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The (l, r)-Lah Numbers

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Abstract

After reviewing the definitions and some properties of the Lah numbers and the Stirling numbers of both kinds, as well as their generalizations (r-Lah numbers, r-Stirling numbers of both kinds and (l, r)-Stirling numbers of both kinds), we define the (l, r)-Lah numbers analogously, prove a recurrence relation that they satisfy, express them explicitly as a multiple sum, and present the difference-differential equations satisfied by their column and row generating functions, respectively. Finally, we pose two conjectures, based on experimental evidence.

1 Introduction

After binomial coefficients, the *Stirling numbers of the first and second kind* are probably the best known triangular arrays of natural numbers, arising in many combinatorial, algebraic, and even analytic contexts. Recognized at least since the 18th century, they were joined and nicely complemented by the *Lah numbers* (a.k.a. *Stirling numbers of the third kind*) in the 1950s. In the 1980s, a third parameter was added to the definition of Stirling numbers, resulting in the *r-Stirling numbers of the first and second kind*. In the last decade, the *r-Lah numbers* were defined and their properties explored. In 2021, a fourth parameter was added to the definition of r-Stirling numbers, yielding the *(l, r)-Stirling numbers of the first and second kind*. Here we define and explore the analogous *(l, r)-Lah numbers*.

The article is organized as follows. Section 2 contains definitions of the Lah and r-Lah numbers, lists some recurrence relations that they satisfy, expresses them explicitly in terms of factorials, and presents some of their applications. With the exception of explicit representation, Section 3 does the same for the Stirling, r-Stirling, and (l,r)-Stirling numbers of both kinds. In Section 4, we define the new (l,r)-Lah numbers, prove a recurrence relation that they satisfy, express them explicitly in terms of a multiple sum, and present the difference-differential equations satisfied by their column and row generating functions, respectively. Finally, in Section 5, we pose two conjectures, based on experimental evidence.

2 The Lah and r-Lah numbers

The *unsigned Lah numbers* $\left\lfloor {n \atop k} \right\rfloor$ count partitions of the set $[n] := \{1,2,\ldots,n\}$ into k non-empty linearly ordered blocks (*lists*, for short).

For $n,k \geq 1$, they satisfy the recurrence

$$\left\lfloor {n \atop k} \right\rfloor = (n+k-1)\left\lfloor {n-1 \atop k} \right\rfloor + \left\lfloor {n-1 \atop k-1} \right\rfloor \tag{1}$$

which, together with the initial conditions $\left\lfloor {0 \atop 0} \right\rfloor = 1$, $\left\lfloor {n \atop 0} \right\rfloor = \left\lfloor {0 \atop k} \right\rfloor = 0$ for $n,k > 0$, can be used to compute $\left\lfloor {n \atop k} \right\rfloor$ for all $n,k \geq 0$.

$n \backslash k$	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0
2	0	2	1	0	0	0	0	0	0
3	0	6	6	1	0	0	0	0	0
4	0	24	36	12	1	0	0	0	0
5	0	120	240	120	20	1	0	0	0
6	0	720	1800	1200	300	30	1	0	0
7	0	5040	15120	12600	4200	630	42	1	0
8	0	40320	141120	141120	58800	11760	1176	56	1

Table 1: Unsigned Lah numbers $\left\lfloor {n \atop k} \right\rfloor$ for $0 \leq n,k \leq 8$.

They can also be expressed explicitly in terms of factorials and binomial coefficients as

$$\left\lfloor {n \atop k} \right\rfloor = \frac{n!}{k!}\binom{n-1}{n-k}. \tag{2}$$

For simple combinatorial proofs of (1) and (2), see, e.g., [13]. Another interesting property of Lah numbers is that they appear in the coefficients in the expansion of rising powers

$$x^{\overline{n}} = x(x+1)(x+2)\cdots(x+n-1)$$

in terms of falling powers

$$x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1)$$

and vice versa, namely:

$$x^{\overline{n}} \;=\; \sum_{k=0}^{n} \left\lfloor {n \atop k} \right\rfloor x^{\underline{k}}, \qquad x^{\underline{n}} \;=\; \sum_{k=0}^{n} (-1)^{n+k} \left\lfloor {n \atop k} \right\rfloor x^{\overline{k}}$$

for all $n \geq 0$ and all x, where $x^{\overline{0}} = x^{\underline{0}} = 1$. These identities can easily be proved by induction on n.

Lah numbers appear not only in combinatorics and algebra, but also in other areas of mathematics, such as mathematical analysis. For instance, the n-th derivative of the function $e^{\frac{1}{x}}$ can be expressed for all $n \geq 0$ as

$$\left(e^{\frac{1}{x}}\right)^{(n)} \;=\; e^{\frac{1}{x}}(-1)^n \sum_{k=0}^{n} \left\lfloor {n \atop k} \right\rfloor x^{-(n+k)} \;=\; e^{\frac{1}{x}} \sum_{k=0}^{n} \left\lfloor {n \atop k} \right\rfloor' x^{-(n+k)}, \tag{3}$$

which again can be proved by induction on n. Here $\left\lfloor {n \atop k} \right\rfloor' = (-1)^n \left\lfloor {n \atop k} \right\rfloor$ are the *signed Lah numbers*, introduced in [10] by Slovenian mathematician, statistician, and actuary Ivo Lah (1896–1979), and later named after him by Riordan in [14, pp. 43–44].

In the last decade, the *r-Lah numbers*, denoted by $\left\lfloor {n \atop k} \right\rfloor_r$ (counting partitions of $[n]$ into k non-empty lists such that the elements $1, 2, \ldots, r$ are in distinct lists) were defined, and their properties explored (see [7, 1, 2, 15, 11, 16, 12]; note however that in [15, 11, 16, 12], the r-Lah number with parameters n, k, r equals our r-Lah number with parameters $n+r, k+r, r$).

