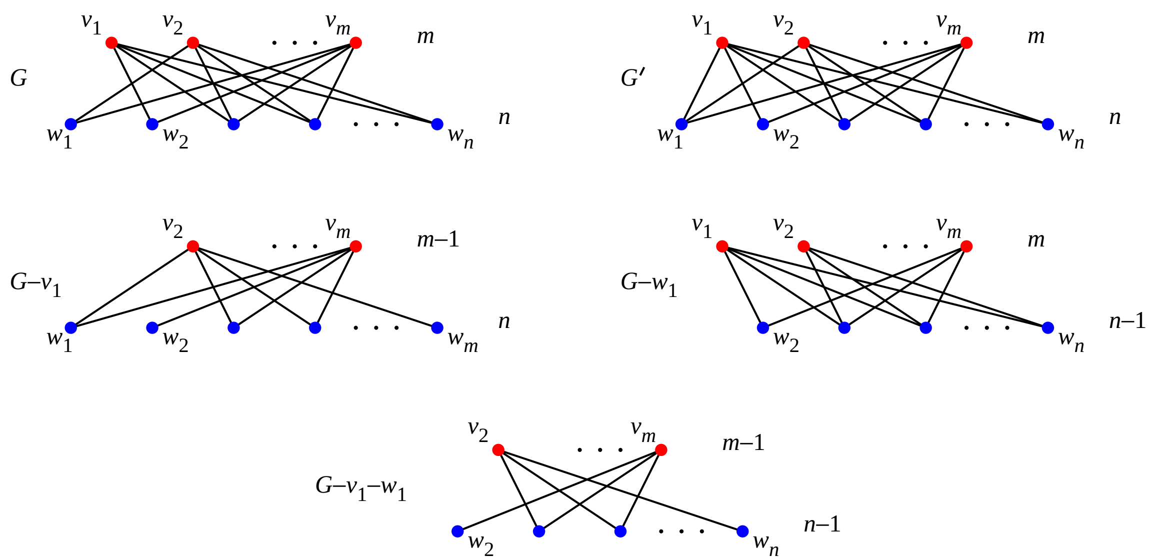


Figure 12: The graphs for case 2.

Since G' is a $K_{m,n}$ complete bipartite graph missing two edges that are incident to different vertices in V_1 and V_2, as seen in Figure 12, we can use E_{2_3} to calculate $\#(G')$, so $\#(G') = E_{2_3}(m,n)$.

When we remove the vertex v_1 from G, the result is a $K_{m-1,n}$ complete bipartite graph missing two edges that are incident to different vertices in V_1 and V_2, as seen in Figure 12. This means we can use E_{2_3} to calculate $\#(G - v_1)$, so $\#(G - v_1) = E_{2_3}(m-1, n)$.

When we remove the vertex w_1 from G, the result is a $K_{m,n-1}$ complete bipartite graph missing two edges that are incident to different vertices in V_1 and V_2, as seen in Figure 12. This means we can use E_{2_3} to calculate $\#(G - w_1)$, so $\#(G - w_1) = E_{2_3}(m, n-1)$.

When we remove vertices v_1 and w_1 from G, the result is a $K_{m-1,n-1}$ complete bipartite graph missing two edges that are incident to different vertices in V_1 and V_2, as seen in Figure 12. This means we can use E_{2_3} to calculate $\#(G-v_1-w_1)$, so $\#(G-v_1-w_1) = E_{2_3}(m-1, n-1)$.

Proposition 9 then gives us

$$\#(G) = \frac{1}{2}(E_{2_3}(m,n) - E_{2_3}(m-1,n) - E_{2_3}(m,n-1) - E_{2_3}(m-1,n-1)).$$

Case 3: In this case, all 3 edges removed are incident to different vertices in V_1 but all 3 removed edges are incident to the same vertex in V_2. An example of a G' missing edges (v_2, w_1) and (v_3, w_1) along with a corresponding G missing the additional edge (v_1, w_1) can be seen in Figure 13. All 3 edges are incident to different vertices in V_1 but are incident to vertex w_1 in V_2.

