Journal of Integer Sequences, Vol. 26 (2023),
Article 23.9.8

Polynomial Identities Involving Binomial Coefficients and Double and Rising Factorials via Probabilistic Interpretations

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Abstract

We formulate several polynomial identities. One side of these identities has a nice simple form, whereas the other has the form of a polynomial whose coefficients contain binomial coefficients, double factorials, or rising factorials. The origins and the proofs of these identities are probabilistic. However, their form suggests universal applications in simplifying expressions. Many useful simplifying formulae are presented in the sequel.

1 Introduction

In this note, we will present some identities involving factorials, binomial coefficients, and the so-called rising factorials (sometimes called Pochhammer symbols). Many of them have the forms of polynomials whose coefficients often have the form of binomial coefficients or rising factorials of some additional variables. These variables appear on both sides of the identity. The domains of these variables can be extended to all complex numbers. All these identities have a common origin, except for the sometimes similar form. Namely, the origins of them are a few, rather deep probabilistic interpretations and sometimes following non-trivial computations.

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The basic idea behind the method of getting these identities is to calculate moments of a probability distributions in two ways and compare the results. One of the ways is straightforward and direct. The other way involves the calculation of some conditional expectation and then calculating the proper moment. This method was already successfully applied to obtain nontrivial relationships between Catalan numbers and the so-called Catalan triangles. In this case, the so-called Kesten distribution was used. For details, see [10]. In this paper, we use the bivariate Normal and Gamma distributions to obtain nontrivial polynomial identities (see Theorem 1). These identities can be the source of an infinite number of relationships between various number sequences important for example in combinatorics. As a corollary, we present several identities involving binomial coefficients, factorials, double factorials, and raising factorials.

The paper is organized as follows. The next section is dedicated to the presentation of the identities and some of the particular, interesting particular cases. The next section is devoted to the presentation of the probabilistic background of the results presented in the previous section and then, finally, the presentation of the calculations leading to the identities.

2 Identities

In this paper, the symbol $n!!$ denotes the so-called double-factorial, i.e.,

$$n!! = \prod_{j=0}^{\lfloor (n-1)/2 \rfloor} (n-2j).$$

We set $(-1)!! = (0)!! = 1$. The symbol $\lfloor x \rfloor$ denotes the largest integer not exceeding x, i.e., the so-called "floor function". Further, let the symbol $\binom{n}{k}$ denote the binomial coefficient and we set $\binom{n}{k} = 0$ when $n < k$. The symbol i always denotes the imaginary unit, i.e., $i = \sqrt{-1} = \exp(i\pi/2)$. In order to simplify the notation, let us introduce also the so-called "rising factorial" (sometimes called also Pochhammer symbol), which is the following function:

$$(x)^{(n)} = x(x+1)\cdots(x+n-1),$$

defined for all complex x. Notice that we have for all $x \neq 0$:

$$(x)^{(n)} = \frac{\Gamma(x+n)}{\Gamma(x)},$$

where $\Gamma(x)$ denotes Euler's gamma function. To learn more about Pochhammer symbols and binomial coefficients, see, e.g., [6]. Notice only that, e.g., $(1)^{(n)} = n!$ and $(1/2)^{(n)} = (2n-1)!!/2^n$. We will use these values below.

Theorem 1. *For all nonegative integers k, n, m and all complex ρ and β we have*

(i)

$$\frac{(2k)!}{k!}(1+\rho)^k = \sum_{j=0}^{k} \binom{2k}{2j} (1+\rho)^{2k-2j}(1-\rho^2)^j(2j-1)!!(2k-2j-1)!!. \tag{1}$$

(ii)

$$(1+\rho)^k = \frac{1}{2^k}\sum_{j=0}^{2k}\sum_{m=0}^{\lfloor j/2\rfloor}(2\rho)^{j-2m}\binom{k}{j-2m}\binom{k-j+2m}{m}. \tag{2}$$

(iii)

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{(\beta)^{(j+m)}}{(\beta)^{(j)}} = (-1)^n n!\binom{m}{n}\frac{(\beta)^{(m)}}{(\beta)^{(n)}}. \tag{3}$$

(iv)

$$\begin{aligned}\sum_{m=0}^{n}(-1)^m\binom{n}{m}(\beta)^{(m)}(\beta)^{(n-m)}\sum_{k=0}^{m}\binom{m}{k}\binom{n-m}{k}\frac{\rho^k k!}{(\beta)^{(k)}} \\ = \begin{cases} 0, & \textit{if } n \textit{ is odd;} \\ \frac{n!}{(n/2)!}(\beta)^{(n/2)}(1-\rho)^{n/2}, & \textit{if } n \textit{ is even.}\end{cases}\end{aligned} \tag{4}$$

(v)

$$\begin{aligned}&\sum_{m=0}^{n}(-1)^{n-m}\binom{n}{m}\sum_{j=0}^{m}\binom{m}{j}(1-\rho)^j\rho^{m-j}(\beta)^{(n-j)}(\beta+m-j)^{(j)} \\ &= \begin{cases} 0, & \textit{if } n \textit{ is odd;} \\ \frac{n!}{(n/2)!}(\beta)^{(n/2)}(1-\rho)^{n/2}, & \textit{if } n \textit{ is even.}\end{cases}\end{aligned} \tag{5}$$

Below we present some particular cases, remarks and corollaries.

