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An Asymptotic Formula for the Number of Stabilized-Interval-Free Permutations

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Abstract

Callan introduced stabilized-interval-free (SIF) permutations, and Ardila, Rincón, and Williams showed that connected positroids bijectively correspond to SIF permutations unless the underlying set is a singleton. In this paper, we derive an approximate formula for the number of SIF permutations on the set $\{1, \ldots, N\}$ that refines the previous result conjectured by Callan and proved by Salvatore and Tauraso.

1 Introduction

Let $[N]$ denote the set $\{1, \ldots, N\}$ for a positive integer N. A *derangement* of the set $[N]$ is a permutation of the set without any fixed point. Let d_N denote the number of derangements on the set $[N]$. The following classical formula for the number d_N

$$d_N = N! \sum_{i=0}^{N} \frac{(-1)^i}{i!},$$

which is the sequence A000166 in the On-Line Encyclopedia of Integer Sequences (OEIS) [3], is well known. Hassani [4, Thm. 1.1] proved that $d_N = \lfloor \frac{N!}{e} + \alpha \rfloor$ for every real number $\alpha \in [\frac{1}{3}, \frac{1}{2}]$. This implies that the following approximate equality

$$d_N = \frac{N!}{e} + O(1)$$

holds. Moreover, Hassani [5] showed that the coefficients of the lower degree terms of this approximation are alternating Bell numbers B_k

$$d_N = \frac{N!}{e} + \sum_{k=1}^{r} (-1)^{N+k+1} \frac{B_k}{N^k} + O(N^{-r-1}).$$

These results all imply that the proportion of derangements among all permutations of the set $[N]$ is given by the following expression

$$\frac{d_N}{N!} = \frac{1}{e} + O(N^{-k})$$

where k is any positive integer.

Callan [2] introduced a subclass of derangements known as *stabilized-interval-free (SIF)* permutations, which satisfy a stronger condition than ordinary derangements. Callan proposed a conjectural asymptotic proportion of SIF permutations among all permutations of the set $[N]$. Specifically, it is conjectured that the proportion of SIF permutations among all permutations of $[N]$ is

$$\frac{1}{e}\left(1 - \frac{1}{N} - \frac{5}{2N^2} + O(N^{-3})\right).$$

Salvatore and Tauraso [8] proved this conjecture using the recursive formula for the number of SIF permutations.

SIF permutations are related to mathematical objects called positroids. Postnikov [7] introduced positroids, which are matroids represented by a real matrix with nonnegative maximal minors, and showed that positroids bijectively correspond to various combinatorial objects such as decorated permutations, Grassmann necklaces, Ꞁ-diagrams, and move-equivalence classes of reduced plabic graphs. Ardila, Rincón, and Williams [1] showed that connected components of positroids form a noncrossing partition of the underlying set. Moreover, they showed that a positroid on a set of cardinality at least 2 is connected if and only if it corresponds to a stabilized-interval-free (SIF) permutation. This result implies that the proportion of connected positroids among all positroids on the set $[N]$ approaches $\frac{1}{e^2}$ as $N \to \infty$. As pointed out in [1], this result stands in stark contrast to the fact, proven by Lowrance, Oxley, Semple, and Welsh in [6], that asymptotically almost all matroids are connected.

We present an alternative method for obtaining an asymptotic formula for the proportion of SIF permutations among all permutations, based on combinatorial arguments. Using our method, we may further compute lower degree terms of the asymptotic formula for the proportion of SIF permutations as follows:

$$\frac{1}{e}\left(1 - \frac{1}{N} - \frac{5}{2N^2} - \frac{32}{3N^3} - \frac{1643}{24N^4} - \frac{23017}{40N^5} + O(N^{-6})\right).$$

2 The number of λ-cyclic intervals

A *cyclic interval* of $[N]$ is a subset of consecutive numbers modulo N. More generally, we define the notion of λ-cyclic interval associated with a partition λ.