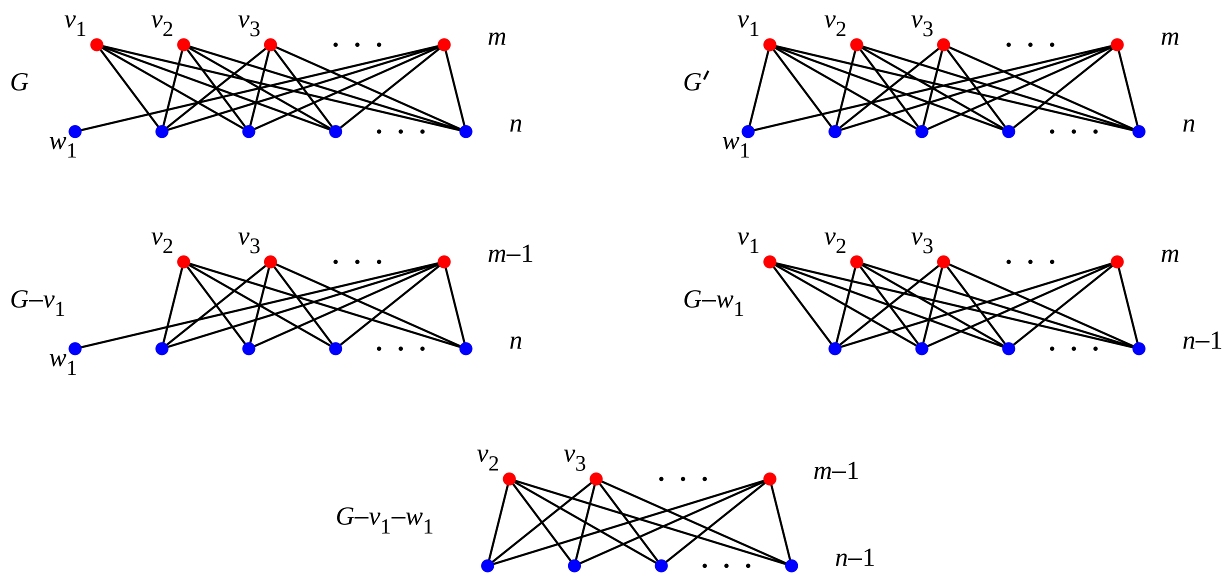


Figure 13: The graphs for case 3.

Since G' is a $K_{m,n}$ complete bipartite graph missing two edges that are incident to different vertices in V_1 but the same vertex in V_2, as seen in Figure 13, we can use E_{2_2} to calculate $\#(G')$, so $\#(G') = E_{2_2}(m,n)$.

When we remove the vertex v_1 from G, the result is a $K_{m-1,n}$ complete bipartite graph missing two edges that are incident to different vertices in V_1 but the same vertex in V_2, as seen in Figure 13. This means we can use E_{2_2} to calculate $\#(G - v_1)$, so $\#(G - v_1) = E_{2_2}(m-1, n)$.

When we remove the vertex w_1 from G, the result is a $K_{m,n-1}$ complete bipartite graph, as seen in Figure 13. This means we can use E to calculate $\#(G - w_1)$, so $\#(G - w_1) = E(m, n - 1)$.

When we remove vertices v_1 and w_1 from G, the result is a $K_{m-1,n-1}$ complete bipartite graph, as seen in Figure 13. This means we can use E to calculate $\#(G - v_1 - w_1)$, so $\#(G - v_1 - w_1) = E(m - 1, n - 1)$.

Proposition 9 then gives us

$$\#(G) = \frac{1}{2}(E_{2_2}(m, n) - E_{2_2}(m - 1, n) - E(m, n - 1) - E(m - 1, n - 1)).$$

Case 4: In this case, exactly 2 of the removed edges are incident to the same vertex in V_1, but none of the removed edges are incident to the same vertex in V_2. An example of a graph G' missing edges (v_1, w_2) and (v_2, w_3) along with a corresponding graph G missing the additional edge (v_1, w_1) can be seen in Figure 14. There are 2 edges incident to v_1 in V_1 while one edge is not. All of the edges are incident to different vertices in V_2.

