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Short Proofs, Generalizations, and Applications of Certain Identities Concerning Multiple Dirichlet Series

László Tóth
Department of Mathematics
University of Pécs
Ifjúság útja 6
7624 Pécs
Hungary
ltoth@gamma.ttk.pte.hu

Abstract

We present other proofs, generalizations, and analogues of the identities concerning multiple Dirichlet series by Tahmi and Derbal (2022). As applications, we obtain asymptotic formulas with remainder terms for certain related sums.

1 Introduction

Recently, Tahmi and Derbal [4] obtained certain identities for the multiple Dirichlet series

$$\sum_{\substack{n_1,\ldots,n_r=1\\ \gcd(n_1,\ldots,n_r)=1}}^{\infty} \frac{f(n_1)\cdots f(n_r)}{n_1^{s_1}\cdots n_r^{s_r}},$$

with $r \geq 2$ an integer, in the cases where $f : \mathbb{N} := \{1, 2, \ldots\} \to \mathbb{C}$ is a completely multiplicative arithmetic function or the Dirichlet convolution of two completely multiplicative functions.

As direct corollaries of their results, they mentioned, among others, the identities

$$\sum_{\substack{n_1,\ldots,n_r=1\\ \gcd(n_1,\ldots,n_r)=1}}^{\infty} \frac{1}{n_1^{s_1}\cdots n_r^{s_r}} = \frac{\zeta(s_1)\cdots\zeta(s_r)}{\zeta(s_1+\cdots+s_r)},$$

with $s_i \in \mathbb{C}$, $\Re s_i > 1$ $(1 \le i \le r)$, concerning the Dirichlet series of the characteristic function of the set of points in $\mathbb{N}^r$, which are visible from the origin (also see Apostol [1, p. 248]), and

$$\sum_{\substack{n_1,n_2=1\\ \gcd(n_1,n_2)=1}}^{\infty} \frac{\tau(n_1)\tau(n_2)}{n_1^{s_1}n_2^{s_2}} = \zeta^2(s_1)\zeta^2(s_2)\prod_p \Big(1 - \frac{4}{p^{s_1+s_2}} + \frac{2}{p^{2s_1+s_2}} + \frac{2}{p^{s_1+2s_2}} - \frac{1}{p^{2s_1+2s_2}}\Big), \quad (1)$$

with $s_i \in \mathbb{C}$, $\Re s_i > 1$ $(1 \le i \le 2)$, where $\tau(n) = \sum_{d\mid n} 1$ is the divisor function.

The proofs given in [4] are by using Euler product expansions of the Dirichlet series of some appropriate multiplicative functions of one variable.

In this paper we present other proofs and generalizations of the results by Tahmi and Derbal [4] by considering Euler product expansions of some multiple Dirichlet series

$$\sum_{n_1,\ldots,n_r=1}^{\infty} \frac{F(n_1,\ldots,n_r)}{n_1^{s_1}\cdots n_r^{s_r}}$$

of multiplicative arithmetic functions $F : \mathbb{N}^r \to \mathbb{C}$ of r variables. Namely, we investigate the functions

$$F(n_1,\ldots,n_r) = \begin{cases} f_1(n_1)\cdots f_r(n_r), & \text{if } \gcd(n_1,\ldots,n_r)=1;\\ 0, & \text{otherwise,} \end{cases} \quad (2)$$

where each of the functions $f_1,\ldots,f_r : \mathbb{N} \to \mathbb{C}$ is the Dirichlet convolution of t $(t \ge 1)$ completely multiplicative functions. Note that if $f_1,\ldots,f_r$ are multiplicative, then F given by (2) is multiplicative, viewed as a function of r variables.

We also make more explicit the formula of [4, Th. 3.2], as applications we obtain asymptotic formulas with remainder terms for certain sums

$$\sum_{\substack{n_1,\ldots,n_r\le x\\ \gcd(n_1,\ldots,n_r)=1}} f_1(n_1)\cdots f_r(n_r),$$

and also derive similar results where the condition $\gcd(n_1,\ldots,n_r)=1$ is replaced by the condition that $n_1,\ldots,n_r$ are pairwise relatively prime. Some basic properties of multiplicative arithmetic functions of r variables are reviewed in Section 2.1. Certain polynomial identities needed in the proofs are given in Section 2.2. Our main results and their proofs on the Dirichlet series are included in Section 3, and some related asymptotic formulas are presented in Section 4. All the identities regarding Dirichlet series and Euler products are considered formally or in the case of absolute convergence.

2 Preliminaries

2.1 Arithmetic functions of several variables

Let $F : \mathbb{N}^r \to \mathbb{C}$ be an arbitrary arithmetic function of r variables ($r \ge 1$). Its Dirichlet series is given by

$$D(F, s_1, \ldots, s_r) := \sum_{n_1,\ldots,n_r=1}^{\infty} \frac{F(n_1, \ldots, n_r)}{n_1^{s_1} \cdots n_r^{s_r}}.$$

The Dirichlet convolution of the functions $F, G : \mathbb{N}^r \to \mathbb{C}$ is defined by

$$(F * G)(n_1, \ldots, n_r) = \sum_{d_1 \mid n_1, \ldots, d_r \mid n_r} F(d_1, \ldots, d_r) G(n_1/d_1, \ldots, n_r/d_r).$$

If $D(F, s_1, \ldots, s_r)$ and $D(G, s_1, \ldots, s_r)$, with $s_1, \ldots, s_r \in \mathbb{C}$, are absolutely convergent, then $D(F * G; s_1, \ldots, s_r)$ is also absolutely convergent and

$$D(F * G, s_1, \ldots, s_r) = D(F, s_1, \ldots, s_r) D(G, s_1, \ldots, s_r).$$

A nonzero arithmetic function $F : \mathbb{N}^r \to \mathbb{C}$ is said to be *multiplicative* if

$$F(m_1 n_1, \ldots, m_r n_r) = F(m_1, \ldots, m_r) F(n_1, \ldots, n_r)$$

holds for every $m_1, \ldots, m_r, n_1, \ldots, n_r \in \mathbb{N}$ such that $\gcd(m_1 \cdots m_r, n_1 \cdots n_r) = 1$. If F is multiplicative, then it is determined by the values $F(p^{a_1}, \ldots, p^{a_r})$, where p is prime and $a_1, \ldots, a_r \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. More exactly, $F(1, \ldots, 1) = 1$ and for every $n_1, \ldots, n_r \in \mathbb{N}$,

$$F(n_1, \ldots, n_r) = \prod_p F\left(p^{a_p(n_1)}, \ldots, p^{a_p(n_r)}\right),$$

by using the notation $n = \prod_p p^{a_p(n)}$ for the prime power factorization of $n \in \mathbb{N}$, the product being over the primes p, where all but a finite number of the exponents $a_p(n)$ are zero.

