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On Inequalities Related to a Generalized Euler Totient Function and Lucas Sequences

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Abstract

Let $\varphi(n)$ be the Euler totient function of n, defined as the number of positive integers less than or equal to n that are co-prime with n. In this paper, we consider the function φ_k, a generalization of φ, and establish some inequalities related to Lucas sequences of the first kind $\left(U_n\right)_{n\geq 1}$ with characteristic equation having real roots. As an application to these inequalities, we further establish inequalities related to Fibonacci, Pell, and balancing sequences.

1 Introduction

Mathematicians have long been interested in studying arithmetic functions and the associated problems in number theory. These problems include solving Diophantine equations, exploring generalizations, and investigating divisibility concerned with arithmetic functions (see [3, 4, 5, 9, 10, 15, 16] and references therein). The Euler totient function $\varphi(n)$ counts the number of positive integers less than or equal to n that are co-prime with n. Additionally, $\sigma_k(n)$ represents the sum of the k^{th} power of the positive divisors of n, where k is a non-negative integer. Also $\sigma_k(n)$ reduces to $\tau(n)$ for $k = 0$, where $\tau(n)$ denotes the number of positive divisors of n. These arithmetic functions are primarily utilized in the field of number theory. In recent times, numerous researchers have worked towards examining arithmetic functions connected to binary recurrence sequences like the Fibonacci sequence, Lucas sequence, Pell sequence, and balancing sequence (see [1, 5, 9, 10, 11, 12, 13, 14]).

The study of inequalities related to arithmetic functions and binary recurrence sequences is one of the interesting problems, and many researchers have contributed in this direction. Before mentioning some recent results, we first define *Lucas sequence of the first kind*.

Let $r = \alpha + \beta$ and $s = -\alpha\beta$ be two non-zero co-prime integers with $\Delta = r^2 + 4s > 0$. Then α and β are roots of the quadratic equation $x^2 - rx - s = 0$. For any non-negative integer n, define

$$U_n \coloneqq U_n(r,s) = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

The sequence $\big(U_n\big)_{n\geq 0}$ is called a *Lucas sequence of the first kind* and quadratic equation $x^2 - rx - s = 0$ is the corresponding characteristic equation of $\big(U_n\big)_{n\geq 0}$. From the definition, it is clear that $U_0 = 0$, $U_1 = 1$ and that

$$U_{n+2} = rU_{n+1} + sU_n,\ n = 0, 1, 2, \ldots.$$

In 1997, Luca [7] showed that the Euler totient function for Lucas sequence of the first kind $\big(U_n\big)_{n\geq 1}$ satisfies the inequality $\varphi(|U_n|) \geq |U_{\varphi(n)}|$ for those sequences whose characteristic equation has real roots, and the inequality is not valid for the sequences with characteristic equation having complex roots. In [8], Luca proved that the n^{th} Fibonacci number satisfies $\sigma_k(F_n) \leq F_{\sigma_k(n)}$ and $\tau(F_n) \geq F_{\tau(n)}$ which are extended to the case of balancing numbers by Sahukar and Panda [17]. There are many generalizations and analogs of the Euler totient function [18]. For example, φ_k is defined as

$$\varphi_k(n) = \sum_{1\leq m<n,(m,n)=1} m^k,$$

where $k \geq 0$ is an integer, and the Jordan totient function J_k is defined as

$$J_k(n) = n^k \prod_{p|n}\left(1 - \frac{1}{p^k}\right),$$

where k is a positive integer and p is a prime number (see [18, 2]). Clearly, we have $\varphi_0(n) = \varphi(n) = J_1(n)$.

