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More Congruences for Central Binomial Sums with Fibonacci and Lucas Numbers

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Abstract

We mainly determine $\sum_{k=1}^{p-1}\binom{2k}{k}h_k x^k$ modulo a prime p with $h_k=\sum_{j=1}^{k}\frac{1}{2j-1}$. We also provide some applications of this polynomial congruence for some special values of x which involve the Fibonacci and Lucas numbers.

1 Introduction

In [4], Lehmer presented a long list of interesting series involving the central binomial coefficient $\binom{2k}{k}$ such as

$$\sum_{k=1}^{\infty}\binom{2k}{k}\frac{x^k}{k}=-2\ln\left(\frac{1+\sqrt{1-4x}}{2}\right). \tag{1}$$

Since then, using many different approaches and methods, several other authors have actively investigated this kind of sums. Among them we would like to mention [3, 1, 2]. In particular, Chen in [2, Theorem 8] obtained

$$\sum_{k=1}^{\infty}\binom{2k}{k}h_k x^k=-\frac{\ln(1-4x)}{2\sqrt{1-4x}} \tag{2}$$

with $h_k=\sum_{j=1}^{k}\frac{1}{2j-1}$. In this work, as we did in [5, 7, 6], we are going to consider the sums of the first $p-1$ terms of the above series, with a prime p, and evaluate them modulo p.

The paper is organized as follows. The next section is devoted to the proofs of two polynomial congruences which are the finite analogues of the infinite series (1) and (2). In the third section, we specialize those polynomial congruences to some particular values $(u_n)_{n\geq 1}$ and $(v_n)_{n\geq 1}$ related to Fibonacci numbers F_n (A000045) and Lucas numbers L_n (A000032):

$$u_n = \frac{(-1)^n}{L_n^2} = \frac{(-1)^n}{L_{2n}+2(-1)^n} \to -1, \frac{1}{9}, -\frac{1}{16}, \frac{1}{49}, -\frac{1}{121}, \frac{1}{324}, -\frac{1}{841}, \dots$$
$$v_n = \frac{(-1)^{n-1}}{5F_n^2} = \frac{(-1)^{n-1}}{L_{2n}-2(-1)^n} \to \frac{1}{5}, -\frac{1}{5}, \frac{1}{20}, -\frac{1}{45}, \frac{1}{125}, -\frac{1}{320}, \frac{1}{845}, \dots$$

Notice that, up to sign, u_n and v_n are the reciprocals of A001254 and A099921 in the *On-Line Encyclopedia of Integer Sequences* (OEIS).

For instance, after extending a result by Williams [8], we show that for any integer $n \geq 1$, and for any prime $p > 2$ not dividing $5F_nL_n$,

$$\sum_{k=1}^{p-1} \binom{2k}{k} \frac{u_n^k}{k} \equiv 2q_p(L_n) - \frac{5nF_n}{L_n} f_p \pmod{p},$$
$$\sum_{k=1}^{p-1} \binom{2k}{k} h_k v_n^k \equiv -\left(\frac{5}{p}\right)\left(\frac{1}{2} q_p\left(\frac{L_n^2}{5F_n^2}\right) + (-1)^n \frac{2n}{F_{2n}} f_p\right) \pmod{p}.$$

Here $\left(\frac{5}{p}\right)$ stands for a Legendre symbol, and let

$$q_p(x) = \frac{x^{p-1}-1}{p}, \qquad f_p = \frac{F_{p-\left(\frac{5}{p}\right)}}{p}$$

denote the *Fermat quotient* and the *Fibonacci quotient*, respectively.

In order to demonstrate the similarities of the above formulas with the corresponding series evaluations, notice that, by (1) and (2), we obtain

$$\sum_{k=1}^{\infty} \binom{2k}{k} \frac{u_n^k}{k} = 2\ln(L_n) - 2n \ln\left(\frac{1+\sqrt{5}}{2}\right),$$
$$\sum_{k=1}^{\infty} \binom{2k}{k} h_k v_n^k = -\frac{\sqrt{5}F_n}{2L_n} \ln\left(\frac{L_n^2}{5F_n^2}\right).$$

2 Polynomial congruences

The next congruence is equivalent to [6, (32)]. Here we give a new and self-contained proof.

