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On the Ternary Purely Exponential Diophantine Equation $(ak)^x + (bk)^y = ((a+b)k)^z$ for Prime Powers a and b

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Abstract

Let k be a positive integer, and let a, b be coprime positive integers with $a, b > 1$. In this paper, using a combination of some elementary number theory techniques with classical results on the Nagell-Ljunggren equation, the Catalan equation, and some new properties of the Lucas sequence, we prove that if $k > 1$ and $a, b > 2$ are both prime powers, then the equation $(ak)^x + (bk)^y = ((a+b)k)^z$ has only one positive integer solution: namely, $(x, y, z) = (1, 1, 1)$. This proves some cases of a conjecture of Yuan and Han.

1 Introduction

Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{P}$ be the sets of all integers, positive integers, and odd primes, respectively. Let A, B, C be fixed positive integers with $A, B, C > 1$. In recent decades, the ternary purely exponential Diophantine equation

$$A^x + B^y = C^z, \quad x, y, z \in \mathbb{N} \tag{1}$$

has yielded very rich results, but some important problems about it are far from being solved (see [7]). In recent 20 years, many authors have considered equation (1) when A, B, and C are Fibonacci A00045 or Lucas A000204 or Pell A000129 numbers in OEIS [14] or when A and B are Fibonacci or Lucas or Pell numbers (see [1, 2, 4, 6, 12, 13, 18]).

Let k be a positive integer, and let a, b be coprime positive integers with $a, b > 1$. In this paper, we discuss (1) for $(A, B, C) = (ak, bk, (a + b)k)$. Then (1) can be written as

$$(ak)^x + (bk)^y = ((a + b)k)^z, \quad x, y, z \in \mathbb{N}. \tag{2}$$

Obviously, for any a, b and k, (2) has the solution $(x, y, z) = (1, 1, 1)$. A solution (x, y, z) of (2) with $(x, y, z) \neq (1, 1, 1)$ is called exceptional. In 2018, Sun and Tang [16] proved that if $k > 1$, $a, b > 2$, and (x, y, z) is an exceptional solution of (2), then either $x > z > y$ or $y > z > x$. On this basis, they further proved that if $k > 1$ and $(a, b) \in \{(3, 5), (5, 8), (8, 13), (13, 21)\}$, then (2) has no exceptional solutions. In the same year, Yuan and Han [17] proposed the following conjecture.

Conjecture 1. For any k, if $a, b > 3$, then (2) has no exceptional solutions.

This conjecture is formally similar to Jeśmanowicz' conjecture concerning Pythagorean triples (see [5, 15]). So far, it has only been solved for some very special cases. For example, the authors of [17] proved that if a and b are squares with $b \equiv 4 \pmod 8$, then (2) has no solutions (x, y, z) with $y > z > x$, in particular, if a is a square and $b = 4$, then Conjecture 1 is true. Afterwards, using Baker's method, Le and Soydan [8] proved that if a and b are squares with $a > 64b^3$, then (2) has no solutions (x, y, z) with $x > z > y$. It implies that if a and b are squares with $a > 64b^3$ and $b \equiv 4 \pmod 8$, then Conjecture 1 is true.

In this paper, using a combination of some elementary number theory techniques with classical results on the Nagell-Ljunggren equation, the Catalan equation and some new properties of the Lucas sequence A000204, we prove the following result.

Theorem 2. *Let r, s be positive integers, and let p, q be distinct odd primes. If $k > 1$ and a, b satisfy one of the following conditions:*

(i) *$(a, b) = (2^r, p^s)$ with $r > 1$; or*

(ii) *$(a, b) = (p^r, 2^s)$ with $s > 1$; or*

(iii) *$(a, b) = (p^r, q^s)$, then (2) has no solutions (x, y, z) with $x > z > y$.*

Since a and b are symmetric in (2), by the first result mentioned in [16], we can obtain the following corollary from Theorem 2 immediately.

Corollary 3. *If $k > 1$ and a, b are prime powers, then Conjecture 1 is true.*

Finally, we briefly analyze the effect of the above mentioned results. For any enough large positive integer N, let $F(N)$ denote the number of pairs (a, b) such that $a, b \leq N$ and they have been proved to hold for Conjecture 1 with $k > 1$. Clearly, by the mentioned results in [8, 16, 17] and Corollary 3, we have $F(N) = 4$, $F(N) = \sqrt{N}$, $N > F(N) > \sqrt{N}$, and $F(N) > N^2/(\log N)^2$, respectively.

2 Preliminaries

Let us now recall that if α and β are roots of a quadratic equation of the form $x^2 - rx - s = 0$ for nonzero coprime integers r and s and such that α/β is not a root of unity, then the sequence $(u_\ell)_{\ell \geq 0}$ with general term

$$u_\ell = \frac{\alpha^\ell - \beta^\ell}{\alpha - \beta} \qquad \text{for all } \ell \geq 0$$

is called a *Lucas sequence* A000204. It can also be defined inductively as $u_0 = 0$, $u_1 = 1$, and $u_{\ell+2} = r \cdot u_{\ell+1} + s \cdot u_\ell$. When $\beta = 1$, Lemmas 9, 10, and 11 below are three new properties we proved about the Lucas sequence.

