Journal of Integer Sequences, Vol. 26 (2023),
Article 23.6.2

Common Values of Generalized Fibonacci and Leonardo Sequences

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Abstract

For an integer $k \geq 2$, let $F_n^{(k)}$ be the k-generalized Fibonacci sequence that starts with $0, \ldots, 0, 1, 1$ (k terms) and each term afterwards is the sum of k preceding terms. In this paper, we find all the k-generalized Fibonacci numbers that are Leonardo numbers. More explicitly, we solve the Diophantine equation $F_n^{(k)} = \text{Le}_m$ in positive integers n, k, m with $k \geq 2$.

1 Introduction

The Fibonacci and Lucas sequence are two fascinating topics in integer sequences. The Leonardo sequence $(\text{Le}_m)_{m\geq 0}$ is an integer sequence that is related to the Fibonacci and Lucas sequences. Leonardo numbers are discussed by Catarino and Borges [9]. It is the sequence A001595 in the OEIS satisfying the recurrence relation

$$\text{Le}_m = \text{Le}_{m-1} + \text{Le}_{m-2} + 1 \tag{1}$$

for $m \geq 2$ with the initial terms $\text{Le}_0 = 1$ and $\text{Le}_1 = 1$. The first few terms of $(\text{Le}_m)_{m\geq 0}$ are

$$1, 1, 3, 5, 9, 15, 25, 41, 67, 109, 177, 287, 465, 753, 1219, 1973, \ldots.$$

In the recent past, many aspects of Leonardo sequence have been studied such as hybrid Leonardo numbers [1], incomplete Leonardo numbers [10], Leonardo Pisano polynomials, hybrinomials [14] and q-Leonardo hybrid numbers [16].

The Fibonacci sequence $(F_n)_{n\geq 0}$ is the binary recurrence sequence given by

$$F_{n+2} = F_{n+1} + F_n \quad \text{for } n \geq 0$$

with the initial terms $F_0 = 0$ and $F_1 = 1$.

Let $k \geq 2$ be an integer. One of numerous generalizations of the Fibonacci sequence, called the k-generalized Fibonacci sequence $(F_n^{(k)})_{n\geq -(k-2)}$ is given by the recurrence

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)} = \sum_{i=1}^{k} F_{n-i}^{(k)} \quad \text{for all } n \geq 2, \tag{2}$$

with the initial conditions $F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \cdots = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$. Here, $F_n^{(k)}$ denotes the nth k-generalized Fibonacci number.

Note that for $k = 2$, we have $F_n^{(2)} = F_n$, the nth Fibonacci number. For $k = 3$, we have $F_n^{(3)} = T_n$, the nth Tribonacci number. They are followed by the Tetranacci numbers for $k = 4$, and so on.

A Leonardo number is called k-Fibonacci Leonardo number if it is a k-generalized Fibonacci number. The aim of this paper is to determine all the k-Fibonacci Leonardo numbers.

Finding the intersection of two recurrent sequences of positive integers is a topic that has been extensively studied in number theory. Currently, several researchers have been interested in finding the intersection of the k-generalized Fibonacci sequence with other number sequences. For instance, one can go through [4, 5, 6, 13, 17, 18].

Motivated by the above literature, we study the Diophantine equation

$$F_n^{(k)} = \mathrm{Le}_m\,. \tag{3}$$

In particular, our main result is the following.

Theorem 1. *All the solutions of the Diophantine equation (3) in positive integers with $k \geq 2$ are given by*

$$(n, k, m) \in \{(1, k, 0), (2, k, 0), (1, k, 1), (2, k, 1), (4, 2, 2), (5, 2, 3), (6, 4, 5)\}.$$

Thus, the only k-Fibonacci Leonardo numbers are 1, 3, 5, *and* 15.

2 Auxiliary results

Our proof of Theorem 1 is mainly based on linear forms in logarithms of algebraic numbers and a reduction algorithm originally introduced by Baker and Davenport [3] (and improved by Dujella and Pethö [12]). Here, we use a variant due to de Weger [19], but first, recall some basic notation from algebraic number theory.