Definition 1. For a partition $\lambda = (\lambda_1, \dots, \lambda_l) \vdash n$, a collection of disjoint cyclic intervals is called a λ-cyclic interval if the partition obtained by sizes of cyclic intervals is λ.

The following proposition provides a formula for the number of λ-cyclic intervals in the set $[N]$.

Proposition 2. *Let $\lambda = (\lambda_1, ..., \lambda_l) \vdash n$ be a partition that is not the partition (N) with only one part. The number of λ-cyclic intervals in the set $[N]$ is expressed as follows:*

$$N \cdot \binom{N-n+l-1}{l-1} \frac{(l-1)!}{\prod_i m_i(\lambda)!}$$

where $m_i(\lambda)$ is the number of indices j such that $\lambda_j = i$.

Proof. Let $\alpha = (\alpha_1, \dots, \alpha_l) \vDash n$ be a composition whose weakly decreasing rearrangement is λ. There are $\frac{l!}{\prod_i m_i(\lambda)!}$ possible number of such compositions α. The number of ways to choose the first cyclic interval of length α_1 is N, and the number of ways to choose other intervals of length $\alpha_2, \dots, \alpha_l$ according to this order is $\binom{N-n+l-1}{l-1}$. Multiplying the number of choices in each step, the total number of choices is

$$N \cdot \binom{N-n+l-1}{l-1} \frac{l!}{\prod_i m_i(\lambda)!}$$

Since each λ-cyclic interval is counted exactly l times, dividing the total count by l gives us the desired result.

□

3 The number of SIF permutations stabilizing a λ-cyclic interval

Callan [2] introduced a subclass of derangements known as *stabilized-interval-free* permutations, which satisfy a stronger condition than ordinary derangements.

Definition 3. A permutation π of the set $[N]$ is stabilized-interval-free (SIF) if for every proper interval $I \subset [N]$, $\pi(I) \neq I$.

We define sif_N to be the number of SIF permutations of the set $[N]$. For each permutation π of the set $[N]$, we let $\mathcal{I}_\pi$ denote the collection of all minimal nonempty cyclic intervals stabilized by π. The collection of lengths of cyclic intervals in $\mathcal{I}_\pi$, arranged in weakly decreasing order, forms a partition of the integer N. By definition, a permutation π is a SIF permutation if and only if the partition obtained from the collection of lengths of cyclic intervals in $\mathcal{I}_\pi$ is (N). A direct consequence of Proposition 2 is a formula for the number of pairs $(\pi, \mathcal{J})$, where π is a derangement on the set $[N]$, and $\mathcal{J}$ is a λ-cyclic interval contained in the collection of cyclic intervals $\mathcal{I}_\pi$.

Corollary 4. *Let $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$ be a partition that is not (N) and let sif_λ denote $\prod_{i=1}^{l} \mathrm{sif}_{\lambda_i}$. Let $D_\lambda(N)$ be the collection of all pairs $(\pi, \mathcal{J})$ of a derangement π of the set $[N]$ and a λ-cyclic interval $\mathcal{J}$ contained in the collection $\mathcal{I}_\pi$. The number of pairs $(\pi, \mathcal{J}) \in D_\lambda(N)$ is obtained by the following formula:*

$$|D_\lambda(N)| = \mathrm{sif}_\lambda \cdot d_{N-n} \cdot N \cdot \binom{N-n+l-1}{l-1} \frac{(l-1)!}{\prod_i m_i(\lambda)!}.$$

This formula can be used to derive an asymptotic formula for the number $|D_\lambda(N)|$ as follows:

$$|D_\lambda(N)| = \frac{N!}{e} \cdot \left(\frac{\mathrm{sif}_\lambda}{\prod_i m_i(\lambda)!} \cdot \frac{1}{(N-1)(N-2)\cdots(N-n+l)} + O(N^{-k}) \right) \tag{1}$$

where k is an arbitrary positive integer.

