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Product of Some Polynomials and Arithmetic Functions

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Abstract

We study the injectivity and noninjectivity of the function fg, where f is a polynomial in a simple form and g is a popular arithmetic function such as the Euler totient function or the sum of divisors function. We also show the connection between our results and Mersenne primes, amicable pairs, and other integer sequences.

1 Introduction

For each $n \in \mathbb{N}$, let $\varphi(n)$ be the number of positive integers that are at most n and relatively prime to n, and let $\sigma(n)$ be the sum of positive divisors of n. The functions φ and σ are connected with some popular topics such as Lehmer's problem, Carmichael's conjecture, perfect numbers, Mersenne primes, amicable pairs, and aliquot sequences. For example, it is easy to see that if n is a prime, then $\varphi(n) = n-1$. Lehmer asked whether $\varphi(n) \mid n-1$ implies that n is a prime, but this question is still open. In addition, Carmichael's long-standing open conjecture on the range of φ states that if $\varphi(x) = n$, then there exists $y \in \mathbb{N}$ distinct from x such that $\varphi(y) = n$ too. Moreover, whether or not there are infinitely many $n \in \mathbb{N}$ with $\sigma(n) = 2n$ and whether or not there exists an odd integer n with $\sigma(n) = 2n$ have been open for a long time.

Many mathematicians including Ford [2, 3], Ford, Luca, and Pomerance [4], Ford and Pollack [5], and Pomerance [7] have contributed to the progress of this area of research. In particular, Ford [3] solved Sierpiński's conjecture and partially solved Carmichael's problem stated above. That is, Ford showed that for each integer $k \geq 2$, there exists a positive integer n for which the equation $\varphi(x) = n$ has exactly k solutions. Furthermore, Ford, Luca, and Pomerance [4] completely answered Erdős' question on the ranges of φ and σ by showing that $\varphi(x) = \sigma(y)$ has infinitely many solutions in $x, y \in \mathbb{N}$. We refer the reader to the sequences A000010 and A007617 in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [9] for more information on the range of φ, the sequences A000396 and A000668 for perfect numbers and Mersenne primes, and the sequences A063990, A063900, A001065, A008888, and A098007 for amicable pairs and aliquot sequences.

It is well known that the function φ is not injective but the function f_0 defined by $f_0(n) = n\varphi(n)$ is injective. Not every function has this property: both σ and the function $A(n) = n\sigma(n)$ are not injective. Nevertheless, we show in Examples 7 and 8 and Theorem 12 that A is injective on squarefree integers and both A and σ are related to Mersenne primes and amicable pairs. Generally speaking, both injectivity and noninjectivity are interesting; if an arithmetic function f is injective, we can conclude that the equation $f(x) = n$ has at most one solution; if f is not injective, then we may like to count the number of solutions to $f(x) = n$, and study the relation between f and each solution.

In this article, we study the injectivity and noninjectivity of the product of polynomials and arithmetic functions. We will replace $n\varphi(n)$ and $n\sigma(n)$ by $g(n)h(n)$ where $g(n)$ is a polynomial and $h(n)$ is an arithmetic function. For simplicity, we focus our attention to polynomials in a simple form such as $g(n) = n^a$ or $g(n) = n + c$ where a, c are any positive integers, while $h(n)$ is a popular arithmetic function such as $\varphi(n)$, $\sigma(n)$, $d(n)$, $s(n)$, $\omega(n)$, $\Omega(n)$, $S_b(n)$, $\psi(n)$, and $J_s(n)$, where $d(n)$ is the number of positive divisors of n, $s(n) = \sigma(n) - n$ is the sum of proper divisors of n, $\omega(n)$ is the number of distinct prime divisors of n, $\Omega(n)$ is the number of prime divisors of n counted with multiplicity, $S_b(n)$ is the sum of digits of n when n is written in base b, $\psi(n)$ is the Dedekind function, and $J_s(n)$ is Jordan's totient function. The functions ψ and J_s are defined by

$$\psi(n) = n\prod_{p|n}(1+\tfrac{1}{p}) \text{ and } J_s(n) = n^s\prod_{p|n}(1-\tfrac{1}{p^s}),$$

where s is a positive integer. For more information about injectivity or noninjectivity of arithmetic functions, see for example in Guy's book [6, Section B], Pongsriiam's recent article [8], and the online database OEIS [9].

We organize this article as follows. In Section 2, we prove some results on injectivity of the function gh where g is a polynomial in a simple form and $h = \varphi, \psi$, and J_s. In Section 3, we show the noninjectivity of gh and a connection to other problems when $h = \sigma, s, d, \omega, \Omega$, and S_b. In Section 4, we study the injectivity of gh where g and h are restricted to squarefree integers. In fact, in Sections 1–4, the function g is of the form $g(n) = n^a$, but in Section 5, we set $g(n) = n + c$ where c is a positive integer. We obtain in Section 5 that the function $n \mapsto (n + c)\varphi(n)$ is not injective for infinitely many $c \in \mathbb{N}$. We also provide some related results in Section 6. Finally, we give a list of open questions in Section 7.

2 Results on injectivity

In this section, we show that the product of φ, ψ, and J_s with a polynomial in a simple form are injective. Recall that an arithmetic function f is said to be multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{N}$ with $(m, n) = 1$. It is well known that φ, σ, d, ψ, and J_s are multiplicative. The following formulas are also well known and may be used throughout this article:

$$\varphi(n) = n\prod_{p|n}(1-\tfrac{1}{p}),\ \ \psi(n) = n\prod_{p|n}(1+\tfrac{1}{p}),\ \ J_s(n) = n^s\prod_{p|n}(1-\tfrac{1}{p^s}),$$

$$d(n) = \prod_{p^\alpha || n}(\alpha+1),\ \ \sigma(n) = \prod_{p^\alpha || n}(1+p+p^2+\cdots+p^\alpha) = \prod_{p^\alpha || n}\left(\frac{p^{\alpha+1}-1}{p-1}\right).$$

We begin our study with φ. Although it is well known that the function $n \mapsto n\varphi(n)$ is injective, we can extend it to the following form.

Theorem 1. *For each $a, b \in \mathbb{N}$, the arithmetic function F defined by $F(n) = n^a\varphi(n)^b$ for all $n \in \mathbb{N}$ is an injective function. In particular, the function f_0 is injective.*

Proof. To show that F is injective, let $m, n \in \mathbb{N}$ and $F(m) = F(n)$. If $m = 1$, then $n^a\varphi(n)^b = F(n) = F(1) = 1$, which implies $n = 1$. Similarly, if $n = 1$, then $m = 1$. Therefore $m = 1$ if and only if $n = 1$. So we assume that $m, n \geq 2$. Let

$$m = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k} \text{ and } n = q_1^{n_1} q_2^{n_2} \cdots q_\ell^{n_\ell},$$

where $p_1 < p_2 < \cdots < p_k$, $q_1 < q_2 < \cdots < q_\ell$ are primes and m_i, n_j are positive integers for all i, j. By the well known formula for φ and the fact that $F(m) = F(n)$, we obtain

$$\prod_{i=1}^{k} p_i^{am_i + b(m_i-1)} \prod_{i=1}^{k} (p_i - 1)^b = \prod_{i=1}^{\ell} q_i^{an_i + b(n_i-1)} \prod_{i=1}^{\ell} (q_i - 1)^b. \tag{1}$$

For convenience, we write LHS and RHS to denote the left-hand side and the right-hand side of (1), respectively. Suppose that $p_k \geq q_\ell$. Since the exponent of p_k in LHS is at least $2m_i - 1 \geq 1$, we see that p_k divides LHS. So p_k divides RHS too. Since p_k does not divide $q_i - 1$ and q_j for any $i = 1, 2, \ldots, \ell$ and $j = 1, 2, \ldots, \ell - 1$, we see that p_k divides q_ℓ. So $p_k = q_\ell$. Similarly, if $p_k \leq q_\ell$, then we start with the fact that q_ℓ divides RHS, and so q_ℓ divides LHS too, which leads to $q_\ell = p_k$. In any case $p_k = q_\ell$. Furthermore, by the unique factorization, the exponent of p_k and q_ℓ are the same. Therefore $am_k + b(m_k - 1) = an_\ell + b(n_\ell - 1)$, which implies $m_k = n_\ell$. Thus (1) reduces to

$$\prod_{i=1}^{k-1} p_i^{am_i + b(m_i-1)} \prod_{i=1}^{k-1} (p_i - 1)^b = \prod_{i=1}^{\ell-1} q_i^{an_i + b(n_i-1)} \prod_{i=1}^{\ell-1} (q_i - 1)^b. \tag{2}$$

We observe that (2) is obtained from (1) by the change of k to $k-1$ and ℓ to $\ell - 1$. So we can use the same argument to conclude that $p_{k-1} = q_{\ell-1}$ and $m_{k-1} = n_{\ell-1}$. Doing this process repeatedly, it will eventually stop. If $k < \ell$, then it leads to the equation $1 = R$ where R is divisible by q_1, which is a contradiction. Similarly, the inequality $k > \ell$ is not possible. Therefore $k = \ell$. So when the process stops, we obtain $k = \ell$, $p_i = q_i$, and $m_i = n_i$ for all i. Therefore $m = n$, as required. □

Since $\psi(n)$ and $\varphi(n)$ are similar, we expect that the function $n \mapsto n\psi(n)$ should also be injective. Nevertheless, there is a little problem with the primes 2 and 3. So we first prove the following lemma.

Lemma 2. *For each $a, b \in \mathbb{N}$, let F be the arithmetic function defined by $F(n) = n^a \psi(n)^b$ for all $n \in \mathbb{N}$. Let $m \in \mathbb{N}$ and $r, s \in \mathbb{N} \cup \{0\}$. If $F(m) = F(2^r 3^s)$, then $m = 2^r 3^s$.*

Proof. If $r = s = 0$, then $F(m) = F(1) = 1$, which implies $m = 1 = 2^r 3^s$. So assume that $r \neq 0$ or $s \neq 0$. By the definition of F and the formula for ψ, we observe that each $x, y \in \mathbb{N}$, we have

$$F(2^x) = 2^{xa+xb-b} 3^b, F(3^y) = 2^{2b} 3^{ya+yb-b}, \text{ and } F(2^x 3^y) = 2^{xa+xb+b} 3^{ya+yb}. \tag{3}$$

Therefore if p is a prime factor of m, then p also divides $F(m) = F(2^r 3^s)$, and so $p \leq 3$. Thus $m = 2^u 3^v$ for some nonnegative integers u and v.