In 2019, Jaidee and Pongsriiam [6] showed that for every natural number n, the n^{th} Fibonacci number satisfies $\varphi_k(F_n) \leq F_{\varphi_k(n)}$ except for $(n,k) = (6,1)$ for which the inequality is reversed, and $J_k(F_n) \leq F_{J_k(n)}$ for all $n \geq 1$. They also established similar results for Lucas sequence in the same article. Motivated by the work of [6] and [7], we establish inequalities related with φ_k and Lucas sequences of the first kind $\big(U_n\big)_{n\geq 1}$ with characteristic equation having real roots. As an application, we also establish inequalities related to Fibonacci, Pell, and balancing sequences. More precisely, we prove the following result:

Theorem 1. *Let n and k be positive integers, φ_k be the generalized Euler totient function, and $\left(U_n\right)_{n\geq 1}$ be Lucas sequence of the first kind with α, β as non-zero real roots of the corresponding characteristic equation. Then the following statements hold.*

(i) For every $k \geq 1$, $\varphi_k(|U_1|) = |U_{\varphi_k(1)}|$, and if $|U_2| = 1$ or 2, then $\varphi_k(|U_2|) = |U_{\varphi_k(2)}|$ otherwise $\varphi_k(|U_2|) > |U_{\varphi_k(2)}|$.

(ii) $\varphi_1(|U_n|) < |U_{\varphi_1(n)}|$ whenever $n \geq 16$.

(iii) $\varphi_k(|U_n|) < |U_{\varphi_k(n)}|$ for every $n \geq 3$ and $k \geq 2$ except for $(n, k) = (3, 2), (3, 3), (3, 4), (4, 2)$.

Remark 2. In Theorem 1, the inequality may or may not hold for $(n, 1)$ where $3 \leq n \leq 15$, and $(n, k) = (3, 2), (3, 3), (3, 4), (4, 2)$. Therefore, the inequality can be checked for every Lucas sequence of the first kind just by computing for the values mentioned above.

As a consequence of Theorem 1, we obtain the result of Jaidee and Pongsriiam [6, Theorem 3.2(i)] concerning the Fibonacci sequence. Additionally, we establish a set of inequalities associated with the Pell sequence and the balancing sequence within the theorem, which can be expressed as follows:

Theorem 3. *Let n and k be positive integers, φ_k be the generalized Euler totient function, and $\left(F_n\right)_{n\geq 1}$, $\left(P_n\right)_{n\geq 1}$, and $\left(B_n\right)_{n\geq 1}$ are Fibonacci sequence, Pell sequence and balancing sequence, respectively. Then the following statements hold.*

(i) $\varphi_k(F_n) \leq F_{\varphi_k(n)}$ for all $n \geq 1$ except for $(n, k) = (6, 1)$ for which the inequality reverses, and equality holds for $(n, k) = (1, k), (2, k), (4, 1)$, where $k \geq 1$.

(ii) $\varphi_k(P_n) \leq P_{\varphi_k(n)}$ for all $n \geq 1$ except for $(n, k) = (3, 1), (4, 1), (6, 1), (3, 2)$ for which the inequality reverses, and equality holds for $(n, k) = (1, k), (2, k)$, where $k \geq 1$.

(iii) $\varphi_k(B_n) \leq B_{\varphi_k(n)}$ for all $n \geq 1$ except for $(n, k) = (2, k), (3, 1), (4, 1), (6, 1), (12, 1), (3, 2)$ for which the inequality reverses, and equality holds for $(n, k) = (1, k)$, where $k \geq 1$.

2 Some examples and related results

In this section, we first give some examples of Lucas sequences of the first kind and some results which are needed to prove our main theorems.

Example 4 (Lucas sequences of the first kind)**.** Some examples of Lucas sequences of the first kind are as follows:

(i) The Fibonacci sequence $\left(F_n\right)_{n\geq 0}$ defined as

$$F_0 = 0, F_1 = 1 \text{ and } F_{n+2} = F_{n+1} + F_n,\ n \geq 0,$$

is the Lucas sequence of the first kind with $r = 1$ and $s = 1$ (see [19, A000045]). The characteristic equation of $\left(F_n\right)_{n\geq 0}$ is $x^2 - x - 1 = 0$ with roots $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

(ii) The Pell sequence $\left(P_n\right)_{n\geq 0}$ defined as

$$P_0 = 0, P_1 = 1 \text{ and } P_{n+2} = 2P_{n+1} + P_n,\ n \geq 0,$$

is the Lucas sequence of the first kind with $r = 2$ and $s = 1$ (see [19, A000129]). The characteristic equation of $\left(P_n\right)_{n\geq 0}$ is $x^2 - 2x - 1 = 0$ with roots $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$.