Proof. Since every cyclic intervals in $\mathcal{I}_\pi$ is minimal stabilized cyclic intervals of π, a pair $(\pi, \mathcal{J})$ satisfying the relation $\mathcal{J} \subseteq \mathcal{I}_\pi$ is determined by a λ-cyclic interval $\mathcal{J}$, SIF permutations of each cyclic intervals in $\mathcal{J}$, and a derangement of the set $[N] \setminus (\bigcup \mathcal{J})$. Therefore the first statement is immediate from Proposition 2. The second statement is obtained by substituting the formula $d_{N-n} = \frac{(N-n)!}{e}(1 + O(N^{-k}))$ for the number of derangements where k is an arbitrary positive integer.

□

4 Asymptotic formula for the number of SIF permutations

We prove the main result of this paper in this section. Using the result, we derive a more precise asymptotic formula for the number of SIF permutations. For a partition λ, let $|\lambda|$ denote the sum of all parts of λ and let $l(\lambda)$ denote the number of parts of λ.

Theorem 5. *For a given positive integer k, the number of SIF permutations of the set $[N]$ is expressed by the following formula:*

$$\mathrm{sif}_N = \frac{N!}{e} \left(\sum_\lambda \frac{\mathrm{sif}_\lambda (-1)^{l(\lambda)}}{(\prod_i m_i(\lambda)!)(N-1)(N-2)\cdots(N-|\lambda|+l)} + O(N^{-k-1}) \right)$$

where the sum runs over all partitions $\lambda = (\lambda_1, \ldots, \lambda_{l(\lambda)})$ *such that* $|\lambda| - l(\lambda) \le k$ *and* $1 < \lambda_{l(\lambda)}$. *If the partition* λ *is the empty partition of the integer* 0, *we treat the term*

$$\frac{\mathrm{sif}_\lambda(-1)^l}{(\prod_i m_i(\lambda)!)(N-1)(N-2)\cdots(N-|\lambda|+l)}$$

as a vacuous product with a value of 1.

Proof. By the inclusion-exclusion principle, the number of SIF permutations on the set $[N]$ is given by the sum $\sum_\lambda (-1)^{l(\lambda)}|D_\lambda(N)|$ for all partitions λ such that $|\lambda| \le N, \lambda \ne (N)$. Corollary 4 implies that $|D_\lambda(N)| = \frac{N!}{e} \cdot O(N^{-k-1})$ unless $|\lambda| - l(\lambda) \le k$. Therefore, the sum can be expressed by using the big O notation

$$\mathrm{sif}_N = \sum_{|\lambda|-l(\lambda)\le k} (-1)^{l(\lambda)}|D_\lambda(N)| + \frac{N!}{e} \cdot O(N^{-k-1}).$$

Substituting the asymptotic formula (1) for the number $|D_\lambda(N)|$, the result follows. □

To obtain an explicit formula for the asymptotic number of SIF permutations, we make use of a well-known partial fraction decomposition formula for the inverse of the falling factorial $(x)_n := x(x-1)\cdots(x-n+1)$. This formula is given by:

$$\frac{(n-1)!}{(x)_n} = \frac{(n-1)!}{x(x-1)\cdots(x-n+1)} = \sum_{i=0}^{n-1} \frac{(-1)^{n-1+i}\binom{n-1}{i}}{x-i}, \tag{2}$$

which can be derived by induction.