Remark 2. First, taking $\rho = \frac{2}{3}$, then $\rho = \frac{1}{3}$, and finally $\rho = \frac{4}{3}$ in (1), we get

$$\sum_{j=0}^{k}\binom{2k}{2j}5^j(2j-1)!!(2k-2j-1)!! = \frac{(2k)!3^k}{k!},$$

$$\sum_{j=0}^{k}\binom{2k}{2j}2^j(2j-1)!!(2k-2j-1)!! = \frac{(2k)!3^k}{k!2^k},$$

$$\sum_{j=0}^{k}\binom{2k}{2j}(-7)^j(2j-1)!!(2k-2j-1)!! = (-1)^k\frac{(2k)!3^k}{k!}.$$

Remark 3. Let us take $\rho = \sqrt{5}$ in (1) and recall that $(1+\sqrt{5})^n = 2^n(\frac{1+\sqrt{5}}{2})^n = 2^{n-1}(L_n + \sqrt{5}F_n)$, where L_n and F_n are, respectively, the n-th Lucas and Fibonacci numbers (also compare Corollary 16, below). Now, equating integer and irrational parts, we end up with the following two identities valid for $k \geq 0$:

$$\frac{(2k)!}{k!}L_k = 2^k \sum_{j=0}^{k} \binom{2k}{2j}(-1)^j L_{2k-2j}(2j-1)!!(2k-2j-1)!!,$$

$$\frac{(2k)!}{k!}F_k = 2^k \sum_{j=0}^{k} \binom{2k}{2j}(-1)^j F_{2k-2j}(2j-1)!!(2k-2j-1)!!$$

Remark 4. Take $\rho = i$ in (1). Further notice that $(1+i)^k = 2^{k/2}\exp(ik\pi/4)$. Now first set $k = 4n$ and cancel $(-4)^n$ on both sides. Then for all $n \geq 0$ we get

$$\frac{(8n)!}{(4n)!} = \sum_{j=0}^{4n} \binom{8n}{2j}(2j-1)!!(8n-2j-1)!!.$$

Next, set $k = 4n+1$ and equate the real and imaginary parts. For all $n \geq 0$ we get

$$\frac{(8n+2)!}{(4n+1)!}(-4)^n = 2^{4n+1}\sum_{m=0}^{2n}(-1)^m \binom{8n+2}{4m+2}(4m+1)!!(8n-4m-1)!!,$$

$$\frac{(8n+2)!}{(4n+1)!}(-4)^n = 2^{4n+1}\sum_{m=0}^{2n}(-1)^m \binom{8n+2}{4m}(4m-1)!!(8n-4m+1)!!.$$

Remark 5. Take $\rho = 1/2$ in (2). We then get the following identity:

$$1 = \sum_{j=0}^{2k}(-1)^j \sum_{m=0}^{\lfloor j/2 \rfloor} \binom{k}{j-2m}\binom{k-j+2m}{m}.$$

Remark 6. Take $\rho = i$ and $k = 4n$ in (2). Further, notice that $(1+i)^{4n} = (-4)^n$ and $(2i)^{j-2m} = i^j(-1)^m$ and finally split the right-hand side into real and imaginary parts. We get

$$4^n = \frac{(-1)^n}{16^n}\sum_{j=0}^{4n}(-1)^j \sum_{m=0}^{j}(-1)^m 4^{j-m}\binom{4n}{2j-2m}\binom{4n-2j+2m}{m},$$

$$0 = \sum_{j=0}^{4n}(-1)^j \sum_{m=0}^{j}(-1)^m 4^{j-m}\binom{4n}{2j-2m+1}\binom{4n-2j-1+2m}{m},$$

for all $n \geq 0$.

Remark 7. First set $\beta = 1$ and then $\beta = 1/2$ in (3). Then for all integers k and n, we get

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{j+k}{j} = (-1)^n\binom{k}{n},$$

$$\sum_{j=0}^{n}(-1)^j\frac{(2n-1)!!(2j+2k-1)!!}{j!(n-j)!(2j-1)!!} = (-1)^n 2^n\binom{k}{n}(2k-1)!!.$$

Remark 8. First set $\beta = 1/2$ and then $\beta = 1$ in (4). Then for all integers k and n and all complex ρ, we get

$$\sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\frac{(2n+2k+2j-1)!!}{(2j-1)!!} = 2^n\frac{(n+k)!(2n+2k-1)!!}{k!(2n-1)!!},$$

$$\sum_{m=0}^{n}(-1)^m\sum_{k=0}^{m}\binom{m}{k}\binom{n-m}{k}\rho^k = \begin{cases}0, & \text{if } n \text{ is odd;}\\ (1-\rho)^{n/2}, & \text{if } n \text{ is even.}\end{cases} \tag{6}$$

Remark 9. Now let us change the order of summation in (6) and compare the coefficients in expansions in powers of ρ. We get

$$\sum_{m=k}^{n}(-1)^{m-k}\binom{m}{k}\binom{n-m}{k} = \begin{cases}0, & \text{if } n \text{ is odd;}\\ \binom{n/2}{k}, & \text{if } n \text{ is even.}\end{cases}$$