Theorem 1. *Let $p > 2$ be a prime. Then*

$$\sum_{k=1}^{p-1} \binom{2k}{k} \frac{x^k}{k} \equiv 2\frac{1-\alpha^p-(1-\alpha)^p}{p} \pmod{p} \tag{3}$$

where $x = \alpha(1-\alpha)$.

Proof. Let $\beta = 1-\alpha$. Then, for $1 \le k \le p-1$, it follows that

$$\binom{p}{k} = \frac{p(p-1)\cdots(p-(k-1))}{k!} \equiv \frac{p(-1)^{k-1}}{k} \pmod{p^2},$$

and we find

$$\begin{aligned}
(1-\alpha z)^p(1-\beta z)^p &= \sum_{k=0}^{p} \binom{p}{k}(-\alpha z)^k \sum_{k=0}^{p} \binom{p}{k}(-\beta z)^k \\
&\equiv \left(1-\alpha^p z^p - p\sum_{k=1}^{p-1}\frac{\alpha^k z^k}{k}\right)\left(1-\beta^p z^p - p\sum_{k=1}^{p-1}\frac{\beta^k z^k}{k}\right) \\
&\equiv 1-(\alpha^p+\beta^p)z^p + (\alpha\beta)^p z^{2p} \\
&\quad - p(1-\alpha^p z^p)\sum_{k=1}^{p-1}\frac{\beta^k z^k}{k} - p(1-\beta^p z^p)\sum_{k=1}^{p-1}\frac{\alpha^k z^k}{k} \pmod{p^2}.
\end{aligned}$$

Hence, we have

$$[z^p](1-\alpha z)^p(1-\beta z)^p \equiv -\alpha^p - \beta^p \pmod{p^2}. \tag{4}$$

On the other hand,

$$\begin{aligned}
[z^p](1-\alpha z)^p(1-\beta z)^p &= [z^p](1-z+xz^2)^p \\
&= [z^p]\sum_{k=0}^{p}\binom{p}{k}x^k z^{2k}(1-z)^{p-k} \\
&= [z^p]\sum_{k=0}^{p}\binom{p}{k}x^k z^{2k}\sum_{j=0}^{p-k}\binom{p-k}{j}(-z)^j \\
&= \sum_{k=0}^{\frac{p-1}{2}}\binom{p}{k}\binom{p-k}{p-2k}(-1)^{p-2k}x^k \\
&= -\sum_{k=0}^{\frac{p-1}{2}}\binom{p}{k}\binom{p-k}{k}x^k.
\end{aligned}$$

Therefore, we get

$$[z^p](1-\alpha z)^p(1-\beta z)^p \equiv -1 + p\sum_{k=1}^{\frac{p-1}{2}}(-1)^k\binom{p-k}{k}\frac{x^k}{k}$$

$$\equiv -1 + \frac{p}{2}\sum_{k=1}^{p-1}\binom{2k}{k}\frac{x^k}{k} \pmod{p^2} \tag{5}$$

where we used the fact that, for $1 \le k \le \frac{p-1}{2}$,

$$\binom{p-k}{k} = \frac{(p-k)\dots(p-(2k-1))}{k!} \equiv \frac{(-1)^k}{2}\binom{2k}{k} \pmod{p},$$

and p divides $\binom{2k}{k}$ for $\frac{p-1}{2} < k < p$. Combining (4) with (5) yields the desired result. □

Once the preliminary congruence (3) is obtained, we are in the position to state the main theorem of this section.

Theorem 2. *If $p > 2$ is a prime then*

$$\sum_{k=1}^{p-1}\binom{2k}{k}h_k x^k \equiv \frac{1}{2}(1-4x)^{\frac{p-1}{2}}\left(\sum_{k=1}^{p-1}\binom{2k}{k}\frac{x^k}{k} - \sum_{k=1}^{p-1}\binom{2k}{k}\frac{y^k}{k}\right) \pmod{p} \tag{6}$$

where $y = -\frac{x}{1-4x}$.