(iii) The balancing sequence $\left(B_n\right)_{n\geq 0}$ defined as

$$B_0 = 0, B_1 = 1 \text{ and } B_{n+2} = 6B_{n+1} - B_n,\ n \geq 0,$$

is the Lucas sequence of the first kind with $r = 6$ and $s = -1$ (see [19, A001109]). The characteristic equation of $\left(B_n\right)_{n\geq 0}$ is $x^2 - 6x + 1 = 0$ with roots $\alpha = 3 + \sqrt{2}$ and $\beta = 3 - \sqrt{2}$.

Throughout this article, we consider the Lucas sequences of the first kind for which α, β are real roots of the corresponding characteristic equation such that $|\alpha| > |\beta| > 0$. Further, we obtain

$$|\alpha| = \frac{|r| + \sqrt{\Delta}}{2},\ |\beta| = \frac{||r| - \sqrt{\Delta}|}{2}.$$

We now give results related with Lucas sequences of the first kind which will be used to prove our main results:

Lemma 5. *[7, Lemma 1] Let $\left(U_n\right)_{n\geq 1}$ be Lucas sequence of the first kind, where n is positive integer and α, β be non-zero real roots of the corresponding characteristic equation. Then the following statements hold.*

(i) If $|\alpha| \neq \frac{1+\sqrt{5}}{2}$, then $|\alpha| \geq 2$.

(ii) $|U_n| < 2|\alpha|^n$.

(iii) If $0 < m < n$ and m is even, then $|\frac{U_n}{U_m}| > |\alpha|^{n-m}$.

Lemma 6. *Let Fibonacci sequence $\left(F_n\right)_{n\geq 0}$ as defined in Example 4 with $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ as real roots of the corresponding characteristic equation. Then the following statements hold.*

(i) (Binet formula) $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ for all $n \geq 0$.

(ii) $\alpha^{n-2} \leq F_n \leq \alpha^{n-1}$ for all $n \geq 0$.

Proof. The proof of statement (i) follows directly from the definition of the Lucas sequence of the first kind and statement (ii) can be proved by mathematical induction. □

Using the Lemma 5 and Lemma 6, we establish the following result:

Lemma 7. *Let $\left(U_n\right)_{n\geq 1}$ be Lucas sequence of the first kind and α, β be non-zero real roots of the corresponding characteristic equation. Then the following statements hold.*

(i) If $|\alpha| = \frac{1+\sqrt{5}}{2}$, then $|U_n| = F_n$ and $|\alpha|^{n-2} \leq |U_n| \leq |\alpha|^{n-1}$.

(ii) If $|\alpha| \geq 2$, then $|\alpha|^{n-2} < |U_n| < |\alpha|^{n+1}$.

Proof.

(i) Let $|\alpha| = \frac{1+\sqrt{5}}{2}$. Since α is a real number, either $\alpha = \frac{1+\sqrt{5}}{2}$ or $\alpha = -\left(\frac{1+\sqrt{5}}{2}\right)$. Now if $\alpha = \frac{1+\sqrt{5}}{2}$, then $\beta = \frac{1-\sqrt{5}}{2}$ is also a root of the characteristic equation. Therefore, $|U_n| = |F_n| = F_n$. Similarly, if $\alpha = -\left(\frac{1+\sqrt{5}}{2}\right)$, then $\beta = \frac{-1+\sqrt{5}}{2} = -\left(\frac{1-\sqrt{5}}{2}\right)$ is also a root of the characteristic equation. Therefore, in this case we have

$$|U_n| = |(-1)^{n-1}F_n| = |F_n| = F_n.$$

Hence, for either case using Lemma 6 (ii), we have $|\alpha|^{n-2} \leq |U_n| \leq |\alpha|^{n-1}$. This completes the proof of (i).

