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Counting Various Classes of Tournament Score Sequences

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Abstract

Based on known methods for computing the number of distinct score sequences for n-vertex tournaments, we develop algorithms for computing the number of distinct score sequences for self-complementary tournaments, strong tournaments, and tournaments that are both self-complementary and strong.

1 Introduction

In graph theory, a *tournament*, sometimes called a *round-robin tournament*, is an oriented complete graph, i.e., a directed graph where for any two vertices v and w there is either a directed edge from v to w or from w to v but not both. For background information about tournaments, see the classic book by Moon [9] or the article by Harary and Moser [5].

If an edge of a tournament is directed from vertex v to vertex w we say that v *dominates* w. The *score* of a vertex v is the number of vertices that v dominates. The *score sequence* of an n-vertex tournament is the sequence $(s_1, s_2, \ldots, s_n)$ of vertex scores, written in nondecreasing order:

$$s_1 \leq s_2 \leq \cdots \leq s_n. \quad (1)$$

For example, we observe that the score sequences for 5-vertex tournaments are $(0, 1, 2, 3, 4)$, $(0, 1, 3, 3, 3)$, $(0, 2, 2, 2, 4)$, $(0, 2, 2, 3, 3)$, $(1, 1, 1, 3, 4)$, $(1, 1, 2, 2, 4)$, $(1, 1, 2, 3, 3)$, $(1, 2, 2, 2, 3)$, and $(2, 2, 2, 2, 2)$.

This paper develops methods for computing the number of score sequences for various classes of n-vertex tournaments. Section 2 reviews the history of computing the total number of n-vertex score sequences, introducing tools that will be used in later sections. Section 3 provides an algorithm for counting self-complementary score sequences, Section 4 contains two different methods for counting strong score sequences, while Section 5 gives two methods for counting score sequences that are both strong and self-complementary. Section 6 contains some asymptotic results and conjectures.

2 Counting score sequences

Sequence A000571 in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [11] gives the number of different score sequences that are possible for an n-vertex tournament. Postulating the existence of a 0-vertex tournament with null score sequence, this sequence begins with $n = 0$ and reads 1, 1, 1, 2, 4, 9, 22, 59, 167, 490, 1486, ... There is no known simple formula for this sequence but various methods have been devised for determining its terms.

The first major effort to catalog score sequences was carried out in 1920 by MacMahon [8], who expanded the product

$$\prod_{1\leq i<j\leq n} (a_i + a_j)$$

to generate the score sequence of each of the $2^{\binom{n}{2}}$ labeled tournaments on n vertices. For example, with $n = 5$, fourteen of the 1024 terms in this product are $a_1^2 a_2^2 a_3^1 a_4^3 a_5^2$, representing some of the labeled tournaments with score sequence $(1, 2, 2, 2, 3)$. By combining these terms with the other terms with the same collection of exponents, MacMahon calculated that there were $14 \times 20 = 280$ labeled tournaments with this score sequence, the most of any of the nine possible score sequences. In a Herculean effort of hand calculation, MacMahon carried out this procedure up through $n = 9$, determining the number of labeled tournaments possessed by each of 490 possible score sequences that arose. The most common sequence turned out to be $(2, 3, 3, 4, 4, 4, 5, 5, 6)$, occurring 5,329,376,640 times among the $2^{36} = 68, 719, 476, 736$ labeled tournaments on nine vertices.

Presumably unaware of the work of MacMahon, David [4] made essentially the same hand calculations in 1959, but only for $n \leq 8$. This was followed by Alway [1], who in 1962 used a computer to catalog all score sequences up to $n = 10$.

Of course it is not necessary to generate all the n-vertex tournament score sequences in order to count them. The following test for a sequence of non-negative integers to be the score sequence of a tournament is usually attributed to Landau [7].

Proposition 1. *A sequence $(s_1, s_2, \ldots, s_n)$ of $n \geq 1$ non-negative integers satisfying (1) is* *the score sequence of a tournament if and only if*

$$\sum_{i=1}^{r} s_i \geq \binom{r}{2} \text{ for } 1 \leq r < n, \text{ and} \tag{2}$$

$$\sum_{i=1}^{n} s_i = \binom{n}{2}. \tag{3}$$

This test was utilized by Bent [2] in a remarkable 1964 Master of Science dissertation, also published by Narayana and Bent [10]. We present Bent's algorithm, which will be used in later sections.

Definition 2. For all positive n and all non-negative T and E, the array entry $F_n[T, E]$ is the number of sequences of length n satisfying (1), (2), $s_n = E$, and

$$\sum_{i=1}^{n} s_i = T.$$

(We can imagine that T stands for *Total* and E for *End*.) These F_n arrays can be computed using the following recursive formulas.

