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The 3-Additive Uniqueness of Generalized Pentagonal Numbers for Multiplicative Functions

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Abstract

The set $\mathcal{P}$ of all nonzero generalized pentagonal numbers is k-additively unique for multiplicative functions for fixed $k \geq 2$. That is, if a multiplicative function f satisfies the condition

$$f(a_1 + a_2 + \cdots + a_k) = f(a_1) + f(a_2) + \cdots + f(a_k)$$

for arbitrary $a_1, a_2, \ldots, a_k \in \mathcal{P}$, then f is the identity function. The known proof for $k = 3$ uses the generalized Riemann hypothesis (GRH). Here, we give a proof bypassing GRH.

1 Introduction

In 1992 Spiro [5] coined the term *additive uniqueness* to describe a curious property of a set $E \subset \mathbb{N}$ for a set S of arithmetic functions. If a function $f \in S$ satisfying the condition

$$f(x + y) = f(x) + f(y) \text{ for all } a, b \in E$$

is uniquely determined, she called E an additive uniqueness set for S.

Many mathematicians have been studying and generalizing her seminal work. The concept of *k-additive uniqueness* is a generalization. A set E is called a k-additive uniqueness set for S, if a function $f \in S$ satisfying the condition

$$f(x_1 + x_2 + \cdots + x_k) = f(x_1) + f(x_2) + \cdots + f(x_k) \text{ for all } x_i \in E$$

is uniquely determined.

In 2018, the author and colleagues [2] proved that the set

$$\mathcal{P} = \left\{ P_m = \frac{m(3m-1)}{2} \,\middle|\, m \in \mathbb{Z}, m \neq 0 \right\} = \{1, 2, 5, 7, 12, 15, \ldots\}$$

of all nonzero generalized pentagonal numbers is an additive uniqueness set for multiplicative functions. Here, a multiplicative function refers to an arithmetic function $f : \mathbb{N} \to \mathbb{C}$ satisfying $f(1) = 1$ and $f(ab) = f(a)f(b)$ for all a, b with $\gcd(a, b) = 1$. In addition, they [3] extended this result to the set

$$\left\{ P_m = \frac{m(3m-1)}{2} \,\middle|\, m \in \mathbb{Z}, m \geq 1 \right\} = \{1, 5, 12, 22, 35, 51, \ldots\}$$

of nonzero ordinary pentagonal numbers.

In 2022, Hasanalizade [1] extended the author's finding for generalized pentagonal numbers to k-additive uniqueness with $k \geq 3$. However, his proof for the case of $k = 3$ relies on the proposition that every integer > 2 can be written as a sum of three nonzero generalized pentagonal numbers [4, Theorem 4.3], which requires the generalized Riemann hypothesis (GRH). In this article, we present an elementary proof that does not rely on GRH.

2 Main result

Theorem 1. *Let f be a multiplicative function and let $\mathcal{P}$ be the set of nonzero generalized pentagonal numbers. If*

$$f(a + b + c) = f(a) + f(b) + f(c)$$

for every $a, b, c \in \mathcal{P}$, then f is the identity function.

Proof. We have $f(1) = 1$ and thus $f(3) = 3$. Note that

$$\begin{aligned} f(5) &= f(1) + f(2) + f(2) = 1 + 2\,f(2), \\ f(7) &= f(1) + f(1) + f(5) = 3 + 2f\,f(2). \end{aligned}$$

Then $f(2) = 2$ by

$$\begin{aligned} f(9) &= f(2) + f(2) + f(5) = 1 + 4\,f(2) \\ &= f(1) + f(1) + f(7) = 5 + 2\,f(2). \end{aligned}$$

So, clearly, $f(n) = n$ for $n \leq 9$.

We use induction. Assume that $f(n) = n$ for all $n < N$.

If $N = ab$ with $a, b \geq 2$ and $\gcd(a, b) = 1$, then $f(N) = f(a)\,f(b) = ab$ by the induction hypothesis. So it suffices to consider the case when N is a power of a prime.

Consider the case of $N = 2^r$. We have that $2^r = 3m \pm 1$ for some m according to the parity of r.

If $N = 2^{2s+1}$, then $N = 3m - 1$ for some $m \in \mathbb{N}$. Note that

$$\begin{aligned}
&P_m + P_{-2m+1} + P_{-2m+1} \\
&= m \cdot \frac{3m-1}{2} + (2m-1)(3m-1) + (2m-1)(3m-1) \\
&= \frac{3m-1}{2} \cdot (9m-4)
\end{aligned}$$

and

$$\begin{aligned}
&P_{2m-1} + P_{-2m+1} + P_{-2m+1} \\
&= (2m-1)(3m-2) + (2m-1)(3m-1) + (2m-1)(3m-1) \\
&= (2m-1)(9m-4).
\end{aligned}$$

Since each factor $< 3m - 1$ is fixed by f by the induction hypothesis, letting $x = f(3m-1)$ and $y = f(9m-4)$, we have that

$$\begin{aligned}
m \cdot \frac{3m-1}{2} + 2(2m-1)\,x &= \frac{3m-1}{2}\,y \\
(2m-1)(3m-2) + 2(2m-1)\,x &= (2m-1)\,y.
\end{aligned}$$

Thus $x = 3m - 1$ and $y = 9m - 4$. So $f(2^{2s+1}) = 2^{2s+1}$.

If we change m to $-m$, then we can show that f fixes $9m + 4$ and $N = 2^{2s} = 3m + 1$ in the same way.

Now suppose $N = 3^r$. Then

$$P_{3^{r-1}} + P_{3^{r-1}} + P_{3^{r-1}} = 3^{r-1} \cdot \frac{3^r - 1}{2} + 3^{r-1} \cdot \frac{3^r - 1}{2} + 3^{r-1} \cdot \frac{3^r - 1}{2} \qquad = 3^r \cdot \frac{3^r - 1}{2},$$

and thus $f(3^r) = 3^r$.

Finally, let $N = p^r$ with odd prime $p \neq 3$. Assume $N = 6m - 1$ for some m. Note that

$$\begin{aligned}
P_m + P_{-m} + P_{2m} &= \frac{m(3m-1)}{2} + \frac{m(3m+1)}{2} + m(6m-1) \\
&= m(9m-1)
\end{aligned}$$

and

$$\begin{aligned}
P_{-m} + P_{2m} + P_{2m} &= \frac{m(3m+1)}{2} + m(6m-1) + m(6m-1) \\
&= \frac{3m(9m-1)}{2}.
\end{aligned}$$

The first expression leads to an equation

$$3m + x = y$$

where $x = f(6m-1)$ and $y = f(9m-1)$.

For the second expression, there are two cases according to m. If $9m-1$ is even, then m is odd and

$$f\left(\frac{3m(9m-1)}{2}\right) = f(3m)\, f\left(\frac{9m-1}{2}\right) = 3m \cdot \frac{9m-1}{2},$$

since $3m$ and $\frac{9m-1}{2}$ are smaller than $N = 6m-1$. If $9m-1$ is odd, then m is even and

$$f\left(\frac{3m(9m-1)}{2}\right) = f\left(\frac{3m}{2}\right) f(9m-1) = \frac{3m}{2} \cdot f(9m-1) = \frac{3m}{2} \cdot y.$$

In both cases, we conclude that $x = f(6m-1) = 6m-1$ and $y = f(9m-1) = 9m-1$.

Similarly, we can show that $f(p^r) = p^r$ by changing m to $-m$, when $p^r = 6m+1$. The proof is complete. $\square$

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