Lemma 4 ([9])**.** *The equation*

$$\frac{X^m - 1}{X - 1} = Y^n, \;\; X, Y, m, n \in \mathbb{N}, \;\; X > 1, \; Y > 1, \; m > 2, \; n > 1$$

has only the solution $(X, Y, m, n) = (3, 11, 5, 2)$ with $2 \mid n$.

Lemma 5 ([10])**.** *The equation*

$$X^m - Y^n = 1, \;\; X, Y, m, n \in \mathbb{N}, \;\; X, Y, m, n > 1$$

has only the solution $(X, Y, m, n) = (3, 2, 2, 3)$.

Let X, ℓ, m, n be positive integers with $X > 1$ and $\ell > 1$, and let p, q be odd primes. Three lemmas for divisibility are given directly below.

Lemma 6 ([11])**.** *If $X^n + 1 \equiv 0 \pmod{X^m + 1}$, then $m \mid n$ and n/m is odd.*

Lemma 7 ([3])**.** *Let ℓ be an odd prime. If $X \not\equiv 1 \pmod{\ell}$, then every prime divisor p of $(X^\ell - 1)/(X - 1)$ satisfies*

$$p \equiv 1 \pmod{2\ell}. \tag{3}$$

If $X \equiv 1 \pmod{\ell}$, then $\ell \mid\mid (X^\ell - 1)/(X - 1)$ and every prime divisor p of $(X^\ell - 1)/\ell(X - 1)$ satisfies (3).

Lemma 8 ([3])**.** *When* $X \equiv 1 \pmod{p}$*,* $p^m \,||\, (X^\ell - 1)/(X - 1)$ *if and only if* $p^m \,||\, \ell$*.*

Lemma 9. *If*

$$\frac{X^\ell - 1}{X - 1} = p^n, \ 2 \nmid \ell, \tag{4}$$

then ℓ *is an odd prime with* (3)*.*

Proof. We now assume that ℓ is not an odd prime. Since $\ell > 1$ and $2 \nmid \ell$, ℓ has an odd prime divisor q with $q < \ell$. Then we have $q \mid \ell$ and $\ell/q > 1$. By (4), we get

$$p^n = \frac{X^\ell - 1}{X - 1} = \left(\frac{X^{\ell/q} - 1}{X - 1}\right)\left(\frac{(X^{\ell/q})^q - 1}{X^{\ell/q} - 1}\right), \tag{5}$$

where $(X^{\ell/q} - 1)/(X - 1)$ and $((X^{\ell/q})^q - 1)/(X^{\ell/q} - 1)$ are positive integers greater than 1. Since $X > 1$, by (5), we have

$$\frac{X^{\ell/q} - 1}{X - 1} = p^f, \ \frac{(X^{\ell/q})^q - 1}{X^{\ell/q} - 1} = p^{n-f}, \ f \in \mathbb{N}, \ f < n. \tag{6}$$

Further, by the first and the second equalities of (6), we get $X^{\ell/q} \equiv 1 \pmod{p}$ and

$$0 \equiv p^{n-f} \equiv \frac{(X^{\ell/q})^q - 1}{X^{\ell/q} - 1} \equiv (X^{\ell/q})^{q-1} + \cdots + X^{\ell/q} + 1 \equiv q \pmod{p}. \tag{7}$$

Since p and q are odd primes, by (7), we obtain $q = p$. Hence by Lemma 7, we see from (6) that $p \,||\, ((X^{\ell/p})^p - 1)/(X^{\ell/p} - 1) = ((X^{\ell/q})^q - 1)/(X^{\ell/q} - 1)$ and

$$p = \frac{(X^{\ell/p})^p - 1}{X^{\ell/p} - 1} = (X^{\ell/p})^{p-1} + \cdots + X^{\ell/p} + 1 > p, \tag{8}$$

a contradiction. It implies that ℓ must be an odd prime. Moreover, using Lemma 7 again, if $X \equiv 1 \pmod{\ell}$, then from (4) we can get $\ell = p$ and $n = 1$, which is the same contradiction as (8). Therefore, we have $x \not\equiv 1 \pmod{\ell}$ and p satisfies (3). The lemma is proved. □

Lemma 10. *If*

$$X^\ell - 1 = 2^m p^n, \tag{9}$$

then one of the following three conclusions holds.