Remark 10. Let us set additionally $\rho = 1$ in (6). Then for all $n \geq 0$, we get

$$\sum_{m=0}^{n}(-1)^m\sum_{k=0}^{m}\binom{m}{k}\binom{n-m}{k} = 0.$$

Remark 11. Let us take $\beta = 1/2$ in (4). Next, let us set $2\rho = x$ and cancel $n!$. We then get

$$\sum_{m=0}^{n}(-1)^m(2m-1)!!(2n-2m-1)!!\sum_{k=0}^{m}\frac{1}{k!(m-k)!(n-m-k)!}\frac{x^k}{(2k-1)!!} \tag{7}$$
$$= \begin{cases}0, & \text{if } n \text{ is odd;}\\ \frac{(n-1)!!}{(n/2)!}(2-x)^{n/2}, & \text{if } n \text{ is even.}\end{cases}$$

for all $n \geq 0$. Additionally, setting $x = 0$ in (7), we get

$$\sum_{m=0}^{n}(-1)^m\binom{n}{m}(2m-1)!!(2n-2m-1)!! = \begin{cases}0, & \text{if } n \text{ is odd;}\\ \frac{n!(n-1)!!}{(n/2)!}2^{n/2}, & \text{if } n \text{ is even.}\end{cases}$$

Remark 12. Let us set $\rho = 1$ in (4). For all complex $\beta \neq 0$ and $n \geq 0$ we get

$$\sum_{m=0}^{n}(-1)^m\binom{n}{m}(\beta)^{(m)}(\beta)^{(n-m)}\sum_{k=0}^{m}\binom{m}{k}\binom{n-m}{k}\frac{k!}{(\beta)^{(k)}} = 0.$$

Remark 13. Let us set $\rho = 0$ in (5). We get

$$\sum_{m=0}^{n}(-1)^{n-m}\binom{n}{m}(\beta)^{(n-m)}(\beta)^{(m)} = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ \frac{n!}{(n/2)!}(\beta)^{(n/2)}, & \text{if } n \text{ is even,} \end{cases}$$

for all $n \geq 0$. This is related to, but not a direct generalization of, the equality

$$\sum_{m=0}^{n}\binom{n}{m}(\beta)^{(n-m)}(\alpha)^{(m)} = (\alpha+\beta)^{(n)}.$$

Remark 14. Let us set $\rho = 1/2$ in (5). Then for all complex β and $n \geq 0$, we get

$$\sum_{m=0}^{n}(-1)^{n-m}\binom{n}{m}2^{-m}\sum_{j=0}^{m}\binom{m}{j}(\beta)^{(n-j)}(\beta+m-j)^{(j)}$$
$$= \begin{cases} 0, & \text{if } n \text{ is odd;} \\ \frac{n!}{2^{n/2}(n/2)!}(\beta)^{(n/2)}, & \text{if } n \text{ is even.} \end{cases}$$

Remark 15. Let us change the order of summation in (5), then compare the coefficients in expansions in powers of ρ, then multiply both sides by $(\beta)^{(k)}(n-k)!$ and finally cancel $n!$. Then for all complex β and $0 \leq k \leq n$ we get

$$\sum_{m=k}^{n}(-1)^{m-k}\frac{(n-k)!}{(m-k)!(n-m-k)!}(\beta)^{(m)}(\beta)^{(n-m)}$$
$$= \begin{cases} 0, & \text{if } n \text{ is odd;} \\ (n/2)!\binom{n-k}{n/2}(\beta)^{(k)}(\beta)^{(n/2)}, & \text{if } n \text{ is even.} \end{cases}$$

We also have the following relationships between Fibonacci $\{F_n\}_{n\geq 0}$ and Lucas $\{L_n\}_{n\geq 0}$ numbers. For Fibonacci and Lucas numbers, see, e.g., [3, 4] and [7, pp. 153–256]. For other relationships between Fibonacci and Lucas numbers and the so-called Catalan triangles (expressed by binomial coefficients), see, e.g., [12].

Corollary 16. *Let ϕ denote the number $(\sqrt{5}-1)/2$, which is the reciprocal of the golden ratio. Take $\rho = -\phi$ in (1). We get*

$$2\frac{(2k)!}{k!}L_k = \sum_{j=0}^{k}(-1)^j\binom{2k}{2j}(2j-1)!!(2k-2j-1)!!(L_{2k-2j}L_j - 5F_{2k-2j}F_j),$$

$$2\frac{(2k)!}{k!}F_k = \sum_{j=0}^{k}(-1)^j\binom{2k}{2j}(2j-1)!!(2k-2j-1)!!(F_{2k-2j}L_j - L_{2k-2j}F_j).$$

Proof. We know that for all $n \geq 0$ we have

$$(1+\phi)^n = ((1+\sqrt{5})/2)^n = L_n/2 + F_n\sqrt{5}/2.$$

Now, also notice that we have $\phi = 1/(1+\phi)$ and that

$$L_n^2 - 5F_n^2 = (-1)^n 4.$$

Hence, we have

$$\phi^n = (-1)^n(L_n/2 - F_n\sqrt{5}/2).$$

Now we take $\rho = -\phi$ and substitute it in (1). Notice that $1-\rho = (1+\phi)$, and hence $(1-\rho)^{2k-2j} = L_{2k-2j}/2 + F_{2k-2j}\sqrt{5}/2$ and $1-\rho^2 = \phi$, consequently $(1-\rho^2)^j = (-1)^j(L_j/2 - F_j\sqrt{5}/2)$. It remains to multiply and separate terms with and without $\sqrt{5}$. □