Case 1: $r = 0$. Then $s \neq 0$ and $F(m) = F(2^r 3^s) = F(3^s) = 2^{2b} 3^{sa+sb-b}$. Suppose, by way of contradiction, that $v = 0$. Then $2^{ua+ub-b} 3^b = 2^{2b} 3^{sa+sb-b}$, which implies that $ua + ub = 3b$

and $sa + sb = 2b$. Since $sa + sb = 2b$, we have $s = 1$ and $a = b$. Then $2ub = ua + ub = 3b$ which is not possible. Thus $v \neq 0$. If $u \neq 0$, then we have that $2^{ua+ub+b}3^{va+vb} = 2^{2b}3^{sa+sb-b}$, which implies $b = ua + ub > b$, a contradiction. So $u = 0$ and $2^{2b}3^{va+vb-b} = 2^{2b}3^{sa+sb-b}$. This implies that $v = s$ and $m = 3^s = 2^r 3^s$.

Case 2: $s = 0$. Then $r \neq 0$. By using an argument similar to Case 1, one can show that $m = 2^r = 2^r 3^s$.

Case 3: $r \neq 0$ and $s \neq 0$. Then $F(m) = 2^{ra+rb+b}3^{sa+sb}$. By considering (3) and the exponents of 2 and 3 in $F(m)$, we see that $u \neq 0$ and $v \neq 0$. This implies that $2^{ua+ub+b}3^{va+vb} = 2^{ra+rb+b}3^{sa+sb}$. Then $u = r$, $v = s$, and so $m = 2^r 3^s$. This completes the proof. □

Theorem 3. *For each $a, b \in \mathbb{N}$, the arithmetic function F defined by $F(n) = n^a \psi(n)^b$ for all $n \in \mathbb{N}$ is an injective function.*

Proof. To show that F is injective, let $m, n \in \mathbb{N}$ and $F(m) = F(n)$. It is easy to see that $m = 1$ if and only if $n = 1$. So we assume that $m, n \geq 2$. Let

$$m = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k} \text{ and } n = q_1^{n_1} q_2^{n_2} \cdots q_\ell^{n_\ell},$$

where $p_1 < p_2 < \cdots < p_k$, $q_1 < q_2 < \cdots < q_\ell$ are primes and m_i, n_j are positive integers for all i, j. Since $F(m) = F(n)$, we obtain

$$\prod_{i=1}^{k} p_i^{am_i + b(m_i - 1)} \prod_{i=1}^{k} (p_i + 1)^b = \prod_{i=1}^{\ell} q_i^{an_i + b(n_i - 1)} \prod_{i=1}^{\ell} (q_i + 1)^b. \tag{4}$$

For simplicity, we write LHS and RHS to denote the left-hand side and the right-hand side of (4), respectively. If $p_k \leq 3$, then Lemma 2 implies that $m = n$. So assume that $p_k > 3$. Suppose that $p_k \geq q_\ell$. Since the exponent of p_k in LHS is at least $2m_k - 1 \geq 1$, we see that p_k divides LHS. So p_k divides RHS too. Since $p_k > 3$, we see that p_k does not divide $q_i + 1$ and q_j for any $i = 1, 2, \ldots, \ell$ and $j = 1, 2, \ldots, \ell - 1$. Then p_k divides q_ℓ, and so $p_k = q_\ell$. Similarly, if $p_k \leq q_\ell$, then we start with the fact that q_ℓ divides RHS, and so q_ℓ divides LHS too, which leads to $q_\ell = p_k$. In any case $p_k = q_\ell$. Furthermore, by the unique factorization, the exponent of p_k and q_ℓ are the same. Therefore $am_k + b(m_k - 1) = an_\ell + b(n_\ell - 1)$, which implies $m_k = n_\ell$. Thus (4) reduces to

$$\prod_{i=1}^{k-1} p_i^{am_i + b(m_i - 1)} \prod_{i=1}^{k-1} (p_i + 1)^b = \prod_{i=1}^{\ell-1} q_i^{an_i + b(n_i - 1)} \prod_{i=1}^{\ell-1} (q_i + 1)^b. \tag{5}$$

We observe that (5) is obtained from (4) by the change of k to $k - 1$ and ℓ to $\ell - 1$. If $p_{k-1} \leq 3$, then we apply Lemma 2 to obtain $m = n$. If $p_{k-1} > 3$, then we repeat the above process and reduce (5) by the change of $k - 1$ to $k - 2$ and $\ell - 1$ to $\ell - 2$. By repeating this process, we eventually obtain $m = n$. This completes the proof. □

Similar result also holds when the function φ is replaced by J_2 as shown below.

Lemma 4. *For each $a, b \in \mathbb{N}$, let F be the arithmetic function defined by $F(n) = n^a J_2(n)^b$ for all $n \in \mathbb{N}$. Let $m \in \mathbb{N}$ and $r, s \in \mathbb{N} \cup \{0\}$. If $F(m) = F(2^r 3^s)$, then $m = 2^r 3^s$.*

Proof. Since this lemma can be proved in the same way as Lemma 2, we skip some details. If $r = s = 0$, then $m = 2^r 3^s$. So assume that $r \neq 0$ or $s \neq 0$. If p is a prime factor of m, then p also divides $F(m) = F(2^r 3^s)$, and so $p \leq 3$. Thus $m = 2^u 3^v$ for some nonnegative integers u and v.

First, assume that $r = 0$. Then $s \neq 0$ and $F(m) = 2^{3b} 3^{sa+2sb-2b}$. If $v = 0$, then $2^{ua+2ub-2b} 3^b = F(m) = 2^{3b} 3^{sa+2sb-2b}$, which implies that $ua + 2ub = 5b$ and $sa + 2sb = 3b$. Since $sa+2sb = 3b$, we have that $s = 1$ and $a = b$. Then $3ub = ua+2ub = 5b$, a contradiction. Thus $v \neq 0$. From this point, we can still consider the exponents of 2 and 3 like the proof of Lemma 2 to obtain $m = n$. For the cases $s = 0$ or ($r \neq 0$ and $s \neq 0$), we can also compare the exponents of 2 and 3 to obtain the desired result. So the proof is completed. □

Next, we show that for certain $a, b \in \mathbb{N}$, the function $n \mapsto n^a J_s(n)^b$ is injective. When $s = 2$, we can use any positive integers a, b as follows.

Theorem 5. *For each $a, b \in \mathbb{N}$, the arithmetic function F defined by $F(n) = n^a J_2(n)^b$ for all $n \in \mathbb{N}$ is injective.*

Proof. Since the proof of this theorem follows the same argument as in Theorem 3, we skip some details. Let $m, n \in \mathbb{N}$ and $F(m) = F(n)$. It is easy to see that $m = 1$ if and only if $n = 1$. So we assume that $m, n \geq 2$. Let

$$m = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k} \text{ and } n = q_1^{n_1} q_2^{n_2} \cdots q_\ell^{n_\ell},$$

where $p_1 < p_2 < \cdots < p_k$, $q_1 < q_2 < \cdots < q_\ell$ are primes and m_i, n_j are positive integers for all i, j. Then

$$\prod_{i=1}^{k} p_i^{am_i+2b(m_i-1)} \prod_{i=1}^{k} (p_i^2 - 1)^b = \prod_{i=1}^{\ell} q_i^{an_i+2b(n_i-1)} \prod_{i=1}^{\ell} (q_i^2 - 1)^b. \tag{6}$$

For convenience, we write LHS and RHS to denote the left-hand side and the right-hand side of (6), respectively. If $p_k \leq 3$, then Lemma 4 implies that $m = n$. So assume that $p_k > 3$. Suppose that $p_k \geq q_\ell$. Since the exponent of p_k in LHS is at least $3m_i - 2 \geq 1$, we see that p_k divides LHS. So p_k divides RHS too. Since p_k does not divide $q_i - 1$, $q_i + 1$, and q_j for any $i = 1, 2, \ldots, \ell$ and $j = 1, 2, \ldots, \ell - 1$, we see that p_k divides q_ℓ. So $p_k = q_\ell$. Similarly, if $p_k \leq q_\ell$, then this leads to $p_k = q_\ell$ and $m_k = n_\ell$. Thus (6) reduces to

$$\prod_{i=1}^{k-1} p_i^{am_i+2b(m_i-1)} \prod_{i=1}^{k-1} (p_i^2 - 1)^b = \prod_{i=1}^{\ell-1} q_i^{an_i+2b(n_i-1)} \prod_{i=1}^{\ell-1} (q_i^2 - 1)^b. \tag{7}$$

We observe that (7) is obtained from (6) by the change of k to $k - 1$ and ℓ to $\ell - 1$. So we can repeat this process like the proof of Theorem 3 to obtain $m = n$, as required. □

When $s \geq 3$, it seems that the function $n \mapsto n^a J_s(n)^b$ is injective for any $a, b \in \mathbb{N}$, but we do not have a proof. In the following theorem, we need to restrict ourselves to the case $a \geq sb$, but we hope to solve the case $a < sb$ in the future. Please see also our comments and the list of other problems in Section 7.

Theorem 6. *For each $a, b, s \in \mathbb{N}$, if $a \geq sb$, then the arithmetic function F defined by $F(n) = n^a J_s(n)^b$ for all $n \in \mathbb{N}$ is injective.*

Proof. For each $n \in \mathbb{N}$, let P_n be the set of all prime factors of n. Let $a, b, s \in \mathbb{N}$ and $a \geq sb$. We remark that we do not need to use the inequality $a \geq sb$ until the calculation in (10). To show that F is injective, let $m, n \in \mathbb{N}$ and $F(m) = F(n)$. It is easy to see that $m = 1$ if and only if $n = 1$. So we assume that $m, n \geq 2$. We will first show that $P_m = P_n$. So suppose, by way of contradiction, that $P_m \neq P_n$.

