Journal of Integer Sequences, Vol. 26 (2023),
Article 23.1.6

# On the Minimal Number of Solutions of the Equation $\phi(n+k) = M\phi(n)$, $M = 1, 2$

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**Abstract**

We fix a positive integer $k$ and look for solutions $n \in \mathbb{N}$ of the equations $\phi(n+k) = \phi(n)$ and $\phi(n+k) = 2\phi(n)$. For $k \leq 12 \cdot 10^{100}$, we prove that Fermat primes can be used to build five solutions for the first equation when $k$ is even, and five for the second one when $k$ is odd. Furthermore, for $k \leq 4 \cdot 10^{58}$, we show that for the second equation there are at least three solutions when $k$ is even. Our work increases the previously known minimal number of solutions for both equations.

# 1 Introduction

Euler's phi function $\phi(n)$ counts the number of positive integers less than or equal to $n$ that are coprime with $n$. In 1956, Sierpiński [7] considered the equation

$$\phi(n+k) = \phi(n), \tag{1}$$

and proved that, for every positive integer $k$, there exists at least one solution. This result was improved by Schinzel [5], who showed that (1) has at least two solutions for every positive integer $k \leq 8 \cdot 10^{47}$. The upper bound on $k$ was extended to $2 \cdot 10^{58}$ by Schinzel and Wakulicz [6] for all $k$, and to $1.38 \cdot 10^{26595411}$ by Holt [3] for even $k$. All these results were obtained exhibiting explicit solutions, built proving (or assuming) the existence of prime numbers satisfying certain conditions. Graham, Holt and Pomerance [1] showed that, for even $k$, if we take $j$ and $r$ such that $j$ and $j+k$ have the same prime factors, and the numbers $jr/g+1, (j+k)r/g+1$, with $g = \gcd(j, j+k)$, are both primes not dividing $j$, then

$$n = j\left(\frac{j+k}{g}r + 1\right)$$

is a solution of (1) [1, Theorem 1]. Despite the lack of general results, numerical evidence suggests that (1) has infinitely many solutions for every fixed $k$. This can be proven for some special values of $k$ (cf. [9]).

In the same spirit of Sierpiński, Makowski [4] considered the equation

$$\phi(n+k) = 2\phi(n), \tag{2}$$

finding one solution for all fixed $k$. More recently, the same equation was studied by Hasanalizade [2], who proved that (2) has at least two solutions for all $k \leq 4 \cdot 10^{58}$ and at least three solutions for some odd $k \leq 4 \cdot 10^{58}$ [2, Theorem 1]. Hasanalizade also gave [2, Lemma 1] a modified version of [1, Theorem 1], valid for odd $k$ multiple of 3 and for equation (2).

In this paper we find new solutions for (1) and (2). These solutions are obtained in two ways. First, we consider Fermat prime numbers and show that for each Fermat prime it is possible to build a solution for (1), when $k$ is even, and for (2), when $k$ is odd. Numerically, we show that these solutions can be actually built for $k \leq 2 \cdot 10^{100}$. Since only five Fermat primes are known, this method provides five different solutions. Second, we exhibit a new solution for equation (2) for even $k$. Together with the results of Hasanalizade, this gives our main theorem.

**Theorem 1.** *Equation (2) has at least three solutions for all $k \leq 4 \cdot 10^{58}$.*

We conclude the paper by providing several ways of building particular solutions when certain conditions are met.

## 2 The equation $\phi(n) = \phi(n+k)$

Let $F_m$ denote the $m$-th Fermat number

$$F_m = 2^{2^m} + 1.$$

It is known that $F_m$ is prime for $m = 0, \ldots, 4$, while it is composite for all $5 \leq m \leq 32$. It is not known whether or not there exist other values of $m$ such that $F_m$ is prime.

Let $a$ and $b$ be positive integers. If the prime factors of $a$ are contained in the prime factors of $b$ we write $a \mid^* b$. If $a \mid^* b$, then it is easy to see that

$$\phi(ab) = a\phi(b).$$

Moreover, $a$ has the same prime factors as $b$ if and only if $a \mid^* b$ and $b \mid^* a$. In this case, we have that

$$a\phi(b) = b\phi(a).$$

We prove the following result:

**Theorem 2.** *Equation (1) has at least six solutions for all even $k \leq 12 \cdot 10^{100}$.*

*Proof.* We build a solution for each $F_m$ prime, exploiting the elementary fact that $(F_m - 1) \mid^* k$ for even $k$.

