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On Degenerate Simsek and Stirling Numbers

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Abstract

In this paper, we introduce the degenerate Simsek and Stirling numbers. We derive some related properties and identities, including the recurrence relation. In addition, we establish some relations between these numbers and certain other special numbers.

1 Introduction

Stirling numbers and other special numbers occur in combinatorial problems, discrete probability, the theory of partitions, mathematical physics, etc. They have also been used to construct computational physics and mathematical models. In recent years, various well-known degenerate versions of these special numbers have been investigated. There are several motivations for defining the degenerate versions. In particular, by specializing the parameters of the degenerate versions, we obtain the ordinary ones. Moreover, these degenerate versions have many combinatorial applications, and they appear in many interesting results in different branches of mathematics, such as umbral calculus, probability theory, differential equations, operator algebras, etc. (cf. [12, 15, 17, 26]).

Stirling numbers of the second kind $S_2(n,k)$ count the number of partitions of a set of size n into k disjoint, non-empty subsets. They appear as coefficients in the expansion of

$$x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k, \tag{1}$$

where

$$(x)_n := \begin{cases} x(x-1)(x-2)\cdots(x-n+1), & \text{if } n \geq 1; \\ 1, & \text{if } n = 0, \end{cases}$$

denotes the falling factorial (cf. [4, 26]). Moreover, Stirling numbers of the first kind $s(n, k)$ count the number of permutations of a set of size n with exactly k cycles. They appear as coefficients in the expansion of the falling factorial $(x)_n$ as follows:

$$(x)_n = \sum_{k=0}^{n} s(n, k)x^k, \tag{2}$$

(cf. [4, 26]).

Throughout this paper, for $z \in \mathbb{C}$, we assume that $\log z$ denotes the principal branch of the many-valued function with the imaginary part $\mathrm{Im}(\log z)$, constrained by

$$-\pi < \mathrm{Im}(\log z) \leq \pi.$$

The generating functions of $s(n, k)$ and $S_2(n, k)$ are given, respectively, by

$$\frac{(\log(1+t))^k}{k!} = \sum_{n \geq k} s(n, k)\frac{t^n}{n!} \tag{3}$$

and

$$\frac{(e^t - 1)^k}{k!} = \sum_{n \geq k} S_2(n, k)\frac{t^n}{n!}. \tag{4}$$

Recently, several authors [1, 10, 11, 14, 16, 18, 20, 21, 22, 23, 27, 29] have studied different modifications of Stirling numbers. In particular, the λ-Stirling numbers of the second kind $S_2(n, k; \lambda)$ are defined by the following generating function:

$$\frac{(\lambda e^t - 1)^k}{k!} = \sum_{n \geq 0} S_2(n, k; \lambda)\frac{t^n}{n!}, \tag{5}$$

(cf. [19, 31]). The degenerate Stirling numbers of the second kind $S_2(n, k|\alpha)$, were introduced by D. S. Kim and T. Kim [14], by means of the generating function

$$\frac{(e^{\frac{\log(1+\alpha t)}{\alpha}} - 1)^k}{k!} = \sum_{n \geq 0} S_2(n, k|\alpha)\frac{t^n}{n!}. \tag{6}$$

They can be expressed in relation to Stirling numbers of the first and second kind as follows [14]:

$$S_2(n, k|\alpha) = \sum_{j=k}^{n} s(n, j)S_2(j, k)\alpha^{n-j}, \text{ where } 0 \leq k \leq n. \tag{7}$$

It is worth mentioning that Eq. (6) was first defined by T. Kim [13], dealing with the degenerate Stirling polynomials of the second kind.

The degenerate Stirling numbers of the second kind $S_2(n,k|\alpha)$ reduce to Stirling numbers of the second kind $S_2(n,k)$, when $\alpha \to 0$. Moreover, Stirling numbers of the second kind $S_2(n,k)$ are a special case of exponential partial Bell polynomials $B_{n,k}$, defined as follows [4]:

$$\frac{1}{k!}\left(\sum_{j\geq 1} x_j \frac{t^j}{j!}\right)^k = \sum_{n\geq k} B_{n,k}(x_1,\ldots,x_{n-k+1})\frac{t^n}{n!}. \tag{8}$$

In particular, for $x_j = 1, \forall j \geq 1$, we obtain the following relation between Stirling numbers of the second kind $S_2(n,k)$ and exponential partial Bell polynomials $B_{n,k}$:

$$B_{n,k}(1,1,\ldots,1) = S_2(n,k),$$

(cf. [4]).