Proof. For $1 \le k \le \frac{p-1}{2}$,

$$\binom{\frac{p-1}{2}}{k} = \frac{(p-1)(p-3)\dots(p-(2k-1))}{2^k k!} \equiv \frac{(-1)^k(2k-1)!!}{2^k k!} = \binom{2k}{k}\frac{1}{(-4)^k} \pmod{p}$$

which implies that

$$\sum_{k=0}^{p-1}\binom{2k}{k}x^k \equiv \sum_{k=0}^{\frac{p-1}{2}}\binom{\frac{p-1}{2}}{k}(-4x)^k \equiv (1-4x)^{\frac{p-1}{2}} \pmod{p}. \tag{7}$$

Furthermore, by [7, (14)], we have that

$$\left(\sum_{k=0}^{p-1}\binom{2k}{k}x^k\right)\cdot\left(\sum_{k=1}^{p-1}\binom{2k}{k}\frac{x^k}{k}\right) \equiv 2\sum_{k=1}^{p-1}\binom{2k}{k}(H_{2k}-H_k)x^k \pmod{p} \tag{8}$$

where $H_k = \sum_{j=1}^{k}\frac{1}{j}$ is the k-th *harmonic number.*

In view of the known identity $\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^k}{k}=-H_n$, it follows that

$$\begin{aligned}\sum_{k=1}^{p-1}\binom{2k}{k}H_kx^k &\equiv \sum_{k=1}^{\frac{p-1}{2}}\binom{\frac{p-1}{2}}{k}H_k(-4x)^k\\ &= -\sum_{k=1}^{\frac{p-1}{2}}\binom{\frac{p-1}{2}}{k}(-4x)^k\sum_{j=1}^{k}\binom{k}{j}\frac{(-1)^j}{j}\\ &= -\sum_{j=1}^{\frac{p-1}{2}}\frac{(-1)^j}{j}\sum_{k=j}^{\frac{p-1}{2}}\binom{\frac{p-1}{2}}{k}\binom{k}{j}(-4x)^k\\ &= -\sum_{j=0}^{\frac{p-1}{2}}\frac{(-1)^j}{j}\binom{\frac{p-1}{2}}{j}(-4x)^j(1-4x)^{\frac{p-1}{2}-j}\\ &\equiv -(1-4x)^{\frac{p-1}{2}}\sum_{j=1}^{p-1}\binom{2j}{j}\frac{y^j}{j} \pmod p. \qquad (9)\end{aligned}$$

Finally, given that $h_k=H_{2k}-\frac{1}{2}H_k$ and combining (7), (8), and (9), we obtain

$$\begin{aligned}\sum_{k=1}^{p-1}\binom{2k}{k}h_kx^k &= \sum_{k=1}^{p-1}\binom{2k}{k}(H_{2k}-H_k)x^k+\frac{1}{2}\sum_{k=1}^{p-1}\binom{2k}{k}H_kx^k\\ &\equiv \frac{1}{2}(1-4x)^{\frac{p-1}{2}}\left(\sum_{k=1}^{p-1}\binom{2k}{k}\frac{x^k}{k}-\sum_{k=1}^{p-1}\binom{2k}{k}\frac{y^k}{k}\right) \pmod p.\end{aligned}$$

□

If $x=\frac{1}{2}$ then $y=-\frac{x}{1-4x}=\frac{1}{2}$ and congruence (6) immediately implies the following corollary.

Corollary 3. *For any prime $p>2$,*

$$\sum_{k=1}^{p-1}\binom{2k}{k}\frac{h_k}{2^k}\equiv 0 \pmod p.$$

It is worth pointing out that, based on a numerical experiment, the above congruence seems to have the following conjectural q-analogue: for any odd integer n,

$$\sum_{k=1}^{n-1}\begin{bmatrix}2k\\k\end{bmatrix}_q\frac{q^k}{\prod_{j=1}^{k}(1+q^j)}\sum_{j=1}^{k}\frac{q^{2j-1}}{[2j-1]_q}\equiv(-1)^{\frac{n-1}{2}}\frac{(n-1)(q-1)}{2}q^{\frac{n^2-1}{4}} \pmod{\Phi_n(q)} \qquad (10)$$

where $[n]_q = (1-q^n)/(1-q)$, for $0 \le k \le n$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$$

is the q-binomial coefficient with $(z;q)_n = (1-z)(1-zq)\cdots(1-zq^{n-1})$, and $\Phi_n(q)$ denotes the n-th cyclotomic polynomial.