**Case 1:** $\gcd(F_m, k) = 1$. We have that

$$n = (F_m - 1)k = 2^{2^m} k$$

is a solution of (1). In fact

$$\phi(n+k) = \phi(F_m k) = \phi(F_m)\phi(k) = (F_m - 1)\phi(k),$$

and

$$\phi(n) = \phi((F_m - 1)k) = (F_m - 1)\phi(k).$$

Note that in this case we have no upper bound on the values of $k$.

**Case 2:** $F_m \mid k$. Assume that there exists a positive integer $r$ such that $(F_m - 1)r + 1$ and $F_m r + 1$ are both primes and do not divide $k$. Then

$$n = (F_m - 1)(F_m r + 1)k$$

is a solution of (1). In fact

$$\phi(n+k) = \phi(((F_m - 1)F_m r + F_m)k) = \phi(F_m((F_m - 1)r + 1)k) = F_m(F_m - 1)r\phi(k),$$

while

$$\phi(n) = \phi((F_m - 1)(F_m r + 1)k) = F_m(F_m - 1)r\phi(k).$$

| $m$ | $r$ |
|---|---|
| 0 | $10^{100} + 9760$ |
| 1 | $10^{100} + 60128$ |
| 2 | $10^{100} + 150326$ |
| 3 | $10^{100} + 51326$ |
| 4 | $10^{100} + 14786$ |

Table 1: Values of $m$ and $r$.

We have that $(F_m - 1)r + 1$ and $F_m r + 1$ are both prime, for the values of $m$ and $r$ given in table 1.

The numbers $(F_m - 1)r + 1$ and $F_m r + 1$ are certainly prime with $k$, taking $k \leq 12 \cdot 10^{100}$. Indeed, since $2F_m \mid k$, taking $k < F_m((F_m - 1)r + 1)$, neither $(F_m - 1)r + 1$ nor $F_m r + 1$ divide $k$. We point out that different choices of $m$ and $r$ provide different solutions. To these we must add the previously known solutions. In particular, Sierpiński's solution might coincide with one of our solutions for some $k$, while Schinzel's solution differs from the ones we provided. Indeed, Schinzel found a solution building a sequence of odd primes $q_1, q_2, \ldots, q_\ell$, such that $2q_i - 1$ is prime for $i = 1, \ldots, \ell$, and $2q_i - 1 \neq q_j$ for $i, j = 1 \ldots \ell$. Then, for even $k < q_1 q_2 \cdots q_\ell$, there exists a prime $q_j$ in the sequence such that both $q_j$ and $2q_j - 1$ do not divide $k$. Schinzel's solution $n = (2q_j - 1)k$ differs from our solutions, since the former is an odd multiple of $k$, while the latter is an even multiple of $k$. This brings the minimal number of known solutions to six for all even $k \leq 2 \cdot 10^{100}$. □

*Remark* 3. It is a well-known conjecture by Dickson that for any fixed $a, b$ there exist infinitely many positive integers $r$ such that both $ar + 1$ and $br + 1$ are prime. Following Graham, Holt and Pomerance [1], we write $\mathcal{P}(a, b)$ if such a property holds. Assuming $\mathcal{P}(F_m, F_m - 1)$ for $m = 0, \ldots, 4$, we can remove the upper bound on $k$, obtaining that (1) has at least six solutions for all even $k$.

*Remark* 4. Assume that $F_m$ divides $k$, for some fixed $m$. Then, for any $r$ such that $(F_m - 1)r + 1$ and $F_m r + 1$ are both prime and do not divide $k$, we find a solution $n$ depending on $r$. Different values of $r$ yield different solutions. As a consequence, if $\mathcal{P}(F_m, F_m - 1)$ is true, then equation (1) has an infinite number of solutions for all even $k$ such that $F_m \mid k$.