Proposition 3 (Bent)**.** *We have*

$$F_1[T, E] = \begin{cases} 1, & \text{if } T = E; \\ 0, & \text{otherwise,} \end{cases}$$

and for $n \geq 2$ we have

$$F_n[T, E] = \begin{cases} \displaystyle\sum_{k=0}^{E} F_{n-1}[T-E, k], & \text{if } T-E \geq \binom{n-1}{2}; \\ 0, & \text{otherwise.} \end{cases}$$

Bent then used the values in the F_n arrays to count score sequences.

Theorem 4 (Bent)**.** *Let $S(n)$ denote the number of distinct score sequences for n-vertex tournaments. Then for $n \geq 1$ we have*

$$S(n) = \sum_{E=\lceil \frac{n-1}{2} \rceil}^{n-1} F_n\left[\binom{n}{2}, E\right].$$

Using Theorem 4 and the computer available to him, Bent computed $S(n)$ for $n \leq 27$. With an alternative method, which allowed him to compute S_{2n} from array F_n and to compute S_{2m+1} from arrays F_m and F_{m+1}, he was able to extend his result up to $n = 36$. The limiting factor was the speed of the computer he was using.

A recent paper by Claesson et al. [3] provides an alternative method for counting score sequences. Today the OEIS entry A000571 gives values of $S(n)$ up to $n = 1,675$. As a test of our programs for this paper, we verified these numbers for $n \leq 500$.

3 Counting self-complementary score sequences

The *complement* of a tournament T_n is the tournament T_n^c obtained from T_n by reversing the direction of all of its edges. Thus a vertex with score s_i in T_n becomes a vertex of score $n-1-s_i$ in T_n^c. A tournament is called *self-complementary* if it is isomorphic to its complement. In this case, for each vertex of score s_i there is a corresponding vertex of score $n-1-s_i$.

Definition 5. A score sequence of length n is called *self-complementary* if $s_{n+1-i} = n-1-s_i$ for $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$.

For example, the self-complementary score sequences of length $n = 6$ are $(0,1,2,3,4,5)$, $(0,2,2,3,3,5)$, $(1,1,1,4,4,4)$, $(1,1,2,3,4,4)$, $(1,2,2,3,3,4)$, and $(2,2,2,3,3,3)$. Clearly self-complementary tournaments have self-complementary score sequences. The converse, however, is not true. Of the four non-isomorphic tournaments with score sequence $(1,1,2,3,4,4)$, two are self-complementary and the other two are complements of each other.

Sequence A345470 in the OEIS, authored by Givner in 2021, lists the number of self-complementary score sequences for n-vertex tournaments. Givner wrote a program that generated all length n score sequences for $n \leq 34$ and then counted those that were self-complementary. As with Bent's program, computer time was the limiting factor in Givner's calculations.

Without actually generating the self-complementary score sequences, we can count them using the F_n arrays of Bent from Section 2.

Theorem 6. *Let* $\mathrm{SCS}(n)$ *denote the number of distinct self-complementary score sequences of length* n*. Then for all* $m \geq 1$ *we have*

$$\mathrm{SCS}(2m) = \sum_{T=\binom{m}{2}}^{m(m-1)} \sum_{E=\left\lceil \frac{T}{m} \right\rceil}^{m-1} F_m[T,E],$$

and

$$\mathrm{SCS}(2m+1) = \sum_{T=\binom{m}{2}}^{m^2} \sum_{E=\left\lceil \frac{T}{m} \right\rceil}^{m} F_m[T,E].$$

Proof. From Definition 5 we know that a self-complementary score sequence of length $2m$ is determined by its first m terms. Therefore $\mathrm{SCS}(2m)$ is a certain sum of values in the array F_m. For the lower summation limit on E we note that $E = s_m$ must be at least as large as the average of s_1 through s_m, or $\frac{T}{m}$. For the upper limit on E we observe that $s_m \leq s_{m+1}$ and $s_m + s_{m+1} = 2m-1$. This implies that $E = s_m \leq \left\lfloor \frac{2m-1}{2} \right\rfloor = m-1$. The lower summation limit on T comes from equation (2) while for the upper limit we note that $T \leq m \cdot s_m \leq m(m-1)$.

For self-complementary score sequences of length $2m+1$ we have that s_{m+1} must be m, and the score sequence is again determined by its first m terms. The only difference between this case and the even case is that $E = s_m$ can now be as large as $s_{m+1} = m$, and T can now be as large as $m \cdot s_m \le m^2$. □

We used Theorem 6 to compute SCS(n) for n from 2 to 500 as displayed in the OEIS entry A345470, confirming the results of Givner for n from 2 to 34.