(i) $(p, X, \ell, m, n) = (3, 5, 2, 3, 1), (3, 7, 2, 4, 1), (5, 9, 2, 4, 1), (5, 3, 4, 4, 1), (3, 17, 2, 5, 2)$ *or* $(7, 15, 2, 5, 1)$*.*

(ii)
$$\ell = 2, \ m \geq 6, \ n = 1, \ X = 2^{m-1} + \zeta, \ p = 2^{m-2} - \zeta, \ \zeta \in \{1, -1\}. \tag{10}$$

(iii)
$$\ell \text{ is an odd prime}, \ X - 1 = 2^m, \ \frac{X^\ell - 1}{X - 1} = p^n, \ p \equiv 1 \pmod{2\ell}. \tag{11}$$

Proof. Obviously, by (9), we have $X > 1$, $2 \nmid X$ and $p \nmid X$. When $2 \mid \ell$, since $X^\ell \equiv 1 \pmod 8$, $\gcd(X^{\ell/2}+1, X^{\ell/2}-1) = 2$ and p is an odd prime, by (9), we have

$$X^{\ell/2} + \zeta = 2p^n,\ X^{\ell/2} - \zeta = 2^{m-1},\ m \geq 3,\ \zeta = ((-1)^{(X-1)/2})^{\ell/2}, \tag{12}$$

where

$$\zeta \in \{1, -1\}. \tag{13}$$

If $3 \leq m \leq 5$, then from (12) and (13) we can easily obtain the conclusion (i). If $m \geq 6$, by Lemma 5, then from the second equality of (12) we get $\ell = 2$. In addition, eliminating $X^{\ell/2}$ from (12), we have

$$p^n - 2^{m-2} = \zeta. \tag{14}$$

Since $m - 2 \geq 4$, using Lemma 5 again, we see from (14) that $n = 1$. Therefore, by (12), (13), and (14), we obtain (10) and conclusion (ii) is proved.

When $2 \nmid \ell$, since $\ell > 1$ and $\dfrac{X^\ell - 1}{X - 1}$ is an odd positive integer greater than 1, by (9), we have

$$X - 1 = 2^m p^f,\ \frac{X^\ell - 1}{X - 1} = p^{n-f},\ f \in \mathbb{Z},\ 0 \leq f < n. \tag{15}$$

If $f > 0$, then from the first equality of (15) we get $X \equiv 1 \pmod p$. Hence, applying Lemma 8 to the second equality of (15), we obtain $p^{n-f} \,||\, \ell$. However, since $\ell \geq p^{n-f}$, we get $p^{n-f} = (X^\ell - 1)/(X - 1) > \ell \geq p^{n-f}$, a contradiction. So $f = 0$. Therefore, by (15), we get

$$X - 1 = 2^m,\ \frac{X^\ell - 1}{X - 1} = p^n. \tag{16}$$

Further, by Lemma 9, we see from the second equality of (16) that ℓ is an odd prime with (3). Thus, we obtain (11) and the conclusion (iii) is proved. The proof is completed. □

Using the same method as in the proof of Lemma 10, we can obtain the following lemma without difficulty

Lemma 11. *If*

$$X^\ell - 1 = p^m q^n, \tag{17}$$

the one of the following six conclusions holds.

(i)

$$2 \mid \ell,\ X^{\ell/2} + \zeta = p^m,\ X^{\ell/2} - \zeta = q^n,\ \zeta \in \{1, -1\}. \tag{18}$$

(ii)

$$2 \nmid \ell,\ X = 2,\ 2^\ell - 1 = p^m q^n. \tag{19}$$

(iii)

$$\ell = p,\ m > 1,\ X - 1 = p^{m-1},\ \frac{X^p - 1}{X - 1} = pq^n,\ q \equiv 1 \pmod{2p}. \tag{20}$$

(iv)
$$\ell = q,\ n > 1,\ X-1 = q^{n-1},\ \frac{X^q-1}{X-1} = p^m q,\ p \equiv 1 \pmod{2q}. \tag{21}$$

(v)
$$\ell \textit{ is an odd prime},\ X-1 = p^m,\ \frac{X^\ell-1}{X-1} = q^n,\ q \equiv 1 \pmod{2\ell}. \tag{22}$$

(vi)
$$\ell \textit{ is an odd prime},\ X-1 = q^n,\ \frac{X^\ell-1}{X-1} = p^m,\ p \equiv 1 \pmod{2\ell}. \tag{23}$$

Proof. Since p and q are distinct odd primes, by (17), we have $2 \mid X$. When $2 \mid \ell$, since $\gcd(X^{\ell/2}+1, X^{\ell/2}-1) = 1$, by (17), we can directly obtain (18) and the conclusion (i) is proved. When $2 \nmid \ell$ and $X = 2$, we see from (17) that (19) is clearly true and the conclusion (ii) is proved. When $2 \nmid \ell$ and $X > 2$, by (17), we have

$$\begin{aligned} &X-1 = p^f q^g,\ \frac{X^\ell-1}{X-1} = p^{m-f}q^{n-g},\ f,g \in \mathbb{Z},\ 0 \le f \le m,\\ &0 \le g \le n,\ (f,g) \ne (0,0) \text{ or } (m,n). \end{aligned} \tag{24}$$