Case 1: $P_m \subseteq P_n$ and $P_n \nsubseteq P_m$. Let

$$m = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k} \text{ and } n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} q_1^{n_{k+1}} q_2^{n_{k+2}} \cdots q_\ell^{n_{k+\ell}},$$

where $p_1 < p_2 < \cdots < p_k$ and $q_1 < q_2 < \cdots < q_\ell$ are primes, $k, \ell \geq 1$, $p_i \neq q_j$, and m_i, n_j are positive integers for all i, j. After dividing both sides of the equation $F(m) = F(n)$ by $\prod_{i=1}^{k} (p_i^s - 1)^b$, we obtain

$$\prod_{i=1}^{k} p_i^{am_i + sb(m_i - 1)} = \prod_{i=1}^{k} p_i^{an_i + sb(n_i - 1)} \prod_{i=1}^{\ell} q_i^{an_{k+i} + sb(n_{k+i} - 1)} \prod_{i=1}^{\ell} (q_i^s - 1)^b. \tag{8}$$

Then q_1 divides the right-hand side of (8) but does not divide the left-hand side. So this case leads to a contradiction.

Case 2: $P_n \subseteq P_m$ and $P_m \nsubseteq P_n$. Similar to Case 1, this leads to a contradiction.

Case 3: $P_m \nsubseteq P_n$ and $P_n \nsubseteq P_m$. Let

$$m = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k} \left(\prod_{i=1}^{t} w_i^{u_i} \right) \text{ and } n = q_1^{n_1} q_2^{n_2} \cdots q_\ell^{n_\ell} \left(\prod_{i=1}^{t} w_i^{v_i} \right),$$

where $p_1 < p_2 < \cdots < p_k$, $q_1 < q_2 < \cdots < q_\ell$, and $w_1 < w_2 < \cdots < w_t$ are primes, p_i, q_j, w_x are distinct for all i, j, x, and m_i, n_j, u_x, v_y are positive integers for all i, j, x, y. In addition, if $P_m \cap P_n = \emptyset$, then we take $t = 0$ and define the empty product to be 1 as usual; if $P_m \cap P_n \neq \emptyset$, then $t \geq 1$. By the fact that $F(m) = F(n)$, we obtain

$$\prod_{i=1}^{t} w_i^{au_i + sb(u_i - 1)} \prod_{i=1}^{k} p_i^{am_i + sb(m_i - 1)} \prod_{i=1}^{k} (p_i^s - 1)^b = \prod_{i=1}^{t} w_i^{av_i + sb(v_i - 1)} \prod_{i=1}^{\ell} q_i^{an_i + sb(n_i - 1)} \prod_{i=1}^{\ell} (q_i^s - 1)^b. \tag{9}$$

Let $L_1 = \prod_{i=1}^{k} p_i^{am_i+sb(m_i-1)}$, $L_2 = \prod_{i=1}^{k}(p_i^s - 1)^b$, $R_1 = \prod_{i=1}^{\ell} q_i^{an_i+sb(n_i-1)}$, and $R_2 = \prod_{i=1}^{\ell}(q_i^s - 1)^b$. Since p_i, q_j, and w_x are all distinct, (9) implies that $L_1|R_2$ and $R_1|L_2$. Recall that $a \geq sb$. Then

$$L_1 \leq R_2 < \prod_{i=1}^{\ell} q_i^{sb} \leq R_1 \leq L_2 < \prod_{i=1}^{k} p_i^{sb} \leq L_1, \tag{10}$$

which is a contradiction.

Therefore we can conclude that $P_m = P_n$. Let

$$m = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k} \text{ and } n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k},$$

where $p_1 < p_2 < \cdots < p_k$ are primes and m_i, n_j are positive integers for all i, j. By the fact that $F(m) = F(n)$, we obtain

$$\prod_{i=1}^{k} p_i^{am_i+sb(m_i-1)} = \prod_{i=1}^{k} p_i^{an_i+sb(n_i-1)}.$$

By the unique factorization, we obtain $m_i = n_i$ for all i, and so $m = n$, as required. □

3 Noninjectivity and a connection with other concepts

Not every arithmetic function has the property like φ, ψ, and J_s. In this section, we give various examples to show that if we replace φ by other arithmetic functions f, the function $n \mapsto nf(n)$ may not be injective. We also give some connections to other problems such as the existence or nonexistence of infinitely many Mersenne primes, amicable pairs, and primitive solutions to certain equations.

Example 7. Let $A(n) = n\sigma(n)$ for all $n \in \mathbb{N}$. A straightforward calculation shows that $A(12) = A(14) = 2^4 \cdot 3 \cdot 7$, and so A is not injective. In fact, we can generate infinitely many $x, y \in \mathbb{N}$ such that $A(x) = A(y)$ using the equality $A(12) = A(14)$. Let $x = 12d$ and $y = 14d$ where $(d, 42) = 1$. It is easy to see that A is multiplicative and $(d, 14) = (d, 12) = 1$, and so $A(x) = A(12d) = A(12)A(d) = A(14)A(d) = A(14d) = A(y)$. Since there are infinitely many $d \in \mathbb{N}$ with $(d, 42) = 1$, we obtain infinitely many $x, y \in \mathbb{N}$ such that $A(x) = A(y)$ too. From this, it is easy to see that if we can find another pair of integers x_0, y_0 such that $A(x_0) = A(y_0)$, then we can generate infinitely many such pairs by following the above method.

Moser [6, p. 102] asked whether there is an infinite number of primitive solutions to the equation $A(x) = A(y)$, that is, the integers $x, y \geq 1$ such that $A(x) = A(y)$ and $A(x/d) \neq A(y/d)$ for any common divisor $d > 1$ of x and y. A conditional answer is known: if $2^p - 1$ and $2^q - 1$ are distinct Mersenne primes, then $x = 2^{p-1}(2^q - 1)$ and $y = 2^{q-1}(2^p - 1)$ is a primitive solution to $A(x) = A(y)$. To see this, recall that if $2^n - 1$ is a prime, then n

is a prime too. Therefore we have $p, q, 2^p - 1, 2^q - 1$ are primes. Without loss of generality, assume that $p > q$. Then

$$\begin{aligned} A(x) &= 2^{p-1}(2^q - 1)\sigma(2^{p-1})\sigma(2^q - 1) \\ &= 2^{p-1}(2^q - 1)(2^p - 1)(2^q) \\ &= 2^{p+q-1}(2^p - 1)(2^q - 1). \end{aligned}$$

By a similar calculation, we see that $A(y) = A(x)$. Since $2^p - 1$ and $2^q - 1$ are distinct odd primes, the greatest common divisor of x and y is 2^{q-1}. So if $d > 1$, $d \mid x$, and $d \mid y$, then $d = 2^\ell$ for some $\ell = 1, 2, \dots, q-1$. By a similar calculation, we obtain

$$A\left(\frac{x}{d}\right) = 2^{p+q-\ell-1}(2^{p-\ell} - 1)(2^q - 1) \quad \text{and} \quad A\left(\frac{y}{d}\right) = 2^{p+q-\ell-1}(2^{q-\ell} - 1)(2^p - 1).$$

From this, we see that $2^p - 1$ divides $A(\frac{y}{d})$ but does not divide $A(\frac{x}{d})$. So $A(\frac{x}{d}) \neq A(\frac{y}{d})$. This shows that x, y is indeed a primitive solution to the equation $A(x) = A(y)$. Nevertheless, since we do not know whether or not there are infinitely many Mersenne primes, this is only a conditional solution to Moser's problem. Without restricting to primitive solutions, Erdős [1] showed that the number of $m, n \in \mathbb{N}$ satisfying $m < n < x$ and $m\sigma(m) = n\sigma(n)$ is asymptotic to cx as $x \to \infty$, where c is a positive constant. For more information on the equation $A(x) = A(y)$, we refer the reader to Guy's book [6, Section B11]. The sequence $(A(n))_{n\geq 1}$ is registered in the OEIS as the sequence A064987. Moreover, the sequence of $n \in \mathbb{N}$ such that $A(x) = n$ has more than one solution is A337873 in the OEIS. Some such integers n and distinct x_1, x_2 such that $A(x_1) = A(x_2) = n$ are shown in Table 2. We remark that n and x_2 in our table are not listed in an increasing order, but the integer x_1 is listed in an increasing order. The reader can also find more related information in the sequence A212490 and our comments in Section 7.

Example 8. Let $s(n) = \sigma(n) - n$ be the sum of proper positive divisors of n and let $B(n) = ns(n)$ for all $n \in \mathbb{N}$. It is not difficult to check that $s(6) = 6$ and $s(9) = 4$, and so $B(6) = B(9)$. So B is not injective. In general, if $x, y \in \mathbb{N}$, $s(x) = y$, and $s(y) = x$, then we have $B(x) = B(y)$. For example, since $s(220) = 284$ and $s(284) = 220$, we have that $B(220) = 220 \cdot 284 = B(284)$. A pair (x, y) satisfying $s(x) = y$ and $s(y) = x$ is called an amicable pair, and mathematicians have found millions such pairs. It is not known whether there are infinitely many amicable pairs, but it is believed that there are. If this is true, then $B(x) = B(y)$ for an infinite number of x, y. For more information on amicable pairs and related concept, we refer the reader to Guy's book [6, Sections B4–B8] and the sequences A063990, A002025, and A002046 in the OEIS [9]. We remark that A063990 gives the list of amicable numbers in an increasing order, but the adjacent numbers are not necessarily the amicable pairs (x, y). The sequences A002025 and A002046 give the list of x and y in the amicable pairs, respectively. The sequence that gives amicable pairs in an increasing order is A259180 in the OEIS. Moreover, the sequence of $n \in \mathbb{N}$ such that $B(x) = n$ has more than one solution is also registered in the OEIS as the sequence A212327. Some values of such n and distinct $x_1, x_2 \in \mathbb{N}$ such that $B(x_1) = B(x_2) = n$ are shown in Table 1.

Example 9. Let $D(n) = nd(n)$ for all $n \in \mathbb{N}$, where $d(n)$ is the number of positive divisors of n. Let $x = 18a$ and $y = 27a$ where $a \in \mathbb{N}$ and $(a, 6) = 1$. Since $D(18) = 108 = D(27)$ and D is multiplicative, we obtain $D(x) = D(18)D(a) = D(27)D(a) = D(y)$. So it may be more interesting to consider only the primitive solutions to the equation $D(x) = D(y)$ where the primitive solutions are defined in a similar way as in Example 7. We leave this problem to the interested reader. The sequence $(D(n))_{n\geq 1}$ is A038040 in the OEIS. The sequence of $n \in \mathbb{N}$ such that $D(x) = n$ has more than one solution is the sequence A338382 in the OEIS. Some values of such integers n, and x_1, x_2 such that $D(x_1) = D(x_2) = n$ are shown in Table 3.