4 Counting strong score sequences

A tournament is called *strong,* or *strongly connected,* if every vertex of the tournament can reach every other vertex along a directed path. A tournament is *reducible* if its vertex set can be partitioned into two nonempty sets A and B with every vertex in B dominating every vertex in A. It is well known that a tournament is strong if and only if it is not reducible.

Harary and Moser [5, Theorem 9] showed that the property of being strong can be determined from a tournament's score sequence.

Proposition 7. *A tournament is strong if and only if its score sequence satisfies equations* (1), (3) *and*

$$\sum_{i=1}^{r} s_i > \binom{r}{2} \text{ for } 1 \le r < n. \tag{4}$$

We call score sequences that satisfy equation (4) *strong* score sequences. For example, the seven strong score sequences for $n = 6$ are $(1, 1, 2, 3, 4, 4)$, $(1, 1, 3, 3, 3, 4)$, $(1, 2, 2, 2, 4, 4)$, $(1, 2, 2, 3, 3, 4)$, $(1, 2, 3, 3, 3, 3)$, $(2, 2, 2, 2, 3, 4)$, and $(2, 2, 2, 3, 3, 3)$.

In this section we present two algorithms for counting strong score sequences. Our first method is a modified version of Bent's algorithm of Section 2, using G_n arrays based on equation (4) instead of the F_n arrays based on equation (2).

Definition 8. For all positive n and all non-negative T and E, let $G_n[T, E]$ be the number of sequences of length n satisfying (1), (4), $s_n = E$, and

$$\sum_{i=1}^{n} s_i = T.$$

Proposition 9. *We have*

$$G_1[T, E] = \begin{cases} 1, & \text{if } T = E; \\ 0, & \text{otherwise,} \end{cases}$$

and for $n \ge 2$ we have

$$G_n[T, E] = \begin{cases} \displaystyle\sum_{k=\lceil \frac{T-E}{n-1} \rceil}^{E} G_{n-1}[T-E, k], & \text{if } T-E > \binom{n-1}{2}; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The case $n = 1$ is clear: a sequence of length one gets counted if and only if its last (and only) element E is equal to the sum T of all its elements. For $n \geq 2$, consider a sequence $S = (s_1, \ldots, s_n)$ counted by $G_n(T, E)$. We have $\sum_{i=1}^{r} s_i > \binom{r}{2}$ for $r < n$, $\sum_{i=1}^{n} s_i = T$, and $s_n = E$. If we delete the last term s_n we have a sequence $\widehat{S} = (s_1, \ldots, s_{n-1})$ with $\sum_{i=1}^{n-1} s_i = T - E$ and s_{n-1} equal to some integer k with $\frac{T-E}{n-1} \leq k \leq E$. The limits arise because s_{n-1} must be at least as large as of the average of the first $n-1$ terms of the sequence and no larger than s_n. Thus for every sequence S counted by array element $G_n[T, E]$, the corresponding sequence $\widehat{S}$ is counted by one of the indicated $G_{n-1}[T-E, k]$ array elements. However, not all strings counted by these array elements satisfy the requirement that $\sum_{i=1}^{n-1} s_i > \binom{n-1}{2}$; our sum only includes those terms for which this requirement is satisfied. □

Our first method for counting strong score sequences now follows.

Theorem 10. *Let* $\mathrm{SS}(n)$ *denote the number of distinct strong score sequences for* n*-vertex tournaments. Then for* $n \geq 1$ *we have*

$$\mathrm{SS}(n) = \sum_{E=\lceil \frac{n-1}{2} \rceil}^{n-2} G_n\left[\binom{n}{2}, E\right].$$

Proof. By definition, $G_n\left[\binom{n}{2}, E\right]$ counts strong score sequences of length n with largest score being E. This score is bounded below by the average score, and bounded above by $n-2$ for strong tournaments. □

Our second method for counting strong score sequences is to count reducible score sequences and subtract that number from the number of all score sequences. Every reducible tournament can be characterized by a non-empty vertex set A which induces a strong subtournament, and a nonempty set B of the remaining vertices, with every vertex in B dominating every vertex in A. The score sequence of such a tournament consists of an initial strong score sequence of length $|A|$, followed by an arbitrary valid score sequence of length $|B|$ with each score increased by $|A|$. Some typical reducible score sequences of length 6 are $(0, 2, 3, 3, 3, 4)$ with $|A| = 1$, $(1, 1, 1, 3, 4, 5)$ with $|A| = 3$, $(1, 1, 2, 2, 4, 5)$ with $|A| = 4$, and $(1, 2, 2, 2, 3, 5)$ with $|A| = 5$. This characterization of reducible score sequences provides us with the following bootstrapping method for counting strong score sequences, using only the series $S(n)$ that counts all score sequences.