For $n, k \geq 1$, the r-Lah numbers satisfy the recurrence

$$\left\lfloor {n \atop k} \right\rfloor_r \;=\; (n+k-1) \left\lfloor {n-1 \atop k} \right\rfloor_r + \left\lfloor {n-1 \atop k-1} \right\rfloor_r \tag{4}$$

which, together with the initial conditions $\left\lfloor {n \atop k} \right\rfloor_r = 0$ if $n < k$ or $k < r$, and $\left\lfloor {n \atop n} \right\rfloor_n = 1$, allows us to compute $\left\lfloor {n \atop k} \right\rfloor_r$ for all $n, k, r \geq 0$. They are expressed explicitly by the formula

$$\left\lfloor {n \atop k} \right\rfloor_r \;=\; \frac{(n+r-1)!}{(k+r-1)!} \binom{n-r}{k-r}. \tag{5}$$

Note that r-Lah numbers have applications in graph theory [12]. They also satisfy identities connecting rising and falling powers such as

$$(x+2r)^{\overline{n}} \;=\; \sum_{k=0}^{n} \left\lfloor {n+r \atop k+r} \right\rfloor_r x^{\underline{k}}, \tag{6}$$

$$(x-r)^{\underline{n}} \;=\; \sum_{k=0}^{n} (-1)^{n+k} \left\lfloor {n+r \atop k+r} \right\rfloor_r (x+r)^{\overline{k}}. \tag{7}$$

For proofs of identities (4)–(7), see [1] and [11].

$n \backslash k$	2	3	4	5	6	7	8
2	1	0	0	0	0	0	0
3	4	1	0	0	0	0	0
4	20	10	1	0	0	0	0
5	120	90	18	1	0	0	0
6	840	840	252	28	1	0	0
7	6720	8400	3360	560	40	1	0
8	60480	90720	45360	10080	1080	54	1

Table 2: 2-Lah numbers $\left\lfloor {n \atop k} \right\rfloor_2$ for $2 \le n, k \le 8$.

3 The Stirling, r-Stirling, and (l, r)-Stirling numbers

Stirling numbers were introduced by the Scottish mathematician James Stirling (1692–1770) in 1730. They come in two flavors: unsigned Stirling numbers of the first kind $\left[{n \atop k} \right]$ count permutations of the set $[n]$ with exactly k cycles (or, equivalently, partitions of the set $[n]$ into k cyclically ordered blocks), while unsigned Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$ count partitions of the set $[n]$ into k unordered blocks.

Unsigned Stirling numbers of the first kind for $n, k \ge 1$ satisfy the recurrence

$$\left[{n \atop k} \right] = (n-1)\left[{n-1 \atop k} \right] + \left[{n-1 \atop k-1} \right] \tag{8}$$

which, together with the initial conditions $\left[{0 \atop 0} \right] = 1$, $\left[{n \atop 0} \right] = \left[{0 \atop k} \right] = 0$ for $n, k > 0$, can be used to compute $\left[{n \atop k} \right]$ for all $n, k \ge 0$. They are also the coefficients in the expansion of rising powers in terms of ordinary powers:

$$x^{\overline{n}} = \sum_{k=0}^{n} \left[{n \atop k} \right] x^k. \tag{9}$$

Similarly, *unsigned Stirling numbers of the second kind* for $n, k \ge 1$ satisfy the recurrence

$$\left\{ {n \atop k} \right\} = k\left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\} \tag{10}$$

which, together with the initial conditions $\left\{ {0 \atop 0} \right\} = 1$, $\left\{ {n \atop 0} \right\} = \left\{ {0 \atop k} \right\} = 0$ for $n, k > 0$, can be used to compute $\left\{ {n \atop k} \right\}$ for all $n, k \ge 0$. They are also the coefficients in the expansion of ordinary powers in terms of falling powers:

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}}. \tag{11}$$

There is a nice identity expressing Lah numbers with Stirling numbers of both kinds, namely

$$\left\lfloor {n \atop k} \right\rfloor = \sum_{j=0}^{n} \left[{n \atop j} \right] \left\{ {j \atop k} \right\}. \tag{12}$$

For proofs of identities (8)–(12), see, e.g., [8] or [13].

In the 1980's, Stirling numbers of both kinds were generalized to r-Stirling numbers (cf. [5, 6, 4]). The *unsigned r-Stirling numbers of the first kind* $\left[{n \atop k} \right]_r$ count permutations of $[n]$ having k cycles such that the numbers $1, 2, \ldots, r$ are in distinct cycles. They satisfy the same recurrence as the numbers $\left[{n \atop k} \right]$

$$\left[{n \atop k} \right]_r = (n-1) \left[{n-1 \atop k} \right]_r + \left[{n-1 \atop k-1} \right]_r, \quad \text{for} \quad n, k \geq 1, \tag{13}$$

which, together with initial conditions $\left[{n \atop k} \right]_r = 0$ for $n < k$ or $k < r$, and $\left[{n \atop n} \right]_r = 1$ for $r \leq n$, can be used to compute $\left[{n \atop k} \right]_r$ for all $n, k, r \geq 0$. Another, so-called 'cross' recurrence (in which r changes but n stays fixed) satisfied by the unsigned r-Stirling numbers of the first kind is

$$\left[{n \atop k} \right]_r = \frac{1}{r-1} \left(\left[{n \atop k-1} \right]_{r-1} - \left[{n \atop k-1} \right]_r \right), \quad \text{for} \quad n, k \geq 1, \ r \geq 2 \tag{14}$$

with initial conditions $\left[{n \atop k} \right]_r = \left[{n \atop k} \right]$ for $r \leq 1$, and $\left[{n \atop k} \right]_r = 0$ for $n < k$ or $k < r$.