Examples of multiplicative functions of r variables are the GCD and LCM functions $\gcd(n_1, \ldots, n_r)$, $\operatorname{lcm}(n_1, \ldots, n_r)$ and the characteristic functions

$$\varrho(n_1, \ldots, n_r) = \begin{cases} 1, & \text{if } \gcd(n_1, \ldots, n_r) = 1; \\ 0, & \text{otherwise,} \end{cases}$$

$$\vartheta(n_1, \ldots, n_r) = \begin{cases} 1, & \text{if } \gcd(n_i, n_j) = 1 \text{ for every } 1 \le i < j \le r; \\ 0, & \text{otherwise.} \end{cases}$$

If $F, G : \mathbb{N}^r \to \mathbb{C}$ are multiplicative, then their Dirichlet convolution $F * G$ is also multiplicative. If F is multiplicative, then its Dirichlet series can be expanded into a (formal) Euler product, that is,

$$D(F, s_1, \ldots, s_r) = \prod_p \sum_{a_1,\ldots,a_r=0}^{\infty} \frac{f(p^{a_1}, \ldots, p^{a_r})}{p^{a_1 s_1 + \cdots + a_r s_r}}, \tag{3}$$

the product being over the primes p. More exactly, if F is multiplicative, then the series $D(F, s_1, \ldots, s_r)$ with $s_1, \ldots, s_r \in \mathbb{C}$ is absolutely convergent if and only if

$$\sum_{p} \sum_{\substack{a_1,\ldots,a_r=0 \\ a_1+\cdots+a_r\geq 1}}^{\infty} \frac{|f(p^{a_1}, \ldots, p^{a_r})|}{p^{a_1\Re s_1+\cdots+a_r\Re s_r}} < \infty$$

and in this case equality (3) holds.

See, e.g., Delange [2] and the survey by the author [5] for these and some related results on arithmetic functions of r variables. If $r = 1$, i.e., in the case of functions of a single variable we recover some familiar properties.

2.2 Some polynomial identities

Let $e_j(x_1, \ldots, x_t) = \sum_{1\leq i_1<\cdots<i_j\leq t} x_{i_1} \cdots x_{i_j}$ denote the elementary symmetric polynomials in $x_1, \ldots, x_t$ of degree j $(1 \leq j \leq t)$. We will use the polynomial identity

$$P(x) := \prod_{j=1}^{t}(x - x_j) = x^t + \sum_{j=1}^{t}(-1)^j e_j(x_1, \ldots, x_t)x^{t-j}. \tag{4}$$

Taking derivatives gives

$$P'(x) = \sum_{j=1}^{t} \prod_{\substack{k=1 \\ k\neq j}}^{t}(x - x_k) = tx^{t-1} + \sum_{j=1}^{t-1}(-1)^j(t - j)e_j(x_1, \ldots, x_t)x^{t-j-1}. \tag{5}$$

We need the following lemma.

Lemma 1. *If $t \in \mathbb{N}$ and $x_1, \ldots, x_t \in \mathbb{C}$, then*

$$(1 - t)\prod_{j=1}^{t}(1 - x_j) + \sum_{j=1}^{t} \prod_{\substack{k=1 \\ k\neq j}}^{t}(1 - x_k) = 1 + \sum_{j=2}^{t}(-1)^{j-1}(j - 1)e_j(x_1, \ldots, x_t).$$

Proof. By using (4) and (5),

$$\begin{aligned}(1 - t)\prod_{j=1}^{t}(1 - x_j) + \sum_{j=1}^{t} \prod_{\substack{k=1 \\ k\neq j}}^{t}(1 - x_k) &= (1 - t)P(1) + P'(1) \\ &= (1 - t)\Big(1 + \sum_{j=1}^{t}(-1)^j e_j(x_1, \ldots, x_t)\Big) \\ &\qquad + t + \sum_{j=1}^{t-1}(-1)^j(t - j)e_j(x_1, \ldots, x_t) \\ &= 1 + \sum_{j=2}^{t}(-1)^{j-1}(j - 1)e_j(x_1, \ldots, x_t).\end{aligned}$$

□

3 Identities for Dirichlet series

Our first result is the following. As above, let $*$ denote the Dirichlet convolution of arithmetic functions and let $D(f,s) := \sum_{n=1}^{\infty} f(n)n^{-s}$ stand for the Dirichlet series of the function $f : \mathbb{N} \to \mathbb{C}$. We recall that a nonzero function $f : \mathbb{N} \to \mathbb{C}$ is *completely multiplicative* if $f(mn) = f(m)f(n)$ holds for all $m, n \in \mathbb{N}$.

Theorem 2. *Let $r \ge 2$, $t \ge 1$ be fixed integers and let $f_i = g_{i1} * \cdots * g_{it}$, where $g_{ij} : \mathbb{N} \to \mathbb{C}$ are nonzero completely multiplicative functions ($1 \le i \le r$, $1 \le j \le t$). Then we have (formally or in the case of absolute convergence),*

$$\sum_{\substack{n_1,\ldots,n_r=1\\ \gcd(n_1,\ldots,n_r)=1}}^{\infty} \frac{f_1(n_1)\cdots f_r(n_r)}{n_1^{s_1}\cdots n_r^{s_r}} = D(f_1,s_1)\cdots D(f_r,s_r)\Delta(f_1,\ldots,f_r,s_1,\ldots,s_r),$$

where

$$\Delta(f_1,\ldots,f_r,s_1,\ldots,s_r) = \prod_p \Big(1 + (-1)^{r-1} \sum_{1\le a_1,\ldots,a_r\le t} \frac{1}{p^{a_1 s_1+\cdots+a_r s_r}} \prod_{i=1}^{r} (-1)^{a_i} G_{i a_i}(p)\Big), \tag{6}$$

and

$$G_{ij}(p) := \sum_{1\le \ell_1<\cdots<\ell_j\le t} g_{i\ell_1}(p)\cdots g_{i\ell_j}(p), \tag{7}$$

with $1 \le i \le r$, $1 \le j \le t$, and p a prime.