To compute the asymptotic formula up to the error term $O(N^{-k-1})$, it suffices to consider partitions λ for which $|\lambda| - l(\lambda) \le k$ by Theorem 5. We will do this for $k = 5$. The condition $|\lambda| - l(\lambda) = 1, 2, 3, 4, 5$ is satisfied by the following collections of partitions:

$$\begin{aligned}
&|\lambda| - l(\lambda) = 1 : \{(2)\} \\
&|\lambda| - l(\lambda) = 2 : \{(3), (2,2)\} \\
&|\lambda| - l(\lambda) = 3 : \{(4), (3,2), (2,2,2)\} \\
&|\lambda| - l(\lambda) = 4 : \{(5), (4,2), (3,3), (3,2,2), (2,2,2,2)\} \\
&|\lambda| - l(\lambda) = 5 : \{(6), (5,2), (4,3), (4,2,2), (3,3,2), (3,2,2,2), (2,2,2,2,2)\}.
\end{aligned}$$

Notice that each partition λ satisfying $|\lambda| - l(\lambda) = k$ corresponds to a partition of k, obtained by subtracting 1 from each part of the partition. For each of these collections of partitions, we can compute the sum

$$\sum_\lambda \frac{\mathrm{sif}_\lambda(-1)^{l(\lambda)}}{\prod_i m_i(\lambda)!}$$

using the sequence sif_N listed as entry A075834 in Sloane's OEIS [3] as follows:

$$
\begin{aligned}
|\lambda| - l(\lambda) = 1: &\ \sum_{\lambda} \frac{\mathrm{sif}_\lambda (-1)^{l(\lambda)}}{\prod_i m_i(\lambda)!} = 1 \\
|\lambda| - l(\lambda) = 2: &\ \sum_{\lambda} \frac{\mathrm{sif}_\lambda (-1)^{l(\lambda)}}{\prod_i m_i(\lambda)!} = -2 + \frac{1^2}{2!} = -\frac{3}{2} \\
|\lambda| - l(\lambda) = 3: &\ \sum_{\lambda} \frac{\mathrm{sif}_\lambda (-1)^{l(\lambda)}}{\prod_i m_i(\lambda)!} = -7 + 2 \cdot 1 - \frac{1^3}{3!} = -\frac{31}{6} \\
|\lambda| - l(\lambda) = 4: &\ \sum_{\lambda} \frac{\mathrm{sif}_\lambda (-1)^{l(\lambda)}}{\prod_i m_i(\lambda)!} = -34 + 7 + \frac{2^2}{2!} - \frac{2 \cdot 1^2}{2!} + \frac{1^4}{4!} = -\frac{623}{24} \\
|\lambda| - l(\lambda) = 5: &\ \sum_{\lambda} \frac{\mathrm{sif}_\lambda (-1)^{l(\lambda)}}{\prod_i m_i(\lambda)!} = -206 + 34 + 7 \cdot 2 - \frac{7 \cdot 1^2}{2!} - \frac{2^2 \cdot 1}{2!} + \frac{2 \cdot 1^3}{3!} - \frac{1^5}{5!} \\
&\qquad\qquad\qquad\quad = -\frac{6527}{40}.
\end{aligned}
$$

Here we used the values of sif_N for $N = 1, \ldots, 6$, which are $1, 1, 2, 7, 34, 206$. Therefore, we have the following asymptotic formula for the number sif_N up to the error term $O(N^{-6})$:

$$
\begin{aligned}
\mathrm{sif}_N = \frac{N!}{e} \Bigg(1 &- \frac{1}{N-1} \\
&- \frac{3}{2} \frac{1}{(N-1)(N-2)} \\
&- \frac{31}{6} \frac{1}{(N-1)(N-2)(N-3)} \\
&- \frac{623}{24} \frac{1}{(N-1)(N-2)(N-3)(N-4)} \\
&- \frac{6527}{40} \frac{1}{(N-1)(N-2)(N-3)(N-4)(N-5)} + O(N^{-6}) \Bigg)
\end{aligned}
$$