The *unsigned r-Stirling numbers of the second kind* $\left\{ {n \atop k} \right\}_r$ count partitions of $[n]$ into k nonempty disjoint subsets such that the numbers $1, 2, \ldots, r$ are in distinct subsets. They satisfy the same recurrence as the numbers $\left\{ {n \atop k} \right\}$

$$\left\{ {n \atop k} \right\}_r = k \left\{ {n-1 \atop k} \right\}_r + \left\{ {n-1 \atop k-1} \right\}_r, \quad \text{for} \quad n, k \geq 1, \tag{15}$$

which, together with initial conditions $\left\{ {n \atop k} \right\}_r = 0$ for $n < k$ or $k < r$, and $\left\{ {n \atop n} \right\}_r = 1$ for $r \leq n$, can be used to compute $\left\{ {n \atop k} \right\}_r$ for all $n, k, r \geq 0$. The 'cross' recurrence satisfied by $\left\{ {n \atop k} \right\}_r$ is

$$\left\{ {n \atop k} \right\}_r = \left\{ {n \atop k} \right\}_{r-1} - (r-1) \left\{ {n-1 \atop k} \right\}_{r-1}, \quad \text{for} \quad n, k, r \geq 1 \tag{16}$$

with initial conditions $\left\{ {n \atop k} \right\}_0 = \left\{ {n \atop k} \right\}$, and $\left\{ {n \atop k} \right\}_r = 0$ for $n < k$ or $k < r$.

For proofs of identities (13)–(16), see [4].

In 2021, Belbachir and Djemmada [3] introduced (l, r)-*Stirling numbers of the first kind* $\left[{n \atop k} \right]_r^{(l)}$ and (l, r)-*Stirling numbers of the second kind* $\left\{ {n \atop k} \right\}_r^{(l)}$ which count ordered l-tuples $(\pi_1, \pi_2, \ldots, \pi_l)$ of partitions of $[n]$ into k cyclically ordered blocks, respectively into k unordered blocks, such that the numbers $1, 2, \ldots, r$ are in distinct cycles, and satisfy

$$\operatorname{bl} \pi_1 = \operatorname{bl} \pi_2 = \cdots = \operatorname{bl} \pi_l$$

where for $i = 1, 2, \ldots, l$ and $\pi_i = \{b_1^{(i)}, b_2^{(i)}, \ldots, b_k^{(i)}\}$, where $b_1^{(i)}, \ldots b_k^{(i)}$ are the blocks of partition π_i, we denote by

$$\operatorname{bl} \pi_i = \{\min b_1^{(i)}, \min b_2^{(i)}, \ldots, \min b_k^{(i)}\}$$

the set of *block leaders*, i.e., of *minima* of the blocks of partition π_i.

Note that the (l, r)-Stirling numbers of the second kind are a generalization of the *central factorial numbers* of the second kind, see, e.g., [9].

Example 1. Let us compute $\left\{ {4 \atop 3} \right\}_2^{(2)}$. Here $n = 4$, $k = 3$, $l = r = 2$, so we need to construct all partitions of the set $[4] = \{1, 2, 3, 4\}$ into three nonempty blocks, such that 1 and 2 are in distinct blocks. There are five such partitions:

$$\begin{aligned}
\pi_1 &= \{\{1\}, \{2\}, \{3, 4\}\}, \\
\pi_2 &= \{\{1\}, \{2, 3\}, \{4\}\}, \\
\pi_3 &= \{\{1\}, \{2, 4\}, \{3\}\}, \\
\pi_4 &= \{\{1, 3\}, \{2\}, \{4\}\}, \\
\pi_5 &= \{\{1, 4\}, \{2\}, \{3\}\},
\end{aligned}$$

and the sets of their block leaders are

$$\begin{aligned}
\operatorname{bl} \pi_1 = \operatorname{bl} \pi_3 = \operatorname{bl} \pi_5 &= \{1, 2, 3\}, \\
\operatorname{bl} \pi_2 = \operatorname{bl} \pi_4 &= \{1, 2, 4\}.
\end{aligned}$$

Now we compute the number of ordered l-tuples (i.e., ordered pairs) of partitions $\pi_1, \pi_2, \pi_3, \pi_4, \pi_5$ such that partitions in the same pair share the same set of block leaders. As there are three partitions with the set of block leaders equal to $\{1, 2, 3\}$, and two partitions with the set of block leaders equal to $\{1, 2, 4\}$, we find that

$$\left\{ {4 \atop 3} \right\}_2^{(2)} = 3^2 + 2^2 = 13.$$

The (l, r)-Stirling numbers of the first kind $\left[{n \atop k} \right]_r^{(l)}$ satisfy the recurrence

$$\left[{n \atop k} \right]_r^{(l)} = (n-1)^l \left[{n-1 \atop k} \right]_r^{(l)} + \left[{n-1 \atop k-1} \right]_r^{(l)}, \quad \text{for } n > k \geq r \geq 0, \tag{17}$$

which, together with initial conditions $\left[{n \atop k} \right]_r^{(l)} = 0$ for $n < k$ or $k < r$, and $\left[{n \atop k} \right]_n^{(l)} = \delta_{n,k}$, where $\delta_{n,k} = 1$ if $n = k$ and $\delta_{n,k} = 0$ if $n \neq k$, can be used to compute the numbers $\left[{n \atop k} \right]_r^{(l)}$ for all $n, k, l, r \geq 0$. They also satisfy the 'cross' recurrence

$$\left[{n \atop k} \right]_r^{(l)} = \frac{1}{(r-1)^l} \left(\left[{n \atop k-1} \right]_{r-1}^{(l)} - \left[{n \atop k-1} \right]_r^{(l)} \right), \quad \text{for } n, k, l \geq 1, \ r \geq 2. \tag{18}$$