If $0 < f < m$ and $0 < g < n$, then from (24) we get $X \equiv 1 \pmod{pq}$ and $0 \equiv p^{m-f}q^{n-g} \equiv (X^\ell-1)(X-1) \equiv X^{\ell-1} + \cdots + X + 1 \equiv \ell \pmod{pq}$. It follows that $pq \mid \ell$ and $(X^{pq}-1)/(X-1) \mid (X^\ell-1)(X-1)$. Hence, by the second equality of (24), we have

$$\begin{aligned} &\frac{X^{pq}-1}{X-1} = \left(\frac{X^p-1}{X-1}\right)\left(\frac{(X^p)^q-1}{X^p-1}\right) = p^{f'}q^{g'},\ f',g' \in \mathbb{Z},\\ &f' \ge 0,\ g' \ge 0,\ (f',g') \ne (0,0), \end{aligned} \tag{25}$$

where $(X^p-1)/(X-1)$ and $((X^p)^q-1)/(X^p-1)$ are positive integers greater than 1. Further, since $X \equiv 1 \pmod{p}$, by Lemma 8, we get $p \,||\, (X^p-1)/(X-1)$. Furthermore, since $(X^p-1)/(X-1) > p$, by (25), we have

$$\frac{X^p-1}{X-1} = pq^{g''},\ g'' \in \mathbb{N}. \tag{26}$$

Recall that $X \equiv 1 \pmod{q}$. By (26), we get $0 \equiv pq^{g''} \equiv (X^p-1)/(X-1) \equiv X^{p-1} + \cdots + X + 1 \equiv p \pmod{q}$ and $p = q$, a contradiction. Therefore, the case $0 < f < m$ and $0 < g < n$ is impossible.

If $0 < f < m$ and $g = n$, then from (24) we get

$$X-1 = p^f q^n,\ \frac{X^\ell-1}{X-1} = p^{m-f}. \tag{27}$$

By the first equality of (27), we have $X \equiv 1 \pmod{p}$. In addition, by Lemma 8, we see from the second equality of (27) that ℓ is an odd prime with $p \equiv 1 \pmod{2\ell}$. But, since $X \equiv 1$

(mod p), we get $0 \equiv p^{m-f} \equiv (X^\ell - 1)/(X-1) \equiv X^{\ell-1} + \cdots + X + 1 \equiv \ell \pmod p$ and $p = \ell$, a contradiction. Therefore, the case $0 < f < m$ and $g = n$ is impossible. Using the same method, we can eliminate the case $f = m$ and $0 < g < n$.

Recall that $(f, g) \neq (0, 0)$ or (m, n). According to the above analysis, we are left with only the cases

$$f = 0,\ 0 < g \leq n \tag{28}$$

and

$$0 < f \leq m,\ g = 0 \tag{29}$$

that have not yet been discussed.

If (28) holds, by (24), then we have

$$X - 1 = q^g,\ \frac{X^\ell - 1}{X - 1} = p^m q^{n-g},\ g \in \mathbb{N},\ g \leq n. \tag{30}$$

When $g = n$, by (30), we get

$$X - 1 = q^n,\ \frac{X^\ell - 1}{X - 1} = p^m. \tag{31}$$

Applying Lemma 9 to the second equality of (31), ℓ is an odd prime with $p \equiv 1 \pmod{2\ell}$. Hence, by (31), we get (23) and the conclusion (vi) is proved.

When $g < n$, by the first equality of (30), we have $X \equiv 1 \pmod q$. Hence, by Lemma 8, we see from the second equality of (30) that $q^{n-g} \,||\, \ell$. It follows that

$$\ell = q^{n-g} \ell_1,\ \ell_1 \in \mathbb{N},\ q \nmid \ell_1. \tag{32}$$

Then we have

$$\frac{X^\ell - 1}{X - 1} = \left(\frac{X^{\ell_1} - 1}{X - 1} \right) \prod_{j=1}^{n-g} \left(\frac{X^{q^j \ell_1} - 1}{X^{q^{j-1} \ell_1} - 1} \right), \tag{33}$$

where $(X^{\ell_1} - 1)/(X - 1)$ and $(X^{q^j \ell_1} - 1)/(X^{q^{j-1}\ell_1} - 1)$ $(j = 1, \cdots, n - g)$ are positive integers with

$$q \nmid \frac{X^{\ell_1} - 1}{X - 1},\ q \,||\, \frac{X^{q^j \ell_1} - 1}{X^{q^{j-1} \ell_1} - 1},\ j = 1, \cdots, n - g. \tag{34}$$

By (33) and (34), we get from the second equality of (30) that

$$\frac{X^{\ell_1} - 1}{X - 1} = p^{m_0},\ m_0 \in \mathbb{Z},\ m_0 \geq 0 \tag{35}$$

and

$$\frac{X^{q^j \ell_1} - 1}{X^{q^{j-1} \ell_1} - 1} = p^{m_j} q,\ m_j \in \mathbb{N},\ j = 1, \cdots, n - g, \tag{36}$$

where

$$m_0 + m_1 + \cdots + m_{n-g} = m. \tag{37}$$

If $n-g>1$, then from (36) we have $X^{q\ell_1} \equiv 1 \pmod p$ and $0 \equiv p^{m_2}q \equiv (X^{q^2\ell_1}-1)/(X^{q\ell_1}-1) \equiv X^{q\ell_1(q-1)} + \cdots + X^{q\ell_1} + 1 \equiv q \pmod p$, whence we get $p=q$, a contradiction. So we have $n-g=1$. Further, if $n-g=1$ and $\ell_1>1$, by (35) and (36), then m_0 is a positive integer, $X^{\ell_1} \equiv 1 \pmod p$, $0 \equiv p^{m_1}q \equiv (X^{q\ell_1}-1)/(X^{\ell_1}-1) \equiv X^{\ell_1(q-1)} + \cdots + X^{\ell_1} + 1 \equiv q \pmod p$ and $p=q$, a contradiction. Therefore, if (28) holds, then we have $n-g=1$ and $\ell_1=1$. By (30),(32),(33), (36) and (37), we get (21) and the conclusion (iv) is proved.