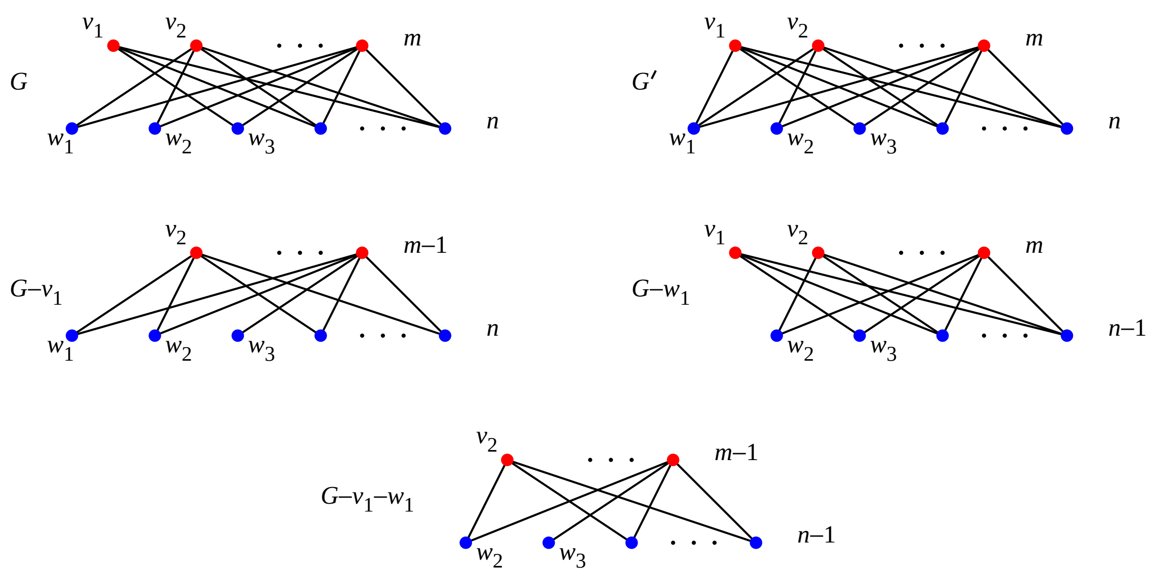


Figure 14: The graphs for case 4.

Since G' is a $K_{m,n}$ complete bipartite graph missing two edges that are incident to different vertices in V_1 and V_2, as seen in Figure 14, we can use E_{2_3} to calculate $\#(G')$, so $\#(G') = E_{2_3}(m, n)$.

When we remove the vertex v_1 from G, the result is a $K_{m-1,n}$ complete bipartite graph missing one edge, as seen in Figure 14. This means we can use E_1 to calculate $\#(G - v_1)$, so $\#(G - v_1) = E_1(m - 1, n)$.

When we remove the vertex w_1 from G, the result is a $K_{m,n-1}$ complete bipartite graph missing two edges that are incident to different vertices in V_1 and different vertices in V_2, as seen in Figure 14. This means we can use E_{2_3} to calculate $\#(G - w_1)$, so $\#(G - w_1) = E_{2_3}(m, n - 1)$.

When we remove vertices v_1 and w_1 from G, the result is a $K_{m-1,n-1}$ complete bipartite graph missing one edge, as seen in Figure 14. This means we can use E_1 to calculate $\#(G - v_1 - w_1)$, so $\#(G - v_1 - v_2) = E_1(m - 1, n - 1)$.

Proposition 9 then gives us

$$\#(G) = \frac{1}{2}(E_{2_3}(m, n) - E_1(m - 1, n) - E_{2_3}(m, n - 1) - E_1(m - 1, n - 1)).$$

Case 5: In this case, all 3 removed edges are incident to different vertices in V_1, but exactly 2 removed edges are incident to the same vertex in V_2. An example of a graph G' missing edges (v_2, w_1) and (v_3, w_3) along with a corresponding graph G missing the additional edge (v_1, w_1) can be seen in Figure 15. All 3 edges are incident to different vertices in V_1, but 2 of the edges are incident to w_1 in V_2.