The (l,r)-Stirling numbers of the second kind $\left\{ {n \atop k} \right\}_r^{(l)}$ satisfy the recurrence

$$\left\{ {n \atop k} \right\}_r^{(l)} = k^l \left\{ {n-1 \atop k} \right\}_r^{(l)} + \left\{ {n-1 \atop k-1} \right\}_r^{(l)}, \quad \text{for} \;\; n > k \geq r \geq 0, \tag{19}$$

which, together with initial conditions $\left\{ {n \atop k} \right\}_r^{(l)} = 0$ for $n < k$ or $k < r$, and $\left\{ {n \atop k} \right\}_n^{(l)} = \delta_{n,k}$, can be used to compute the numbers $\left\{ {n \atop k} \right\}_r^{(l)}$ for all $n, k, l, r \geq 0$. They also satisfy the 'cross' recurrence

$$\left\{ {n \atop k} \right\}_r^{(l)} = \left\{ {n \atop k} \right\}_{r-1}^{(l)} - (r-1)^l \left\{ {n-1 \atop k} \right\}_{r-1}^{(l)}, \quad \text{for} \;\; n, k, l, r \geq 1. \tag{20}$$

For proofs of identities (17)–(20), see [3].

4 The (l,r)-Lah numbers

Definition 2. *(l,r)-Lah numbers* $\left\lfloor {n \atop k} \right\rfloor_r^{(l)}$ count ordered l-tuples $(\pi_1, \pi_2, \ldots, \pi_l)$ of partitions of $[n]$ into k linearly ordered blocks (lists, for short) such that the numbers $1, 2, \ldots, r$ are in distinct lists, and

$$\operatorname{bl} \pi_1 = \operatorname{bl} \pi_2 = \cdots = \operatorname{bl} \pi_l$$

where for $i = 1, 2, \ldots, l$ and $\pi_i = \{b_1^{(i)}, b_2^{(i)}, \ldots, b_k^{(i)}\}$, where $b_1^{(i)}, b_2^{(i)}, \ldots, b_k^{(i)}$ are the blocks of partition π_i,

$$\operatorname{bl} \pi_i = \{\min b_1^{(i)}, \min b_2^{(i)}, \ldots, \min b_k^{(i)}\}$$

is the set of block leaders, i.e., of minima of the lists in partition π_i.

Example 3. Let us compute $\left\lfloor {4 \atop 3} \right\rfloor_2^{(2)}$. Here $n = 4$, $k = 3$, $l = r = 2$, so we need to construct all partitions of the set $[4] = \{1, 2, 3, 4\}$ into three nonempty disjoint lists, such that 1 and 2 are in distinct lists. There are 10 such partitions:

$$\begin{aligned}
\pi_1 &= \{(1), (2), (3,4)\}, \\
\pi_2 &= \{(1), (2), (4,3)\}, \\
\pi_3 &= \{(1), (2,3), (4)\}, \\
\pi_4 &= \{(1), (3,2), (4)\}, \\
\pi_5 &= \{(1), (2,4), (3)\}, \\
\pi_6 &= \{(1), (4,2), (3)\}, \\
\pi_7 &= \{(1,3), (2), (4)\}, \\
\pi_8 &= \{(3,1), (2), (4)\}, \\
\pi_9 &= \{(1,4), (2), (3)\}, \\
\pi_{10} &= \{(4,1), (2), (3)\},
\end{aligned}$$

and the sets of their block leaders are

$$\begin{aligned}
&\operatorname{bl}\pi_1 = \operatorname{bl}\pi_2 = \operatorname{bl}\pi_5 = \operatorname{bl}\pi_6 = \operatorname{bl}\pi_9 = \operatorname{bl}\pi_{10} = \{1,2,3\},\\
&\operatorname{bl}\pi_3 = \operatorname{bl}\pi_4 = \operatorname{bl}\pi_7 = \operatorname{bl}\pi_8 = \{1,2,4\}.
\end{aligned}$$

Now we compute the number of ordered l-tuples (i.e., ordered pairs) of partitions $\pi_1, \pi_2, \ldots, \pi_{10}$ such that partitions in the same pair share the same set of block leaders. As there are six partitions with the set of block leaders equal to $\{1,2,3\}$, and four partitions with the set of block leaders equal to $\{1,2,4\}$, we find that

$$\left\lfloor {4 \atop 3} \right\rfloor_2^{(2)} \;=\; 6^2 + 4^2 \;=\; 52.$$

Theorem 4. *For $n \geq k \geq r \geq 1$, (l,r)-Lah numbers satisfy the recurrence relation*

$$\left\lfloor {n \atop k} \right\rfloor_r^{(l)} = (n+k-1)^l \left\lfloor {n-1 \atop k} \right\rfloor_r^{(l)} + \left\lfloor {n-1 \atop k-1} \right\rfloor_r^{(l)}. \tag{21}$$

Proof. Let $\pi = (\pi_1, \pi_2, \ldots, \pi_l)$ be an l-tuple of partitions of $[n]$ into k nonempty lists such that the numbers $1, 2, \ldots, r$ are in distinct lists, and $\operatorname{bl}\pi_1 = \operatorname{bl}\pi_2 = \cdots = \operatorname{bl}\pi_l$. Each such l-tuple π is of exactly one of the following two types:

Type A: There is an $i \in [l]$ such that n is alone in its list in partition π_i. Then n is a block leader in each of $\pi_1, \pi_2, \ldots, \pi_l$; hence n is alone in its list in each of the π_i. By deleting this list from each of the π_i, we obtain an l-tuple of partitions $\rho = (\rho_1, \rho_2, \ldots, \rho_l)$ of $[n-1]$ into $k-1$ nonempty lists such that the numbers $1, 2, \ldots, r$ are in distinct lists, and $\operatorname{bl}\rho_1 = \operatorname{bl}\rho_2 = \cdots = \operatorname{bl}\rho_l$. Going back, by appending the list (n) to each partition ρ_i we recover the initial l-tuple π, hence the number of l-tuples π of partitions of type A is $\left\lfloor {n-1 \atop k-1} \right\rfloor_r^{(l)}$.