# 3 The equation $\phi(n + k) = 2\phi(n)$

Our main results concern the number of solutions of (2). As for equation (1), there is substantial difference in the solutions that can be found for even $k$ and odd $k$. For equation (1), it has already been observed [1, Table 1] that solutions abound for even $k$, while they are scarce for odd $k$. After an extensive numerical investigation, we find that the same phenomenon holds for equation (2), with the parity of $k$ inverted. Indeed, we have an analogous version of Theorem 2 (with a similar proof), that holds for odd $k$ instead of even $k$.

**Theorem 5.** *Equation* (2) *has at least five solutions for all odd* $k \leq 6 \cdot 10^{100}$.

*Proof.* We build a solution for each $F_m$ prime, noting that $\gcd(F_m - 1, k) = 1$, and that

$$F_m - 1 = 2\phi(F_m - 1).$$

**Case 1:** $\gcd(F_m, k) = 1$. We have that

$$n = (F_m - 1)k = 2^{2^m}k$$

is a solution of (2). Again, in this case we have no upper bound on the values of $k$ for which we find solutions.

**Case 2:** $F_m \mid k$. Assume that there exists a positive integer $r$ such that $(F_m - 1)r + 1$ and $F_m r + 1$ are both primes and do not divide $k$. Then

$$n = (F_m - 1)(F_m r + 1)k$$

is a solution of (1). Reasoning similarly to Theorem 2, we have five solutions for all odd $k \leq 6 \cdot 10^{100}$. Indeed, since $F_m \mid k$, taking $k < F_m((F_m - 1)r + 1)$, neither $(F_m - 1)r + 1$ nor $F_m r + 1$ divide $k$. □

*Remark* 6. Remarks 3 and 4 are also valid in this case, replacing even $k$ with odd $k$. In particular for $m = 0$, Remark 4 applied to equation (2) has, as a consequence, the second part of [2, Lemma 1].

When $k$ is even, equation (2) has at least two solutions. The solution $n = k$ was noticed by Makowski [4]. Another solution was found by Hasanalizade [2], building a sequence of primes [8, A001259]. Here we recall the construction. Take a sequence of primes $3 = p_1 < p_2 < \cdots < p_m$ satisfying, for all $i = 2, \ldots, m$, the following conditions:

- $(p_i - 2) \mid \prod_{j \leq i-1} p_j$,
- $(p_i - 1) \mid^* 2 \prod_{j \leq i-1} p_j$,

and such that $\prod_j p_j$ does not divide $k$. Let $p_\ell$ be the smallest prime of the sequence such that $\gcd(p_\ell, k) = 1$. Then $n = p_\ell/(p_\ell - 2)k$ is a solution of equation (2). Building the explicit sequence of primes $\{3, 5, 7, 17, 19, 37, 97, 113, 257, 401, 487, 631, 971, 1297, 1801, 19457, 22051, 28817, 65537, 157303, 160001\}$, we have one solution different from $n = k$ for all even $k \leq 4 \cdot 10^{58}$.

Now we find a new third solution for even $k$.

**Theorem 7.** *Equation* (2) *has at least three solutions for all even* $k \leq 4 \cdot 10^{58}$.

*Proof.* Assume that there exists a sequence of prime numbers $2 = p_1 < p_2 < \cdots < p_m$ and of positive integers $a_2 < \cdots < a_m$ such that, for all $i = 2, \ldots, m$, the following conditions hold:

- $p_i = 2a_i + 1$,
- $a_i \mid^* \prod_{j \le i-1} p_j$,
- $(a_i + 1) \mid \prod_{j \le i-1} p_j$,

and such that $\prod_j p_j$ does not divide $k$. Let $p_\ell$ be the smallest prime such that $\gcd(p_\ell, k) = 1$. Then $n = a_\ell k/(a_\ell + 1)$ is a solution of (2). In fact

$$\phi(n+k) = \phi\left(\frac{2a_\ell + 1}{a_\ell + 1}k\right) = 2a_\ell \phi\left(\frac{k}{a_\ell + 1}\right),$$

and

$$2\phi(n) = 2\phi\left(\frac{a_\ell}{a_\ell + 1}k\right) = 2a_\ell \phi\left(\frac{k}{a_\ell + 1}\right).$$