3 Congruences involving Fibonacci and Lucas numbers

We first establish the next two congruences modulo p^2, which extend [8, (2.9) and (2.10)].

Lemma 4. *Let $\varepsilon = \left(\frac{5}{p}\right)$ and let p be a prime different from 2 and 5. Then for any integer $n \ge 0$,*

$$L_{np} \equiv L_n + \frac{5n}{2} F_n F_{p-\varepsilon} \pmod{p^2}, \tag{11}$$

$$F_{np} \equiv \varepsilon \left(F_n + \frac{n}{2} L_n F_{p-\varepsilon} \right) \pmod{p^2}. \tag{12}$$

Proof. We show both congruences by induction with respect to n. They are are clearly true for $n = 0$. For $n = 1$, by [8, (2.9)], we have $L_{p-\varepsilon} \equiv 2\varepsilon \pmod{p^2}$. Therefore

$$L_p = \frac{1}{2}\left(L_{p-\varepsilon} L_\varepsilon + 5F_{p-\varepsilon} F_\varepsilon\right) \equiv \varepsilon L_\varepsilon + \frac{5}{2} F_{p-\varepsilon} F_\varepsilon = L_1 + \frac{5}{2} F_1 F_{p-\varepsilon} \pmod{p^2}$$

and

$$F_p = \frac{1}{2}\left(L_{p-\varepsilon} F_\varepsilon + F_{p-\varepsilon} L_\varepsilon\right) \equiv \varepsilon F_\varepsilon + \frac{\varepsilon}{2} L_1 F_{p-\varepsilon} \equiv \varepsilon \left(F_1 + \frac{1}{2} L_1 F_{p-\varepsilon} \right) \pmod{p^2}$$

where we applied the identities

$$2L_{n+m} = L_n L_m + 5F_n F_m, \qquad 2F_{n+m} = L_n F_m + F_n L_m,$$

and $\varepsilon L_\varepsilon = 1 = L_1$, $F_\varepsilon = 1 = F_1$.

Inductive step for (11): for $n \ge 3$, by the multiple-angle recurrence for L_{np}, and because p divides $F_{p-\varepsilon}$, we get

$$\begin{aligned} L_{np} &= L_p L_{(n-1)p} - (-1)^p L_{(n-2)p} \\ &\equiv \left(1 + \frac{5}{2} F_{p-\varepsilon}\right)\left(L_{n-1} + \frac{5(n-1)}{2} F_{n-1} F_{p-\varepsilon}\right) + \left(L_{n-2} + \frac{5(n-2)}{2} F_{n-2} F_{p-\varepsilon}\right) \\ &\equiv (L_{n-1} + L_{n-2}) + \frac{5}{2} F_{p-\varepsilon}\left(L_{n-1} + (n-1)F_{n-1} + (n-2)F_{n-2}\right) \\ &\equiv L_n + \frac{5}{2} F_{p-\varepsilon}\left(F_n + (n-1)F_{n-1} + (n-1)F_{n-2}\right) \\ &\equiv L_n + \frac{5n}{2} F_n F_{p-\varepsilon} \pmod{p^2}. \end{aligned}$$

A similar argument shows the inductive step for (12):

$$\begin{aligned}
F_{np} &= L_p F(n-1)p - (-1)^p F_{(n-2)p} \\
&\equiv \left(1+\frac{5}{2}F_{p-\varepsilon}\right)\left(F_{n-1}+\frac{(n-1)}{2}L_{n-1}F_{p-\varepsilon}\right)+\left(F_{n-2}+\frac{(n-2)}{2}L_{n-2}F_{p-\varepsilon}\right) \\
&\equiv (F_{n-1}+F_{n-2})+\frac{5}{2}F_{p-\varepsilon}\left(F_{n-1}+(n-1)L_{n-1}+(n-2)L_{n-2}\right) \\
&\equiv F_n+\frac{5}{2}F_{p-\varepsilon}\left(L_n+(n-1)L_{n-1}+(n-1)L_{n-2}\right) \\
&\equiv F_n+\frac{5n}{2}L_nF_{p-\varepsilon} \pmod{p^2}.
\end{aligned}$$

□

In the next two theorems we specialize the polynomial congruences (3) and (6) obtained in the previous section to the numbers u_n and v_n.