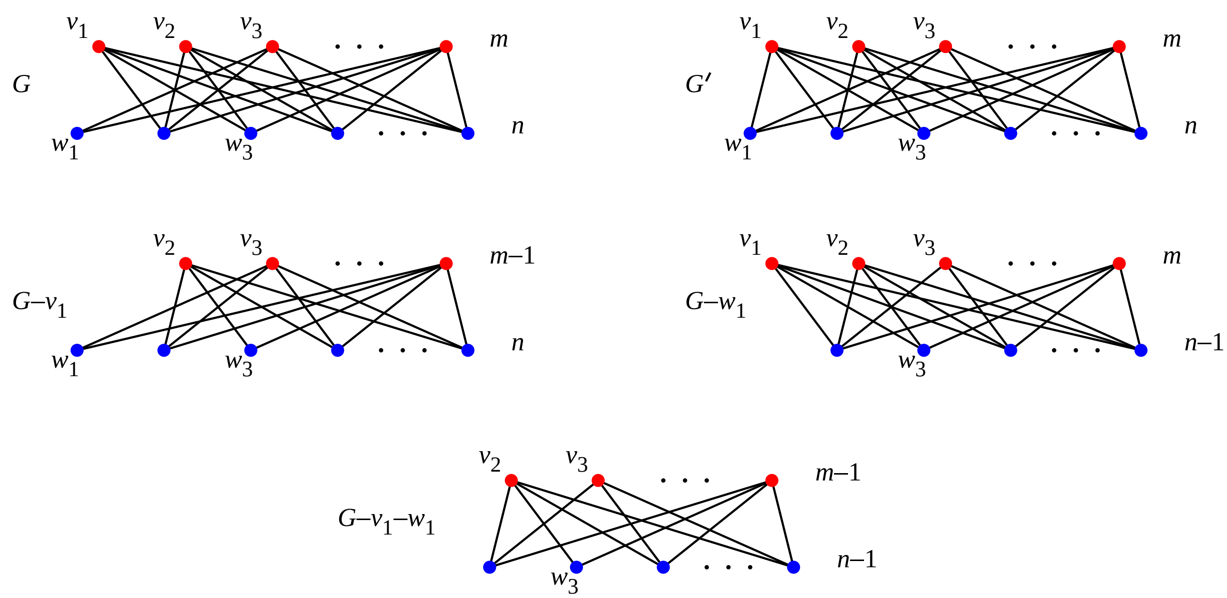


Figure 15: The graphs for case 5.

Since G' is a $K_{m,n}$ complete bipartite graph missing two edges that are incident to different vertices in V_1 and V_2, as seen in Figure 15, we can use E_{2_3} to calculate $\#(G')$, so $\#(G') = E_{2_3}(m, n)$.

When we remove the vertex v_1 from G, the result is a $K_{m-1,n}$ complete bipartite graph missing two edges that are incident to different vertices in V_1 and V_2, as seen in Figure 15. This means we can use E_{2_3} to calculate $\#(G - v_1)$, so $\#(G - v_1) = E_{2_3}(m - 1, n)$.

When we remove the vertex w_1 from G, the result is a $K_{m,n-1}$ complete bipartite graph missing one edge, as seen in Figure 15. This means we can use E_1 to calculate $\#(G - w_1)$, so $\#(G - w_1) = E_1(m, n - 1)$.

When we remove vertices v_1 and w_1 from G, the result is a $K_{m-1,n-1}$ complete bipartite graph missing one edge, as seen in Figure 15. This means we can use E_1 to calculate $\#(G - v_1 - w_1)$, so $\#(G - v_1 - w_1) = E_1(m - 1, n - 1)$.

Proposition 9 then gives us

$$\#(G) = \frac{1}{2}(E_{2_3}(m, n) - E_{2_3}(m - 1, n) - E_1(m, n - 1) - E_1(m - 1, n - 1)).$$

Case 6: In this case, exactly 2 removed edges are incident to the same vertex in V_1 and exactly 2 removed edges are incident to the same vertex in V_2. An example of a graph G' missing edges (v_2, w_1) and (v_1, w_2) along with a corresponding graph G missing the additional edge (v_1, w_1) can be seen in Figure 16. Exactly two edges are incident to v_1 in V_1, and exactly two edges are incident to w_1 in V_2.