Example 10. Let $W_1(n) = n\omega(n)$ and $W_2(n) = n\Omega(n)$ for all $n \in \mathbb{N}$. Let $x = 30 \cdot 5^k$ and $y = 45 \cdot 5^k$ where $k \in \mathbb{N}$. Then $W_1(x) = 30 \cdot 5^k \cdot 3 = 45 \cdot 5^k \cdot 2 = W_1(y)$. So W_1 is not injective. In general, if $W_1(m) = W_1(n)$ and $(m, n) > 1$, then we can generate infinitely many $x, y \in \mathbb{N}$ such that $W_1(x) = W_1(y)$, namely, $x = m \cdot p^k$ and $y = n \cdot p^k$ where k is any positive integer and p is any prime divisor of (m, n). For W_2, we observe that if p is any odd prime, then $W_2(16p) = (16p)(5) = (20p)(4) = W_2(20p)$. So W_2 is not injective and there are infinitely many $m, n \in \mathbb{N}$ such that $W_2(m) = W_2(n)$. Some values of $n \in \mathbb{N}$ such that $W_1(x) = n$ or $W_2(y) = n$ have more than one solution are shown in Table 4 and Table 5, respectively.

Example 11. For each positive integer $b \geq 2$, let $S_b(n)$ be the sum of digits of n when n is written in base b, and let $H_b(n) = nS_b(n)$ for all $n \in \mathbb{N}$. If $b = 2$, then it is easy to see that $H_2(22) = 66 = H_2(33)$. For $b > 2$, we have

$$H_b(b^3+1) = 2(b^3+1) = (b^2+(b-2)b+2)(b+1) = H_b(b^2+(b-2)b+2) = H_b(2(b^2-b+1)),$$

and $b^3+1 \neq 2(b^2-b+1)$. So H_b is not injective for any $b \geq 2$. In general, if $H_b(m) = H_b(n)$, then there are infinitely many $x, y \in \mathbb{N}$ satisfying the equation $H_b(x) = H_b(y)$, namely, $x = b^t m$ and $y = b^t n$ where t is an arbitrary positive integer. Some values of $n \in \mathbb{N}$ such that $H_{10}(x) = n$ has more than one solution are shown in Table 6.

4 Restricted injectivity

Since the functions defined in Example 7 to Example 11 are not injective on $\mathbb{N}$, it is natural to consider the injectivity of these functions on other infinite proper subsets of $\mathbb{N}$. In 1959, Erdős [1] observed that although the function $n \mapsto n\sigma(n)$ is not injective on $\mathbb{N}$, it is injective on the set of squarefree integers. In fact, Erdős' observation is a special case of the next theorem.

Theorem 12. *For each $a, b \in \mathbb{N}$, let F be defined by $F(n) = n^a\sigma(n)^b$ for all $n \in \mathbb{N}$. Then F is injective on squarefree integers. That is, if $m, n \in \mathbb{N}$ are squarefree and $m^a\sigma(m)^b = n^a\sigma(n)^b$, then $m = n$. In particular, the function A in Example 7 is injective on squarefree integers.*

Proof. If m and n are squarefree, then $m^a\sigma(m)^b = m^a\psi(m)^b$ and $n^a\sigma(n)^b = n^a\psi(n)^b$. So the assumption that $m^a\sigma(m)^b = n^a\sigma(n)^b$ implies $m^a\psi(m)^b = n^a\psi(n)^b$, and so $m = n$ by Theorem 3. □

We remark that the integer a in Theorem 12 cannot be zero since σ is not injective on squarefree integers. For instance, we have $\sigma(6) = \sigma(11)$ and 6, 11 are squarefree. However, if we replace σ by d in Theorem 12, the resulting function is also injective on squarefree integers.

Theorem 13. *For each $a, b \in \mathbb{N}$, let F be defined by $F(n) = n^a d(n)^b$ for all $n \in \mathbb{N}$. Then F is injective on squarefree integers. In particular, the function D, defined in Example 9, is injective on the set of squarefree integers.*

Proof. Let $m, n \in \mathbb{N}$ be squarefree and $F(m) = F(n)$. It is easy to see that $m = 1$ if and only if $n = 1$, so we assume that $m, n > 1$. Let $m = \prod_{i=1}^{k} p_i$ and $n = \prod_{i=1}^{\ell} q_i$, where $p_1, p_2, \ldots, p_k$ and $q_1, q_2, \ldots, q_\ell$ are distinct primes. Then we have

$$2^{kb} \prod_{i=1}^{k} p_i^a = 2^{\ell b} \prod_{i=1}^{\ell} q_i^a. \tag{11}$$

We denote the left-hand side and the right-hand side of (11) by LHS and RHS, respectively. If $k \geq \ell + 1$, then after dividing both sides of (11) by $2^{\ell b}$, LHS has at least k distinct prime factors while RHS has $\ell \leq k - 1$ distinct prime factors, a contradiction. Similarly, the inequality $\ell > k$ leads to a contradiction. So $k = \ell$, and (11) reduces to $m = n$, as required. □

Not every arithmetic function has the property like σ and d in Theorems 12 and 13. This is shown in the following examples.

Example 14. Let B, W_1, W_2, and H_b be the functions defined in Example 8, 10, and 11. We show that these functions are not injective on the set of squarefree integers. For the function B, we have 1955 and 2093 are squarefree, but

$$B(1955) = 1955s(1955) = 1955 \cdot 637 = 2093 \cdot 595 = 2093s(2093) = B(2093).$$

For W_1 and W_2, let $p_1, p_2, \ldots, p_9$ be distinct primes and $(2, p_i) = (5, p_i) = (11, p_i) = 1$ for all $1 \leq i \leq 9$. Let

$$a = 11 \prod_{i=1}^{9} p_i \quad \text{and} \quad b = 10 \prod_{i=1}^{9} p_i.$$

Then $W_1(a) = W_1(b) = W_2(a) = W_2(b)$. Therefore W_1 and W_2 are not injective on the set of squarefree integers.

It is shown in Example 11 that H_2 is not injective on the set of squarefree integers. Moreover, we have

$$H_3(51) = 255 = H_3(85), \quad H_4(26) = 130 = H_4(65), \quad H_5(21) = 105 = H_5(35),$$
$$H_6(26) = 156 = H_6(39), \quad H_7(55) = 385 = H_7(77), \quad H_8(26) = 130 = H_8(65),$$
$$H_9(15) = 105 = H_9(21), \quad \text{and} \quad H_{10}(15) = 90 = H_{10}(30).$$

So H_b is not injective on squarefree integers for $2 \leq b \leq 10$. One can use a computer to verify that H_b is not injective for other values of b too. We believe that H_b is not injective for any $b \geq 11$ but we do not have a proof.

Some products of a polynomial and two arithmetic functions are also injective on the set of squarefree integers. This can be proved by applying our theorems as follows.

Corollary 15. *For each $a, b \in \mathbb{N}$, the functions F_1 and F_2 defined by*

$$F_1(n) = n^a \sigma(n)^b \varphi(n)^b \text{ and } F_2(n) = n^a \sigma(n)^b \psi(n)^b \text{ for all } n \in \mathbb{N}$$

are injective on the set of squarefree integers.

Proof. Let $n \in \mathbb{N}$ be squarefree. We observe that $\sigma(n) = \psi(n)$ and it is easy to see that $\psi(n)\varphi(n) = J_2(n)$. So $F_1(n) = n^a J_2(n)^b$ and $F_2(n) = n^a \psi(n)^{2b}$. Therefore F_1 and F_2 are injective on the set of squarefree integers by Theorems 5 and 3, respectively. □

We remark that F_1 and F_2 are not injective on $\mathbb{N}$. For example, when $a = b = 1$, we have $F_1(56) = F_1(60)$ and $F_2(12) = F_2(14)$.

5 Other results on noninjectivity

In this section, we study the product of a different simple polynomial and φ. So for each nonnegative integer c, let f_c be the arithmetic function given by

$$f_c(n) = (n + c)\varphi(n) \text{ for all } n \in \mathbb{N}.$$

Although f_0 is injective, we believe that f_c is not injective for any $c \geq 1$, and we will provide some supporting evidence. If c is fixed and is given explicitly, we can always use a computer to search for distinct positive integers a, b such that $f_c(a) = f_c(b)$. For each $c = 1, 2, 3$, Tables 7, 8 and 9 show distinct positive integers $x_1, x_2 \leq 2000$ such that $f_c(x_1) = f_c(x_2)$. Therefore we immediately see that f_1, f_2, f_3 are not injective. However, this only gives us a small number of c for which f_c is not injective.

In what follows, we will develop a tool and use it together with Table 7 to generate an infinite number of c such that f_c is not injective, and then use the results that we obtain to show that f_c is not injective for any positive integer $c \leq 1000$, and that there are at least 98 percent of $c \leq N$ such that f_c is not injective when N is any large positive integer. To do so, we define for each $c, n \in \mathbb{N}$, the product

$$\alpha(c, n) = \prod_{p \mid c \text{ and } p \nmid n} \left(1 - \frac{1}{p}\right)$$

where the product is taken over all primes p that are a factor of c and do not divide n. As usual, the empty product is defined to be 1. So if $c = 1$ or every prime divisor of c is a divisor of n, then $\alpha(c, n) = 1$.

The following lemmas are simple but they are the key to the construction of an infinite number of $c \in \mathbb{N}$ such that f_c is not injective.

Lemma 16. *Let c and n be positive integers. Then the following statements hold.*

(i) $\varphi(cn) = c\varphi(n)\alpha(c, n)$.

(ii) $\varphi(cn) = c\varphi(n)$ *if and only if* $c = 1$ *or every prime divisor of* c *is a divisor of* n.