In the cases $t = 1$ and $t = 2$, with $f_1 = \cdots = f_r = f$, Theorem 2 recovers [4, Ths. 3.1, 3.2].

Proof. As mentioned above, the characteristic function ϱ of the r-tuples with relatively prime components is multiplicative, viewed as a functions of r variables. Note that for primes p and $a_1,\ldots,a_r \ge 0$ we have $\varrho(p^{a_1},\ldots,p^{a_r}) = 1$ if and only if there is at least one $a_i = 0$. Also, if $f_1,\ldots,f_r : \mathbb{N} \to \mathbb{C}$ are arbitrary multiplicative functions of a single variable, then their product $f_1(n_1)\cdots f_r(n_r)$ is multiplicative as a function of r variables. We deduce the Euler product expansion

$$\begin{aligned} D &:= \sum_{\substack{n_1,\ldots,n_r=1\\ \gcd(n_1,\ldots,n_r)=1}}^{\infty} \frac{f_1(n_1)\cdots f_r(n_r)}{n_1^{s_1}\cdots n_r^{s_r}} \\ &= \sum_{n_1,\ldots,n_r=1}^{\infty} \frac{f_1(n_1)\cdots f_r(n_r)\varrho(n_1,\ldots,n_r)}{n_1^{s_1}\cdots n_r^{s_r}} \end{aligned}$$

$$
\begin{aligned}
&= \prod_p \sum_{a_1,\ldots,a_r=0}^{\infty} \frac{f_1(p^{a_1})\cdots f_r(p^{a_r})\varrho(p^{a_1},\ldots,p^{a_r})}{p^{a_1s_1+\cdots+a_rs_r}}\\
&= \prod_p \sum_{\substack{a_1,\ldots,a_r=0\\ a_1\cdots a_r=0}}^{\infty} \frac{f_1(p^{a_1})\cdots f_r(p^{a_r})}{p^{a_1s_1+\cdots+a_rs_r}}\\
&= \prod_p \Big(\sum_{a_1,\ldots,a_r=0}^{\infty} - \sum_{a_1,\ldots,a_r=1}^{\infty} \Big) \frac{f_1(p^{a_1})\cdots f_r(p^{a_r})}{p^{a_1s_1+\cdots+a_rs_r}}.
\end{aligned}
$$

Now if $f_i = g_{i1} * \cdots * g_{it}$, where $g_{ij} : \mathbb{N} \to \mathbb{C}$ are completely multiplicative functions ($1 \le i \le r$, $1 \le j \le t$), then

$$
\sum_{n=1}^{\infty} \frac{f_i(n)}{n^s} = \prod_{j=1}^{t} \sum_{n=1}^{\infty} \frac{g_{ij}(n)}{n^s} = \prod_{j=1}^{t} \prod_p \Big(1 - \frac{g_{ij}(p)}{p^s}\Big)^{-1} = \prod_p \prod_{j=1}^{t} \Big(1 - \frac{g_{ij}(p)}{p^s}\Big)^{-1}.
$$

At the same time, since the functions f_i ($1 \le i \le r$) are multiplicative, we have

$$
\sum_{n=1}^{\infty} \frac{f_i(n)}{n^s} = \prod_p \sum_{a=0}^{\infty} \frac{f_i(p^a)}{p^{as}},
$$

showing that

$$
\sum_{a=0}^{\infty} \frac{f_i(p^a)}{p^{as}} = \prod_{j=1}^{t} \Big(1 - \frac{g_{ij}(p)}{p^s}\Big)^{-1},
$$

and

$$
\sum_{a=1}^{\infty} \frac{f_i(p^a)}{p^{as}} = \prod_{j=1}^{t} \Big(1 - \frac{g_{ij}(p)}{p^s}\Big)^{-1} - 1. \tag{8}
$$

It follows that

$$
\begin{aligned}
D &= \prod_p \Big(\prod_{i=1}^{r} \prod_{j=1}^{t} \Big(1 - \frac{g_{ij}(p)}{p^{s_i}}\Big)^{-1} - \prod_{i=1}^{r} \Big(\prod_{j=1}^{t} \Big(1 - \frac{g_{ij}(p)}{p^{s_i}}\Big)^{-1} - 1 \Big) \Big)\\
&= \prod_p \prod_{i=1}^{r} \prod_{j=1}^{t} \Big(1 - \frac{g_{ij}(p)}{p^{s_i}}\Big)^{-1} \prod_p \Big(1 - \prod_{i=1}^{r} \Big(1 - \prod_{j=1}^{t} \Big(1 - \frac{g_{ij}(p)}{p^{s_i}}\Big) \Big) \Big).
\end{aligned} \tag{9}
$$

Using identity (4) for $x = 1$ and $x_j = g_{ij}(p)p^{-s_i}$ ($1 \le j \le t$) we have

$$
1 - \prod_{j=1}^{t} \Big(1 - \frac{g_{ij}(p)}{p^{s_i}}\Big) = \sum_{j=1}^{t} \frac{(-1)^{j-1}}{p^{js_i}} \sum_{1 \le \ell_1 < \cdots < \ell_j \le t} g_{i\ell_1}(p) \cdots g_{i\ell_j}(p),
$$

and inserting into (9) we deduce

$$D = D(f_1, s_1) \cdots D(f_r, s_r) \prod_p \left(1 - \prod_{i=1}^r \sum_{j=1}^t \frac{(-1)^{j-1}}{p^{js_i}} G_{ij}(p)\right),$$