2.1 Linear forms in logarithms

Let γ be an algebraic number of degree d with minimal primitive polynomial

$$f(X) := a_0X^d + a_1X^{d-1} + \cdots + a_d = a_0\prod_{i=1}^{d}(X - \gamma^{(i)}) \in \mathbb{Z}[X],$$

where the a_i are relatively prime integers, $a_0 > 0$, and the $\gamma^{(i)}$ are conjugates of γ. Then

$$h(\gamma) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d}\log\Big(\max\{|\gamma^{(i)}|, 1\}\Big)\right) \tag{4}$$

is called the *logarithmic height* of γ.

With the established notation, Matveev (see [15] or [8, Theorem 9.4]) proved the following result.

Theorem 2. *Assume that $\eta_1, \ldots, \eta_t$ are positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D, $b_1, \ldots, b_t$ are rational integers, and*

$$\Lambda := \eta_1^{b_1}\cdots\eta_t^{b_t} - 1,$$

is not zero. Then

$$|\Lambda| \geq \exp\Big(-1.4\cdot 30^{t+3}\cdot t^{4.5}\cdot D^2(1+\log D)(1+\log B)A_1\cdots A_t\Big),$$

where

$$B \geq \max\{|b_1|, \ldots, |b_t|\},$$

and

$$A_i \geq \max\{Dh(\eta_i), |\log\eta_i|, 0.16\}, \text{ for all } i = 1, \ldots, t.$$

2.2 The de Weger reduction algorithm

Here we present a variant of the reduction method of Baker and Davenport [3] (and improved by Dujella and Pethö [12]) due to de Weger [19].

Let $\vartheta_1, \vartheta_2, \delta \in \mathbb{R}$ be given and let $x_1, x_2 \in \mathbb{Z}$ be unknowns. Let

$$\Lambda = \delta + x_1\vartheta_1 + x_2\vartheta_2. \tag{5}$$

Set $X = \max\{|x_1, |x_2|\}$. Let X_0, Y be positive. Assume that

$$|\Lambda| < c\exp(-\rho Y) \tag{6}$$

and

$$Y \leq X \leq X_0, \tag{7}$$

where c, ρ be positive constants. When $\delta = 0$ in (5), we get

$$\Lambda = x_1\vartheta_1 + x_2\vartheta_2.$$

Put $\vartheta = -\vartheta_1/\vartheta_2$. We assume that x_1 and x_2 are coprime. Let the continued fraction expansion of ϑ be given by

$$[a_0, a_1, a_2, \dots],$$

and let the kth convergent of ϑ be p_k/q_k for $k = 0, 1, 2, \dots$. We may assume without loss of generality that $|\vartheta_1| < |\vartheta_2|$ and $x_1 > 0$. We have the following results.

Lemma 3. *[19, Lemma 3.1] If (6) and (7) hold for x_1, x_2 with $X \geq 1$ and $\delta = 0$, then $(-x_2, x_1) = (p_k, q_k)$ for an index k that satisfies*

$$k \leq -1 + \frac{\log(1 + X_0\sqrt{5})}{\log\left(\frac{1+\sqrt{5}}{2}\right)} := Y_0.$$

Lemma 4. *[19, Lemma 3.2] Let*

$$A = \max_{0 \leq k \leq Y_0} a_{k+1}.$$

If (6) and (7) hold for x_1, x_2 with $X \geq 1$ and $\delta = 0$, then

$$Y < \frac{1}{\rho}\log\left(\frac{c(A+2)}{|\vartheta_2|}\right) + \frac{1}{\rho}\log X < \frac{1}{\rho}\log\left(\frac{c(A+2)X_0}{|\vartheta_2|}\right).$$

When $\delta \neq 0$ in (5), put $\vartheta = -\vartheta_1/\vartheta_2$ and $\psi = \delta/\vartheta_2$. Then we have

$$\frac{\Lambda}{\vartheta_2} = \psi - x_1\vartheta + x_2.$$

Let p/q be a convergent of ϑ with $q > X_0$. For a real number x, we let $\|x\| = \min\{|x - n| : n \in \mathbb{Z}\}$ be the distance from x to the nearest integer. We have the following result.