(ii) Consider $|\alpha| \geq 2$. Using Lemma 5 (ii), we have $|U_n| < 2|\alpha|^n \leq |\alpha||\alpha|^n = |\alpha|^{n+1}$. Now using Lemma 5 (iii) for $m = 2$, we have $|U_n| \geq |\frac{U_n}{U_2}| > |\alpha|^{n-2}$. On combining the above two inequalities, we conclude that $|\alpha|^{n-2} < |U_n| < |\alpha|^{n+1}$.

□

Now we establish some results related with the Euler totient function and the generalized Euler totient function φ_k.

Lemma 8. *[6, Lemma 2.5] Let $n \geq 3$. Then the following statements hold.*

(i) If $k = 1$, then $\varphi_k(n) = \dfrac{n\varphi(n)}{2}$.

(ii) If $k \geq 2$, then $\dfrac{n^k\varphi(n)}{2^k} < \varphi_k(n) < \dfrac{n^k\varphi(n)}{2}$.

Lemma 9. *[7, Lemma 3 (iv)] Let φ be the Euler totient function and n be any natural number. Then $\varphi(n) \geq 2\sqrt{n/3}$ whenever $n \neq 1, 2, 6$.*

Lemma 10. *Let n and k be positive integer such that $3 \leq n \leq 6$ and $k \geq 2$. Then for the generalized Euler totient function φ_k, the following statements hold.*

(i) $\varphi_k(3) \geq 4(k+1) + 2$ whenever $k \geq 5$.

(ii) $\varphi_k(4) \geq 5(k+1) + 2$ whenever $k \geq 3$.

(iii) $\varphi_k(5) \geq 6(k+1)+2$ *whenever* $k \geq 2$.

(iv) $\varphi_k(6) \geq 7(k+1)+2$ *whenever* $k \geq 2$.

Proof. By definition of φ_k, $\varphi_k(3) = 1 + 2^k$. Therefore, $\varphi_k(3) \geq 4(k+1)+2$ if and only if $2^k \geq 4k+5$ and by induction, $2^k \geq 4k+5$ is true whenever $k \geq 5$. This implies that $\varphi_k(3) \geq 4(k+1)+2$ whenever $k \geq 5$. This completes the proof of (i). The proof of (ii), (iii), and (iv) follows similarly using definition of function φ_k and mathematical induction. □

3 Proof of main results

3.1 Proof of Theorem 1

(i) Using the definition of φ_k, we have $\varphi_k(1) = 1$ and $\varphi_k(2) = 1$ for every $k \geq 1$. Therefore, using the definition of the Lucas sequence of the first kind for $n = 1$, we have $\varphi_k(|U_1|) = \varphi_k(1) = 1 = |U_1| = |U_{\varphi_k(1)}|$ for every $k \geq 1$. Similarly, using the definition of the Lucas sequence of the first kind for $n = 2$, we have $|U_2| = |\alpha + \beta| = |r| \geq 1$. Therefore, $\varphi_k(|U_2|) = \varphi_k(|r|) \geq \varphi_k(1) = 1 = |U_1| = |U_{\varphi_k(2)}|$ for every $k \geq 1$. Hence, if $|U_2| = |r| = 1$ or 2, then $\varphi_k(|U_2|) = |U_{\varphi_k(2)}|$ otherwise $\varphi_k(|U_2|) = \varphi_k(|r|) > 1 = |U_{\varphi_k(2)}|$ for every $k \geq 1$.

(ii) Let n be positive integer such that $n \geq 16$. By Lemma 5(i), we divide our problem in two cases:

Case I: $|\alpha| = \frac{1+\sqrt{5}}{2}$**.**

If $|\alpha| = \frac{1+\sqrt{5}}{2}$, then using Lemma 8 and Lemma 7, we obtain

$$\varphi_1(|U_n|) = \frac{|U_n|\varphi(|U_n|)}{2} < \frac{|U_n||U_n|}{2} = \frac{|U_n|^2}{2} \leq \frac{|\alpha|^{2(n-1)}}{2} \leq |\alpha|^{2(n-1)-1},$$

and $|U_{\varphi_1(n)}| \geq |\alpha|^{\varphi_1(n)-2}$. Therefore, in order to prove $\varphi_1(|U_n|) < |U_{\varphi_1(n)}|$, it is sufficient to show that $\varphi_1(n) - 2 \geq 2(n-1) - 1$, i.e.,