where $G_{ij}(p)$ is defined by (7). Here the product over the primes p is

$$\prod_p \left(1 - \left(\sum_{a_1=1}^t \frac{(-1)^{a_1-1}}{p^{a_1 s_1}} G_{1a_1}(p)\right) \cdots \left(\sum_{a_r=1}^t \frac{(-1)^{a_r-1}}{p^{a_r s_r}} G_{ra_r}(p)\right)\right)$$

$$= \prod_p \left(1 + (-1)^{r-1} \sum_{1 \le a_1, \ldots, a_r \le t} \frac{1}{p^{a_1 s_1 + \cdots + a_r s_r}} \prod_{i=1}^r (-1)^{a_i} G_{ia_i}(p)\right),$$

finishing the proof. □

Remark 3. Identity (6) shows that under the assumptions of Theorem 2 we have the convolutional identity

$$f_1(n_1) \cdots f_r(n_r) \varrho(n_1, \ldots, n_r) = \sum_{d_1 e_1 = n_1, \ldots, d_r e_r = n_r} f_1(d_1) \cdots f_r(d_r) F_{f_1, \ldots, f_r}(e_1, \ldots, e_r), \tag{10}$$

where $F_{f_1, \ldots, f_r}$ is the multiplicative function defined for prime powers $p^{a_1}, \ldots, p^{a_r}$ ($a_1, \ldots, a_r \ge 0$, not all zero) by

$$F_{f_1, \ldots, f_r}(p^{a_1}, \ldots, p^{a_r}) = \begin{cases} (-1)^{r-1} \prod_{i=1}^r (-1)^{a_i} G_{ia_i}(p), & \text{if } 1 \le a_1, \ldots, a_r \le t; \\ 0, & \text{otherwise.} \end{cases}$$

Let $\tau_t(n) = \sum_{d_1 \cdots d_t = n} 1$ denote the Piltz divisor function of order t.

Corollary 4. *Let $r \ge 2$ and let $t_i \ge 2$ $(1 \le i \le r)$ be fixed integers. If $s_i \in \mathbb{C}$, $\Re s_i > 1$ $(1 \le i \le r)$, then*

$$\sum_{\substack{n_1, \ldots, n_r = 1 \\ \gcd(n_1, \ldots, s_r) = 1}}^{\infty} \frac{\tau_{t_1}(n_1) \cdots \tau_{t_r}(n_r)}{n_1^{s_1} \cdots n_r^{s_r}} = \zeta^{t_1}(s_1) \cdots \zeta^{t_r}(s_r) \Delta(\tau_{t_1}, \ldots, \tau_{t_r}, s_1, \ldots, s_r),$$

with

$$\Delta(\tau_{t_1}, \ldots, \tau_{t_r}, s_1, \ldots, s_r) = \prod_p \left(1 + (-1)^{r-1} \sum_{1 \le a_1, \ldots, a_r \le t} \frac{1}{p^{a_1 s_1 + \cdots + a_r s_r}} \prod_{i=1}^r (-1)^{a_i} \binom{t_i}{a_i}\right), \tag{11}$$

where $\binom{t_i}{a_i}$ are binomial coefficients with the usual convention that $\binom{t_i}{a_i} = 0$ for $a_i > t_i$ $(1 \le i \le r)$.

Proof. Choose $g_{ij}(p)=1$ for $1\le j\le t_i$ and $g_{ij}(p)=0$ for $t_i+1\le j\le t$ $(1\le i\le r)$. Then $f_i(n)=\tau_{t_i}(n)$ $(1\le i\le r)$, and use that $G_{ij}(p)=\binom{t_i}{j}$ $(1\le i\le r,\ 1\le j\le t)$ for a prime p. ☐

Corollary 5. *Let $r\ge 2$. If $s_i\in\mathbb{C}$, $\Re s_i>1$ $(1\le i\le r)$, then*

$$\sum_{\substack{n_1,\ldots,n_r=1\\ \gcd(n_1,\ldots,n_r)=1}}^{\infty} \frac{\tau(n_1)\cdots\tau(n_r)}{n_1^{s_1}\cdots n_r^{s_r}} =$$

$$\zeta^2(s_1)\cdots\zeta^2(s_r)\times\prod_p\left(1+(-1)^{r-1}\sum_{1\le a_1,\ldots,a_r\le 2}\frac{(-2)^{\#\{1\le i\le r:\, a_i=1\}}}{p^{a_1s_1+\cdots+a_rs_r}}\right).$$

Proof. Apply Corollary 4 for $t_i=2$ $(1\le i\le r)$. ☐

If $r=2$, then this recovers identity (1) and for $r=3$ we have

$$\sum_{\substack{n_1,n_2,n_3=1\\ \gcd(n_1,n_2,n_3)=1}}^{\infty} \frac{\tau(n_1)\tau(n_2)\tau(n_3)}{n_1^{s_1}n_2^{s_2}n_3^{s_3}} = \zeta^2(s_1)\zeta^2(s_2)\zeta^2(s_3)$$

$$\times\prod_p\Big(1-\frac{8}{p^{s_1+s_2+s_3}}+\frac{4}{p^{2s_1+s_2+s_3}}+\frac{4}{p^{s_1+2s_2+s_3}}+\frac{4}{p^{s_1+s_2+2s_3}}$$

$$-\frac{2}{p^{2s_1+2s_2+s_3}}-\frac{2}{p^{2s_1+s_2+2s_3}}-\frac{2}{p^{s_1+2s_2+2s_3}}+\frac{1}{p^{2s_1+2s_2+2s_3}}\Big). \tag{12}$$

Now we consider Dirichlet series with $\gcd(n_1,\ldots,n_r)=1$ replaced by the condition that $n_1,\ldots,n_r$ are pairwise relatively prime.