Lemma 5. *[19, Lemma 3.3] Suppose that*

$$\|q\psi\| > \frac{2X_0}{q}.$$

Then the solutions of (6) and (7) satisfy

$$Y < \frac{1}{\rho}\log\left(\frac{q^2 c}{|\vartheta_2| X_0}\right).$$

2.3 Properties of the Leonardo sequence

The characteristic equation of $(\mathrm{Le}_m)_{m\geq 0}$ is $x^3 - 2x^2 - 1 = 0$, which has roots $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{-1}{\alpha}$ (see [2]). The Binet formula for Le_m is

$$\mathrm{Le}_m = 2\left(\frac{\alpha^{m+1} - \beta^{m+1}}{\alpha - \beta}\right) - 1 = \frac{\alpha(2\alpha^m - 1) - \beta(2\beta^m - 1)}{\alpha - \beta} \quad \text{for all } m \geq 0. \tag{8}$$

Lemma 6. *The inequality*

$$\alpha^m \leq \mathrm{Le}_m \leq \alpha^{m+1}, \tag{9}$$

holds for all positive integers $m \geq 2$.

Proof. This can be easily proved by the method of induction on m. □

Lemma 7. *[9, Lemma 2.1] For all $m \geq 0$, the m-th Leonardo number Le_m is an odd number.*

2.4 Properties of the k-generalized Fibonacci sequence

In this subsection, we recall some facts and properties of the k-generalized Fibonacci sequence which will be used later. The characteristic polynomial of the k-generalized Fibonacci sequence is

$$\Psi_k(x) = x^k - x^{k-1} - \cdots - x - 1.$$

The polynomial $\Psi_k(x)$ is irreducible over $\mathbb{Q}[x]$ and has just one root outside the unit circle. It is real and positive, so it satisfies $\varphi(k) > 1$. The other roots are strictly inside the unit circle. Throughout this paper, $\varphi := \varphi(k)$ denotes that single root, which is located between $2(1 - 2^{-k})$ and 2 (see [20]). To simplify the notation, in general, we omit the dependence of k on φ.

Dresden and Du [11] gave the following simplified Binet-like formula for $F_n^{(k)}$:

$$F_n^{(k)} = \sum_{i=1}^{k} f_k(\varphi_i)\varphi_i^{n-1} = \sum_{i=1}^{k} \frac{\varphi_i - 1}{2 + (k+1)(\varphi_i - 2)}\varphi_i^{n-1}, \tag{10}$$

where $\varphi := \varphi_1, \varphi_2, \ldots, \varphi_k$ are the roots of the characteristic polynomial $\Psi_k(x)$. It was also proved in [11] that the contribution of the roots that lie inside the unit circle to the formula (10) is very small, namely that the approximation

$$\left|F_n^{(k)} - f_k(\varphi)\varphi^{n-1}\right| < \frac{1}{2} \quad \text{holds for all} \quad n \geq 2 - k. \tag{11}$$

Furthermore, it was shown by Bravo and Luca in [7] that the inequality

$$\varphi^{n-2} \leq F_n^{(k)} \leq \varphi^{n-1} \text{ holds for all } n \geq 1 \text{ and } k \geq 2. \tag{12}$$

The first direct observation is that the first $k+1$ non-zero terms in $F_n^{(k)}$ are powers of 2, namely

$$F_1^{(k)} = 1, F_2^{(k)} = 1, F_3^{(k)} = 2, F_4^{(k)} = 4, \ldots, F_{k+1}^{(k)} = 2^{k-1},$$

while the next term in the above sequence is $F_{k+2}^{(k)} = 2^k - 1$. Thus, we have that

$$F_n^{(k)} = 2^{n-2} \text{ holds for all } 2 \le n \le k+1.$$

We also observe that the recursion (2) implies the three-term recursion

$$F_n^{(k)} = 2F_{n-1}^{(k)} - F_{n-k-1}^{(k)} \text{ for all } n \ge 3,$$

which shows that the k-generalized Fibonacci sequence grows at a rate less than 2^{n-2}. In fact, the inequality $F_n^{(k)} < 2^{n-2}$ holds for all $n \ge k+2$ (see [7, Lemma 2]).

The following result was proved by Bravo and Luca [7].

Lemma 8. *Let $k \ge 2$, φ be the dominant root of $(F_n^{(k)})_{n \ge -(k-2)}$, and consider the function defined in (10). Then the inequalities*

$$\frac{1}{2} < f_k(\varphi) < \frac{3}{4} \quad \textit{and} \quad \left| f_k(\varphi^{(i)}) \right| < 1 \quad \textit{hold for all} \quad 2 \le i \le k.$$

In addition, they proved that the logarithmic height of f satisfies

$$h(f_k(\varphi)) < \log(k+1) + \log 4 \quad \text{for all} \quad k \ge 2. \tag{13}$$

We finish this subsection with the following estimate due to Bravo, Gómez, and Luca [5], which will be used later.