Type B: Otherwise, for each $i \in [l]$, the number n is not alone in its list in π_i, hence n is not a block leader in any of $\pi_1, \pi_2, \ldots, \pi_l$. By deleting n from each of the π_i, we obtain an l-tuple of partitions $\rho = (\rho_1, \rho_2, \ldots, \rho_l)$ of the set $[n-1]$ into k nonempty lists such that the numbers $1, 2, \ldots, r$ are in distinct lists, and $\operatorname{bl}\rho_1 = \operatorname{bl}\rho_2 = \cdots = \operatorname{bl}\rho_l$. Going back, by inserting n into some position of some list in each of the ρ_i, we obtain an l-tuple of partitions of $[n]$ into k nonempty lists such that the numbers $1, 2, \ldots, r$ are in distinct lists, and the sets of block leaders of these partitions are the same. As the number of possible insertion points in a list equals its length plus one, there are $(n-1)+k = n+k-1$ possible insertion points for n in a single partition ρ_i, and $(n+k-1)^l$ possibilities for the l-tuple of partitions ρ. Hence the number of l-tuples π of partitions of type B is $(n+k-1)^l \left\lfloor {n-1 \atop k} \right\rfloor_r^{(l)}$.

It follows that the number $\left\lfloor {n \atop k} \right\rfloor_r^{(l)}$ of all such l-tuples π of partitions of $[n]$ equals the sum of the number of partitions of type A and the number of partitions of type B, which is $\left\lfloor {n-1 \atop k-1} \right\rfloor_r^{(l)} + \; (n+k-1)^l \left\lfloor {n-1 \atop k} \right\rfloor_r^{(l)}$. $\square$

This recurrence, together with initial conditions $\left\lfloor {n \atop k} \right\rfloor_r^{(l)} = 0$ for $n < k$ or $k < r$, $\left\lfloor {n \atop n} \right\rfloor_r^{(l)} = 1$, and $\left\lfloor {r \atop k} \right\rfloor_r^{(l)} = \delta_{k,r}$ can be used to compute the numbers $\left\lfloor {n \atop k} \right\rfloor_r^{(l)}$ for all $n, k, l, r \geq 0$.

$n \backslash k$	2	3	4	5	6	7	8
2	1	0	0	0	0	0	0
3	16	1	0	0	0	0	0
4	400	52	1	0	0	0	0
5	14400	2948	116	1	0	0	0
6	705600	203072	12344	216	1	0	0
7	45158400	17154432	1437472	38480	360	1	0
8	3657830400	1760601600	191088544	6978592	99320	556	1

Table 3: (2, 2)-Lah numbers $\left\lfloor {n \atop k} \right\rfloor_2^{(2)}$ for $2 \leq n, k \leq 8$.

$n \backslash k$	3	4	5	6	7	8	9
3	1	0	0	0	0	0	0
4	36	1	0	0	0	0	0
5	1764	100	1	0	0	0	0
6	112896	9864	200	1	0	0	0
7	9144576	1099296	34064	344	1	0	0
8	914457600	142159392	6004512	92200	540	1	0
9	110649369600	21385410048	1156921920	24075712	213700	796	1

Table 4: (2, 3)-Lah numbers $\left\lfloor {n \atop k} \right\rfloor_3^{(2)}$ for $3 \leq n, k \leq 9$.

Proposition 5 (some special cases)**.**

$$\left\lfloor {n \atop r} \right\rfloor_r^{(l)} = \left(\left\lfloor {n \atop r} \right\rfloor_r \right)^l = \frac{(n+r-1)!^l}{(2r-1)!^l}, \tag{22}$$

$$\left\lfloor {n \atop n-1} \right\rfloor_r^{(l)} = 2^l \sum_{j=r}^{n-1} j^l. \tag{23}$$

Proof. By Definition 2, $\left\lfloor {n \atop r} \right\rfloor_r^{(l)}$ is the number of ordered l-tuples $(\pi_1, \pi_2, \ldots, \pi_l)$ of partitions of $[n]$ into r lists such that the numbers $1, 2, \ldots, r$ are in distinct lists, and $\operatorname{bl} \pi_1 = \operatorname{bl} \pi_2 = \cdots = \operatorname{bl} \pi_l$. As there are r lists in each partition, and $1, 2, \ldots, r$ are in distinct lists, we have

$\operatorname{bl}\pi = [r]$ for all partitions π, so $\left\lfloor {n \atop r} \right\rfloor_r^{(l)}$ is simply the l-th power of the number of partitions of $[n]$ into r lists such that the numbers $1, 2, \ldots, r$ are in distinct lists. Hence $\left\lfloor {n \atop r} \right\rfloor_r^{(l)} = \left(\left\lfloor {n \atop r} \right\rfloor_r \right)^l$, proving the first equality in (22), while the second equality in (22) follows from (5).