Theorem 6. *Let $r\ge 2$, $t\ge 1$ be fixed integers and let $f_i=g_{i1}*\cdots*g_{it}$, where $g_{ij}:\mathbb{N}\to\mathbb{C}$ are nonzero completely multiplicative functions $(1\le i\le r,\ 1\le j\le t)$. Then we have (formally or in the case of absolute convergence),*

$$\sum_{\substack{n_1,\ldots,n_r=1\\ \gcd(n_i,n_j)=1\,(i\ne j)}}^{\infty} \frac{f_1(n_1)\cdots f_r(n_r)}{n_1^{s_1}\cdots n_r^{s_r}} = D(f_1,s_1)\cdots D(f_r,s_r)\overline{\Delta}(f_1,\ldots,f_r,s_1,\ldots,s_r),$$

where

$$\overline{\Delta}(f_1,\ldots,f_r,s_1,\ldots,s_r)=$$

$$\prod_p\left(1-\sum_{i=2}^{r}(i-1)\sum_{1\le \ell_1<\cdots<\ell_i\le r}\ \sum_{a_{\ell_1},\ldots,a_{\ell_i}=1}^{t}\frac{1}{p^{a_{\ell_1}s_{\ell_1}+\cdots+a_{\ell_i}s_{\ell_i}}}\prod_{m=1}^{i}(-1)^{a_{\ell_m}}G_{ia_{\ell_m}}(p)\right), \tag{13}$$

with $G_{ij}(p)$ $(1\le i\le r,\ 1\le j\le t,\ p$ prime$)$ defined by (7).

Proof. The characteristic function ϑ of the r-tuples with pairwise relatively prime components is multiplicative, viewed as a functions of r variables. Note that for primes p and $a_1,\ldots,a_r \ge 0$ we have $\vartheta(p^{a_1},\ldots,p^{a_r}) = 1$ if and only if there is at most one $a_i \ge 1$. If $f_1,\ldots,f_r : \mathbb{N} \to \mathbb{C}$ are arbitrary multiplicative functions of a single variable, then we have the Euler product expansion

$$\begin{aligned}\overline{D} := & \sum_{\substack{n_1,\ldots,n_r=1\\ \gcd(n_i,n_j)=1\,(i\neq j)}}^{\infty} \frac{f_1(n_1)\cdots f_r(n_r)}{n_1^{s_1}\cdots n_r^{s_r}} \\ &= \sum_{n_1,\ldots,n_r=1}^{\infty} \frac{f_1(n_1)\cdots f_r(n_r)\vartheta(n_1,\ldots,n_r)}{n_1^{s_1}\cdots n_r^{s_r}} \\ &= \prod_p \sum_{a_1,\ldots,a_r=0}^{\infty} \frac{f_1(p^{a_1})\cdots f_r(p^{a_r})\vartheta(p^{a_1},\ldots,p^{a_r})}{p^{a_1s_1+\cdots+a_rs_r}} \\ &= \prod_p \Big(1+\sum_{i=1}^{r}\sum_{a_i=1}^{\infty}\frac{f_i(p^{a_i})}{p^{a_is_i}}\Big) \\ &= \prod_p \Big(1+\sum_{i=1}^{r}\Big(\prod_{j=1}^{t}\Big(1-\frac{g_{ij}(p)}{p^{s_i}}\Big)^{-1}-1\Big)\Big)\end{aligned}$$

by using (8). We deduce that

$$\begin{aligned}\overline{D} = \prod_p \prod_{i=1}^{r}\prod_{j=1}^{t}\Big(1-\frac{g_{ij}(p)}{p^{s_i}}\Big)^{-1} \prod_p \Big((1-r)\prod_{i=1}^{r}\prod_{j=1}^{t}\Big(1-\frac{g_{ij}(p)}{p^{s_i}}\Big) \\ +\sum_{i=1}^{r}\prod_{\substack{k=1\\ k\neq i}}^{r}\prod_{j=1}^{t}\Big(1-\frac{g_{kj}(p)}{p^{s_k}}\Big)\Big) \\ = \prod_{i=1}^{r} D(f_i,s_i)\prod_p K(p), \qquad (14)\end{aligned}$$

say. Let $x_{ij} = g_{ij}(p)p^{-s_i}$ $(1 \le i \le r,\ 1 \le j \le t)$. Then by (4),

$$\prod_{j=1}^{t}\Big(1-\frac{g_{ij}(p)}{p^{s_i}}\Big) = \prod_{j=1}^{t}(1-x_{ij}) = 1-\sum_{j=1}^{t}(-1)^{j-1}e_j(x_{i1},\ldots,x_{it}) = 1-y_i,$$

with $1 \le i \le r$, where

$$\begin{aligned}y_i &:= \sum_{j=1}^{t}(-1)^{j-1}e_j(x_{i1},\ldots,x_{it}) \\ &= \sum_{j=1}^{t}(-1)^{j-1}\sum_{1\le \ell_1<\cdots<\ell_j\le r} x_{i\ell_1}\cdots x_{i\ell_j}\end{aligned}$$

$$= \sum_{j=1}^{t} \frac{(-1)^{j-1}}{p^{js_i}} \sum_{1\le \ell_1<\cdots<\ell_j\le r} g_{i\ell_1}(p)\cdots g_{i\ell_j}(p)$$

$$= \sum_{j=1}^{t} \frac{(-1)^{j-1}}{p^{js_i}} G_{ij}(p). \tag{15}$$

Therefore, by applying Lemma 1 for $y_1, \ldots, y_r$ we obtain that the expression $K(p)$ under the product $\prod_p$ in (14) is

$$\begin{aligned}
K(p) &= (1-r)\prod_{i=1}^{r}(1-y_i) + \sum_{i=1}^{r}\prod_{\substack{k=1\\k\ne i}}^{r}(1-y_k)\\
&= 1+\sum_{i=2}^{r}(-1)^{i-1}(i-1)e_i(y_1,\ldots,y_r)\\
&= 1+\sum_{i=2}^{r}(-1)^{i-1}(i-1)\sum_{1\le\ell_1<\cdots<\ell_i\le r} y_{\ell_1}\cdots y_{\ell_i}\\
&= 1+\sum_{i=2}^{r}(-1)^{i-1}(i-1)\sum_{1\le\ell_1<\cdots<\ell_i\le r}\left(\sum_{a_{\ell_1}=1}^{t}\frac{(-1)^{a_{\ell_1}-1}}{p^{a_{\ell_1}s_{\ell_1}}}G_{ia_{\ell_1}}(p)\right)\times\cdots\\
&\qquad\times\left(\sum_{a_{\ell_i}=1}^{t}\frac{(-1)^{a_{\ell_i}-1}}{p^{a_{\ell_i}s_{\ell_i}}}G_{ia_{\ell_i}}(p)\right)\\
&= 1-\sum_{i=2}^{r}(i-1)\sum_{1\le\ell_1<\cdots<\ell_i\le r}\ \sum_{a_{\ell_1},\ldots,a_{\ell_i}=1}^{t}\frac{1}{p^{a_{\ell_1}s_{\ell_1}+\cdots+a_{\ell_i}s_{\ell_i}}}\prod_{m=1}^{i}(-1)^{a_{\ell_m}}G_{ia_{\ell_m}}(p),
\end{aligned}$$