Lemma 9. *Let $k \ge 2$ and suppose that $n < 2^{k/2}$. Then*

$$F_n^{(k)} = 2^{n-2}(1+\xi) \quad \textit{where} \quad |\xi| < \frac{1}{2^{k/2}}.$$

3 Proof of Theorem 1

Proof. First, note that $F_1^{(k)} = F_2^{(k)} = 1 = \mathrm{Le}_0 = \mathrm{Le}_1$. Therefore, we may assume that $n \ge 3$. For $3 \le n \le k+1$, we have that $F_n^{(k)} = 2^{n-2}$, but Le_m is an odd number for $m \ge 0$. Thus, there is no solution of (3) in this range. From now, we assume that $n \ge k+2$ and $k \ge 2$.

Combining the inequalities (9) and (12) together with equation (3), we have

$$\varphi^{n-2} \le \alpha^{m+1} \quad \text{and} \quad \alpha^m \le \varphi^{n-1}.$$

Then, we deduce that

$$(n-2)\left(\frac{\log \varphi}{\log \alpha}\right) - 1 \le m \le (n-1)\left(\frac{\log \varphi}{\log \alpha}\right).$$

Using the fact $2(1-2^{-k}) < \varphi(k) < 2$ for all $k \ge 2$, it follows that

$$0.8n - 2.6 < m < 1.5n - 1.5. \tag{14}$$

3.1 An inequality for n and m in terms of k

By using (3), (8), (11) and taking absolute value, we obtain

$$\left| f_k(\varphi)\varphi^{n-1} - \frac{2\alpha^{m+1}}{\sqrt{5}} \right| < \frac{1}{2} + \frac{2|\beta|^{m+1}}{\sqrt{5}} + \frac{|-\alpha|}{\sqrt{5}} + \frac{|\beta|}{\sqrt{5}}. \tag{15}$$

Dividing both sides of the above inequality by $\frac{2\alpha^{m+1}}{\sqrt{5}}$, we conclude that

$$\left| \left(\frac{\sqrt{5} f_k(\varphi)}{2} \right) \varphi^{n-1}\alpha^{-(m+1)} - 1 \right| < 4\alpha^{-m}. \tag{16}$$

Let

$$\Lambda_1 := \left(\frac{\sqrt{5} f_k(\varphi)}{2} \right) \varphi^{n-1}\alpha^{-(m+1)} - 1, \tag{17}$$

and inequality (16) becomes

$$|\Lambda_1| < 4\alpha^{-m}. \tag{18}$$

Before applying Theorem 2, we need to prove that $\Lambda_1 \neq 0$. Assume that $\Lambda_1 = 0$, then we get

$$f_k(\varphi) = \frac{2}{\sqrt{5}} \varphi^{-(n-1)} \alpha^{(m+1)},$$

and so $f_k(\varphi)$ is an algebraic integer, which is impossible. Thus, $\Lambda_1 \neq 0$. Therefore, we apply Theorem 2 to get a lower bound for Λ_1 given by (17) with the parameters:

$$\eta_1 := \frac{\sqrt{5} f_k(\varphi)}{2}, \quad \eta_2 := \varphi, \quad \eta_3 := \alpha,$$

and

$$b_1 := 1, \quad b_2 := n-1, \quad b_3 := -(m+1).$$

Note that η_1, η_2, η_3 are positive real numbers and belong to the field $\mathbb{K} := \mathbb{Q}(\varphi, \sqrt{5})$. So we can take $D := [\mathbb{K} : \mathbb{Q}] \leq 2k$. Since $h(\eta_2) = (\log \varphi)/k < (\log 2)/k$ and $h(\eta_3) = (\log \alpha)/2$, we choose

$$\max\{2kh(\eta_2), |\log \eta_2|, 0.16\} = 2\log 2 := A_2$$

and

$$\max\{2kh(\eta_3), |\log \eta_3|, 0.16\} = k\log \alpha := A_3.$$

By using the estimate (13) and the properties of logarithmic height, we get that for all $k \geq 2$

$$\begin{aligned} h(\eta_1) &\leq h(f_k(\varphi)) + h\left(\frac{\sqrt{5}}{2} \right) \\ &< \log(k+1) + \log 4 + \log\left(2\sqrt{5}\right) \\ &< 5.8 \log k. \end{aligned}$$