By (21) with $k = n - 1$, we have for $n \geq 1$

$$\begin{aligned} \left\lfloor {n \atop n-1} \right\rfloor_r^{(l)} &= (2n-2)^l \left\lfloor {n-1 \atop n-1} \right\rfloor_r^{(l)} + \left\lfloor {n-1 \atop n-2} \right\rfloor_r^{(l)} \\ &= \left\lfloor {n-1 \atop n-2} \right\rfloor_r^{(l)} + (2n-2)^l \end{aligned} \tag{24}$$

since $\left\lfloor {n-1 \atop n-1} \right\rfloor_r^{(l)} = 1$. Using (24) repeatedly, we obtain

$$\begin{aligned} \left\lfloor {n \atop n-1} \right\rfloor_r^{(l)} &= \underline{\left\lfloor {n-1 \atop n-2} \right\rfloor_r^{(l)}} + (2n-2)^l \\ \underline{\left\lfloor {n-1 \atop n-2} \right\rfloor_r^{(l)}} &= \underline{\left\lfloor {n-2 \atop n-3} \right\rfloor_r^{(l)}} + (2n-4)^l \\ &\vdots \\ \underline{\left\lfloor {r+1 \atop r} \right\rfloor_r^{(l)}} &= \left\lfloor {r \atop r-1} \right\rfloor_r^{(l)} + (2r)^l. \end{aligned}$$

Summing these equations and cancelling all the underlined terms, we find

$$\left\lfloor {n \atop n-1} \right\rfloor_r^{(l)} = \left\lfloor {r \atop r-1} \right\rfloor_r^{(l)} + \sum_{j=r}^{n-1} (2j)^l = 2^l \sum_{j=r}^{n-1} j^l,$$

which is (23).

$\square$

Theorem 6. *(l, r)-Lah numbers can be expressed explicitly as*

$$\left\lfloor {n \atop k} \right\rfloor_r^{(l)} = \sum_{r+1 \leq j_1 < j_2 < \cdots < j_{n-k} \leq n} (2j_1 - 2)^l (2j_2 - 3)^l \cdots (2j_{n-k} - (n-k+1))^l. \tag{25}$$

Proof. We will compute $\left\lfloor {n \atop k} \right\rfloor_r^{(l)}$ by counting in how many ways one can construct an l-tuple $\pi = (\pi_1, \pi_2, \ldots, \pi_l)$ of partitions of the set $[n]$ into k lists such that $1, 2, \ldots, r$ are in distinct lists, and $\operatorname{bl}\pi_1 = \operatorname{bl}\pi_2 = \cdots = \operatorname{bl}\pi_l$. In π_1 there will be k leading elements $1, 2, \ldots, r, \lambda_{r+1}, \lambda_{r+2}, \ldots, \lambda_k \in [n]$ with $r + 1 \leq \lambda_{r+1} < \lambda_{r+2} < \cdots < \lambda_k \leq n$, and $n - k$ non-leading elements $\nu_1, \nu_2, \ldots, \nu_{n-k} \in [n]$ with $r + 1 \leq \nu_1 < \nu_2 < \cdots < \nu_{n-k} \leq n$.

Starting with a set of k lists of length 1 containing their leaders, i.e., with

$$\{(1), (2), \ldots, (r), (\lambda_{r+1}), (\lambda_{r+2}), \ldots, (\lambda_k)\},$$

we insert the non-leading elements $\nu_1, \nu_2, \ldots, \nu_{n-k}$ one after another into the above lists, counting the ways to do this as we go along. A non-leading element ν_i can be inserted into a list iff this list's leader is smaller than ν_i; on the other hand, ν_i can be inserted into any position within any such list. We now prove that for $i = 1, 2, \ldots, n-k$, *there are exactly* $2\nu_i - (i+1)$ *possible insertion points for* ν_i. Recall that when inserting an element into any of k given lists in any position, where the sum of the list lengths is ℓ, the total number of possible insertion points is $k + \ell$.

$i = 1$: Since ν_1 is the least non-leader, all the smaller elements $1, 2, \ldots, \nu_1 - 1$ are leaders, hence there $\nu_1 - 1$ lists of length 1 into which ν_1 can be inserted. Hence there are $(\nu_1 - 1) + (\nu_1 - 1) = 2\nu_1 - 2$ possible insertion points for ν_1.

$i > 1$: Assume that all the $\nu_i - 1$ elements smaller than ν_i have already been inserted into the lists. All of them, except for $\nu_1, \nu_2, \ldots, \nu_{i-1}$, are leaders, hence there are $\nu_i - 1 - (i-1) = \nu_i - i$ lists into which ν_i can be inserted. These lists jointly contain $\nu_i - 1$ elements, so there are $(\nu_i - i) + (\nu_i - 1) = 2\nu_i - (i+1)$ possible insertion points for ν_i as claimed.

In all, we thus have $(2\nu_1 - 2)(2\nu_2 - 3)\cdots(2\nu_{n-k} - (n-k+1))$ ways to construct partition π_1, and the same number of ways also for each of $\pi_2, \pi_3, \ldots, \pi_l$ independently. Hence the number of ways to construct an l-tuple $\pi = (\pi_1, \pi_2, \ldots, \pi_l)$ for a fixed choice of $\nu_1, \nu_2, \ldots, \nu_{n-k}$ is $(2\nu_1 - 2)^l (2\nu_2 - 3)^l \cdots (2\nu_{n-k} - (n-k+1))^l$, and the total number of such l-tuples is obtained by summing this product over all possible choices of $\nu_1, \nu_2, \ldots, \nu_{n-k}$. Hence