Proof. For (i), we apply the well known formula to obtain

$$\frac{\varphi(cn)}{\varphi(n)} = \frac{cn\prod_{p|cn}(1-\frac{1}{p})}{n\prod_{p|n}(1-\frac{1}{p})} = c\alpha(c, n),$$

which implies (i). Then (ii) follows immediately from (i). □

Lemma 17. *The value of the finite product of the form* $\prod_{p}(1-\frac{1}{p})^{a_p}$ *uniquely determines the set of primes in the product. More precisely, if* $p_1 < p_2 < \cdots < p_m$ *and* $q_1 < q_2 < \cdots < q_k$ *are primes,* $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_k$ *are positive integers, and*

$$\prod_{i=1}^{m}\left(1-\frac{1}{p_i}\right)^{a_i} = \prod_{i=1}^{k}\left(1-\frac{1}{q_i}\right)^{b_i}, \tag{12}$$

then $m = k$, $p_i = q_i$, *and* $a_i = b_i$ *for every* $i = 1, 2, \ldots, m$.

Proof. The idea of proof is the same as that of Theorem 1. The equality (12) leads to

$$\prod_{i=1}^{k} q_i^{b_i} \prod_{i=1}^{m}(p_i - 1)^{a_i} = \prod_{i=1}^{m} p_i^{a_i} \prod_{i=1}^{k}(q_i - 1)^{b_i}. \tag{13}$$

Let LHS and RHS denote the left-hand side and the right-hand side of (13). If $q_k \geq p_m$, then we start with LHS, which is divisible by q_k, and so $q_k \mid$ RHS, which implies $q_k = p_m$. Similarly, if $p_m \geq q_k$, then we start with $p_m \mid$ RHS, which eventually leads to $p_m = q_k$. By the unique factorization, the exponents of p_m and q_k are also equal, that is, $a_m = b_k$. Therefore (13) reduces to an equation that is similar to (13) but k becomes $k-1$ and m becomes $m-1$. So we can repeat this process like the proof of Theorem 1 to obtain the desired result. □

Lemma 18. *Let* a, b, c *be positive integers and* $\varphi(a) = \varphi(b)$. *Then the following statements are equivalent.*

(i) $\varphi(ca) = \varphi(cb)$.

(ii) $\alpha(c, a) = \alpha(c, b)$.

(iii) $\{p \in \mathbb{N} : p \text{ is prime, } p \mid c, \text{ and } p \nmid a\} = \{p \in \mathbb{N} : p \text{ is prime, } p \mid c, \text{ and } p \nmid b\}$.

(iv) $\{p \in \mathbb{N} : p \text{ is prime, } p \mid c, \text{ and } p \mid a\} = \{p \in \mathbb{N} : p \text{ is prime, } p \mid c, \text{ and } p \mid b\}$.

Proof. By Lemma 16, we see that (i) and (ii) are equivalent. Lemma 17 implies that (ii) and (iii) are equivalent. Clearly, the sets in (iv) are the complement of the corresponding sets in (iii) with respect to the set of prime divisors of c. So (iii) and (iv) are equivalent. This completes the proof. □

Lemma 19. *Let a, b, c, d be positive integers. Then the following statements hold.*

(i) *If $d \mid c$ and $f_{\frac{c}{d}}(a) = f_{\frac{c}{d}}(b)$, then $f_c(da) = f_c(db)$ if and only if $\alpha(d, a) = \alpha(d, b)$.*

(ii) *If $f_1(a) = f_1(b)$, then $f_c(ca) = f_c(cb)$ if and only if $\alpha(c, a) = \alpha(c, b)$.*

(iii) *If $f_1(a) = f_1(b)$ and $(c, ab) = 1$, then $f_c(ca) = f_c(cb)$.*

Proof. For (i), suppose that $d \mid c$ and $f_{\frac{c}{d}}(a) = f_{\frac{c}{d}}(b)$. By Lemma 16, we obtain

$$f_c(da) = (da + c)\varphi(da) = d^2\left(a + \frac{c}{d}\right)\varphi(a)\alpha(d, a) = d^2 f_{\frac{c}{d}}(a)\alpha(d, a).$$

Similarly, we have $f_c(db) = d^2 f_{\frac{c}{d}}(b)\alpha(d, b)$. From this, we immediately obtain (i). Then (ii) follows from (i) by the substitution $d = c$. In addition, if $(c, ab) = 1$, then $\alpha(c, a) = \alpha(c, b)$, and so (iii) follows from (ii). This completes the proof. □

We are now ready to show that there are infinitely many $c \in \mathbb{N}$ such that f_c is not injective.

Theorem 20. *Let c be a positive integer. Then the following statements hold.*

(i) *If $(c, 130) = 1$, then f_c is not injective.*

(ii) *If the set of prime divisors of c is a subset of $\{2, 5, 13\}$, then f_c is not injective.*

(iii) *If $c = p^k$ where p is a prime and k is a positive integer, then f_c is not injective.*

Proof. For (i), let $(c, 130) = 1$. Let $a = 13$ and $b = 20$. From Table 1, we know that $f_1(a) = f_1(b) = 168$. Since $(c, ab) = 1$, we obtain by Lemma 19 that $f_c(ca) = f_c(cb)$. So f_c is not injective, and so (i) is proved.

For (ii), let $c = 2^{c_1}5^{c_2}13^{c_3}$ where c_1, c_2, c_3 are nonnegative integers. Let $a = 649$ and $b = 753$. We know from Table 1 that $f_1(a) = f_1(b) = 377000$ and $(c, ab) = 1$. By Lemma 19, we obtain $f_c(ca) = f_c(cb)$, and so f_c is not injective.

For (iii), let $c = p^k$ where p is a prime and k is a positive integer. If $p \notin \{2, 5, 13\}$, then the result follows from (i). If $p \in \{2, 5, 13\}$, then the result can be obtained from (ii). So the proof is complete. □

By a similar method, we can generate more $c \in \mathbb{N}$ such that f_c is not injective. We give one more similar theorem and then use it to show that f_c is not injective for any positive integers $c \leq 1000$. We remark that the integers $2, 157, 443, 17, 47, \ldots, 331$ appearing in the statement of the next theorem are prime numbers.

Theorem 21. *Let c be a positive integer. Then the following statements hold.*

(i) *If $(c, 2) = (c, 157) = (c, 443) = 1$, then f_c is not injective.*

(ii) *If $(c, 2) = (c, 17) = (c, 47) = 1$, then f_c is not injective.*

(iii) *If $(c, 13) = (c, 71) = (c, 881) = 1$, then f_c is not injective.*

(iv) *If $(c, 2) = (c, 23) = 1$, then f_c is not injective.*

(v) *If $(c, 5) = (c, 61) = (c, 271) = 1$, then f_c is not injective.*

(vi) *If $(c, 3) = (c, 13) = (c, 31) = 1$, then f_c is not injective.*

(vii) *If $(c, 5) = (c, 37) = (c, 41) = (c, 331) = 1$, then f_c is not injective.*

Proof. The proof of this theorem is similar to that of Theorem 20. We only need to choose an appropriate choice of $a, b \in \mathbb{N}$. For (i), we choose $a = 443$ and $b = 628$ to obtain from Table 7 and Lemma 19 that $f_1(a) = f_1(b) = 196248$, $(c, ab) = 1$, and $f_c(ca) = f_c(cb)$. In the same way, for (ii), (iii), (iv), (v), (vi), and (vii), we choose $(a, b) = (47, 68), (881, 923), (23, 32), (271, 305), (31, 39)$, and $(1517, 1655)$, respectively, to obtain that $f_c(ca) = f_c(cb)$. This completes the proof. □

Corollary 22. *For each positive integer $c \leq 1000$, the function f_c is not injective.*

Proof. Let $1 \leq c \leq 1000$ be a positive integer. If c is odd and is divisible by neither 157 nor 443, then the result follows from Theorem 21(i). Suppose c is odd and is divisible by 157 or 443. Since $c \leq 1000$, the possible values of c are $c = 157, 157 \cdot 3, 157 \cdot 5, 443$. If $c \neq 157 \cdot 5$, then $(c, 130) = 1$ and the result follows from Theorem 20(i). If $c = 157 \cdot 5$, we apply Theorem 21(ii) to obtain the desired result.

Therefore it remains to consider the case that c is even. Let $c = 2^k d$ where $k \geq 1$ and d is odd. If $(d, 13) = (d, 71) = d(881) = 1$, then the result follows from Theorem 21(iii). So we only need to consider the case that $13 \mid d$, $71 \mid d$, or $881 \mid d$. Since $c \leq 1000$, we see that $d \leq 500$. So $881 \mid d$ is not possible.

Case 1: $71 \mid d$. Then $d = 71, 71 \cdot 3, 71 \cdot 5, 71 \cdot 7$. If $d \neq 71 \cdot 5$, then the result can be obtained from Theorem 21(v). If $d = 71 \cdot 5$, then we apply Theorem 21(vi) to obtain the desired result.

Case 2: $13 \mid d$. Then $d = 13, 13 \cdot 3, 13 \cdot 5, \ldots, 13 \cdot 37$. If $d \neq 13 \cdot 5, 13 \cdot 15, 13 \cdot 25, 13 \cdot 35$, then we use Theorem 21(v); if $d = 13 \cdot 5, 13 \cdot 25$, then we apply Theorem 20(ii) to obtain the desired result. So it remains to consider the case $d = 13 \cdot 15$ or $d = 13 \cdot 35$.

Suppose $d = 13 \cdot 35$. Then we use Table 8 to solve it as follows. Let $a = 173$ and $b = 213$. Then $f_2(a) = f_2(b)$. Since $d = 13 \cdot 35$, we have $c = 2^k \cdot 5 \cdot 7 \cdot 13$. Since $c \leq 1000$, we have $k = 1$ and $c = 2 \cdot 5 \cdot 7 \cdot 13$. Then $\left(\frac{c}{2}, a\right) = \left(\frac{c}{2}, b\right) = 1$ and

$$f_c\left(\frac{c}{2}a\right) = \left(\frac{c}{2}a + c\right)\varphi\left(\frac{c}{2}a\right) = \frac{c}{2}(a+2)\varphi\left(\frac{c}{2}\right)\varphi(a) = \frac{c}{2}\varphi\left(\frac{c}{2}\right)f_2(a).$$

Similarly, we have $f_c\left(\frac{c}{2}b\right) = \frac{c}{2}\varphi\left(\frac{c}{2}\right) f_2(b)$, and so $f_c\left(\frac{c}{2}a\right) = f_c\left(\frac{c}{2}b\right)$. This shows that f_c is not injective.