We can expand the above formula using the partial fraction decomposition formula 2 for the inverse of the falling factorial and the series expansion

$$
\frac{1}{N-m} = \sum_{i=1}^{\infty} \frac{m^{i-1}}{N^i}.
$$

$$
\begin{aligned}
\mathrm{sif}_N = \frac{N!}{e}\Bigg(1 &- \left(\frac{1}{N}+\frac{1}{N^2}+\frac{1}{N^3}+\cdots\right) \\
&- \frac{3}{2}\left(\left(\frac{1}{N}+\frac{2}{N^2}+\frac{2^2}{N^3}+\cdots\right)-\left(\frac{1}{N}+\frac{1}{N^2}+\frac{1}{N^3}+\cdots\right)\right) \\
&- \frac{31}{6}\frac{1}{2!}\left(\left(\frac{1}{N}+\frac{3}{N^2}+\frac{3^2}{N^3}+\cdots\right)-\binom{2}{1}\left(\frac{1}{N}+\frac{2}{N^2}+\frac{2^2}{N^3}+\cdots\right)\right. \\
&\left.+\left(\frac{1}{N}+\frac{1}{N^2}+\frac{1}{N^3}+\cdots\right)\right) \\
&- \frac{623}{24}\frac{1}{3!}\left(\left(\frac{1}{N}+\frac{4}{N^2}+\frac{4^2}{N^3}+\cdots\right)-\binom{3}{2}\left(\frac{1}{N}+\frac{3}{N^2}+\frac{3^2}{N^3}+\cdots\right)\right. \\
&\left.+\binom{3}{1}\left(\frac{1}{N}+\frac{2}{N^2}+\frac{2^2}{N^3}+\cdots\right)-\left(\frac{1}{N}+\frac{1}{N^2}+\frac{1}{N^3}+\cdots\right)\right) \\
&- \frac{6527}{40}\frac{1}{4!}\left(\left(\frac{1}{N}+\frac{5}{N^2}+\frac{5^2}{N^3}+\cdots\right)-\binom{4}{3}\left(\frac{1}{N}+\frac{4}{N^2}+\frac{4^2}{N^3}+\cdots\right)\right. \\
&+\binom{4}{2}\left(\frac{1}{N}+\frac{3}{N^2}+\frac{3^2}{N^3}+\cdots\right)-\binom{4}{1}\left(\frac{1}{N}+\frac{2}{N^2}+\frac{2^2}{N^3}+\cdots\right) \\
&\left.+\left(\frac{1}{N}+\frac{1}{N^2}+\frac{1}{N^3}+\cdots\right)\right)+O(N^{-6})\Bigg)
\end{aligned}
$$

Therefore, the coefficients of the terms $\frac{1}{N^k}$ for $k=1,2,3,4,5$ are given by:

$$
\begin{aligned}
k=1 &: 1 \\
k=2 &: -1-\frac{3}{2}(2-1) = -\frac{5}{2} \\
k=3 &: -1-\frac{3}{2}(2^2-1)-\frac{31}{6\cdot 2!}(3^2-\binom{2}{1}2^2+1) = -\frac{32}{3} \\
k=4 &: -1-\frac{3}{2}(2^3-1)-\frac{31}{6\cdot 2!}(3^3-\binom{2}{1}2^3+1)-\frac{623}{24\cdot 3!}(4^3-\binom{3}{2}3^3+\binom{3}{1}2^3-1) \\
&= -\frac{1643}{24} \\
k=5 &: -1-\frac{3}{2}(2^4-1)-\frac{31}{6\cdot 2!}(3^4-\binom{2}{1}2^4+1)-\frac{623}{24\cdot 3!}(4^4-\binom{3}{2}3^4+\binom{3}{1}2^4-1) \\
&-\frac{6527}{40\cdot 4!}(5^4-\binom{4}{3}4^4+\binom{4}{2}3^4-\binom{4}{1}2^4+1) = -\frac{23017}{40}.
\end{aligned}
$$

This gives the following asymptotic formula for sif_N:

$$\mathrm{sif}_N = \frac{N!}{e}\left(1 - \frac{1}{N} - \frac{5}{2N^2} - \frac{32}{3N^3} - \frac{1643}{24N^4} - \frac{23017}{40N^5} + O(N^{-6})\right).$$

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