Simsek [28] considered a new family of special numbers $y_1(n,k;\lambda)$ as coefficients of the generating function

$$F_{y_1}(t,k;\lambda) := \frac{(\lambda e^t+1)^k}{k!} = \sum_{n\geq 0} y_1(n,k;\lambda)\frac{t^n}{n!}, \tag{9}$$

which are essentially related to the many well-known special numbers, for instance, Bernoulli numbers, Fibonacci numbers, Stirling numbers of the second kind, Lucas numbers and the central numbers. Using Eq. (9), the numbers $y_1(n,k;\lambda)$ can be expressed explicitly by the following formula:

$$y_1(n,k;\lambda) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j} j^n \lambda^j. \tag{10}$$

Substituting $\lambda = 1$ in (10) and setting

$$B(n,k) := k! y_1(n,k;1) = \sum_{j=0}^{k}\binom{k}{j} j^n, \tag{11}$$

the numbers $B(n,k)$ are related to the numbers of the form $a_k 2^k$, with a_k being a sequence of positive integers. We have the following first cases [28]:

$$B(0,k) = 2^k,$$

$$B(1,k) = k2^{k-1},$$

$$B(2,k) = k(k+1)2^{k-2}.$$

The numbers $B(n,k)$ have many combinatorial applications. For instance, Golombek [5] established the following formula:

$$B(n,k) = \frac{d^n}{dt^n}(e^t+1)^k\Big|_{t=0}. \tag{12}$$

Spivey [30] proved a relation between the numbers $B(n,k)$ and Stirling numbers of the second kind $S_2(n,k)$ as follows:

$$B(n,k) = \sum_{j=0}^{k} \binom{k}{j} j! 2^{k-j} S_2(n,j). \tag{13}$$

On the other hand, Simsek [28] derived the following recursive formula:

$$B(n,k) = \frac{2^{k-n}}{m_0}\binom{k}{n} - \sum_{j=1}^{n} \frac{m_j}{m_0} B(n-j,k), \tag{14}$$

where $m_j \in \mathbb{Q}$, for $j = 0, 1, \ldots, n$. Furthermore, Simsek [28] conjectured that

$$B(n,k) = 2^{k-n} \sum_{j=0}^{n-1} x_j k^{n-j} \tag{15}$$

holds for some positive integers $x_0, x_1, \ldots, x_{n-1}$, and therefore, stated the following problems:

1. How can we compute the coefficients $x_j, j = 0, 1, 2, \ldots, n-1$?
2. Is it possible to find functions f_n, such that $f_n(x) = \sum_{k\geq 0} B(n,k)x^k,$ for $|x| > r$?

In 2019, Xu [32] solved the above problems using the standard Stirling numbers of the first and second kind, and generalized the result to an arbitrary λ, that is, to the family of numbers $B(n,k;\lambda) := k! y_1(n,k;\lambda)$.

In this paper, we define the degenerate Simsek numbers $y_1(n,k;\lambda|\alpha)$ and a different version of degenerate Stirling numbers of the second kind $S_2(n,k;\lambda|\alpha)$ (which will be called (α,λ)-Stirling numbers) other than the numbers in (6), as a modification and generalization of Simsek numbers $y_1(n,k;\lambda)$ and Stirling numbers of the second kind $S_2(n,k)$. We establish a relation between these numbers and the well-known special numbers: Stirling numbers of the first and second kind, the degenerate Stirling numbers of the second kind $S_2(n,k|\alpha)$, first and second kind Apostol-Euler numbers, and λ-Stirling numbers of the second kind. We investigate some basic properties, including a recursive formula for these numbers. Moreover, we extend Xu's solution for Simsek's questions [28] to the numbers $B(n,k;\lambda|\alpha) := k! y_1(n,k;\lambda|\alpha)$.

2 Main results

In this section, we introduce the degenerate Simsek and Stirling numbers by using generating functions. We establish some properties and recurrence relations regarding these numbers. We also introduce the degenerate first and second kind Apostol-type Euler numbers and their relation with the degenerate Simsek numbers.

2.1 Degenerate Simsek and Stirling numbers

For $k \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$, $\lambda \in \mathbb{C}$ and $\alpha \in \mathbb{R} \setminus \{0\}$, we define the degenerate Simsek numbers $y_1(n, k; \lambda|\alpha)$ by means of the following generating function:

$$F_{y_1}(t, k; \lambda|\alpha) := \frac{1}{k!}(\lambda e^{\frac{\log(1+\alpha t)}{\alpha}} + 1)^k = \sum_{n\geq 0} y_1(n, k; \lambda|\alpha)\frac{t^n}{n!}. \tag{16}$$

It follows that, for $\alpha \to 0$, the above identity reduces to (9).

Theorem 1. *The degenerate Simsek numbers $y_1(n, k; \lambda|\alpha)$ can be expressed explicitly as follows:*

$$y_1(n, k; \lambda|\alpha) = \frac{1}{k!}\sum_{m=0}^{n}\sum_{j=0}^{k}\binom{k}{j} j^m \lambda^j \alpha^{n-m} s(n, m) \tag{17}$$

or

$$y_1(n, k; \lambda|\alpha) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\lambda^j \alpha^n \left(\frac{j}{\alpha}\right)_n. \tag{18}$$

Proof. Using (16), we obtain

$$\sum_{n\geq 0} y_1(n, k; \lambda|\alpha)\frac{t^n}{n!} = \frac{1}{k!}(\lambda e^{\frac{\log(1+\alpha t)}{\alpha}} + 1)^k \tag{19}$$

$$= \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\lambda^j e^{\frac{j}{\alpha}\log(1+\alpha t)} \tag{20}$$

$$= \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\lambda^j \sum_{m\geq 0}\frac{j^m(\log(1+\alpha t))^m}{\alpha^m m!}. \tag{21}$$