Finally, let $d = 13 \cdot 15$. Then $c = 2^k \cdot 3 \cdot 5 \cdot 13$ where $k = 1, 2$. We first consider the case $k = 1$. Let $a = 391 = 17{\cdot}23$ and $b = 526 = 2{\cdot}263$. From Table 8, we know that $f_2(a) = f_2(b)$. In addition, we have $\left(\frac{c}{2}, a\right) = \left(\frac{c}{2}, b\right) = 1$, and by the same calculation as above, we obtain $f_c\left(\frac{c}{2}a\right) = f_c\left(\frac{c}{2}b\right)$, as desired. Next, let $k = 2$. Let $a = 301 = 7 \cdot 43$ and $b = 339 = 3 \cdot 113$. By Table 9, we have $f_3(a) = f_3(b)$. In addition, the equality $\left(\frac{c}{3}, a\right) = \left(\frac{c}{3}, b\right) = 1$ also holds. Therefore

$$f_c\left(\frac{c}{3}a\right) = \frac{c}{3}\varphi\left(\frac{c}{3}\right) f_3(a) = \frac{c}{3}\varphi\left(\frac{c}{3}\right) f_3(b) = f_c\left(\frac{c}{3}b\right).$$

Hence f_c is not injective, as required. This completes the proof. □

Finally, we show that there are at least 98 percent of positive integers $c \leq N$ such that f_c is not injective for every large positive integer N.

Theorem 23. *For each $N \in \mathbb{N}$, let $A = A(N)$ be the set of all positive integers $c \leq N$ such that f_c is not injective. Then*

$$|A(N)| \geq (0.981728)N + O(1).$$

Proof. For each $d \in \mathbb{N}$, let $A_d = A_d(N)$ be the set of all positive integers $c \leq N$ such that $(c, d) = 1$. Let

$$B_1 = A_5 \cap A_{61} \cap A_{271},\ B_2 = A_{13} \cap A_{71} \cap A_{881},\ B_3 = A_5 \cap A_{37} \cap A_{41} \cap A_{331}.$$

Therefore B_1, B_2, and B_3 are the set of positive integers $c \leq N$ satisfying conditions (v), (iii), and (vii) in Theorem 21, respectively. Therefore $B_1 \cup B_2 \cup B_3 \subseteq A$. By the inclusion-exclusion principle, we obtain

$$|A| \geq |B_1| + |B_2| + |B_3| - |B_1 \cap B_2| - |B_1 \cap B_3| - |B_2 \cap B_3| + |B_1 \cap B_2 \cap B_3|. \tag{14}$$

Let N be a large positive integer. Each block of integers

$$[1, d], (d, 2d], (2d, 3d], \ldots, ((k-1)d, kd],$$

where $k = \lfloor N/d \rfloor$, contains exactly $\varphi(d)$ integers c such that $(c, d) = 1$. Therefore

$$A_d = \sum_{\substack{c \leq N \\ (c,d)=1}} 1 = \varphi(d) \left\lfloor \frac{N}{d} \right\rfloor + r_d,$$

where $0 \leq r_d < d$. Thus

$$A_d \geq \frac{\varphi(d)}{d} N + O_d(1), \tag{15}$$

where the implied constant depends at most on d but not on N.

We also observe that $A_{d_1} \cap A_{d_2} = A_{d_1 d_2}$. Therefore $B_1 = A_x$, $B_2 = A_y$, $B_3 = A_z$, and $B_1 \cap B_2 \cap B_3 = A_{xyz}$, where $x = 5 \cdot 61 \cdot 271$, $y = 13 \cdot 71 \cdot 881$, and $z = 5 \cdot 37 \cdot 41 \cdot 331$. By (14) and (15), we obtain

$$|A| \geq \left(\frac{\varphi(x)}{x} + \frac{\varphi(y)}{y} + \frac{\varphi(z)}{z} - \frac{\varphi(xy)}{xy} - \frac{\varphi(xz)}{xz} - \frac{\varphi(yz)}{yz} + \frac{\varphi(xyz)}{xyz} \right) N + O(1), \tag{16}$$

where the implied constant depends at most on x, y, z but not on N. We have

$$\frac{\varphi(x)}{x} = \prod_{p|x} \left(1 - \frac{1}{p}\right) = \left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{61}\right)\left(1 - \frac{1}{271}\right) \geq 0.783981,$$

$$\frac{\varphi(y)}{y} = \prod_{p|y} \left(1 - \frac{1}{p}\right) = \left(1 - \frac{1}{13}\right)\left(1 - \frac{1}{71}\right)\left(1 - \frac{1}{881}\right) \geq 0.909042.$$

Similarly, we also have

$$\frac{\varphi(z)}{z} \geq 0.757099, \quad \frac{\varphi(xy)}{xy} \leq 0.712673, \quad \frac{\varphi(xz)}{xz} \leq 0.741940,$$
$$\frac{\varphi(yz)}{yz} \leq 0.688236, \quad \text{and} \quad \frac{\varphi(xyz)}{xyz} \geq 0.674455.$$

Applying these estimates in (16), we obtain the desired result. This completes the proof. □

6 Notes on some related results

We also obtain a result that looks interesting and seem to be related to the Euler function. We record it here for a possibility of future reference. We observe that

$$\frac{1}{2} = 1 - \frac{1}{2}, \quad \frac{1}{3} = \frac{1}{2}\left(1 - \frac{1}{3}\right) = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right),$$
$$\frac{1}{4} = \left(\frac{1}{2}\right)^2 = \left(1 - \frac{1}{2}\right)^2, \quad \frac{1}{5} = \frac{1}{4}\left(1 - \frac{1}{5}\right) = \left(1 - \frac{1}{2}\right)^2\left(1 - \frac{1}{5}\right), \text{ and so on.}$$

In general, we have the following result.

Theorem 24. *For each integer $n \geq 2$, there exists a unique set of primes $p_1 > p_2 > \cdots > p_k$ and positive integers $a_1, a_2, \ldots, a_k$ such that*

$$\frac{1}{n} = \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right)^{a_i}. \tag{17}$$

Proof. We call the product in the form of the right-hand side of (17) a good form, and we observe that if $1/a$ and $1/b$ are written in a good form, then $1/ab = (1/a)(1/b)$ can also be written in a good form. We use this observation and a strong induction on n to prove this theorem. It is easy to check that the result holds when $n = 2$. So assume that $n \geq 3$ and the result holds for $2, 3, \ldots, n-1$. If n is a prime, then we write

$$\frac{1}{n} = \left(1 - \frac{1}{n}\right)\left(\frac{1}{n-1}\right),$$

and then write $1/(n-1)$, by the induction hypothesis, in a good form to obtain a good form for $1/n$. So assume that n is a composite. Then we write $n = q_1^{n_1} q_2^{n_2} \cdots q_k^{n_k}$ where $q_1, q_2, \ldots q_k$ are distinct primes and $n_1, n_2, \ldots, n_k$ are positive integers. Then $2 \leq q_i \leq n-1$ for every $1 \leq i \leq k$. By the induction hypothesis, the number $1/q_i$ can be written in a good form, and so $1/q_i^{n_i} = (1/q_i)^{n_i}$ can also be written in a good form for every i. This gives a good form for $1/n$, as required. □

Remark 25. Let $a(1) = 0$, and for $n \geq 2$, let $a(n)$ be the number of factors counted with multiplicity in writing $1/n$ in the form of (17). Then it is easy to see that $a(p) = 1 + a(p-1)$ for every prime p and $a(mn) = a(m) + a(n)$ for every $m, n \in \mathbb{N}$. In fact, the sequence $(a(n))_{n\geq 1}$ is the same as A064097 in the OEIS. So, perhaps, Theorem 24 is known, but as far as we are aware, it is not widely known. We did not know about this before Ruankong sent us the statement of Theorem 24 sometime ago. He did not give a proof and did not publish the result either, but he allowed us to include it in this paper. So our idea and proof may be different from what he had in mind.

7 Open questions

In this section, we propose some problems related to our results. We do not claim that these problems are difficult or interesting. They are not important and may even be trivial. However, we would merely like to record them for ourselves and to share them among interested readers. We do not plan to solve them soon and we do not mind if the readers solve them.

Question 26. We show that the function $n \mapsto (n+c)\varphi(n)$ is not injective for positive integers $c \leq 1000$, and also for more than 98 percent of positive integers $c \leq N$ when N is large. Can one show that the function is not injective for any $c \in \mathbb{N}$?

Question 27. By Ford's result [3], we know that if $k \geq 2$ is a fixed positive integer, there exists $m \in \mathbb{N}$ such that the equation $\varphi(x) = m$ has exactly k solutions. Since $n \mapsto n\varphi(n)$ is injective, the equation $x\varphi(x) = m$ has at most one solution. What are the answers if we replace $\varphi(n)$ by $\sigma(n)$ or other arithmetic functions. For example, if $m \in \mathbb{N}$ is given, how many solutions in $x \in \mathbb{N}$ to the equations $x\sigma(x) = m$, $x\psi(x) = m$, and $xd(x) = m$? Makowski [6, p. 102] observed that if $M_1 = 2^{p_1} - 1, M_2 = 2^{p_2} - 1, \ldots, M_k = 2^{p_k} - 1$ are distinct Mersenne primes,

$$M = \prod_{i=1}^{k} M_i, \text{ and } n_i = \frac{M}{M_i} \text{ for each } i = 1, 2, \ldots, k,$$

then $n_i\sigma(n_i)$ is a constant. This shows that if k is less than or equal to the number of Mersenne primes, then there exists $m \in \mathbb{N}$ such that the equation $x\sigma(x) = m$ has at least k solutions in $x \in \mathbb{N}$. What are the answers if we replace $\sigma(x)$ by other arithmetic functions?

Question 28. For each $k \geq 2$, does there exist $m \in \mathbb{N}$ for which the equation $x\sigma(x) = m$ has exactly k solutions? For example, $x\sigma(x) = 6$, $x\sigma(x) = 336$, and $x\sigma(x) = 333312$ have exactly one, two, and three solutions, respectively, namely, $x = 2$ for the first equation, $x = 12, 14$ for the second equation, and $x = 336, 372, 434$ for the third equation, respectively. The smallest m such that $x\sigma(x) = m$ has exactly n solutions is the sequence A212490 in the OEIS. It should be observed that $6 \mid 336$ and $336 \mid 333312$. If a_n is the nth term of the sequence A212490, is it true that a_{n+1} is always divisible by a_n?