by (15), ending the proof. □

Remark 7. Identity (13) shows that under the assumptions of Theorem 6 we have the convolutional identity

$$f_1(n_1)\cdots f_r(n_r)\vartheta(n_1,\ldots,n_r) = \sum_{d_1e_1=n_1,\ldots,d_re_r=n_r} f_1(d_1)\cdots f_r(d_r)\overline{F}_{f_1,\ldots,f_r}(e_1,\ldots,e_r), \tag{16}$$

where $\overline{F}_{f_1,\ldots,f_r}$ is the multiplicative function defined for prime powers $p^{a_1},\ldots,p^{a_r}$ ($a_1,\ldots,a_r\ge 0$, not all zero) by

$$\overline{F}_{f_1,\ldots,f_r}(p^{a_1},\ldots,p^{a_r}) = \begin{cases} (1-i)\prod_{m=1}^{i}(-1)^{a_{\ell_m}}G_{ia_{\ell_m}}(p), & \text{if there exists } 2\le i\le r \text{ and there exist } 1\le \ell_1,\ldots,\ell_i\le r \text{ such that } 1\le a_{\ell_1},\ldots,a_{\ell_i}\le t;\\ 0, & \text{otherwise.}\end{cases}$$

Corollary 8. *Let $r \geq 2$ and let $t_i \geq 2$ $(1 \leq i \leq r)$ be fixed integers. If $s_i \in \mathbb{C}$, $\Re s_i > 1$ $(1 \leq i \leq r)$, then*

$$\sum_{\substack{n_1,\dots,n_r=1\\ \gcd(n_i,n_j)=1\,(i\neq j)}}^{\infty} \frac{\tau_{t_1}(n_1)\cdots\tau_{t_r}(n_r)}{n_1^{s_1}\cdots n_r^{s_r}} = \zeta^{t_1}(s_1)\cdots\zeta^{t_r}(s_r)\overline{\Delta}(\tau_{t_1},\dots,\tau_{t_r},s_1,\dots,s_r),$$

where

$$\overline{\Delta}(\tau_{t_1},\dots,\tau_{t_r},s_1,\dots,s_r) = \prod_p \Bigg(1-\sum_{i=2}^{r}(i-1)\sum_{1\leq \ell_1<\cdots<\ell_i\leq r}\ \sum_{a_{\ell_1},\dots,a_{\ell_i}=1}^{t} \frac{1}{p^{a_{\ell_1}s_{\ell_1}+\cdots+a_{\ell_i}s_{\ell_i}}}\prod_{m=1}^{i}(-1)^{a_{\ell_m}}\binom{t_i}{a_{\ell_m}}\Bigg). \tag{17}$$

Proof. Apply Theorem 6 in the case $g_{ij}(p)=1$ for $1 \leq j \leq t_i$ and $g_{ij}(p)=0$ for $t_i+1 \leq j \leq t$ $(1 \leq i \leq r)$. □

Corollary 9. *Let $r \geq 2$. If $s_i \in \mathbb{C}$, $\Re s_i > 1$ $(1 \leq i \leq r)$, then*

$$\sum_{\substack{n_1,\dots,n_r=1\\ \gcd(n_i,n_j)=1\,(i\neq j)}}^{\infty} \frac{\tau(n_1)\cdots\tau(n_r)}{n_1^{s_1}\cdots n_r^{s_r}} = \zeta^2(s_1)\cdots\zeta^2(s_r)$$
$$\times\prod_p\Bigg(1-\sum_{i=2}^{r}(i-1)\sum_{1\leq \ell_1<\cdots<\ell_i\leq r}\ \sum_{1\leq a_{\ell_1},\dots,a_{\ell_i}\leq 2}\frac{(-2)^{\#\{1\leq m\leq i:\, a_{\ell_m}=1\}}}{p^{a_{\ell_1}s_{\ell_1}+\cdots+a_{\ell_i}s_{\ell_i}}}\Bigg).$$

Proof. Apply Corollary 8 for $t_i = 2$ $(1 \leq i \leq r)$. □

For $r = 2$ this gives (1) and for $r = 3$ we have

$$\sum_{\substack{n_1,n_2,n_3=1\\ \gcd(n_1,n_2)=\gcd(n_1,n_3)=\gcd(n_2,n_3)=1}}^{\infty} \frac{\tau(n_1)\tau(n_2)\tau(n_3)}{n_1^{s_1}n_2^{n_2}n_3^{s_3}} =$$
$$\zeta^2(s_1)\zeta^2(s_2)\zeta^2(s_3)\times\prod_p\Big(1-\frac{4}{p^{s_1+s_2}}+\frac{2}{p^{2s_1+s_2}}+\frac{2}{p^{s_1+2s_2}}-\frac{1}{p^{2s_1+2s_2}}-\frac{4}{p^{s_1+s_3}}+\frac{2}{p^{2s_1+s_3}}$$
$$+\frac{2}{p^{s_1+2s_3}}-\frac{1}{p^{2s_1+2s_3}}-\frac{4}{p^{s_2+s_3}}+\frac{2}{p^{2s_2+s_3}}+\frac{2}{p^{s_2+2s_3}}-\frac{1}{p^{2s_2+2s_3}}+\frac{16}{p^{s_1+s_2+s_3}}-\frac{8}{p^{2s_1+s_2+s_3}}$$
$$-\frac{8}{p^{s_1+2s_2+s_3}}-\frac{8}{p^{s_1+s_2+2s_3}}+\frac{4}{p^{2s_1+2s_2+s_3}}+\frac{4}{p^{2s_1+s_2+2s_3}}+\frac{4}{p^{s_1+2s_2+2s_3}}-\frac{2}{p^{2s_1+2s_2+2s_3}}\Big). \tag{18}$$

If we compare the infinite product (18) to (12), then we can see that in (12) we only have exponents of p of form $a_1s_1+a_2s_2+a_3s_3$ with $1 \leq a_1,a_2,a_3 \leq 2$, while in (18) the exponents of p are $a_1s_1 + a_2s_2 + a_3s_3$ with $0 \le a_1, a_2, a_3 \le 2$ and with at least two nonzero values a_1, a_2, a_3. Similar in the general case, according to Theorems 2 and 6.