Theorem 11. *Terms in the sequence* $\mathrm{SS}(n)$ *with* $n \geq 1$ *can be computed recursively from the sequence* $S(n)$ *using the formula*

$$\mathrm{SS}(n) = S(n) - \sum_{i=1}^{n-1} \mathrm{SS}(i) S(n-i).$$

Claesson et al. [3] observed that this method also works when using their method for computing $S(n)$.

We computed $\mathrm{SS}(n)$ for n from 1 to 500 by both of these methods with identical results, as displayed in the OEIS entry A351822.

5 Counting strong self-complementary score sequences

In this section we sketch two methods for computing the number of score sequences for n-vertex tournaments that are both strong and self-complementary. The first is based on our method of counting self-complementary score sequences, but using the G_n arrays rather than the F_n arrays.

Theorem 12. *Let* $\mathrm{SSCS}(n)$ *denote the number of distinct score sequences of length* n *that are both strong and self-complementary. Then for all* $m \geq 1$ *we have*

$$\mathrm{SSCS}(2m) = \sum_{T=\binom{m}{2}+1}^{m(m-1)} \sum_{E=\lceil \frac{T}{m} \rceil}^{m-1} G_m\,[T, E]$$

and

$$\mathrm{SSCS}(2m+1) = \sum_{T=\binom{m}{2}+1}^{m^2} \sum_{E=\lceil \frac{T}{m} \rceil}^{m} G_m\,[T, E]\,.$$

Our second method for counting strong self-complementary score sequences is similar to our second method for counting strong score sequences: subtract the number of self-complementary reducible score sequences from the number of self-complementary score sequences. Now a self-complementary reducible score sequence of length n begins with a strong score sequence of length i with $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ and ends with the complement of this initial subsequence. In between we must have a self-complementary sequence of length $n-2i$, with each term increased by i. For example, the three self-complementary reducible score sequences of length $n = 6$ are $(0, 1, 2, 3, 4, 5)$ and $(0, 2, 2, 3, 3, 5)$ with $i = 1$, and $(1, 1, 1, 4, 4, 4)$ with $i = 3$. Then the sequence that counts strong self-complementary score sequences can be computed from the sequences for strong sequences and self-complementary sequences.

Theorem 13. *For all* $n \geq 1$ *we have*

$$\mathrm{SSCS}(n) = \mathrm{SCS}(n) - \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \mathrm{SS}(i)\,\mathrm{SCS}(n-2i).$$

We observe that for $n = 2m$ and $i = m$ the last term in the above summation is $\mathrm{SS}(m)\,\mathrm{SCS}(0)$. For this to be correct we must have $\mathrm{SCS}(0) = 1$, i.e., we must assume the existence of a self-complementary tournament on zero vertices.

We computed $\mathrm{SSCS}(n)$ for n from 1 to 500 by both of these methods with identical results, as displayed in the OEIS entry A351869.

6 Asymptotic Results and Conjectures

Moon [9] reports that Erdős and Moser, in an unpublished work, showed that there exist constants c_1 and c_2 such that

$$\frac{c_1 \cdot 4^n}{n^{9/2}} < S(n) < \frac{c_2 \cdot 4^n}{n^{3/2}}.$$

Winston and Kleitman [12] improved these bounds, showing that

$$\frac{c_1 \cdot 4^n}{n^{5/2}} < S(n) < \frac{c_2 \cdot 4^n}{n^2}$$

and conjecturing that

$$S(n) = \Theta\left(\frac{4^n}{n^{5/2}}\right).$$

This conjecture was confirmed by Kim and Pittel [6]. Vaclav Kotesovec asserts in the OEIS entry A000571 that

$$S(n) \sim \frac{c \cdot 4^n}{n^{5/2}},$$

with $c = 0.392478\ldots$, but with no apparent proof.

For strong score sequences, Kotesovec asserts in A351822 that

$$\mathrm{SS}(n) \sim \frac{c \cdot 4^n}{n^{5/2}},$$

with $c = 0.202756\ldots$. It would be nice to at least have a proof of the order of magnitude of $\mathrm{SS}(n)$. Assuming that the assertions of Kotesovec are correct, we have that slightly more than half of all tournament score sequences are strong.

As for self-complementary score sequences, we conjecture that both

$$\mathrm{SCS}(n) = \Theta\left(\frac{2^n}{n^{3/4}}\right) \text{ and } \mathrm{SSCS}(n) = \Theta\left(\frac{2^n}{n^{3/4}}\right).$$

We have no theoretical evidence for these formulas, but they are consistent with the data available. Results suggest that over seventy percent of all self-complementary score sequences are strong.

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