Question 29. Let B be the function defined in Example 8 by $B(x) = xs(x)$. Recall that if (x, y) is an amicable pair, then $B(x) = B(y)$. Nevertheless, if $B(x) = B(y)$, then (x, y) may or may not be an amicable pair. For instance, we know from Table 1 that $B(6) = B(9) = 36$, $B(320) = B(340) = 141440$, and $B(1280) = B(1504) = 2286080$, but they are not amicable pairs. Are there infinitely many $x, y \in \mathbb{N}$ such that $B(x) = B(y)$? Are there infinitely many such $x, y \in \mathbb{N}$ that are not an amicable pair?

Question 30. We show in Examples 11 and 14 that the function H_b, which is defined by $H_b(n) = nS_b(n)$, is not injective on $\mathbb{N}$ for any $b \geq 2$ and is not injective on squarefree integers for $2 \leq b \leq 10$. Can one show that H_b is not injective on squarefree integers for any $b \geq 11$?

Question 31. We show that the function $n \mapsto n^a J_s(n)^b$ is injective if $s = 2$ or $a \geq sb$. Is the function $n \mapsto n^a J_s(n)^b$ injective if $s \geq 3$ and $a < sb$?

Question 32. It is not difficult to show that an analogue of Lemma 17 where $1 - \frac{1}{p}$ is replaced by $1 + \frac{1}{p}$ also holds. That is, the product $\prod_p \left(1 + \frac{1}{p}\right)^{a_p}$ uniquely determines the primes p and the exponents a_p in the product. However, it is not clear how an analogue of Theorem 24 should look like. Can one determine the set of all rational numbers q that can be written as

$$q = \prod_p \left(1 + \frac{1}{p}\right)^{a_p}$$

for some primes p and positive integers a_p?

Question 33. Other questions stated in Guy's book [6] are the following:

(i) Among all $m, n \in \mathbb{N}$ such that $m\sigma(m) = n\sigma(n)$, is m/n bounded?

(ii) Are there relatively prime positive integers m and n satisfying $m\sigma(n) = n\sigma(m)$?

Question 34. Let $(a(n))_{n\geq 1}$ be the sequence A064097 in the OEIS. This sequence is related to Theorem 24 and is also mentioned in Remark 25. Some conjectures regarding $a(n)$ stated in the OEIS are as follows.

(i) (Cloitre) $\log n < a(n) < (5/2)\log n$ for $n \geq 2$, and there exists a positive constant c such that

$$\sum_{1\leq k\leq n} a(k) \sim cn\log n.$$

(ii) (Wilson) $\lfloor \log 2n \rfloor < a(n) < (5/2)\log n$ for $n \geq 2$.

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9 Tables

$B(x_1) = B(x_2)$	x_1	x_2
36	6	9
62480	220	284
141440	320	340
1432640	1184	1210
2286080	1280	1504
1245335	1955	2093
6680960	2080	2288
7660880	2620	2924
27931280	5020	5564
39685376	6232	6368

Table 1: Distinct integers $x_1, x_2 \in (1, 10000]$ such that $B(x_1) = B(x_2)$ where B is defined in Example 8 by $B(x) = xs(x)$ for all $x \in \mathbb{N}$. The values of x_1 are increasing, but x_2 and $B(x_1) = B(x_2)$ are not listed in an increasing order.

$A(x_1) = A(x_2)$	x_1	x_2
336	12	14
5952	48	62
10080	60	70
27776	112	124
44352	132	154
61152	156	182
60480	160	189
97536	192	254
102816	204	238
127680	228	266
178560	240	310
185472	276	322
260400	300	350
196560	315	351
333312	336	372
333312	336	434
292320	348	406
333312	372	434
472416	444	518
455168	448	508
578592	492	574
635712	516	602
785664	528	682
833280	560	620
758016	564	658
1083264	624	806
1179360	630	702
961632	636	742
1330560	660	770
1189440	708	826
1270752	732	854
1530816	804	938
1717632	852	994

$A(x_1) = A(x_2)$	x_1	x_2
1834560	780	910
1821312	816	1054
1815072	876	1022
2261760	912	1178
2123520	948	1106
2926080	960	1270
2342592	996	1162
3249792	1008	1116
3084480	1020	1190
2691360	1068	1246
3285504	1104	1426
3830400	1140	1330
3194016	1164	1358
4612800	1200	1550
3461472	1212	1414
3666432	1232	1364
3599232	1236	1442
5503680	1260	1404
3882816	1284	1498
4028640	1308	1526
5462016	1344	1524
5462016	1344	1778
4328352	1356	1582
5564160	1380	1610
5178240	1392	1798
5407248	1452	1694
5055232	1456	1612
6552000	1500	1750
5462016	1524	1778
5810112	1572	1834
6352416	1644	1918
6538560	1668	1946
9999360	1680	1860

Table 2: Distinct integers $x_1, x_2 \in (1, 2000]$ such that $A(x_1) = A(x_2)$ where A is defined in Example 7 by $A(x) = x\sigma(x)$ for all $x \in \mathbb{N}$. The values of x_1 are increasing, but x_2 and $A(x_1) = A(x_2)$ are not listed in an increasing order.

$D(x_1) = D(x_2)$	x_1	x_2
108	18	27
192	24	32
448	56	64
1080	90	135
1920	120	160
1512	126	189
2688	168	192
2688	168	224
2688	192	224
2376	198	297
2808	234	351
4224	264	352
4480	280	320
3672	306	459
4992	312	416
4104	342	513
8640	360	432
6000	400	500
6528	408	544
4968	414	621
8100	450	675
7296	456	608
12096	504	576
6264	522	783
12960	540	648
8832	552	736
6696	558	837
14400	600	800
9856	616	704
15120	630	945
7992	666	999
11136	696	928
11648	728	832
8856	738	1107

$D(x_1) = D(x_2)$	x_1	x_2
11904	744	992
9288	774	1161
26880	840	960
26880	840	1120
10152	846	1269
15876	882	1323
14208	888	1184
15232	952	1088
11448	954	1431
26880	960	1120
15744	984	1312
23760	990	1485
16512	1032	1376
12744	1062	1593
17024	1064	1216
13176	1098	1647
18048	1128	1504
28080	1170	1755
28224	1176	1568
36000	1200	1500
14472	1206	1809
20352	1272	1696
15336	1278	1917
20608	1288	1472
32400	1296	1350
15768	1314	1971
42240	1320	1760
33600	1400	1600
22656	1416	1888
20412	1458	1701
23424	1464	1952
48384	1512	1728
25984	1624	1856
27776	1736	1984

Table 3: Distinct integers $x_1, x_2 \in (1, 2000]$ such that $D(x_1) = D(x_2)$ where D is defined in Example 9 by $D(x) = xd(x)$ for all $x \in \mathbb{N}$. The values of x_1 are increasing, but x_2 and $D(x_1) = D(x_2)$ are not listed in an increasing order.

$W_1(x_1) = W_1(x_2)$	x_1	x_2
90	30	45
126	42	63
198	66	99
234	78	117
270	90	135
306	102	153
342	114	171
378	126	189
414	138	207
450	150	225
522	174	261
558	186	279
594	198	297
840	210	280
666	222	333
702	234	351
738	246	369
774	258	387
810	270	405
846	282	423
882	294	441
918	306	459
954	318	477
1320	330	440
1026	342	513
1062	354	531
1098	366	549

$W_1(x_1) = W_1(x_2)$	x_1	x_2
1134	378	567
1560	390	520
1206	402	603
1242	414	621
1680	420	560
1278	426	639
1314	438	657
1350	450	675
1848	462	616
1422	474	711
1494	498	747
2040	510	680
1566	522	783
1602	534	801
2184	546	728
1674	558	837
2280	570	760
1746	582	873
1782	594	891
1818	606	909
1854	618	927
1926	642	963
1962	654	981
2640	660	880
1998	666	999
2760	690	920
2856	714	952

Table 4: Distinct integers $x_1, x_2 \in (1, 1000]$ such that $W_1(x_1) = W_1(x_2)$ where W_1 is defined in Example 10 by $W_1(x) = x\omega(x)$ for all $x \in \mathbb{N}$. The values of x_1 are increasing, but x_2 and $W_1(x_1) = W_1(x_2)$ are not listed in an increasing order.

$W_2(x_1) = W_2(x_2)$	x_1	x_2
160	32	40
240	48	60
360	72	90
400	80	100
540	108	135
560	112	140
600	120	150
840	168	210
880	176	220
900	180	225
1344	192	224
1000	200	250
1040	208	260
1260	252	315
1320	264	330
1360	272	340
1400	280	350
2016	288	336
1500	300	375
1520	304	380
1560	312	390
1840	368	460
1960	392	490
1980	396	495
2040	408	510
2100	420	525
3024	432	504
2200	440	550
2280	456	570

$W_2(x_1) = W_2(x_2)$	x_1	x_2
2320	464	580
2340	468	585
3360	480	560
2480	496	620
2500	500	625
4608	512	576
2600	520	650
2760	552	690
2940	588	735
2960	592	740
3060	612	765
3080	616	770
4536	648	756
3280	656	820
3300	660	825
4704	672	784
3400	680	850
3420	684	855
3440	688	860
3480	696	870
3500	700	875
5040	720	840
3640	728	910
3720	744	930
3760	752	940
3800	760	950
6912	768	864
3900	780	975

Table 5: Distinct integers $x_1, x_2 \in (1, 1000]$ such that $W_2(x_1) = W_2(x_2)$ where W_2 is defined in Example 10 by $W_2(x) = x\Omega(x)$ for all $x \in \mathbb{N}$. The values of x_1 are increasing, but x_2 and $W_2(x_1) = W_2(x_2)$ are not listed in an increasing order.