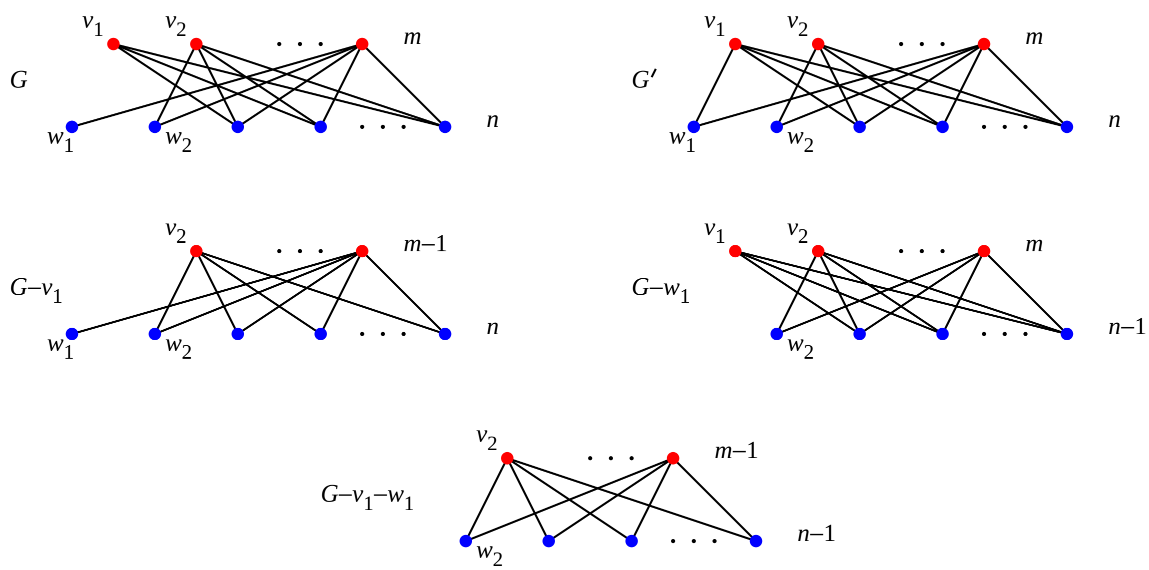


Figure 16: The graphs for case 6.

Since G' is a $K_{m,n}$ complete bipartite graph missing two edges that are incident to different vertices in V_1 and V_2, as seen in Figure 16, we can use E_{2_3} to calculate $\#(G')$, so $\#(G') = E_{2_3}(m, n)$.

When we remove the vertex v_1 from G, the result is a $K_{m-1,n}$ complete bipartite graph missing one edge, as seen in Figure 16. This means we can use E_1 to calculate $\#(G - v_1)$, so $\#(G - v_1) = E_1(m - 1, n)$.

When we remove the vertex w_1 from G, the result is a $K_{m,n-1}$ complete bipartite graph missing one edge, as seen in Figure 16. This means we can use E_1 to calculate $\#(G - w_1)$, so $\#(G - w_1) = E_1(m, n-1)$.

When we remove vertices v_1 and w_1 from G, the result is a $K_{m-1,n-1}$ complete bipartite graph, as seen in Figure 16. This means we can use E to calculate $\#(G - v_1 - w_1)$, so $\#(G - v_1 - w_1) = E(m-1, n-1)$.

Proposition 9 then gives us

$$\#(G) = \frac{1}{2}(E_{2_3}(m,n) - E_1(m-1,n) - E_1(m,n-1) - E(m-1,n-1)).$$

□

A few of the sequences given by Theorem 12 are shown in Table 3. The first entry is given in sequences A024023 and A103453—the remainder had not been previously recorded in the OEIS.

Theorem 12 once again informs us about the number of sets that lie on certain line segments in $\mathcal{H}(\mathbb{R}^N)$ at a fixed distance from one of the endpoints. For example, if $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ and $B = \{b_1, b_2\}$, then $\#([A, B]) = E_{3_1}(2, 6) = 26$ when $[A, B]$ is the configuration shown at left in Figure 17, and $\#([A, B]) = E_{3_4}(2, 6) = 27$ when $[A, B]$ is the configuration shown at right in Figure 17.

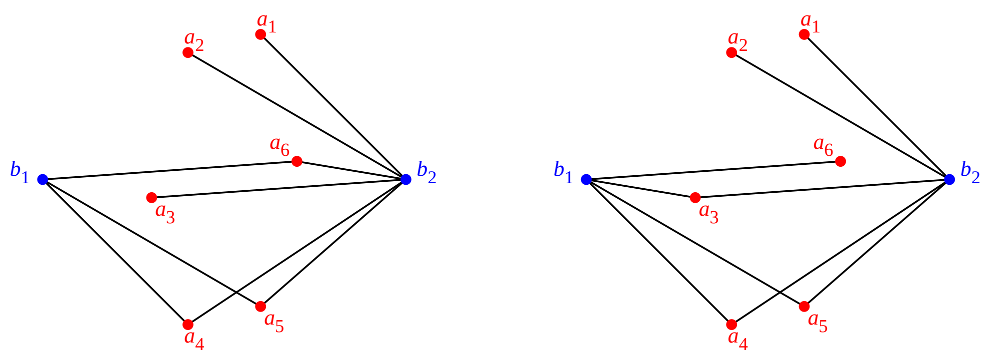


Figure 17: Two complete configurations missing three edges.

9 Conclusion

All of the sequences $(E_1(k, n))$ for k from 3 to 6, $(E_{2_i}(k, n))$ for i from 1 to 3 and k from 3 to 6, and $(E_{3_i}(k, n))$ for i from 1 to 6 and k from 4 to 6 in this investigation were previously uncatalogued integer sequences in the OEIS. It is likely that line segments in $\mathcal{H}(\mathbb{R}^N)$ (and edge covers of graphs) can provide many more examples of new integer sequences.