Thus, we obtain

$$\max\{2kh(\eta_1), |\log \eta_1|, 0.16\} = 11.6k \log k := A_1.$$

In addition, by (14) we take $B := 1.5n$. Then by Theorem 2, we have

$$|\Lambda_1| > \exp\left(-1.432 \times 10^{11}(2k)^2(1 + \log 2k)(1 + \log 1.5n)(11.6k \log k)(2 \log 2)(k \log \alpha)\right). \quad (19)$$

Comparing (18) and (19), taking logarithms and then performing the respective calculations, we get that

$$m \log \alpha - \log 4 < 4.43 \times 10^{12} k^4 \log k(1 + \log 2k)(1 + \log 1.5n).$$

Taking into consideration the facts $1 + \log 2k < 3.5 \log k$ for all $k \geq 2$ and $1 + \log 1.5n < 2.1 \log n$ for all $n \geq 4$, we conclude that

$$m < 6.77 \times 10^{13} k^4 \log^2 k \log n.$$

Using (14), the last inequality becomes

$$\frac{n}{\log n} < 8.47 \times 10^{13} k^4 \log^2 k. \quad (20)$$

Since the function $x \mapsto x/\log x$ is increasing for all $x > e$, it is easy to check that

$$\frac{x}{\log x} < S \implies x < 2S \log S \quad \text{whenever} \quad S \geq 3. \quad (21)$$

Then, taking $x := n$ and $S := 8.47 \times 10^{13} k^4 \log^2 k$, inequality (21) together with $32.07 + 4 \log k + 2 \log \log k < 49.3 \log k$ for all $k \geq 2$, yields

$$\begin{aligned} n &< 2\left(8.47 \times 10^{13} k^4 \log^2 k\right) \log\left(8.47 \times 10^{13} k^4 \log^2 k\right) \\ &< (1.69 \times 10^{14} k^4 \log^2 k)(32.07 + 4 \log k + 2 \log \log k) \\ &< 8.35 \times 10^{15} k^4 \log^3 k. \end{aligned}$$

By combining the above results, we obtain the following lemma.

Lemma 10. *If (n, k, m) is a solution in integers of equation (3) with $k \geq 2$ and $n \geq k + 2$, then the inequalities*

$$0.6m < n < 8.35 \times 10^{15} k^4 \log^3 k \quad (22)$$

hold.

3.2 The case of small k

Suppose now that $k \in [2, 230]$. In order to apply Lemma 5, we let

$$\Gamma_1 := (n-1)\log\varphi - (m+1)\log\alpha + \log\left(\frac{\sqrt{5}f_k(\varphi)}{2}\right). \tag{23}$$

Then $e^{\Gamma_1} - 1 := \Lambda_1$, where Λ_1 is defined by (17). Therefore, (18) can be written as

$$|e^{\Gamma_1} - 1| < 4\alpha^{-m}. \tag{24}$$

Note that $\Gamma_1 \neq 0$. Since $\Lambda_1 \neq 0$, we distinguish the following cases. If $\Gamma_1 > 0$, then $e^{\Gamma_1}-1 > 0$. Using the fact $x \le e^x - 1$ for all $x \in \mathbb{R}$ and the inequality (24), we obtain

$$0 < \Gamma_1 < 4\alpha^{-m}.$$

If, on the contrary, $\Gamma_1 < 0$, then $4\alpha^{-m} < 1/2$ holds for all $m \ge 5$. Thus, from (24), we have $|e^{\Gamma_1} - 1| < 1/2$ and therefore $e^{|\Gamma_1|} < 2$. Since $\Gamma_1 < 0$, we have

$$0 < |\Gamma_1| \le e^{|\Gamma_1|} - 1 = e^{|\Gamma_1|}|e^{\Gamma_1} - 1| < 8\alpha^{-m}.$$