Using the same method, as in the proof about the case (28), we can deduce that if (29) holds, then there can only obtain (20) or (22), and the conclusions (iii) and (v) are proved. To sum up, the proof is complete. □

For any positive integer m, let $\operatorname{rad}(m)$ denote the product of all distinct prime divisors of m, and let $\operatorname{rad}(1)=1$. Obviously, $\operatorname{rad}(m)$ is equal to the largest squarefree divisor of m.

Lemma 12 ([16]). *If $k>1$ and (x,y,z) is a solution of (2) with $x>z>y$, then we have*

$$\operatorname{rad}(k) \mid b,\ b=b_1b_2,\ b_1,b_2 \in \mathbb{N},\ b_1>1,\ \gcd(b_1,b_2)=1,$$

$$b_1^y = k^{z-y}$$

and

$$a^x k^{x-z} + b_2^y = (a+b)^z.$$

By Lemma 12, we can obtain the following lemma immediately.

Lemma 13. *If $k>1$, b is a prime power and (x,y,z) is a solution of (2) with $x>z>y$, then we have*

$$b^y = k^{z-y}$$

and

$$a^x k^{x-z} + 1 = (a+b)^z.$$

3 Proof of Theorem 2

We now assume that $k>1$ and (x,y,z) is a solution of (2) with $x>z>y$. We will prove that this solution does not exist in the following three cases.

3.1 Case (i): $(a,b)=(2^r,p^s)$ with $r>1$.

Since $k>1$ and p is an odd prime, Lemma 13, we have

$$p^{sy} = k^{z-y} \tag{38}$$

and

$$2^{rx}k^{x-z} + 1 = (2^r+p^s)^z. \tag{39}$$

We see from (38) that k is a power of p. So we have

$$k^{x-z} = p^t,\ t \in \mathbb{N}. \tag{40}$$

Substituting (40) into (39), we get

$$(2^r + p^s)^z - 1 = 2^{rx} p^t. \tag{41}$$

We find from (41) that equation

$$X^\ell - 1 = 2^m p^n,\ X, \ell, m, n \in \mathbb{N},\ X > 1, \ell > 1 \tag{42}$$

has a solution

$$(X, \ell, m, n) = (2^r + p^s, z, rx, t). \tag{43}$$

Since $r \geq 2$ and $x > z > y \geq 1$, we have $x \geq 3$ and $rx \geq 6$. Hence, by Lemma 10, the solution (43) must satisfy the conclusion (ii) or (iii) in this lemma.

When (43) satisfies the conclusion (ii) in Lemma 10, by (10) and (43), we have

$$z = 2,\ t = 1,\ 2^r + p^s = 2^{rx-1} + \zeta,\ p = 2^{rx-2} + \zeta,\ \zeta \in \{1, -1\}. \tag{44}$$

Since $z = 2$ and $z > y \geq 1$, we get $y = 1$. Hence, by (38), we have

$$k = p^s. \tag{45}$$

Further, by (40), (43) and (45), we get $p = p^t = k^{x-z} = p^{s(x-z)} = p^{s(x-2)}$, whence we obtain $s = 1$ and $x = 3$. Therefore, by the third and fourth equalities of (44), we have

$$2^{3r-1} + \zeta = 2^{rx-1} + \zeta = 2^r + p^s = 2^r + p = 2^r + (2^{rx-2} + \zeta) = 2^{3r-2} + 2^r + \zeta,$$

whence we get $2^r = 2^{3r-1} - 2^{3r-2} = 2^{3r-2}$ and $r = 1$, a contradiction.

When (43) satisfies the conclusion (iii) in Lemma 10, by (11) and (43), we have

$$z \text{ is an odd prime},\ 2^r + p^s - 1 = 2^{rx},\ \frac{(2^r + p^s)^z - 1}{2^r + p^s - 1} = p^t,\ p \equiv 1 \pmod{2z}. \tag{46}$$

By the second equality of (46), we get

$$p^s = 2^{rx} - 2^r + 1. \tag{47}$$

If $t \geq s$, then from the third equality of (46) we have

$$0 \equiv p^t \equiv \frac{(2^r + p^s)^z - 1}{2^r + p^s - 1} = \frac{2^{rz} - 1}{2^r - 1} \pmod{p^s}.$$

It implies that $(2^{rz} - 1)/(2^r - 1)$ is a positive integer satisfying

$$\frac{2^{rz} - 1}{2^r - 1} \geq p^s. \tag{48}$$

Since $x > z$, by (47) and (48), we have

$$2^{r(z-1)+1} > \frac{2^{rz}-1}{2^r-1} \geq p^s = 2^{rx} - 2^r + 1 \geq 2^{r(z+1)} - 2^r + 1 > 2^{r(z+1)} - 2^r$$

and $r \geq 2r - 1$, a contradiction. So we have $t < s$. Then, since $z > 2$, by the third equality of (46), we get

$$p^s > p^t = \frac{(2^r + p^s)^z - 1}{2^r + p^s - 1} > \frac{(2^r + p^s)^2 - 1}{2^r + p^s - 1} = 2^r + p^s + 1 > p^s,$$

a contradiction. Thus, the theorem holds for this case.