Theorem 5. *Let $n \geq 1$ and let $p > 2$ be a prime not dividing $5L_nF_n$. Then*

$$\sum_{k=1}^{p-1}\binom{2k}{k}\frac{u_n^k}{k} \equiv -q_p(u_n) - \frac{5nF_n}{L_n}f_p \pmod{p} \tag{13}$$

$$\sum_{k=1}^{p-1}\binom{2k}{k}\frac{v_n^k}{k} \equiv -q_p(v_n) - \frac{nL_n}{F_n}f_p \pmod{p}. \tag{14}$$

Proof. Let $\varphi_{\pm} = (1 \pm \sqrt{5})/2$. For $\alpha = \varphi_+^n/L_n$, we have that

$$\alpha(1-\alpha) = \frac{\varphi_+^n(L_n-\varphi_+^n)}{L_n^2} = \frac{\varphi_+^n(\varphi_-^n)}{L_n^2} = \frac{(-1)^n}{L_n^2} = u_n.$$

Let $x = u_n$ in (3). Then we find

$$\begin{aligned}
\sum_{k=1}^{p-1}\binom{2k}{k}\frac{u_n^k}{k} &\equiv 2\frac{1-\alpha^p-(1-\alpha)^p}{p} \\
&= \frac{2}{pL_n^p}\left(L_n^p-\varphi_+^{np}-(L_n-\varphi_+^n)^p\right) \\
&= \frac{2}{L_n^p}\cdot\frac{L_n^p-L_{np}}{p} \\
&= \frac{2}{L_n^{p-1}}q_p(L_n)-\frac{2}{L_n^p}\cdot\frac{L_{np}-L_n}{p} \\
&\equiv 2q_p(L_n)-\frac{2}{L_n}\cdot\frac{5nF_nF_{p-\varepsilon}}{2p} \\
&\equiv -q_p(u_n)-\frac{5nF_n}{L_n}f_p \pmod{p}
\end{aligned}$$

where we applied Fermat's little theorem, (11), and $F_nL_n = F_{2n}$. Also recall that $q_p(x\cdot y) \equiv q_p(x) + q_p(y) \pmod p$.

For $\alpha = \varphi_+^n/(\sqrt{5}F_n)$, we have

$$\alpha(1-\alpha) = \frac{\varphi_+^n(\sqrt{5}F_n - \varphi_+^n)}{5F_n^2} = \frac{\varphi_+^n(-\varphi_-^n)}{5F_n^2} = \frac{(-1)^{n-1}}{5F_n^2} = v_n.$$

In the same manner, setting $x = v_n$ in (3), we obtain

$$\begin{aligned}\sum_{k=1}^{p-1}\binom{2k}{k}\frac{v_n^k}{k} &\equiv 2\frac{1-\alpha^p-(1-\alpha)^p}{p}\\ &= \frac{2}{p5^{p/2}F_n^p}\left(5^{p/2}F_n^p - \varphi_+^{np} - (\sqrt{5}F_n - \varphi_+^n)^p\right)\\ &= \frac{2}{5^{p/2}F_n^p}\cdot\frac{5^{p/2}F_n^p - \sqrt{5}F_{np}}{p}\\ &= \frac{2}{F_n^{p-1}}q_p(F_n) - \frac{2}{5^{\frac{p-1}{2}}F_n^p}\cdot\frac{F_{np} - 5^{\frac{p-1}{2}}F_n}{p}\\ &\equiv 2q_p(F_n) + q_p(5) - \frac{2\varepsilon}{F_n}\cdot\frac{F_{np}-\varepsilon F_n}{p}\\ &\equiv -q_p(v_n) - \frac{nL_n}{F_n}f_p \pmod p,\end{aligned}$$

where we used (12), and

$$5^{\frac{p-1}{2}} \equiv \left(\frac{5}{p}\right)\left(1 + \frac{p}{2}q_p(5)\right) \pmod{p^2},$$