$$\varphi_1(n) \geq 2n - 1. \tag{1}$$

Case II: $|\alpha| \geq 2$**.**

If $|\alpha| \geq 2$, again using Lemma 8 and Lemma 7, we obtain

$$\varphi_1(|U_n|) = \frac{|U_n|\varphi(|U_n|)}{2} < \frac{|U_n||U_n|}{2} = \frac{|U_n|^2}{2} < \frac{|\alpha|^{2(n+1)}}{2} \leq |\alpha|^{2(n+1)},$$

and $|U_{\varphi_1(n)}| > |\alpha|^{\varphi_1(n)-2}$. Hence, in order to prove $\varphi_1(|U_n|) < |U_{\varphi_1(n)}|$, it is sufficient to show that $\varphi_1(n) - 2 \geq 2(n+1)$, i.e.,

$$\varphi_1(n) \geq 2n + 4. \tag{2}$$

Since $2n+4 > 2n-1$ for every $n \geq 16$, therefore, in order to prove (1) and (2), it is sufficient to prove (2), i.e., $\varphi_1(n) \geq 2n+4$. By Lemma 8, $\varphi_1(n) = \dfrac{n\varphi(n)}{2}$ and using Lemma 9, for $n \geq 16$, $\varphi(n) \geq 2\sqrt{n/3}$. Hence, in order to prove (2), it is sufficient to prove that

$$n\sqrt{n/3} \geq 2n+4. \tag{3}$$

Consider the function $f : [16, \infty) \to \mathbb{R}$ defined by

$$f(x) = x\sqrt{x/3} - 2x - 4.$$

If $x \geq 16$, then $f'(x) = \sqrt{3x}/2 - 2 > 2\sqrt{3} - 2 > 0$, and therefore, f is an increasing function on $[16, \infty)$. Therefore, $f(x) \geq f(16) = 16\sqrt{16/3} - 2 \cdot 16 - 4 = 0.95 > 0$. This implies that

$$x\sqrt{x/3} - 2x - 4 \geq 0 \textit{ for } x \geq 16.$$

Hence, (3) holds and this proves that $\varphi_1(|U_n|) < |U_{\varphi_1(n)}|$ whenever $n \geq 16$.

(iii) We proceed in similar manner as in proof of (i). Let n and k be positive integer such that $n \geq 7$ and $k \geq 2$. By Lemma 5(i), we divide our problem in two cases:

Case I: $|\alpha| = \frac{1+\sqrt{5}}{2}$**.**

If $|\alpha| = \frac{1+\sqrt{5}}{2}$, then using Lemma 8 and Lemma 7, we obtain

$$\begin{aligned}
\varphi_k(|U_n|) &< \frac{|U_n|^k \varphi(|U_n|)}{2} \\
&< \frac{|U_n|^k |U_n|}{2} = \frac{|U_n|^{k+1}}{2} \\
&\leq \frac{|\alpha|^{(n-1)(k+1)}}{2} \\
&< |\alpha|^{(n-1)(k+1)-1},
\end{aligned}$$

and $|U_{\varphi_k(n)}| \geq |\alpha|^{\varphi_k(n)-2}$. Therefore, in order to prove $\varphi_k(|U_n|) < |U_{\varphi_k(n)}|$, it is sufficient to show that $\varphi_k(n) - 2 \geq (n-1)(k+1) - 1$, i.e.,

$$\varphi_k(n) \geq (n-1)(k+1) + 1. \tag{4}$$

Case II: $|\alpha| \geq 2$**.**

If $|\alpha| \geq 2$, again using Lemma 8 and Lemma 7, we obtain

$$\begin{aligned}
\varphi_k(|U_n|) &< \frac{|U_n|^k \varphi(|U_n|)}{2} \\
&< \frac{|U_n|^k |U_n|}{2} = \frac{|U_n|^{k+1}}{2} \\
&< \frac{|\alpha|^{(n+1)(k+1)}}{2} \leq |\alpha|^{(n+1)(k+1)},
\end{aligned}$$