A sequence of primes [8, A358717], up to $10^8$, satisfying the above assumptions is given by

$$\{2, 3, 5, 11, 19, 37, 73, 109, 1459, 2179, 2917, 4357, 8713\}.$$

The product of the prime numbers in the above sequence is of the order of $6 \cdot 10^{26}$, and $\prod_j p_j$ does not divide $k$, for $k < \prod_j p_j$. To improve the upper bound on $k$, we slightly modify our argument. We proceed as follows:

- If $k$ is not divisible by $2 \cdot 3 \cdot 5 \cdot 11 \cdot 19$, then we simply apply our argument with no changes.
- If $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \mid k$, we take the sequence starting with $2, 3, 5, 7$ and then built using the rules as explained before. The solution to (2) will be again $n = a_\ell k/(a_\ell + 1)$, where $\ell$ is the smallest index such that $\gcd(p_\ell, k) = 1$. The sequence [8, A358718] continues as

  $$\{2, 3, 5, 7, 11, 13, 19, 29, 37, 41, 43, 59, 73, 83, 109, 113, 131, 163, 173, 181, \\ 227, 257, 331, 347, 353, 379, 419, 491, 523, 571, 601, 653, 661, 677, \dots, 12011\}.$$

  In this case, since the sequence contains the number 7, it becomes dense enough in the primes to give a large upper bound. We find that such upper bound can be taken to be of the order of $2 \cdot 10^{310}$.
- If $2 \cdot 3 \cdot 5 \cdot 11 \cdot 19 \mid k, \gcd(7, k) = 1$, and $\gcd(13, k) = 1$, then $n = 36k/55$ is a solution of (2).
- If $2 \cdot 3 \cdot 5 \cdot 11 \cdot 13 \cdot 19 \mid k, \gcd(7, k) = 1$, and $\gcd(23, k) = 1$, then $n = 66k/95$ is a solution of (2).

- If $2 \cdot 3 \cdot 5 \cdot 11 \cdot 13 \cdot 19 \cdot 23 \mid k$ and $\gcd(7, k) = 1$, we find, proceeding as before, the following sequence [8, A358719]:

$$\{2, 3, 5, 11, 13, 23, 19, 37, 73, 109, 131, 229, 457, 571, 1459, 1481, 2179, 2621,$$
$$2917, 2963, 4357, 8713, 49921, 1318901, 3391489, 6782977, 13565953\},$$

that gives the upper bound of $2 \cdot 10^{83}$.

We conclude the proof by observing that our solution differs from the previously known solutions, since in our case $n < k$. Indeed, Hasanalizade's solutions [2] satisfy $n \geq k$. □

*Remark* 8. Our trick can be repeated as many times as desired to improve the upper bound on $k$, as long as we can find particular solutions.

As a consequence we obtain our main theorem.

*Proof of Theorem 1.* Theorem 5 proves that for all odd $k \leq 2 \cdot 10^{100}$ there are at least five solutions to equation (2), while Theorem 7 proves that for all even $k \leq 4 \cdot 10^{58}$ we have at least three solutions. Overall, we get three solutions for all $k \leq 4 \cdot 10^{58}$. □

*Remark* 9. We recall that while the upper bound on $k$ for our new solution is fairly high, increasing the upper bound of $4 \cdot 10^{58}$ for Hasanalizade's solution would require a lot of effort, as pointed out by Holt [3].

# 4 Special solutions

We open the section with a modified version of the first part of [2, Lemma 1]. Our result removes the assumption $3 \mid k$.