3 Probabilistic background and the proofs

3.1 Probabilistic background

All identities presented in Theorem 1 stem from calculating the moments $E(X-Y)^n$, $n \geq 0$, where X and Y are two normalized random variables. The joint distribution of these random variables has one parameter and is either the bivariate Normal (Gaussian) or bivariate Gamma distribution. More precisely, in the first case, the joint distribution of (X, Y) has the following well-known density, valid for all $x, y \in \mathbb{R}$ and $|\rho| < 1$:

$$f_N(x, y|\rho) = \frac{1}{2\pi(1-\rho^2)} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right), \tag{8}$$

while in the second case

$$f_G(x, y|\rho) = f_g(x|\beta) f_g(y|\beta) \frac{\exp\left(-\frac{\rho(x+y)}{1-\rho}\right)}{(1-\rho)\,(xy\rho)^{(\beta-1)/2}} I_{\beta-1}\left(\frac{2\sqrt{xy\rho}}{1-\rho}\right),$$

valid for all $x, y, \beta > 0$ and $|\rho| < 1$, where

$$f_g(x|\beta) = x^{\beta-1}\exp(-x)/\Gamma(\beta),$$

for $x > 0$ and 0 otherwise. We let I_α denote, here and below, the modified Bessel function of the first kind. The most important property of these joint densities is that they allow the so-called Lancaster expansions. To learn more about Lancaster expansions, their probabilistic interpretations, and convergence problems associated with them, see, e.g., [1, 8, 9, 13, 12, 11]. The orthogonal polynomials that we are using are well presented, for example, in two monographs [5, 13].

In the first case, we have the so-called Poisson-Mehler expansion

$$f_N(x, y|\rho) = f_N(x) f_N(x) \sum_{n\geq 0} \rho^n h_n(x) h_n(y), \tag{9}$$

where we write

$$f_N(x) = \exp(-x^2/2)/\sqrt{2\pi}$$

for simplicity, and

$$h_n(x) = H_n(x)/\sqrt{n!}.$$

$h_n(x)$ is the orthonormal modification of the so-called probabilistic Hermite polynomials $\{H_n(x)\}$, i.e., polynomials defined by the following three-term recurrence:

$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x),$$

with $H_{-1}(x) = 0$ and $H_0(x) = 1$. It is also known that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h_n(x) h_m(x) \exp(-x^2/2) dx = \delta_{nm}, \tag{10}$$

where δ_{nm} denotes Kronecker's delta.

In the second case we have

$$f_G(x, y|\rho) = f_g(x|\beta) f_g(y|\beta) \sum_{j\geq 0} \rho^n l_n(x|\beta) l_n(y|\beta), \tag{11}$$

where $l_n(x|\beta) = \sqrt{\frac{n!}{(\beta)^{(n)}}} L_n(x|\beta)$ is the orthonormal modification of the so-called generalized Laguerre polynomials $\{L_n(x|\beta)\}$, defined by the following expansions:

$$L_n(x|\beta) = \sum_{k=0}^{n} (-1)^k \frac{(\beta)^{(n)}}{(n-k)!(\beta)^{(k)}} x^k/k!. \tag{12}$$

Let us remark that the expansion (11) is known under the name Hardy-Hille formula (see, e.g., [13], p. 102). One knows that

$$\int_0^{\infty} l_n(x|\beta) l_m(x|\beta) f_g(x|\beta) dx = \delta_{n,m}. \tag{13}$$

Our aim is to calculate $\{E(X-Y)^n\}_{n\geq 0}$. We will do it in two ways.

The first way is to calculate the generating function of the set of these numbers, i.e., the function

$$g(t) = \sum_{n\geq 0} t^n E(X-Y)^n/n!.$$

Since it is known that all moments exist, it can be calculated by exchanging integration and summation. Namely, we have

$$g(t) = \int\int \exp\left(tx - ty\right) f(x, y|\rho) dxdy.$$

Obviously here and below the integral depends on the case of integration over all of $\mathbb{R}^2$, or just over $\mathbb{R}^+ \times \mathbb{R}^+$.

The second way is to calculate the number using an expansion:

$$E(-X+Y)^n = \sum_{m=0}^{n} (-1)^m E X^m Y^{n-m}.$$

Now

$$EX^m Y^{n-m} = \int\int x^m y^{n-m} f(x, y|\rho) dxdy.$$

where f is either f_N or f_G presented above. Using one of the expansions (9) or (11) we will find these moments using the numbers

$$H_{m,j} = EX^m k_j(X),$$

where $k_i(x)$ is either $h_j(x)$, if we consider the Normal case, or $l_j(x)$, if we consider the Gamma case. Since both expansions (9) and (11) have a similar structure, we have

$$EX^m Y^{n-m} = \sum_{j=0}^{\min(m,n-m)} \rho^j H_{m,j} H_{n-m,j}. \tag{14}$$

This is so since X and Y have the same distributions and since $H_{m,j} = 0$ if $j > m$ because of the orthogonality of polynomials h or l. We will calculate this function assuming either expansion (9) or (11).