which is a consequence of [5, Lemma 4.1]. □

Theorem 6. *Let $n \geq 1$ and let $p > 2$ be a prime not dividing $5L_nF_n$. Then*

$$\sum_{k=1}^{p-1}\binom{2k}{k}h_ku_n^k \equiv \left(\frac{5}{p}\right)\left(\frac{1}{2}q_p\left(\frac{v_n}{u_n}\right) + (-1)^n\frac{2n}{F_{2n}}f_p\right) \pmod p \tag{15}$$

and

$$\sum_{k=1}^{p-1}\binom{2k}{k}h_kv_n^k \equiv -\sum_{k=1}^{p-1}\binom{2k}{k}h_ku_n^k \pmod p. \tag{16}$$

Proof. Setting $x = u_n$, the identity $L_n^2 - 5F_n^2 = 4(-1)^n$ yields

$$1 - 4x = \frac{L_n^2 - 4(-1)^n}{L_n^2} = \frac{5F_n^2}{L_n^2},$$

and therefore

$$y = \frac{-x}{1-4x} = -\frac{(-1)^n}{L_n^2} \cdot \frac{L_n^2}{5F_n^2} = \frac{(-1)^{n-1}}{5F_n^2} = v_n.$$

Moreover

$$(1-4x)^{\frac{p-1}{2}} = 5^{\frac{p-1}{2}} \frac{F_n^{p-1}}{L_n^{p-1}} \equiv \left(\frac{5}{p}\right) \pmod p.$$

Thus, by (6), (13), and (14), we obtain (15). Indeed,

$$\begin{aligned}
\sum_{k=1}^{p-1} \binom{2k}{k} h_k u_n^k &\equiv \frac{1}{2}(1-4u_n)^{\frac{p-1}{2}} \left(\sum_{k=1}^{p-1} \binom{2k}{k} \frac{u_n^k}{k} - \sum_{k=1}^{p-1} \binom{2k}{k} \frac{v_n^k}{k} \right) \\
&\equiv \frac{1}{2} \left(\frac{5}{p}\right) \left(q_p(v_n) - q_p(u_n) + \left(\frac{L_n}{F_n} - \frac{5F_n}{L_n} \right) n f_p \right) \\
&\equiv \frac{1}{2} \left(\frac{5}{p}\right) \left(q_p\left(\frac{v_n}{u_n}\right) + \frac{4(-1)^n}{F_{2n}} n f_p \right) \pmod p.
\end{aligned}$$

It remains to show (16). Since

$$1 - 4v_n = 1 - 4y = 1 + \frac{4x}{1-4x} = (1-4x)^{-1} = (1-4u_n)^{-1},$$

it follows that

$$(1-4v_n)^{\frac{p-1}{2}} = (1-4u_n)^{-\frac{p-1}{2}} \equiv \left(\frac{5}{p}\right)^{-1} \equiv \left(\frac{5}{p}\right) \pmod p$$

and we are done because of (6). □

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References

[1] K. N. Boyadzhiev, Series with central binomial coefficients, Catalan numbers, and harmonic numbers, *J. Integer Sequences* **15** (2012), Article 12.1.7.

[2] H. Chen, Interesting series associated with central binomial coefficients, Catalan numbers and harmonic numbers, *J. Integer Sequences* **19** (2016), Article 16.1.5.

[3] W. Chu and D. Zheng, Infinite series with harmonic numbers and central binomial coefficients, *Int. J. Number Theory* **5** (2009), 429–448.

[4] D. H. Lehmer, Interesting series involving the central binomial coefficient, *Amer. Math. Monthly* **92** (1985), 449–457.

[5] S. Mattarei and R. Tauraso, Congruences for central binomial sums and finite polylogarithms, *J. Number Theory* **133** (2013), 131–157.

[6] S. Mattarei and R. Tauraso, From generating series to polynomial congruences, *J. Number Theory* **182** (2018), 179–205.

[7] R. Tauraso, Some congruences for central binomial sums involving Fibonacci and Lucas numbers, *J. Integer Sequences* **19** (2016), Article 16.5.4.

[8] H. C. Williams, A note on the Fibonacci quotient $F_{p-\varepsilon}/p$, *Can. Math. Bull.* **25** (1982), 366–370.

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