and $|U_{\varphi_k(n)}| > |\alpha|^{\varphi_k(n)-2}$. Similar to Case I, in order to prove $\varphi_k(|U_n|) < |U_{\varphi_k(n)}|$, it suffices to show that $\varphi_k(n) - 2 \geq (n+1)(k+1)$, i.e.,

$$\varphi_k(n) \geq (n+1)(k+1) + 2. \tag{5}$$

Since $(n+1)(k+1)+2 \geq (n-1)(k+1)+1$ for every $n \geq 1$ and $k \geq 2$, therefore in order to prove (4) and (5), it is sufficient to prove (5), i.e., $\varphi_k(n) \geq (n+1)(k+1)+2$. Now we claim that $\varphi(n) \geq 4$ for every $n \geq 7$. For this, suppose that $n \geq 7$. If there exists a prime $p \geq 5$ dividing n, then by multiplicativity of φ, we obtain

$$\varphi(n) \geq \varphi(p) \geq p - 1 \geq 4.$$

Suppose $n = 2^a 3^b$ for some $a, b \in \mathbb{N} \cup \{0\}$. If $a \geq 3$, then $\varphi(n) \geq \varphi(2^3) = 2^3 - 2^2 = 4$. If $a = 2, b \geq 1$, then $\varphi(n) \geq \varphi(2^2)\phi(3) = (2^2 - 2)(3-1) = 4$. If $a \leq 1, b \geq 2$, then $\varphi(n) \geq \varphi(3^2) = 6$. Thus, $\varphi(n) \geq 4$ for every $n \geq 7$.

Now by Lemma 8, we have

$$\varphi_k(n) \geq \varphi(n)\left(\frac{n}{2}\right)^k \geq 4\left(\frac{n}{2}\right)^k.$$

Hence, in order to prove (5), it is sufficient to prove that

$$4\left(\frac{n}{2}\right)^k \geq (n+1)(k+1) + 2. \tag{6}$$

Consider the function $f : [7, \infty) \to \mathbb{R}$ defined by

$$f(x) = 4\left(\frac{x}{2}\right)^k - (x+1)(k+1) - 2.$$

If $x \geq 7$, then $f'(x) = 2k\left(\frac{x}{2}\right)^{k-1} - (k+1) > 6k - (k+1) > 0$, and so f is an increasing function on $[7, \infty)$. Therefore, $f(x) \geq f(7) = 4\left(\frac{7}{2}\right)^k - 8(k+1) - 2 > 4(3)^k - (8)(k+1) - 2 > 0$, where $k \geq 2$. This implies that

$$4\left(\frac{x}{2}\right)^k \geq (x+1)(k+1) + 2 \textit{ for } x \geq 7.$$

Hence, (6) holds and this proves that $\varphi_k(|U_n|) < |U_{\varphi_k(n)}|$ for every $n \geq 7$ and $k \geq 2$. By Lemma 10, for $3 \leq n \leq 6$ and $k \geq 2$, (5) holds except for for $(n, k) = (3, 2), (3, 3), (3, 4), (4, 2)$. This completes the proof of (iii).

3.2 Proof of Theorem 3

(i) The Fibonacci sequence $\left(F_n\right)_{n\geq 1}$ is the Lucas sequence of the first kind with $r = 1$ and $s = 1$. The characteristic equation of $\left(F_n\right)_{n\geq 1}$ is $x^2 - x - 1 = 0$ with roots $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Since both α and β are real roots of the characteristic equation, all hypothesis of Theorem 1 holds. Hence, $|U_n| = F_n$ and by using Theorem 1 (i), for every $k \geq 1$, $\varphi_k(F_1) = F_{\varphi_k(1)}$ and since $F_2 = 1$, this implies that $\varphi_k(F_2) = F_{\varphi_k(2)}$. Now, using Theorem 1 (ii), $\varphi_1(F_n) < F_{\varphi_1(n)}$ whenever $n \geq 16$. Using definition of φ and a straightforward calculation for $(n, 1)$ where $3 \leq n \leq 15$, we obtain that $\varphi_1(F_n) \leq F_{\varphi_1(n)}$ except for $(n, k) = (6, 1)$ for which the inequality is reversed, and equality holds for $(n, k) = (4, 1)$. Using Theorem 1 (iii), $\varphi_k(F_n) < F_{\varphi_k(n)}$ for every $n \geq 3$ and $k \geq 2$ except for $(n, k) = (3, 2), (3, 3), (3, 4), (4, 2)$. Again using definition of φ and a straightforward calculation, we obtain $\varphi_k(F_n) < F_{\varphi_k(n)}$ for $(n, k) = (3, 2), (3, 3), (3, 4), (4, 2)$. This completes the proof of (i).