It is possible to derive a common generalization of Theorems 2 and 6 by considering k-wise relatively prime integers. Let $r \ge k \ge 2$ be fixed integers. The positive integers $n_1, \ldots, n_r$ are called k-wise relatively prime if any k of them are relatively prime, that is, $\gcd(n_{i_1}, \ldots, n_{i_k}) = 1$ for every $1 \le i_1 < \cdots < i_k \le r$. In particular, in the case $k = 2$ the integers are pairwise relatively prime and for $k = r$ they are mutually relatively prime. Let ϱ_k denote the characteristic function of the set of r-tuples of positive integers with k-wise relatively prime components. Hence, $\varrho_r = \varrho$ and $\varrho_2 = \vartheta$, with our previous notation.

Here we confine ourselves to the case $t = 1$, that is, the functions $f_1, \ldots, f_r$ are completely multiplicative.

Theorem 10. *Let $r \ge k \ge 2$ and let $f_1, \ldots, f_r : \mathbb{N} \to \mathbb{C}$ be completely multiplicative functions. Then*

$$\sum_{n_1,\ldots,n_r=1}^{\infty} \frac{f_1(n_1)\cdots f_r(n_r)\varrho_k(n_1,\ldots,n_r)}{n_1^{s_1}\cdots n_r^{s_r}} = D(f_1,s_1)\cdots D(f_r,s_r)$$

$$\times \prod_p \left(1 - \sum_{i=k}^{r} (-1)^{i-k} \binom{i-1}{k-1} \sum_{1\le \ell_1 < \cdots < \ell_i \le r} \frac{f_{\ell_1}(p)\cdots f_{\ell_i}(p)}{p^{s_{\ell_1}+\cdots+s_{\ell_i}}}\right),$$

Proof. For fixed k the function ϱ_k is multiplicative. Also, for prime powers $p^{a_1}, \ldots, p^{a_r}$ $(a_1 \ldots, a_r \ge 0)$ we have $\varrho_k(p^{a_1}, \ldots, p^{a_r}) = 1$ if and only there are at most $k-1$ values $a_i \ge 1$. Now the proof is similar to the proofs of Theorems 2 and 6. In the case $f_1(n) = \cdots = f_r(n) = 1$ $(n \in \mathbb{N})$ this result and its detailed proof are given in [6, Th. 2.1]. □

4 Related asymptotic formulas

The above identities can be used to obtain asymptotic formulas with remainder terms for certain related sums. As examples, we point out the following formulas.

Theorem 11. *Let $r \ge 2$ and let $t_i \ge 2$ $(1 \le i \le r)$ be fixed integers. Then for every $\varepsilon > 0$,*

$$\sum_{\substack{n_1,\ldots,n_r \le x \\ \gcd(n_1,\ldots,n_r)=1}} \tau_{t_1}(n_1)\cdots\tau_{t_r}(n_r) = x^r Q(\log x) + O(x^{r-1+\max_{1\le i\le r}\vartheta_{t_i}+\varepsilon}),$$

where $Q(u)$ is a polynomial in u of degree $t_1 + \cdots + t_r - r$ having the leading coefficient

$$\frac{\Delta(\tau_{t_1},\ldots,\tau_{t_r},1,\ldots,1)}{(t_1-1)!\cdots(t_r-1)!},$$

where $\Delta(\tau_{t_1},\ldots,\tau_{t_r},1,\ldots,1)$ is obtained from (11) for $s_1 = \cdots = s_r = 1$, and ϑ_{t_i} are the exponents in the Piltz divisor problems for τ_{t_i}, namely

$$\sum_{n\le x} \tau_{t_i}(n) = xP_{t_i}(\log x) + O(x^{\vartheta_{t_i}+\varepsilon}), \tag{19}$$

with some polynomials $P_{t_i}(u)$ in u of degree $t_i - 1$ having the leading coefficients $1/(t_i-1)!$ $(1 \le i \le r)$.

Proof. We have, according to the convolutional identity (10),

$$\sum_{\substack{n_1,\ldots,n_r\le x\\ \gcd(n_1,\ldots,n_r)=1}} \tau_{t_1}(n_1)\cdots\tau_{t_r}(n_r) = \sum_{d_1e_1=n_1\le x,\ldots,d_re_r=n_r\le x} \tau_{t_1}(d_1)\cdots\tau_{t_r}(d_r)F_{\tau_{t_1},\ldots,\tau_{t_r}}(e_1,\ldots,e_r)$$

$$= \sum_{e_1,\ldots,e_r\le x} F_{\tau_{t_1},\ldots,\tau_{t_r}}(e_1,\ldots,e_r) \sum_{d_1\le x/e_1} \tau_{t_1}(d_1)\cdots \sum_{d_r\le x/e_r} \tau_{t_r}(d_r).$$

Now by using formulas (19) and the fact that the infinite product $\Delta(\tau_{t_1},\ldots,\tau_{t_r},s_1,\ldots,s_r)$ given by (11) is absolutely convergent provided that $s_i \in \mathbb{C}$, $\Re s_i > 0$ $(1\le i\le r)$, $\Re(s_1+\cdots+s_r) > 1$, we obtain the desired formula. For the details see the proof of [7, Th. 3.3], which is a generalization of the present result. □

Corollary 12. *Let $r \ge 2$. Then for every $\varepsilon > 0$,*

$$\sum_{\substack{n_1,\ldots,n_r\le x\\ \gcd(n_1,\ldots,n_r)=1}} \tau(n_1)\cdots\tau(n_r) = x^r T(\log x) + O(x^{r-1+\theta+\varepsilon})$$

where $T(u)$ is a polynomial in u of degree r having the leading coefficient K_r, where

$$K_r = \prod_p \Big(1 - \Big(\frac{2p-1}{p^2}\Big)^r\Big) = \prod_p \Big(1 - \sum_{i=0}^r (-1)^i \binom{r}{i} \frac{2^{r-i}}{p^{r+i}}\Big);$$

in particular,

$$K_2 = \prod_p \Big(1 - \frac{4}{p^2} + \frac{4}{p^3} - \frac{1}{p^4}\Big), \tag{20}$$

$$K_3 = \prod_p \Big(1 - \frac{8}{p^3} + \frac{12}{p^4} - \frac{6}{p^5} + \frac{1}{p^6}\Big),$$

and θ is the exponent in Dirichlet's divisor problem.