3.2 Case (ii): $(a, b) = (p^r, 2^s)$ with $s > 1$.

By Lemma 13, we have

$$2^{sy} = k^{z-y} \tag{49}$$

and

$$p^{rx}k^{x-z} + 1 = (p^r + 2^s)^z. \tag{50}$$

We see from (49) that k is a power of 2. So we have

$$k^{x-z} = 2^t,\ t \in \mathbb{N}. \tag{51}$$

Substituting (51) into (50), we get

$$(p^r + 2^s)^z - 1 = 2^t p^{rx}. \tag{52}$$

We find from (52) that (42) has a solution

$$(X, \ell, m, n) = (p^r + 2^s, z, t, rx). \tag{53}$$

Since $rx \geq x \geq 3$, by Lemma 10, the solution (53) only satisfies the conclusion (iii) in this lemma. Then, by (11) and (53), we have

$$z \text{ is an odd prime},\ p^r + 2^s - 1 = 2^t,\ \frac{(p^r + 2^s)^z - 1}{p^r + 2^s - 1} = p^{rx},\ p \equiv 1 \pmod{2z}. \tag{54}$$

Further, since $rx \geq x > 2$, applying Lemma 4 to the third equality of (54), we get

$$2 \nmid x. \tag{55}$$

By the first equality of (54), z is an odd prime. Since $z > y \geq 1$, we have $\gcd(y, z) = 1$ and

$$\gcd(y, z - y) = 1. \tag{56}$$

We see from (49) and (56) that $s \equiv 0 \pmod{z-y}$,

$$s = (z-y)s_1, \ s_1 \in \mathbb{N} \tag{57}$$

and

$$k = 2^{s_1 y}. \tag{58}$$

Further, by (51) and (58), we have

$$t = s_1 y(x-z), \tag{59}$$

whence we get

$$y \mid t. \tag{60}$$

By the second equality of (54), we have

$$p^r + 2^s = 2^t + 1 \tag{61}$$

and

$$p^r \equiv -2^s \pmod{2^t+1}. \tag{62}$$

Substituting (61) into the third equality of (54), we get

$$p^{rx} = \frac{(2^t+1)^z - 1}{(2^t+1)-1} = (2^t+1)^{z-1} + \cdots + (2^t+1) + 1 \tag{63}$$

and

$$p^{rx} \equiv 1 \pmod{2^t+1}. \tag{64}$$

Further, by (55), (62) and (64), we have

$$2^{sx} + 1 \equiv 0 \pmod{2^t+1}. \tag{65}$$

Applying Lemma 6 to (65), we get $sx \equiv 0 \pmod{t}$ and

$$sx = tt_1, \ t_1 \in \mathbb{N}, \ 2 \nmid t_1. \tag{66}$$

Hence, by (57), (59) and (66), we have

$$(z-y)x = y(x-z)t_1. \tag{67}$$

Furthermore, by (56) and (67), we get

$$y \mid x. \tag{68}$$

Therefore, by (60), (63) and (68), we have

$$1 = (2^t+1)^z - 2^t p^{rx} = (2^t+1)^z - (2^{t/y} p^{rx/y})^y, \tag{69}$$

where $2^{t/y}p^{rx/y}$ is a positive integer greater than 6. Since $z \geq 3$, by Lemma 5, we see from (69) that

$$y = 1. \tag{70}$$

Thus, by (57) and (70), we get

$$s \geq z - 1. \tag{71}$$

On the other hand, we see from (61) that

$$t > s \tag{72}$$

and

$$p^r \equiv 1 \pmod{2^s}. \tag{73}$$

Further, by (63), (72) and (73), we have

$$1 \equiv p^{rx} \equiv \frac{(2^t+1)^z - 1}{(2^t+1) - 1} = (2^t+1)^{z-1} + \cdots + (2^t+1) + 1 \equiv z \pmod{2^s}. \tag{74}$$

Since $z > 1$, by (74), we get

$$z \geq 2^s + 1. \tag{75}$$

Therefore, the combination of (71) and (75) yields $s \geq z - 1 \geq 2^s$, a contradiction. Thus, the theorem holds in this case.

3.3 Case (iii): $(a, b) = (p^r, q^s)$

By Lemma 13, we have

$$q^{sy} = k^{z-y} \tag{76}$$

and

$$p^{rx}k^{x-z} + 1 = (p^r + q^s)^z. \tag{77}$$

We see from (76) that k is a power of q. So we have

$$k^{x-z} = q^t, \ t \in \mathbb{N}. \tag{78}$$

Substituting (78) into (77), we get

$$(p^r + q^s)^z - 1 = p^{rx}q^t. \tag{79}$$

We find from (79) that the equation

$$X^\ell - 1 = p^m q^n, \ X, \ell, m, n \in \mathbb{N}, \ X > 1, \ell > 1 \tag{80}$$

has a solution

$$(X, \ell, m, n) = (p^r + q^s, z, rx, t). \tag{81}$$

Since $p^r+q^s \geq 8$, applying Lemma 11 to (80) and (81), we only need to consider the following five subcases.