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{(l)} = \sum_{r+1 \le \nu_1 < \nu_2 < \cdots < \nu_{n-k} \le n} (2\nu_1 - 2)^l (2\nu_2 - 3)^l \cdots (2\nu_{n-k} - (n-k+1))^l,$$

which proves the theorem. $\square$

Theorem 7. *For $k \ge r \ge 1$ and $l \ge 1$, the column generating function*

$$F_k^{l,r}(z) = \sum_{n=0}^{\infty} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{(l)} z^n$$

of (l,r)-Lah numbers, as a formal power series, satisfies the difference-differential equation

$$F_k^{l,r}(z) = zF_{k-1}^{l,r}(z) + \sum_{j=0}^{l} \binom{l}{j} (k-1)^{l-j} \sum_{i=0}^{j} \left\{ \begin{matrix} j \\ i \end{matrix} \right\} z^i \frac{d^i}{dz^i} \left(zF_k^{l,r}(z) \right). \tag{26}$$

Proof. Multiplying recurrence (21) by z^n and formally summing it on n from 2 to ∞ yields

$$\sum_{n=2}^{\infty} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{(l)} z^n \;=\; \sum_{n=2}^{\infty} (n+k-1)^l \left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rfloor_r^{(l)} z^n + \sum_{n=2}^{\infty} \left\lfloor \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rfloor_r^{(l)} z^n. \tag{27}$$

Rewrite the left-hand side of (27) as

$$\sum_{n=2}^{\infty} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{(l)} z^n \;=\; F_k^{l,r}(z) - \left\lfloor \begin{matrix} 0 \\ k \end{matrix} \right\rfloor_r^{(l)} - \left\lfloor \begin{matrix} 1 \\ k \end{matrix} \right\rfloor_r^{(l)} z \;=\; \begin{cases} F_k^{l,r}(z), & \text{if } k \geq 2; \\ F_k^{l,r}(z) - z, & \text{if } k = 1. \end{cases}$$

Rewrite the second term on the right-hand side of (27) as

$$\begin{aligned} \sum_{n=2}^{\infty} \left\lfloor \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rfloor_r^{(l)} z^n \;&=\; \sum_{n=1}^{\infty} \left\lfloor \begin{matrix} n \\ k-1 \end{matrix} \right\rfloor_r^{(l)} z^{n+1} \;=\; z \left(F_{k-1}^{l,r}(z) - \left\lfloor \begin{matrix} 0 \\ k-1 \end{matrix} \right\rfloor_r^{(l)} \right) \\ &=\; z \begin{cases} F_{k-1}^{l,r}(z), & \text{if } k \geq 2; \\ F_{k-1}^{l,r}(z) - 1, & \text{if } k = 1, \end{cases} \;=\; \begin{cases} zF_{k-1}^{l,r}(z), & \text{if } k \geq 2; \\ zF_{k-1}^{l,r}(z) - z, & \text{if } k = 1, \end{cases} \end{aligned}$$

and the first term on the right-hand side of (27) as

$$\begin{aligned} \sum_{n=2}^{\infty} (n+k-1)^l \left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rfloor_r^{(l)} z^n \;&=\; \sum_{n=2}^{\infty} \sum_{j=0}^{l} \binom{l}{j} n^j (k-1)^{l-j} \left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rfloor_r^{(l)} z^n \\ &=\; \sum_{j=0}^{l} \binom{l}{j} (k-1)^{l-j} \sum_{n=2}^{\infty} \left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rfloor_r^{(l)} n^j z^n \\ &=\; \sum_{j=0}^{l} \binom{l}{j} (k-1)^{l-j} \sum_{n=2}^{\infty} \left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rfloor_r^{(l)} \sum_{i=0}^{j} \left\{ \begin{matrix} j \\ i \end{matrix} \right\} n^{\underline{i}} z^n \\ &=\; \sum_{j=0}^{l} \binom{l}{j} (k-1)^{l-j} \sum_{i=0}^{j} \left\{ \begin{matrix} j \\ i \end{matrix} \right\} \sum_{n=1}^{\infty} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{(l)} (n+1)^{\underline{i}} z^{n+1} \\ &=\; \sum_{j=0}^{l} \binom{l}{j} (k-1)^{l-j} \sum_{i=0}^{j} \left\{ \begin{matrix} j \\ i \end{matrix} \right\} \sum_{n=1}^{\infty} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{(l)} z^i \frac{d^i}{dz^i} z^{n+1} \\ &=\; \sum_{j=0}^{l} \binom{l}{j} (k-1)^{l-j} \sum_{i=0}^{j} \left\{ \begin{matrix} j \\ i \end{matrix} \right\} z^i \frac{d^i}{dz^i} \left(z \sum_{n=1}^{\infty} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{(l)} z^n \right) \\ &=\; \sum_{j=0}^{l} \binom{l}{j} (k-1)^{l-j} \sum_{i=0}^{j} \left\{ \begin{matrix} j \\ i \end{matrix} \right\} z^i \frac{d^i}{dz^i} \left(zF_k^{l,r}(z) \right). \end{aligned}$$

Putting the three rewritten terms of equation (27) together again, we obtain

$$\begin{cases} F_k^{l,r}(z), & \text{if } k \geq 2; \\ F_k^{l,r}(z) - z, & \text{if } k = 1, \end{cases} = \begin{cases} zF_{k-1}^{l,r}(z), & \text{if } k \geq 2; \\ zF_{k-1}^{l,r}(z) - z, & \text{if } k = 1, \end{cases}$$
$$+ \sum_{j=0}^{l} \binom{l}{j} (k-1)^{l-j} \sum_{i=0}^{j} \left\{ {j \atop i} \right\} z^i \frac{d^i}{dz^i} \left(zF_k^{l,r}(z) \right),$$

which is equivalent to equation (26). □

Theorem 8. *For $n-1 \geq r \geq 1$ and $l \geq 1$, the row generating function*

$$P_n^{l,r}(x) = \sum_{k=0}^{n} \left\lfloor {n \atop k} \right\rfloor_r^{(l)} x^k \tag{28}$$

of (l,r)-Lah numbers, which is a polynomial in x of degree n, satisfies the difference-differential equation

$$P_n^{l,r}(x) \;=\; x\, P_{n-1}^{l,r}(x) + \sum_{j=0}^{l} \binom{l}{j} (n-1)^{l-j} \sum_{i=0}^{j} \left\{ {j \atop i} \right\} x^i \frac{d^i}{dx^i} \left(P_{n-1}^{l,r}(x) \right). \tag{29}$$