$H_{10}(x_1) = H_{10}(x_2)$	x_1	x_2
36	6	12
90	15	30
160	32	40
280	28	35
306	51	102
684	57	114
360	60	120
640	64	80
792	66	132
900	75	150
900	75	300
1105	85	221
1204	86	301
1408	88	128
1440	96	240
520	104	130
630	105	210
1360	136	170
900	150	300
1872	156	312
1980	165	330
2520	168	420
1720	172	215
2992	187	272
2080	208	260
2440	244	305
3060	255	510
4140	276	345
2800	280	350
4576	286	416
1600	320	400
4732	338	364
5220	348	435
3520	352	440
5760	384	480
6160	385	560
4240	424	530
6370	455	490
6840	456	570
7744	484	704
7380	492	615
7920	528	660
8460	564	705
8008	572	616
11305	595	665
11920	596	745
6400	640	800
13360	668	835
10080	672	840
11160	744	930
14212	748	836
15520	776	970

Table 6: Distinct integers $x_1, x_2 \in (1, 1000]$ such that $H_{10}(x_1) = H_{10}(x_2)$ where H_{10} is defined in Example 11 by $H_{10}(x) = xS_{10}(x)$ for all $x \in \mathbb{N}$. The values of x_1 are increasing, but x_2 and $H_{10}(x_1) = H_{10}(x_2)$ are not listed in an increasing order.

$f_1(x_1) = f_1(x_2)$	x_1	x_2	Factorization of x_1	Factorization of x_2
168	13	20	13	$2^2 \cdot 5$
528	23	32	23	2^5
960	31	39	31	$3 \cdot 13$
1368	37	56	37	$2^3 \cdot 7$
2208	47	68	47	$2^2 \cdot 17$
5040	71	104	71	$2^3 \cdot 13$
18720	155	194	$5 \cdot 31$	$2 \cdot 97$
73440	271	305	271	$5 \cdot 61$
78880	289	492	17^2	$2^2 \cdot 3 \cdot 41$
144072	413	666	$7 \cdot 59$	$2 \cdot 3^2 \cdot 37$
196248	443	628	443	$2^2 \cdot 157$
131328	455	512	$5 \cdot 7 \cdot 13$	2^9
212520	461	804	461	$2^2 \cdot 3 \cdot 67$
199080	473	710	$11 \cdot 43$	$2 \cdot 5 \cdot 71$
210528	515	730	$5 \cdot 103$	$2 \cdot 5 \cdot 73$
253440	527	575	$17 \cdot 31$	$5^2 \cdot 23$
256320	533	800	$13 \cdot 41$	$2^5 \cdot 5^2$
226800	539	674	$7^2 \cdot 11$	$2 \cdot 337$
218120	573	664	$3 \cdot 191$	$2^3 \cdot 83$
361200	601	902	601	$2 \cdot 11 \cdot 41$
320544	635	741	$5 \cdot 127$	$3 \cdot 13 \cdot 19$
377000	649	753	$11 \cdot 59$	$3 \cdot 251$
776160	881	923	881	$13 \cdot 71$
863040	929	1239	929	$3 \cdot 7 \cdot 59$
820800	949	1025	$13 \cdot 73$	$5^2 \cdot 41$
1585080	1259	1784	1259	$2^3 \cdot 223$
708048	1340	1638	$2^2 \cdot 5 \cdot 67$	$2 \cdot 3^2 \cdot 7 \cdot 13$
2185920	1517	1655	$37 \cdot 41$	$5 \cdot 331$

Table 7: Distinct integers $x_1, x_2 \in (1, 2000]$ such that $f_1(x_1) = f_1(x_2)$ where f_1 is the function defined in Section 5 by $f_1(x) = (x+1)\varphi(x)$ for all $x \in \mathbb{N}$. The values of x_1 are increasing, but x_2 and $f_1(x_1) = f_1(x_2)$ are not listed in an increasing order.

$f_2(x_1) = f_2(x_2)$	x_1	x_2	Factorization of x_1	Factorization of x_2
1920	62	78	$2 \cdot 31$	$2 \cdot 3 \cdot 13$
30100	173	213	173	$3 \cdot 71$
37440	193	310	193	$2 \cdot 5 \cdot 31$
78480	325	434	$5^2 \cdot 13$	$2 \cdot 7 \cdot 31$
89040	369	422	$3^2 \cdot 41$	$2 \cdot 211$
138336	391	526	$17 \cdot 23$	$2 \cdot 263$
126480	525	618	$3 \cdot 5^2 \cdot 7$	$2 \cdot 3 \cdot 103$
146880	542	610	$2 \cdot 271$	$2 \cdot 5 \cdot 61$
254016	565	754	$5 \cdot 113$	$2 \cdot 13 \cdot 29$
363456	629	1260	$17 \cdot 37$	$2^2 \cdot 3^2 \cdot 5 \cdot 7$
219120	662	828	$2 \cdot 331$	$2^2 \cdot 3^2 \cdot 23$
288144	665	826	$5 \cdot 7 \cdot 19$	$2 \cdot 7 \cdot 59$
453600	673	1078	673	$2 \cdot 7^2 \cdot 11$
294528	765	942	$3^2 \cdot 5 \cdot 17$	$2 \cdot 3 \cdot 157$
290880	806	1008	$2 \cdot 13 \cdot 31$	$2^4 \cdot 3^2 \cdot 7$
320256	832	1110	$2^6 \cdot 13$	$2 \cdot 3 \cdot 5 \cdot 37$
469440	976	1302	$2^4 \cdot 61$	$2 \cdot 3 \cdot 7 \cdot 31$
506880	1054	1150	$2 \cdot 17 \cdot 31$	$2 \cdot 5^2 \cdot 23$
761376	1131	1234	$3 \cdot 13 \cdot 29$	$2 \cdot 617$
796320	1262	1420	$2 \cdot 631$	$2^2 \cdot 5 \cdot 71$
641088	1270	1482	$2 \cdot 5 \cdot 127$	$2 \cdot 3 \cdot 13 \cdot 19$
754000	1298	1506	$2 \cdot 11 \cdot 59$	$2 \cdot 3 \cdot 251$
907200	1348	1510	$2^2 \cdot 337$	$2 \cdot 5 \cdot 151$
1552320	1762	1846	$2 \cdot 881$	$2 \cdot 13 \cdot 71$

Table 8: Distinct integers $x_1, x_2 \in (1, 2000]$ such that $f_2(x_1) = f_2(x_2)$ where f_2 is the function defined in Section 5 by $f_2(x) = (x+2)\varphi(x)$ for all $x \in \mathbb{N}$. The values of x_1 are increasing, but x_2 and $f_2(x_1) = f_2(x_2)$ are not listed in an increasing order.

$f_3(x_1) = f_3(x_2)$	x_1	x_2	Factorization of x_1	Factorization of x_2
60	7	12	7	$2^2 \cdot 3$
560	25	32	5^2	2^5
540	27	42	3^3	$2 \cdot 3 \cdot 7$
1008	39	60	$3 \cdot 13$	$2^2 \cdot 3 \cdot 5$
3168	69	96	$3 \cdot 23$	$2^5 \cdot 3$
7056	95	144	$5 \cdot 19$	$2^4 \cdot 3^2$
6120	99	150	$3^2 \cdot 11$	$2 \cdot 3 \cdot 5^2$
8208	111	168	$3 \cdot 37$	$2^3 \cdot 3 \cdot 7$
13248	141	204	$3 \cdot 47$	$2^2 \cdot 3 \cdot 17$
21360	175	264	$5^2 \cdot 7$	$2^3 \cdot 3 \cdot 11$
30240	213	312	$3 \cdot 71$	$2^3 \cdot 3 \cdot 13$
54000	247	297	$13 \cdot 19$	$3^3 \cdot 11$
76608	301	339	$7 \cdot 43$	$3 \cdot 113$
112320	465	582	$3 \cdot 5 \cdot 31$	$2 \cdot 3 \cdot 97$
186912	469	646	$7 \cdot 67$	$2 \cdot 17 \cdot 19$
324896	569	920	569	$2^3 \cdot 5 \cdot 23$
376992	613	1068	613	$2^2 \cdot 3 \cdot 89$
378000	753	872	$3 \cdot 251$	$2^3 \cdot 109$
396144	783	1176	$3^3 \cdot 29$	$2^3 \cdot 3 \cdot 7^2$
580320	803	933	$11 \cdot 73$	$3 \cdot 311$
440640	813	915	$3 \cdot 271$	$3 \cdot 5 \cdot 61$
879840	937	1830	937	$2 \cdot 3 \cdot 5 \cdot 61$
808128	973	1101	$7 \cdot 139$	$3 \cdot 367$
822400	1025	1282	$5^2 \cdot 41$	$2 \cdot 641$
1254960	1159	1491	$19 \cdot 61$	$3 \cdot 7 \cdot 71$
1177488	1329	1884	$3 \cdot 443$	$2^2 \cdot 3 \cdot 157$
787968	1365	1536	$3 \cdot 5 \cdot 7 \cdot 13$	$2^9 \cdot 3$
1520640	1581	1725	$3 \cdot 17 \cdot 31$	$3 \cdot 5^2 \cdot 23$
2537472	1649	1885	$17 \cdot 97$	$5 \cdot 13 \cdot 29$

Table 9: Distinct integers $x_1, x_2 \in (1, 2000]$ such that $f_3(x_1) = f_3(x_2)$ where f_3 is the function defined in Section 5 by $f_3(x) = (x+3)\varphi(x)$ for all $x \in \mathbb{N}$. The values of x_1 are increasing, but x_2 and $f_3(x_1) = f_3(x_2)$ are not listed in an increasing order.

References

[1] P. Erdős, Remarks on number theory II: some problems on the σ function, *Acta Arith.* **5** (1959), 171–177.

[2] K. Ford, The distribution of totients, *Ramanujan J.* **2** (1998), 67–151.

[3] K. Ford, The number of solutions of $\varphi(x) = m$, *Ann. Math.* **150** (1999), 1–29.

[4] K. Ford, F. Luca, and C. Pomerance, Common values of the arithmetic functions ϕ and σ, *Bull. Lond. Math. Soc.* **42** (2010), 478–488.

[5] K. Ford and P. Pollack, On common values of $\varphi(n)$ and $\sigma(n)$ II, *Algebra Number Theory* **6** (2012), 1669–1696.

[6] R. K. Guy, *Unsolved Problems in Number Theory*, 3rd edition, Springer, 2004.

[7] C. Pomerance, Popular values of Euler's function, *Mathematatika* **27** (1980), 84–89.

[8] P. Pongsriiam, Quasi-injectivity of some arithmetic functions, *J. Integer Sequences* **24** (2021), Article 21.10.1.

[9] N. J. A. Sloane et al., The On-line Encyclopedia of Integer Sequences, 2022. Available at `https://oeis.org/`.

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