Subcase (iii)-1:

$$2 \mid z,\ (p^r+q^s)^{z/2}+\zeta=p^{rx},\ (p^r+q^s)^{z/2}-\zeta=q^t,\ \zeta \in \{1,-1\}. \tag{82}$$

Since $rx \geq 3$, by Lemma 5, we see from the second equality of (82) that $z=2$. So we have

$$p^r+q^s+\zeta=p^{rx},\ p^r+q^s-\zeta=q^t. \tag{83}$$

By the first equality of (83), we get

$$q^s=p^{rx}-p^r-\zeta=p^r(p^{r(x-1)}-1)-\zeta \geq p^r(p^{2r}-1)-1>p^r. \tag{84}$$

However, by the second equality of (83), we have $t>s$ and

$$p^r=q^t-q^s+\zeta=q^s(q^{t-s}-1)-1 \geq q^s(q-1)-1>q^s,$$

which contradicts (84). Therefore, this subcase can be eliminated.

Subcase (iii)-2:

$$z=p,\quad p^r+q^s-1=p^{rx-1},\ \frac{(p^r+q^s)^p-1}{p^r+q^s-1}=pq^t,\ q \equiv 1 \pmod{2p}. \tag{85}$$

If $t \geq s$, then from the third equality of (85) we get

$$0 \equiv pq^t \equiv \frac{(p^r+q^s)^p-1}{p^r+q^s-1} \equiv \frac{p^{rp}-1}{p^r-1} \pmod{q^s}, \tag{86}$$

where $(p^{rp}-1)/(p^r-1)$ is a positive integer. However, by (85) and (86), we have

$$\begin{aligned}2p^{r(z-1)}=2p^{r(p-1)}=p^{r(p-1)}\sum_{i=0}^{\infty}\frac{1}{2^i}>p^{r(p-1)}\sum_{j=0}^{p-1}\frac{1}{p^{rj}}=\frac{p^{rp}-1}{p^r-1}\\ \geq q^s=p^{rx-1}-p^r+1 \geq p^{r(z+1)-1}-p^r+1,\end{aligned}$$

whence we get

$$\begin{aligned}p^r>p^r-1 &\geq p^{r(z+1)-1}-2p^{r(z-1)}=p^{r(z-1)}(p^{2r-1}-2)\\ &\geq p^{r(z-1)}(p-2) \geq p^{r(z-1)} \geq p^r,\end{aligned}$$

a contradiction. So we have $t<s$. Then, since p is an odd prime, by the third equality of (85), we get

$$pq^s>pq^t=\frac{(p^r+q^s)^p-1}{p^r+q^s-1} \geq \frac{(p^r+q^s)^3-1}{p^r+q^s-1}>(p^r+q^s)^2>pq^s,$$

a contradiction. Thus, this subcase can be eliminated.

Subcase (iii)-3:

$$z = q,\ p^r + q^s - 1 = q^{t-1},\ \frac{(p^r + q^s)^q - 1}{p^r + q^s - 1} = p^{rx} q,\ p \equiv 1 \pmod{2q}. \tag{87}$$

By (87), we have $p^r + q^s = q^{t-1} + 1$ and

$$p^{rx} q = \frac{(p^r + q^s)^q - 1}{p^r + q^s - 1} = \frac{(q^{t-1} + 1)^q - 1}{q^{t-1}} = \sum_{i=1}^{q-1} \binom{q}{i} q^{(t-1)(i-1)},$$

whence we get

$$p^{rx} = 1 + \sum_{i=2}^{q-1} \binom{q}{i} q^{(t-1)(i-1)-1}. \tag{88}$$

On the other hand, by the second equality of (87), we have

$$s < t - 1 \tag{89}$$

and

$$p^r = q^s (q^{t-s-1} - 1) + 1. \tag{90}$$

Hence, by (90), we get

$$p^{rx} = 1 + \sum_{j=1}^{x} \binom{x}{j} (q^s (q^{t-s-1} - 1))^j. \tag{91}$$

The combination of (88) and (91) yields

$$\sum_{i=2}^{q-1} \binom{q}{i} q^{(t-1)(i-1)-1} = \sum_{j=1}^{x} \binom{x}{j} (q^s (q^{t-s-1} - 1))^j. \tag{92}$$

Let $q^\alpha \mid\mid x$, where α is a nonnegative integer. Notice that

$$q^{t-1} \mid\mid \sum_{i=2}^{q-1} \binom{q}{i} q^{(t-1)(i-1)-1},\ q^{\alpha+s} \mid\mid \sum_{j=1}^{x} \binom{x}{j} (q^s (q^{t-s-1} - 1))^j. \tag{93}$$