In both cases, we have

$$0 < |\Gamma_1| < 8\alpha^{-m} < 8\exp(-0.48 \times m). \tag{25}$$

Let

$$c := 8, \quad \rho := 0.48, \quad \psi := \frac{\log\left(\frac{\sqrt{5}f_k(\varphi)}{2}\right)}{\log\varphi},$$

and

$$\vartheta := \frac{\log\alpha}{\log\varphi}, \quad \vartheta_1 := -\log\alpha, \quad \vartheta_2 := \log\varphi, \quad \delta := \log\left(\frac{\sqrt{5}f_k(\varphi)}{2}\right).$$

For each $k \in [2, 230]$, we find a good approximation of φ and a convergent p_l/q_l of the continued fraction of ϑ such that $q_l > X_0$, where $X_0 = \lfloor 8.35 \times 10^{15} k^4 \log^3 k \rfloor$, which is an upper bound of $\max\{n-1, m\}$ from Lemma 10. After doing this, we use Lemma 5 on inequality (25). A computer search with *Mathematica* revealed that if $k \in [2, 230]$, then the maximum value of $\left\lfloor \frac{1}{\rho}\log\left(q^2 c/|\vartheta_2| X_0\right)\right\rfloor$ is 318, which is an upper bound on m according to Lemma 5. Hence, we deduce that the possible solutions (n, m, k) of the equation (3) for which $k \in [2, 230]$ have $m \le 318$, therefore we use inequalities (14) to obtain $n \le 401$.

Finally, we used *Mathematica* to compare $F_n^{(k)}$ and Le_m for the range $4 \le n \le 401$ and $2 \le m \le 318$, with $m < n/0.6$ and checked that the solutions of equation (3) are

$$F_4^{(2)} = 3 = \mathrm{Le}_2, \quad F_5^{(2)} = 5 = \mathrm{Le}_3, \quad \text{and} \quad F_6^{(4)} = 15 = \mathrm{Le}_5\,.$$

3.3 The case of large k

From now on, we assume that $k > 230$. Here, it follows from Lemma 10 that

$$0.6m < n < 8.35 \times 10^{15} k^4 \log^3 k < 2^{k/2}.$$

Using (8) and Lemma 9, we can write (3) as

$$2^{n-2} - \frac{2\alpha^{m+1}}{\sqrt{5}} = 2^{n-2}\xi - \frac{2\beta^{m+1}}{\sqrt{5}} - \frac{\alpha}{\sqrt{5}} + \frac{\beta}{\sqrt{5}}. \tag{26}$$

Taking absolute values on both sides of (26), we have that

$$\left| 2^{n-2} - \frac{2\alpha^{m+1}}{\sqrt{5}} \right| < \frac{2^{n-2}}{2^{k/2}} + \frac{5}{\sqrt{5}}.$$

Dividing both sides of the above inequality by 2^{n-2} and taking into account that $1/2^{n-2} < 1/2^{k/2}$ for $n \geq k+2$, we obtain

$$\left| 1 - 2^{-n} \frac{8}{\sqrt{5}} \alpha^{m+1} \right| < \frac{3.23}{2^{k/2}}. \tag{27}$$

Applying Theorem 2 for the left-hand side, we set

$$\Lambda_2 := 2^{-n} \frac{8}{\sqrt{5}} \alpha^{m+1} - 1.$$

Note that $\Lambda_2 \neq 0$. Indeed, if $\Lambda_2 = 0$, then $\alpha^{2(m+1)}$ is a rational number, which is not possible for all positive integers m. Therefore $\Lambda_2 \neq 0$. We take $t := 3$,

$$\eta_1 := 2, \quad \eta_2 := \frac{8}{\sqrt{5}}, \quad \eta_3 := \alpha,$$

and

$$b_1 := -n, \quad b_2 := 1, \quad b_3 := m+1.$$

Note that $\mathbb{K} := \mathbb{Q}(\alpha)$ contains η_1, η_2, η_3 and has $D := 2$. Since $m < 1.5n$, we deduce that $B := \max\{|b_1|, |b_2|, |b_3|\} = 1.5n$. The logarithmic heights for η_1, η_2, and η_3 are calculated as follows:

$$h(\eta_1) = \log 2, \quad h(\eta_2) = \log\left(8\sqrt{5}\right) \quad \text{and} \quad h(\eta_3) = \frac{\log \alpha}{2}.$$