Proof. Apply Theorem 11 in the case $t_i = 2$ $(\le i \le r)$. The representation of K_r follows from (9) for $g_{ij}(p) = 1$ $(1\le i\le r,\ 1\le j\le 2)$. □

We note that in the case $r = 2$ this result has been proved in [3, Lemma 3.3] by analytic methods, with a weaker error term.

Theorem 13. *Let $r \ge 2$ and let $t_i \ge 2$ $(1\le i\le r)$ be fixed integers. Then for every $\varepsilon > 0$,*

$$\sum_{\substack{n_1,\ldots,n_r\le x\\ \gcd(n_i,n_j)=1\,(i\ne j)}} \tau_{t_1}(n_1)\cdots\tau_{t_r}(n_r) = x^r \overline{Q}(\log x) + O(x^{r-1+\max_{1\le i\le r}\vartheta_{t_i}+\varepsilon}),$$

where $\overline{Q}(u)$ *is a polynomial in* u *of degree* $t_1 + \cdots + t_r - r$ *having the leading coefficient*

$$\frac{\overline{\Delta}(\tau_{t_1}, \ldots, \tau_{t_r}, 1, \ldots, 1)}{(t_1-1)! \cdots (t_r-1)!},$$

where $\overline{\Delta}(\tau_{t_1}, \ldots, \tau_{t_r}, 1, \ldots, 1)$ *is obtained from* (17) *for* $s_1 = \cdots = s_r = 1$, *and* ϑ_{t_i} *are the exponents in the Piltz divisor problems for* τ_{t_i} $(1 \le i \le r)$.

Proof. Similar to the proof of Theorem 11. By the convolutional identity (16) we have

$$\begin{aligned}\sum_{\substack{n_1,\ldots,n_r \le x\\ \gcd(n_i,n_j)=1\,(i\ne j)}} \tau_{t_1}(n_1)\cdots\tau_{t_r}(n_r) &= \sum_{d_1e_1=n_1\le x,\ldots,d_re_r=n_r\le x} \tau_{t_1}(d_1)\cdots\tau_{t_r}(d_r)\overline{F}_{\tau_{t_1},\ldots,\tau_{t_r}}(e_1,\ldots,e_r),\\ &= \sum_{e_1,\ldots,e_r\le x} \overline{F}_{\tau_{t_1},\ldots,\tau_{t_r}}(e_1,\ldots,e_r) \sum_{d_1\le x/e_1} \tau_{t_1}(d_1) \times \cdots\\ &\quad \times \sum_{d_r \le x/e_r} \tau_{t_r}(d_r).\end{aligned}$$

Now use formulas (19) and the fact that the infinite product $\overline{\Delta}(\tau_{t_1}, \ldots, \tau_{t_r}, s_1, \ldots, s_r)$ given by (17) is absolutely convergent provided that $s_i \in \mathbb{C}$, $\Re s_i > 0$ $(1 \le i \le r)$, $\Re(s_i+s_j) > 1$ $(1 \le i < j \le r)$. This is also a special case of [7, Th. 3.3]. □

Corollary 14. *Let* $r \ge 2$. *Then for every* $\varepsilon > 0$,

$$\sum_{\substack{n_1,\ldots,n_r \le x\\ \gcd(n_i,n_j)=1\,(i\ne j)}} \tau(n_1)\cdots\tau(n_r) = x^r\overline{T}(\log x) + O(x^{r-1+\theta+\varepsilon})$$

where $\overline{T}(u)$ *is a polynomial in* u *of degree* r *having the leading coefficient* $\overline{K}_r$, *where*

$$\overline{K}_r = \prod_p \Big(1-\frac{1}{p}\Big)^{2(r-1)}\Big(1+\frac{(r-1)(2p-1)}{p^2}\Big),$$

in particular, $\overline{K}_2 = K_2$ *given by* (20),

$$\overline{K}_3 = \prod_p \Big(1 - \frac{12}{p^2} + \frac{28}{p^3} - \frac{27}{p^4} + \frac{12}{p^5} - \frac{2}{p^6}\Big),$$

and θ *is the exponent in Dirichlet's divisor problem.*

Proof. Apply Theorem 13 in the case $t_i = 2$ $(\le i \le r)$. The representation of $\overline{K}_r$ follows from (14) for $g_{ij}(p) = 1$ $(1 \le i \le r,\ 1 \le j \le 2)$. □

References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, 1976.

[2] H. Delange, On some sets of pairs of positive integers, *J. Number Theory* **1** (1969), 261–279.

[3] W. G. Nowak and L. Tóth, On the average number of subgroups of the group $\mathbb{Z}_m \times \mathbb{Z}_n$, *Int. J. Number Theory* **10** (2014), 363–374.

[4] N. Tahmi and A. Derbal, Some multiple Dirichlet series of completely multiplicative arithmetic functions, *Notes Number Theory Discrete Math.* **28** (2022), 603–616.

[5] L. Tóth, Multiplicative arithmetic functions of several variables: a survey, in *Mathematics Without Boundaries, Surveys in Pure Mathematics*, Springer, 2014, pp. 483–514.

[6] L. Tóth, Counting r-tuples of positive integers with k-wise relatively prime components, *J. Number Theory* **166** (2016), 105–116.

[7] L. Tóth and W. Zhai, On multivariable averages of divisor functions, *J. Number Theory* **192** (2018), 251–269.

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