**Lemma 10.** *Let $k$ be an odd positive integer. Suppose that $j$ is a positive integer such that $j$ and $2j + k$ have the same prime factors. Take $g = \gcd(j, 2j + k)$ and consider a positive integer $r$ such that $2jr/g + 1$ and $(2j + k)r/g + 1$ are both prime and coprime with $k$. Then*

$$n = 2j\left(\frac{2j + k}{g}r + 1\right)$$

*is a solution of equation (2).*

*Proof.* We have that

$$\phi(n + k) = \phi\left(2j\left(\frac{2j + k}{g}r + 1\right) + k\right) = \phi\left((2j + k)\left(\frac{2j}{g}r + 1\right)\right) = 2j\phi(2j + k)\frac{r}{g},$$

and

$$2\phi(n) = 2\phi\left(2j\left(\frac{2j + k}{g}r + 1\right)\right) = 2\phi(2j)\phi\left(\frac{2j + k}{g}r + 1\right) = 2\phi(2j)(2j + k)\frac{r}{g}.$$

We conclude the result using that

$$2j\phi(2j+k) = 2j\phi(2(2j+k)) = \phi(4j(2j+k)) = 2\phi(2j)(2j+k).$$

$\square$

Motivated by explicit computations, we would like to find more general formulas that provide new families of solutions to (2). In practice, we always find at least four solutions in the range $n \le 10^6$, for $k \le 10^4$. The only value of $k$ for which we find exactly 4 solutions is $k = 6$. In that case, the solutions are

$$n \in \{4, 6, 7, 10\}.$$

A computer search yielded no other solutions $n \le 10^8$ for $k = 6$. We notice that $n = 6$ is the solution given by Makowski and $n = 10$ is the solution from Hasanalizade, while $n = 4$ is obtained via our method with $a_\ell = 2$. There is no known family of solutions providing $n = 7$. In an attempt to find such a family, we prove the following result.

**Proposition 11.** *Assume that there exists a prime $p$ such that $2p-1$ is prime, $\gcd(p,k) = 1, \gcd(2p-1,k) = 1$, and $(p-1) \mid k$. Then*

$$n = \frac{p}{p-1}k$$

*is a solution of* (2).

*Proof.* It is immediate to check that

$$\phi(n+k) = \phi\left(k\frac{p}{p-1} + k\right) = \phi\left(k\frac{2p-1}{p-1}\right) = (2p-2)\phi\left(\frac{k}{p-1}\right),$$

and

$$2\phi(n) = 2\phi\left(k\frac{p}{p-1}\right) = 2(p-1)\phi\left(\frac{k}{p-1}\right).$$

$\square$

We can take $p = 7$ and obtain the solution $n = 7$, for $k = 6$. Unfortunately, this cannot be generalized to all even $k$, since the condition $(p-1) \mid k$ is satisfied only for a finite number of $p$ and it might happen that none of them satisfies the other requirements (e.g., $k = 10$).

We observe that for all odd $k$, thanks to Proposition 11, we can take $p = 2$ and recover the solution $n = 2k$ given by Makowski.

Finally, we conclude the paper by giving one more way of building solutions in another special case.

**Proposition 12.** *Let $k$ be an even positive integer. Assume that there exists $m$ such that $m \mid k, \gcd(m+2,k)=1, \gcd(m+4,k)=1,$ and $\phi(m+2)=\phi(m+4)$. Then*

$$n = \frac{m+4}{m}k$$

*is a solution of* (2).

*Proof.* Again, it is immediate to check that

$$\phi(n+k) = \phi\left(k\frac{m+4}{m}+k\right) = 2\phi\left(k\frac{m+2}{m}\right) = 2\phi(m+2)\phi\left(\frac{k}{m}\right),$$

and

$$2\phi(n) = 2\phi\left(k\frac{m+4}{m}\right) = 2\phi(m+4)\phi\left(\frac{k}{m}\right).$$

□

# 5 Acknowledgments

We thank the anonymous referee for suggesting an argument that improved the upper bound on $k$ in Theorems 2 and 5. We also thank the referee for very helpful comments that improved the exposition of the paper. The first author is partially supported by MIUR grant *"Dipartimenti di Eccellenza 2018-2022"*, CUP E11G18000350001. The second author is partially supported by GNSAGA of INdAM. The numerical computations supporting our paper have been performed using Magma V2.27-5, and are available upon request to the first author.

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2020 *Mathematics Subject Classification*: Primary 11N25. Secondary 11Y99.
*Keywords:* Euler's phi function.

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(Concerned with sequences A001259, A358717, A358718, and A358719.)

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Received September 7 2022; revised versions received December 27 2022; January 14 2023. Published in *Journal of Integer Sequences*, January 15 2023.

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