(ii) The Pell sequence $\left(P_n\right)_{n\geq 1}$ is the Lucas sequence of the first kind with $r = 2$ and $s = 1$. The characteristic equation of $\left(P_n\right)_{n\geq 1}$ is $x^2 - 2x - 1 = 0$ with roots $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. Since both α and β are real roots of the characteristic equation, all hypothesis of Theorem 1 holds. Hence, $|U_n| = P_n$ and by using Theorem 1 (i), for every $k \geq 1$, $\varphi_k(P_1) = P_{\varphi_k(1)}$ and since $P_2 = 2$, this implies that $\varphi_k(P_2) = P_{\varphi_k(2)}$. Now, using Theorem 1 (ii), $\varphi_1(P_n) < P_{\varphi_1(n)}$ whenever $n \geq 16$. Using definition of φ and a straightforward calculation for $(n, 1)$ where $3 \leq n \leq 15$, we obtain that $\varphi_1(P_n) \leq P_{\varphi_1(n)}$ except for $(n, k) = (3, 1), (4, 1), (6, 1)$ for which the inequality is reversed. Using Theorem 1 (iii), we have $\varphi_k(P_n) < P_{\varphi_k(n)}$ for every $n \geq 3$ and $k \geq 2$ except for $(n, k) = (3, 2), (3, 3), (3, 4), (4, 2)$. Again using definition of φ and a straightforward calculation , we obtain $\varphi_k(P_n) < P_{\varphi_k(n)}$ for $(n, k) = (3, 3), (3, 4), (4, 2)$ and $\varphi_k(P_n) > P_{\varphi_k(n)}$ for $(n, k) = (3, 2)$. This completes the proof of (ii).

(iii) The balancing sequence $\left(B_n\right)_{n\geq 1}$ is the Lucas sequence of the first kind with $r = 6$ and $s = -1$. The characteristic equation of $\left(B_n\right)_{n\geq 1}$ is $x^2 - 6x + 1 = 0$ with roots $\alpha = 3 + \sqrt{2}$ and $\beta = 3 - \sqrt{2}$. Since both α and β are real roots of the characteristic equation, all hypothesis of Theorem 1 holds. Hence, $|U_n| = B_n$ and by using Theorem 1 (i), for every $k \geq 1$, $\varphi_k(B_1) = B_{\varphi_k(1)}$ and since $B_2 = 6$, this implies that $\varphi_k(B_2) > B_{\varphi_k(2)}$. Now, using Theorem 1 (ii), $\varphi_1(B_n) < B_{\varphi_1(n)}$ whenever $n \geq 16$. Using definition of φ and a straightforward calculation for $(n, 1)$ where $3 \leq n \leq 15$, we obtain that $\varphi_1(B_n) \leq B_{\varphi_1(n)}$ except for $(n, k) = (3, 1), (4, 1), (6, 1), (12, 1)$ for which the inequality is reversed. Using Theorem 1 (iii), $\varphi_k(B_n) < B_{\varphi_k(n)}$ for every $n \geq 3$ and $k \geq 2$ except for $(n, k) = (3, 2), (3, 3), (3, 4), (4, 2)$. Again using definition of φ and a straightforward calculation, we obtain $\varphi_k(F_n) < F_{\varphi_k(n)}$ for $(n, k) = (3, 3), (3, 4), (4, 2)$ and $\varphi_k(B_n) > B_{\varphi_k(n)}$ for $(n, k) = (3, 2)$. This completes the proof of (iii).

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