Proof. Multiplying recurrence (21) by x^k and summing it on k from 2 to n yields

$$\sum_{k=2}^{n} \left\lfloor {n \atop k} \right\rfloor_r^{(l)} x^k \;=\; \sum_{k=2}^{n} (n+k-1)^l \left\lfloor {n-1 \atop k} \right\rfloor_r^{(l)} x^k + \sum_{k=2}^{n} \left\lfloor {n-1 \atop k-1} \right\rfloor_r^{(l)} x^k. \tag{30}$$

Rewrite the left-hand side of (30) as

$$\sum_{k=2}^{n} \left\lfloor {n \atop k} \right\rfloor_r^{(l)} x^k \;=\; P_n^{l,r}(x) - \left\lfloor {n \atop 0} \right\rfloor_r^{(l)} - \left\lfloor {n \atop 1} \right\rfloor_r^{(l)} x.$$

Rewrite the second term on the right-hand side of (30) as

$$\sum_{k=2}^{n} \left\lfloor {n-1 \atop k-1} \right\rfloor_r^{(l)} x^k \;=\; \sum_{k=1}^{n-1} \left\lfloor {n-1 \atop k} \right\rfloor_r^{(l)} x^{k+1} \;=\; x \left(P_{n-1}^{l,r}(x) - \left\lfloor {n-1 \atop 0} \right\rfloor_r^{(l)} \right)$$

and the first term on the right-hand side of (30) as

$$\sum_{k=2}^{n} (n+k-1)^l \left\lfloor {n-1 \atop k} \right\rfloor_r^{(l)} x^k = \sum_{k=2}^{n} \sum_{j=0}^{l} \binom{l}{j} k^j (n-1)^{l-j} \left\lfloor {n-1 \atop k} \right\rfloor_r^{(l)} x^k$$

$$
\begin{aligned}
&= \sum_{j=0}^{l} \binom{l}{j} (n-1)^{l-j} \sum_{k=2}^{n} \left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rfloor_r^{(l)} k^j x^k \\
&= \sum_{j=0}^{l} \binom{l}{j} (n-1)^{l-j} \sum_{k=2}^{n} \left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rfloor_r^{(l)} \sum_{i=0}^{j} \left\{ \begin{matrix} j \\ i \end{matrix} \right\} k^{\underline{i}} x^k \\
&= \sum_{j=0}^{l} \binom{l}{j} (n-1)^{l-j} \sum_{i=0}^{j} \left\{ \begin{matrix} j \\ i \end{matrix} \right\} \sum_{k=2}^{n} \left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rfloor_r^{(l)} k^{\underline{i}} x^k \\
&= \sum_{j=0}^{l} \binom{l}{j} (n-1)^{l-j} \sum_{i=0}^{j} \left\{ \begin{matrix} j \\ i \end{matrix} \right\} \sum_{k=2}^{n} \left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rfloor_r^{(l)} x^i \frac{d^i}{dx^i} x^k \\
&= \sum_{j=0}^{l} \binom{l}{j} (n-1)^{l-j} \sum_{i=0}^{j} \left\{ \begin{matrix} j \\ i \end{matrix} \right\} x^i \frac{d^i}{dx^i} \left(\sum_{k=2}^{n} \left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rfloor_r^{(l)} x^k \right) \\
&= \sum_{j=0}^{l} \binom{l}{j} (n-1)^{l-j} \sum_{i=0}^{j} \left\{ \begin{matrix} j \\ i \end{matrix} \right\} x^i \frac{d^i}{dx^i} \left(P_{n-1}^{l,r}(x) - \left\lfloor \begin{matrix} n-1 \\ 0 \end{matrix} \right\rfloor_r^{(l)} - \left\lfloor \begin{matrix} n-1 \\ 1 \end{matrix} \right\rfloor_r^{(l)} x \right).
\end{aligned}
$$

Putting the three rewritten terms of equation (30) together again, and checking that

$$
\begin{aligned}
\left\lfloor \begin{matrix} n \\ 0 \end{matrix} \right\rfloor_r^{(l)} + \left\lfloor \begin{matrix} n \\ 1 \end{matrix} \right\rfloor_r^{(l)} = {} & \left\lfloor \begin{matrix} n-1 \\ 0 \end{matrix} \right\rfloor_r^{(l)} x \\
& + \sum_{j=0}^{l} \binom{l}{j} (n-1)^{l-j} \sum_{i=0}^{j} \left\{ \begin{matrix} j \\ i \end{matrix} \right\} x^i \frac{d^i}{dx^i} \left(\left\lfloor \begin{matrix} n-1 \\ 0 \end{matrix} \right\rfloor_r^{(l)} + \left\lfloor \begin{matrix} n-1 \\ 1 \end{matrix} \right\rfloor_r^{(l)} x \right)
\end{aligned}
$$

for all $n-1 \geq r \geq 1$ and $l \geq 1$, we obtain equation (29). □

5 Conclusion

We conclude the paper with two conjectures, based on experimental evidence.

Conjecture 9. For $n \geq k \geq r \geq 1$, the sequence $\left(\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{(l)} \right)_{k=0}^{n}$ is strictly log-concave.

Conjecture 10. All the roots of polynomials $P_n^{l,r}(x)$, defined in (28), are real and non-positive.

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