So let us start with the expansion (9). First let us calculate the auxiliary numbers $H_{j,n}$. We have

$$H_{j,n} = EX^j h_n(X) = \frac{1}{\sqrt{n!}} EX^j H_n(X) = \begin{cases} 0, & \text{if } n > j \text{ or } j - n \text{ odd;} \\ \frac{j!}{2^{(j-n)/2}((j-n)/2)!\sqrt{n!}}, & \text{if } j - n \text{ is even.} \end{cases} \tag{15}$$

This is so, since polynomials h_n are orthogonal and also since it is common knowledge that

$$x^j = j! \sum_{m=0}^{\lfloor j/2 \rfloor} \frac{1}{2^m m!(j-2m)!} H_{j-2m}(x)$$

for all $n \geq 0$. Secondly, let us calculate the following auxiliary function that will simplify many further calculations.

$$\begin{aligned} m_n(t) = Eh_n(X)\exp(tX) &= \frac{1}{\sqrt{2\pi n!}}\int_{-\infty}^{\infty} H_n(x)\exp(tx)\exp(-x^2/2)dx \\ &= \frac{\exp(t^2/2)}{\sqrt{2\pi n!}}\int_{-\infty}^{\infty} H_n(x)\exp(-(x-t)^2/2)dx. \end{aligned}$$

Now let us change the variable under the integral by setting $y = x - t$. Then we get

$$m_n(t) = \frac{\exp(t^2/2)}{\sqrt{2\pi n!}}\int_{-\infty}^{\infty} H_n(y+t)\exp(-y^2/2)dy.$$

Next, we utilize the following the expansion:

$$H_n(x+y) = \sum_{j=0}^{n}\binom{n}{j}H_j(x)y^{n-j}.$$

Now, since we have (10), we see that $m_n(t) = t^n\frac{\exp(t^2/2)}{\sqrt{n!}}$ and we get

$$\begin{aligned} E\exp(tX - tY) &= \sum_{n\geq 0} E\exp(tX - tY) \\ &= \frac{1}{2\pi}\sum_{n\geq 0}\rho^n\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(tx - ty)\exp\left(-\left(x^2+y^2\right)/2\right)h_n(x)h_n(y)dxdy \end{aligned}$$

and

$$\sum_{n\geq 0}\rho^n m_n(t)m_n(-t) = \exp(t^2)\sum_{n\geq 0}(-\rho)^n t^{2n}/n! = \exp(t^2(1-\rho))\sum_{n\geq 0}t^{2n}(1-\rho)^n/n!.$$

Thus, we deduce that

$$E(X-Y)^n = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ \frac{n!}{(n/2)!}(1-\rho)^{n/2}, & \text{if } n \text{ is even.} \end{cases} \tag{16}$$

Now let us consider expansion (11). We need to recall some simple facts (see any textbook on probability). The Gamma distribution with rate parameter zero and shape parameter $\beta > 0$ is the distribution with the density f_g presented above. It is easy to see, recalling the definition of the Euler's Gamma function that

$$EX^n = \int_0^{\infty} x^n f_g(x|\beta)dx = (\beta)^{(n)}. \tag{17}$$

As before, let us calculate the set of auxiliary quantities and functions. We start with the numbers

$$H_{j,n} = EX^j l_n(X).$$

We have

$$H_{j,n} = \frac{\sqrt{n!}}{\Gamma(\beta)\sqrt{(\beta)^{(n)}}} \int_0^\infty x^j L_n^{(\beta)}(x) x^{\beta-1} \exp(-x) dx. \tag{18}$$

Now, we use the following expansion of x^j in powers of Laguerre polynomials (see [5, 13]):

$$x^j = j! \sum_{k=0}^{j} (-1)^k \frac{(\beta)^{(j)}}{(j-k)!(\beta)^{(k)}} L_k^{(\beta)}(x), \tag{19}$$

and then we use the orthogonality of the Laguerre polynomials. Hence, we have

$$H_{j,n} = (-1)^n \binom{j}{n} (\beta)^{(j)} \sqrt{\frac{n!}{(\beta)^{(n)}}}. \tag{20}$$

Let us now calculate the following auxiliary functions:

$$\begin{aligned} m_n(t) = El_n(X)\exp(tX) &= \sqrt{\frac{n!}{(\beta)^{(n)}}} E\exp(tX) L_n(X|\beta) \\ &= \sqrt{\frac{n!}{(\beta)^{(n)}}} \frac{1}{\Gamma(\beta)} \int_0^\infty \exp(xt) L_n(x|\beta) x^{\beta-1} \exp(-x) dx \\ &= \sqrt{\frac{n!}{(\beta)^{(n)}}} \frac{1}{\Gamma(\beta)} \int_0^\infty L_n(x|\beta) x^{\beta-1} \exp(-x(1-t)) dx. \end{aligned}$$

Now, let us change variables under the integral by considering $y = x(1-t)$. Then we get

$$m_n(t) = \sqrt{\frac{n!}{(\beta)^{(n)}}} \frac{1}{1-t} \frac{1}{\Gamma(\beta)} \int_0^\infty L_n(\frac{y}{1-t}|\beta) \left(\frac{y}{1-t}\right)^{\beta-1} \exp(-y) dy.$$

We will now utilize the following recurrence relation for Laguerre polynomials (see, e.g., [2, Chap. 22]):

$$L_n(y|\beta) = \sum_{j=0}^{n} L_{n-j}(x|\beta+j)(y-x)^j/j!.$$

So we get

$$m_n(t) = \sqrt{\frac{n!}{(\beta)^{(n)}}} \frac{1}{(1-t)^\beta} \frac{1}{\Gamma(\beta)} \times \int_0^\infty \left(\sum_{j=0}^{n} \left(\frac{-ty}{1-t}\right)^j L_{n-j}(x|(\beta+j))/j! \right) y^{\beta-1} \exp(-y) dy.$$