By (92) and (93), we have $t - 1 = \alpha + s$. Further, by (89), we get $\alpha = (t-1) - s > 0$. It implies that

$$q \mid x. \tag{94}$$

Recall that $z > y$ and $z = q$ is an odd prime. Hence, y and z satisfy (56). By (56) and (76), we have $s \equiv 0 \pmod{z - y}$ and (57). Further, by (76) and (78) we get $k = q^{s_1 y}$ and

$$t = s_1 y (x - z) = s_1 y (x - q). \tag{95}$$

Furthermore, by (94) and (95), we obtain

$$q \mid t. \tag{96}$$

Therefore, by (87), (94) and (96), we have

$$\begin{aligned} p^{rx/q}q^{t/q} \in \mathbb{N},\ 1 = (p^r+q^s)^q - p^{rx}q^t &= (p^r+q^s)^q - (p^{rx/q}q^{t/q})^q \\ &= (p^r+q^s-p^{rx/q}q^{t/q})\sum_{i=0}^{q-1}(p^r+q^s)^{q-i-1}(p^{rx/q}q^{t/q})^i > 1, \end{aligned}$$

a contradiction. Thus, this subcase can be eliminated.

Subcase (iii)-4:

$$z \text{ is an odd prime},\ p^r+q^s-1 = p^{rx},\ \frac{(p^r+q^s)^z-1}{p^r+q^s-1} = q^t,\ q \equiv 1 \pmod{2z}. \tag{97}$$

We can eliminate this subcase by using the same method as in the proof of Subcase (iii)-2. An outline of this process is given below. If $t \geq s$, then from the second and third equalities of (97) we have

$$0 \equiv q^t \equiv \frac{(p^r+q^s)^z-1}{p^r+q^s-1} \equiv \frac{p^{rz}-1}{p^r-1} \pmod{q^s},$$

whence we get $(p^{rz}-1)/(p^r-1) \geq q^s$. So we have

$$p^{r(z-1)+1} > \frac{p^{rz}-1}{p^r-1} \geq q^s = p^{rx} - p^r + 1 \geq p^{r(z+1)} - p^r + 1 > p^{r(z+1)-1},$$

a contradiction. It implies that $t < s$. Then, by (97), we get

$$q^s > q^t = \frac{(p^r+q^s)^z-1}{p^r+q^s-1} > p^r+q^s+1 > q^s,$$

a contradiction. Thus, this subcase can be eliminated.

Subcase (iii)-5:

$$z \text{ is an odd prime},\ p^r+q^s-1 = q^t,\ \frac{(p^r+q^s)^z-1}{p^r+q^s-1} = p^{rx},\ p \equiv 1 \pmod{2z}. \tag{98}$$

We see from the second equality of (98) that $t > s$. So we have $p^r = q^s(q^{t-s}-1)+1$ and

$$p^r \equiv 1 \pmod{q^s}. \tag{99}$$

By the second and third equalities of (98), we get

$$p^r + q^s = q^t + 1 \tag{100}$$

and

$$p^{rx} = \frac{(p^r+q^s)^z - 1}{p^r+q^s-1} = \frac{(q^t+1)^z-1}{(q^t+1)-1}. \tag{101}$$

Further, by (101), we have

$$p^{rx} \equiv (q^t+1)^{z-1} + \cdots + (q^t+1) + 1 \equiv z \pmod{q^t}. \tag{102}$$

Since $t > s$, by (99) and (102), we get $z \equiv 1 \pmod{q^s}$. Furthermore, since $z > 1$, we have

$$z \geq q^s + 1. \tag{103}$$

On the other hand, since z is an odd prime with $z > y \geq 1$, y and z satisfy (56). Hence, by (56) and (76), s satisfies (57), whence we can obtain (60).

Since $p^r + q^s > 3$, applying Lemma 4 to the third equality of (98), we have

$$2 \nmid x. \tag{104}$$

By (101), we have

$$p^{rx} \equiv \frac{(q^t+1)^z-1}{q^t+1} \equiv (q^t+1)^{z-1} + \cdots + (q^t+1) + 1 \equiv 1 \pmod{q^t+1}. \tag{105}$$

Since $p^r \equiv -q^s \pmod{q^t+1}$ by (100), we get from (104) and (105) that

$$0 \equiv p^{rx} - 1 \equiv (-q^s)^x - 1 \equiv -(q^{sx}+1) \pmod{q^t+1}. \tag{106}$$

Hence, by Lemma 6, we see from (106) that x satisfies (66) and (67). Further, by (56) and (67), x satisfies (68). Therefore, by (60) and (68), we get from the third equality of (98) that

$$p^{rx/y}q^{t/y} \in \mathbb{N},\ (p^r+q^s)^z - (p^{rx/y}q^{t/y})^y = 1. \tag{107}$$

Since $z \geq 3$ and $p^r + q^s > 3$, by Lemma 5, we find from (107) that y satisfies (70). Furthermore, by (57) and (70), s satisfies (71). However, the combination of (71) and (103) yields $s \geq z - 1 \geq q^s$, a contradiction. Thus, this subcase can be eliminated.

To sum up, the theorem is proved.

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