$E_{3_1}(2,n)$	$3^{n-3}-1$
$E_{3_1}(3,n)$ (A340898)	$27\cdot 7^{n-3}-29\cdot 3^{n-3}+2$
$E_{3_1}(4,n)$ (A340899)	$343\cdot 15^{n-3}-424\cdot 7^{n-3}+28\cdot 3^{n-2}-3$
$E_{3_1}(5,n)$ (A342580)	$3375\cdot 31^{n-3}-4747\cdot 15^{n-3}-166\cdot 3^{n-3}+1534\cdot 7^{n-3}+4$
$E_{3_1}(6,n)$ (A342796)	$29791\cdot 63^{n-3}-46666\cdot 31^{n-3}+20305\cdot 15^{n-3}-3700\cdot 7^{n-3}+275\cdot 3^{n-3}-5$
$E_{3_2}(3,n)$ (A342850)	$27\cdot 7^{n-3}-3^{n-1}$
$E_{3_2}(4,n)$ (A340403)	$343\cdot 15^{n-3}-216\cdot 7^{n-3}+4\cdot 3^{n-1}-1$
$E_{3_2}(5,n)$ (A340404)	$3375\cdot 31^{n-3}-2891\cdot 15^{n-3}+846\cdot 7^{n-3}-10\cdot 3^{n-1}+2$
$E_{3_2}(6,n)$ (A340405)	$29791\cdot 63^{n-3}-31050\cdot 31^{n-3}+12369\cdot 15^{n-3}-2260\cdot 7^{n-3}+19\cdot 3^{n-1}-3$
$E_{3_3}(4,n)$ (A340433)	$15^{n-1}-3\cdot 7^{n-1}+3^{n}-1$
$E_{3_3}(5,n)$ (A340434)	$3\cdot 31^{n-1}-11\cdot 15^{n-1}+15\cdot 7^{n-1}-3^{n-1}+2$
$E_{3_3}(6,n)$ (A340435)	$7\cdot 63^{n-1}-30\cdot 31^{n-1}+51\cdot 15^{n-1}-43\cdot 7^{n-1}+6\cdot 3^{n}-3$
$E_{3_4}(3,n)$ (A340436)	$27\cdot 7^{n-3}-13\cdot 3^{n-3}+1$
$E_{3_4}(4,n)$ (A340437)	$343\cdot 15^{n-3}-264\cdot 7^{n-3}+52\cdot 3^{n-3}-2$
$E_{3_4}(5,n)$ (A340438)	$3375\cdot 31^{n-3}-3339\cdot 15^{n-3}+1054\cdot 7^{n-3}-118\cdot 3^{n-3}+3$
$E_{3_4}(6,n)$ (A341551)	$29791\cdot 63^{n-3}-34890\cdot 31^{n-3}+14673\cdot 15^{n-3}-2740\cdot 7^{n-3}+211\cdot 3^{n-3}-4$
$E_{3_5}(3,n)$ (A341552)	$3\cdot 7^{n-2}-2\cdot 3^{n-2}$
$E_{3_5}(4,n)$ (A341553)	$21\cdot 15^{n-2}-4\cdot 7^{n-1}+11\cdot 3^{n-2}-1$
$E_{3_5}(5,n)$ (A342327)	$105\cdot 31^{n-2}-185\cdot 15^{n-2}+116\cdot 7^{n-2}-29\cdot 3^{n-2}+2$
$E_{3_5}(6,n)$ (A342328)	$465\cdot 63^{n-2}-982\cdot 31^{n-2}+807\cdot 15^{n-2}-316\cdot 7^{n-2}+56\cdot 3^{n-2}-3$
$E_{3_6}(3,n)$ (A343372)	$3\cdot 7^{n-2}-4\cdot 3^{n-2}+1$
$E_{3_6}(4,n)$ (A343373)	$21\cdot 15^{n-2}-36\cdot 7^{n-2}+17\cdot 3^{n-2}-2$
$E_{3_6}(5,n)$ (A343374)	$105\cdot 31^{n-2}-217\cdot 15^{n-2}+148\cdot 7^{n-2}-13\cdot 3^{n-1}+3$
$E_{3_6}(6,n)$ (A343800)	$465\cdot 63^{n-2}-1110\cdot 31^{n-2}+967\cdot 15^{n-2}-388\cdot 7^{n-2}+70\cdot 3^{n-2}-4$

Table 3: Sequences $(E_{3_k}(m,n))_{n\geq n}$.

10 Acknowledgments

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