Further, we have

$$m_n(t) = \sqrt{\frac{n!}{(\beta)^{(n)}}} \frac{1}{(1-t)^\beta} \frac{1}{\Gamma(\beta)} \times \sum_{j=0}^{n} \int_0^\infty (-1)^j \frac{t^j}{(1-t)^j} L_{n-j}(y\,|(\beta+j)|) y^{j+\beta-1} \exp(-y) dy$$

$$= (-1)^n \sqrt{\frac{n!}{(\beta)^{(n)}}} \frac{t^n \Gamma(\beta+n)}{\Gamma(\beta) n! (1-t)^{\beta+n}}$$

$$= (-1)^n \sqrt{\frac{n!}{(\beta)^{(n)}}} \frac{t^n \, (\beta)^{(n)}}{n!(1-t)^{\beta+n}}.$$

Hence, we now have

$$g(t) = E \exp(tX - tY) = \sum_{n\geq 0} \rho^n \int_0^\infty \exp(tx - ty) l_n(x|\beta) l_n(y|\beta) f_g(x|\beta) f_g(y|\beta) dx dy$$

$$= \sum_{n\geq 0} \rho^n m_n(t) m_n(-t) = \sum_{n\geq 0} \rho^n \frac{n!(-1)^n t^{2n} \, (\beta)^{(n)} \, (\beta)^{(n)}}{(\beta)^{(n)} \, n! n! (1-t^2)^{n+\beta}}$$

$$= \frac{1}{(1-t^2)^\beta} \sum_{n\geq 0} \rho^n \frac{(-1)^n t^{2n} \, (\beta)^{(n)}}{n!(1-t^2)^n}$$

$$= \frac{1}{(1-t^2)^\beta} \sum_{n\geq 0} (-1)^n \left(\frac{\rho t^2}{1-t^2} \right)^n (\beta)^{(n)} / n!,$$

and further we have

$$g(t) = \frac{1}{(1-t^2)^\beta} \left(1 + \frac{\rho t^2}{1-t^2} \right)^{-\beta}$$

$$= \frac{1}{(1 - t^2(1-\rho))^\beta}$$

$$= \sum_{n\geq 0} t^{2n} (1-\rho)^n \, (\beta)^{(n)} / n!.$$

Here we twice used the formula for the binomial series. Thus, we deduce

$$E(X-Y)^n = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ \frac{n!}{(n/2)!} \, (\beta)^{(n/2)} \, (1-\rho)^{n/2}, & \text{if } n \text{ is even.} \end{cases} \tag{21}$$

3.2 Proofs

The basis of all proofs is the consideration of different methods for calculating $\{E(X-Y)^n\}_{n\geq 1}$.

Proof of assertion (i) of Theorem 1. Let (X, Y) have bivariate Gaussian distribution given by (8) and let $N(m, \sigma^2)$ denote Normal (or Gaussian) distribution with mean value m and variance σ^2. Then it is well known that every linear transformation of such a variable also has Gaussian distribution. In particular we deduce that $X - Y \sim N(0, (1-\rho)^2)$. Hence, keeping in mind that

$$EZ^j = \begin{cases} 0, & \text{if } j \text{ is odd;} \\ 1, & \text{if } j = 0; \\ \sigma^j (2j-1)!!, & \text{if } j \text{ is even,} \end{cases}$$

if $Z \sim N(0, \sigma^2)$. Thus, the left-hand side of assertion (i) is equal to $(2k-1)!!(1-\rho)^k = \frac{(2k)!}{k!}(1-\rho)^k$. In order to get the right-hand side, we calculate

$$\begin{aligned} E(X-Y)^{2k} &= E(X - \rho Y - (1-\rho)Y)^{2k} \\ &= \sum_{j=0}^{2k} \binom{2k}{j} (-1)^j (1-\rho)^{2k-j} E(X-\rho Y)^j Y^{2k-j} \\ &= \sum_{j=0}^{2k} \binom{2k}{j} (-1)^j (1-\rho)^{2k-j} E(E(X-\rho Y)^j | Y) Y^{2k-j}. \end{aligned}$$

Now, we use the known fact that random variables $(X - \rho Y)$ and Y are independent. Hence we get

$$(E(X-\rho Y)^j | Y) = \begin{cases} 0, & \text{if } j \text{ is odd;} \\ 1, & \text{if } j = 0; \\ (2j-1)!!(1-\rho^2)^{j/2}, & \text{if } j \text{ is even.} \end{cases}$$

Thus, we get

$$E(X-Y)^{2k} = \sum_{j=0}^{k} \binom{2k}{2j} (1-\rho)^{2k-2j} (1-\rho^2)^j (2j-1)!!(2k-2j-1)!!.$$

Now it is enough to change ρ to $-\rho$ to get the assertion. □

Proof of assertion (ii) of Theorem 1. We use (14) with (15), getting

$$\frac{(2k)!}{k!}(1-\rho)^k = 2\sum_{j=0}^{k-1} (-1)^j \binom{2k}{j} \sum_{n=0}^{j} \frac{\rho^n}{n!} H_{j,n} H_{2k-j,n} + (-1)^k \binom{2k}{k} \sum_{n=0}^{k} \frac{\rho^n}{n!} H_{k,n}^2,$$

where

$$H_{j,n} = \begin{cases} 0, & \text{if } n > j \text{ or } j-n \text{ odd;} \\ \frac{j!}{2^{(j-n)/2}((j-n)/2)!}, & \text{if } j-n \text{ is even.} \end{cases}$$