Thus, we can take

$$A_1 := 2\log 2, \quad A_2 := 5.8 \quad \text{and} \quad A_3 := \log \alpha.$$

As before, by applying Theorem 2, we have

$$|\Lambda_2| > \exp\left(-8.27 \times 10^{12} \log n\right), \tag{28}$$

where $1+\log 1.5n < 2.1\log n$ holds for all $n \geq 4$. Comparing (27) and (28), we obtain

$$k < 2.39 \times 10^{13} \log n.$$

By Lemma 10 and using the fact $36.66 + 4\log k + 3\log\log k < 11.7\log k$ for all $k > 230$, we get

$$\begin{aligned} k &< 2.39 \times 10^{13} \log\left(8.35 \times 10^{15} k^4 \log^2 k\right) \\ &< (2.39 \times 10^{13})(36.66 + 4\log k + 3\log\log k) \\ &< 2.8 \times 10^{14} \log k. \end{aligned}$$

Solving the above inequality by using the relation (21) gives

$$k < 1.87 \times 10^{16}.$$

Again from Lemma 10, we obtain

$$n < 5.38 \times 10^{85} \quad \text{and} \quad m < 8.97 \times 10^{85}. \tag{29}$$

Let

$$\Gamma_2 := (m+1)\log\alpha - n\log 2 + \log\left(\frac{8}{\sqrt{5}}\right). \tag{30}$$

Using a similar method to show the inequality (25), one can see that

$$0 < |\Gamma_2| < \frac{6.46}{2^{k/2}} < 6.46 \times \exp(-0.34 \times k) \tag{31}$$

holds for all $k > 230$. The inequality (29) implies that we can take $X_0 := 8.97 \times 10^{85}$. Further, we choose

$$c := 6.46, \quad \rho := 0.34, \quad \psi := -\frac{\log\left(\frac{8}{\sqrt{5}}\right)}{\log 2},$$

and

$$\vartheta := \frac{\log\alpha}{\log 2}, \quad \vartheta_1 := \log\alpha, \quad \vartheta_2 := -\log 2, \quad \delta := \log\left(\frac{8}{\sqrt{5}}\right).$$

Using Lemma 3 with $c := 6.46$, $\rho := 0.34$ and $X_0 := 8.97 \times 10^{85}$, we get $Y_0 := 411.954\ldots$. Let

$$[a_0, a_1, a_2, \ldots] := [0, 1, 2, 3, 1, 2, 3, 2, 4, 2, 1, 2, 11, 2, 1, 11, 1, 1, 134, 2, 2, 2, 1, 4, 1, 1, 3, 1, \ldots]$$

be the continued fraction expansion of $\log\alpha/\log 2$. With the help of *Mathematica*, we find that

$$\max_{0 \leq k \leq Y_0} a_{k+1} = 880 := A.$$

Then by Lemma 4, we have

$$k < \frac{1}{0.34} \cdot \log\left(\frac{6.46 \cdot 882 \cdot 8.97 \cdot 10^{85}}{\log 2}\right) < 609.$$

With the above upper bound on k and by Lemma 10, we have

$$n < 3.1 \times 10^{29} \quad \text{and} \quad m < 5.2 \times 10^{29}. \tag{32}$$

We apply again Lemma 4 with $X_0 := 5.2 \times 10^{29}$. Hence by Lemma 3, we obtain $Y_0 := 142.863\ldots$ and $A := 134$. According to Lemma 4, it becomes

$$k < \frac{1}{0.34} \cdot \log\left(\frac{6.46 \cdot 136 \cdot 5.2 \cdot 10^{29}}{\log 2}\right) < 223,$$

which contradicts our assumption that $k > 230$. Hence, we have shown that there are no solutions (n, k, m) to equation (3) with $k > 230$. This completes the proof of Theorem 1. $\square$

4 Acknowledgments

The authors would like to thank the anonymous referee for their useful comments and suggestions that improved the exposition of the paper.

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2020 *Mathematics Subject Classification*: Primary 11B39; Secondary 11J86, 11R52.
Keywords: k-generalized Fibonacci number, Leonardo number, linear form in logarithms, reduction method.

(Concerned with sequences A000032, A000045, and A001595.)

Received January 31 2023; revised versions received February 8 2023; May 17 2023; May 26 2023. Published in *Journal of Integer Sequences*, June 12 2023.

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