Notice that, if $j-n$ is even, then

$$\binom{2k}{j} H_{j,n} H_{2k-j,n} = 2^n \binom{2k}{k}\binom{k}{n}\binom{k-n}{(j-n)/2}/2^k,$$

and when $k-n$ is even, we have

$$H_{k,n}^2 = 2^n \binom{k}{n}\binom{k-n}{(k-n)/2}/2^k.$$

Hence (2) can be reformulated in the following way:

$$\frac{1}{2^{k-1}} \sum_{j=0}^{k-1} (-1)^j \sum_{n=0-}^{j} (2\rho)^n \binom{k}{n} mh_{j,n,k} + \frac{(-1)^k}{2^k} \sum_{n=0}^{k} (2\rho)^n \binom{k}{n} mh_{k,n,k}, = (1-\rho)^k,$$

where

$$mh_{j,n,k} = \begin{cases} 0, & \text{if } j > k-1 \text{ or } j-n \text{ is odd;} \\ \binom{k-n}{(j-n)/2}, & \text{if } j-n \text{ is even.} \end{cases}$$

Now it remains to change the order of summation in the internal sums and ρ to $-\rho$. □

Proof of assertion (iii) of Theorem 1. Let us recall Eqs. (12) and (18) and let us calculate $H_{j,n}$ directly, getting

$$\begin{aligned} H_{j,n} &= \frac{\sqrt{n!}}{\Gamma(\beta)\sqrt{\Gamma(n+\beta)}} \sum_{k=0}^{n} \frac{(-1)^k}{k!} \frac{(\beta)^{(n)}}{(n-k)!(\beta)^{(k)}} \int_0^\infty x^{k+j} x^{\beta-1} \exp(-x) dx \\ &= \frac{\sqrt{n!}}{\sqrt{\Gamma(n+\beta)}} \sum_{k=0}^{n} \frac{(-1)^k}{k!} \frac{(\beta)^{(n)}(\beta)^{(k+j)}}{(n-k)!(\beta)^{(k)}}. \end{aligned}$$

Now, we compare it with the already-calculated $H_{j,n}$ for the Gamma distribution, i.e., $(-1)^n \binom{j}{n} (\beta)^{(j)} \sqrt{\frac{n!}{(\beta)^{(n)}}}$. □

Proof of assertion (iv) of Theorem 1. Recall Eq. (20). We have

$$\begin{aligned} E(-X+Y)^n &= \sum_{m=0}^{n} (-1)^m \binom{n}{m} EX^m Y^{n-m} \\ &= \begin{cases} 0, & \text{if } n \text{ is odd;} \\ \frac{n!}{(n/2)!} (\beta)^{(n/2)} (1-\rho)^{n/2}, & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Now we use (14) with $H_{j,m}$ given by (20). By (21) we have

$$\begin{aligned}
E(-X+Y)^n &= \sum_{m=0}^{n}(-1)^m\binom{n}{m}\sum_{j=0}^{\min(m,n-m)}\rho^j H_{m,j}H_{n-m,j} \\
&= \sum_{m=0}^{n}(-1)^{n-m}\binom{n}{m}\sum_{j=0}^{\min(m,n-m)}\rho^j\binom{m}{j}\binom{n-m}{j}(\beta)^{(m)}(\beta)^{(n-m)}\frac{j!}{(\beta)^{(j)}} \\
&= \sum_{m=0}^{n}(-1)^{n-m}\binom{n}{m}(\beta)^{(m)}(\beta)^{(n-m)}\sum_{j=0}^{m}\binom{m}{j}\binom{n-m}{j}\frac{j!\rho^j}{(\beta)^{(j)}} \\
&= \begin{cases} 0, & \text{if } n \text{ is odd;} \\ \frac{n!}{(n/2)!}(\beta)^{(n/2)}(1-\rho)^{n/2}, & \text{if } n \text{ is even.} \end{cases}
\end{aligned}$$

This is so, since $\binom{m}{j}\binom{n-m}{j}$ is zero whenever $m > j$ or $n-m > j$. □

Proof of assertion (v) of Theorem 1. One can easily notice, based on (11), that $E(l_n(X)|Y) = \rho^n l_n(Y)$. Hence, let us first calculate the conditional moments $\eta_n(y|\beta,\rho) = E(X^n|Y=y)$. By (19) we have

$$\begin{aligned}
\eta_j(y|\beta,\rho) = E(X^j|Y=y) &= j!\sum_{k=0}^{j}(-1)^k\frac{(\beta)^{(j)}}{(j-k)!(\beta)^{(k)}}\rho^k L_k^{(\beta)}(y) \\
&= j!\sum_{k=0}^{j}(-1)^k\frac{(\beta)^{(j)}}{(j-k)!(\beta)^{(k)}}\rho^k\sum_{m=0}^{k}(-1)^m\frac{(\beta)^{(k)}}{(k-m)!(\beta)^{(m)}}y^m/m! \\
&= j!\sum_{m=0}^{j}\frac{(\beta)^{(j)}}{(\beta)^{(m)}(j-m)!m!}(\rho y)^m\sum_{k=m}^{j}(-1)^{k-m}\frac{(j-m)!}{(k-m)!(j-k)!}\rho^{k-m} \\
&= \sum_{m=0}^{j}\binom{j}{m}\frac{(\beta)^{(j)}}{(\beta)^{(m)}}(\rho y)^m(1-\rho)^{j-m} = j!(1-\rho)^j L_j(-\frac{\rho y}{1-\rho}).
\end{aligned}$$

Further, we get

$$\begin{aligned}
E\left((X-Y)^n|Y=y\right) &= \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}\eta_j(y|\beta,\rho)y^{n-j} \\
&= \sum_{j=0}^{n}(-1)^{n-j}y^{n-j}\binom{n}{j}\sum_{m=0}^{j}\binom{j}{m}\frac{(\beta)^{(j)}}{(\beta)^{(m)}}(\rho y)^m(1-\rho)^{j-m}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \frac{(\rho y)^m}{(\beta)^{(m)}} \sum_{j=m}^{n} (-1)^{n-j} \frac{(n-m)!(\beta)^{(j)} y^{n-j}(1-\rho)^{j-m}}{(n-j)!(j-m)!} \\
&\qquad \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} (\rho y)^m \sum_{s=0}^{n-m} (-1)^{n-m-s} \binom{n-m}{s} y^{n-m-s} (1-\rho)^s (\beta+m)^{(s)} \\
&= \sum_{t=0}^{n} (-1)^{n-t} y^{n-t} \frac{n!}{t!(n-t)!} \sum_{s=0}^{t} \binom{t}{s} (\rho y)^{t-s} (1-\rho)^s (\beta+t-s)^{(s)} \\
&= \sum_{s=0}^{n} \binom{n}{s} y^{n-s} (1-\rho)^s \sum_{t=s}^{n} (-1)^{n-t} \frac{(n-s)!}{(t-s)!t!} \rho^{t-s} (\beta+t-s)^{(s)} \\
&= \sum_{s=0}^{n} \binom{n}{s} y^{n-s} (1-\rho)^s \sum_{m=0}^{n-s} (-1)^{n-m-s} \binom{n-s}{m} \rho^m (\beta+m)^{(s)}.
\end{aligned}
$$

Hence, using (17), we get

$$
\begin{aligned}
E(X-Y)^n &= \int_0^\infty (\sum_{s=0}^{n} \binom{n}{s} y^{n-s} (1-\rho)^s \times \sum_{m=0}^{n-s} (-1)^{n-m-s} \binom{n-s}{m} \rho^m (\beta+m)^{(s)}) f_g(x|\beta) dx \\
&= \sum_{s=0}^{n} \binom{n}{s} (\beta)^{(n-s)} (1-\rho)^s \sum_{m=0}^{n-s} (-1)^{n-m-s} \binom{n-s}{m} \rho^m (\beta+m)^{(s)} \\
&= \sum_{t=0}^{n} (-1)^{n-t} \frac{n!(\beta)^{(n-t)}}{t!(n-t)!} \sum_{s=0}^{t} \frac{t!}{s!(t-s)!} (1-\rho)^s \rho^{t-s} (\beta+n-t)^{(t-s)} (\beta+t-s)^{(s)}.
\end{aligned}
$$

In the last line, we changed the order of summation. To get the assertion, we use the formula

$$
(\beta)^{(n)} (\beta+n)^{(m)} = (\beta)^{(n+m)},
$$

true for all complex β. □

References

[1] G. Alexits, *Convergence Problems of Orthogonal Series*, in International Series of Monographs in Pure and Applied Mathematics, Vol. 20, Pergamon Press, 1961.

[2] M. Abramowitz and I. A. Stegun, eds., *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Applied Mathematics Series, Vol. 55, Dover, 1983.

[3] K. Ball, *Strange Curves, Counting Rabbits, and Other Mathematical Explorations*, Princeton University Press, 2003.

[4] P. Cull, M. Flahive, and R. Robson, *Difference Equations. From Rabbits to Chaos.* Springer, 2005.

[5] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Mathematics and its Applications, Vol. 13, Gordon and Breach, 1978.

[6] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 8th edition, Academic Press, 2014.

[7] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*, 2nd ed., Addison-Wesley, 1994.

[8] H. O. Lancaster, The structure of bivariate distributions, *Ann. Math. Statistics*, **29** (1958), 719–736.

[9] H. O. Lancaster, Joint probability distributions in the Meixner classes. *J. Roy. Statist. Soc. Ser. B* **37** (1975), 434–443.

[10] P. J. Szabłowski, Yet another way of calculating moments of the Kesten's distribution and its consequences for Catalan numbers and Catalan triangles, *Discrete Math.* **345** (2022), Paper No. 112891.

[11] P. J. Szabłowski, On stationary Markov processes with polynomial conditional moments. *Stoch. Anal. Appl.* **35** (2017), 852–872.

[12] P. J. Szabłowski, On positivity of orthogonal series and its applications in probability, *Positivity* **26** (2022), Article 19.

[13] G. Szegő, *Orthogonal Polynomials*, 4th edition, *Amer. Math. Soc. Colloq. Publ.*, Vol. **23**, Amer. Math. Soc., 1975.

2020 *Mathematics Subject Classification*: Primary 05A10; Secondary 05D40, Secondary 11B65, 11B73.
Keywords: binomial coefficient, Pochhammer symbol, bivariate normal distribution, bivariate gamma distribution, Hermite polynomial, Laguerre polynomial, Fibonacci number, Lucas number.

(Concerned with sequences A000032, A000045, A001147, A007318, and A221954.)

Received July 7 2023; revised versions received November 7 2023; November 13 2023. Published in *Journal of Integer Sequences*, January 3 2024.

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