By using (3) in (21), we get

$$\sum_{n\geq 0} y_1(n, k; \lambda|\alpha)\frac{t^n}{n!} = \frac{1}{k!}\sum_{n\geq 0}\sum_{m=0}^{n}\sum_{j=0}^{k}\binom{k}{j} j^m \lambda^j \alpha^{n-m} s(n, m)\frac{t^n}{n!}. \tag{22}$$

Comparing the coefficient of $\frac{t^n}{n!}$ yields (17). Similarly, one can prove (18) by writing $e^{\frac{\log(1+\alpha t)}{\alpha}} = (1 + \alpha t)^{\frac{1}{\alpha}}$, and using the binomial expansion.

Remark 2. One can also prove (18) using the combination of (17) and (2).

□

Example 3. The special cases of the degenerate Simsek numbers are as follows:

- $y_1(0,k;\lambda|\alpha) = \frac{(\lambda+1)^k}{k!}$
- $y_1(n,0;\lambda|\alpha) = \delta_{n,0}$
- $y_1(n,1;\lambda|\alpha) = \delta_{n,0} + \lambda\left(\frac{1}{\alpha}\right)_n \alpha^n$

Remark 4.

1. If we set $\lambda = -1$ in (17) or (18), we obtain the degenerate Stirling numbers of the second kind
$$S_2(n,k|\alpha) = (-1)^k y_1(n,k;-1|\alpha).$$
2. For $\alpha \to 0$, we obtain
$$y_1(n,k;\lambda|0) := \lim_{\alpha\to 0} y_1(n,k;\lambda|\alpha) = y_1(n,k;\lambda),$$
which are Simsek numbers given by Eq. (10). If moreover we take $\lambda = -1$, then we obtain a relation with Stirling numbers of the second kind as follows:
$$S_2(n,k) = (-1)^k y_1(n,k;-1|0).$$

Theorem 5. *For any non-negative integers n and k, we have*
$$\lambda^k \left(\frac{k}{\alpha}\right)_n \alpha^n = \sum_{l=0}^{k} (-1)^{k-l} l! \binom{k}{l} y_1(n,l;\lambda|\alpha). \tag{23}$$

Proof. We have
$$\begin{aligned}
\lambda^k \sum_{n\geq 0} \left(\frac{k}{\alpha}\right)_n \alpha^n \frac{t^n}{n!} = \lambda^k (1+\alpha t)^{\frac{k}{\alpha}} &= \left((\lambda e^{\frac{\log(1+\alpha t)}{\alpha}} + 1) - 1\right)^k \\
&= \sum_{l=0}^{k} \binom{k}{l} \left(\lambda e^{\frac{\log(1+\alpha t)}{\alpha}} + 1\right)^l (-1)^{k-l} \\
&= \sum_{n\geq 0} \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} l! y_1(n,l;\lambda|\alpha) \frac{t^n}{n!},
\end{aligned}$$
where in the last equality, we use (16). Hence, the desired result follows from the comparison of the coefficient of $\frac{t^n}{n!}$. □

In the following theorem, we establish another explicit formula for the numbers $y_1(n,k;\lambda|\alpha)$ in relation with the degenerate Stirling numbers of the second kind $S_2(n,k|\alpha)$.

Theorem 6. *Let n be a non-negative integer. Then*

$$y_1(n,k;\lambda|\alpha) = \frac{1}{k!}\sum_{j=0}^{n}\binom{k}{j} j!\lambda^j(\lambda+1)^{k-j}S_2(n,j|\alpha). \tag{24}$$

Proof. By virtue of (6) and (16), we obtain

$$\begin{aligned}\sum_{n\geq 0} y_1(n,k;\lambda|\alpha)\frac{t^n}{n!} &= \frac{1}{k!}\left(\lambda(e^{\frac{\log(1+\alpha t)}{\alpha}}-1)+\lambda+1\right)^k \\ &= \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\lambda^j(\lambda+1)^{k-j}(e^{\frac{\log(1+\alpha t)}{\alpha}}-1)^j \\ &\overset{\text{by using (6)}}{=} \frac{1}{k!}\sum_{n\geq 0}\sum_{j=0}^{k}\binom{k}{j}\lambda^j j!(\lambda+1)^{k-j}S_2(n,j|\alpha)\frac{t^n}{n!},\end{aligned}$$

and the desired result follows from the comparison of coefficient of $\frac{t^n}{n!}$. □

Now, from the viewpoint of Eqs. (5) and (6), we consider the (α,λ)-Stirling numbers of the second kind $S_2(n,k;\lambda|\alpha)$, which are given by the generating function

$$G_{S_2}(t,k;\lambda|\alpha) := \frac{(\lambda e^{\frac{\log(1+\alpha t)}{\alpha}}-1)^k}{k!} = \sum_{n\geq 0} S_2(n,k;\lambda|\alpha)\frac{t^n}{n!}. \tag{25}$$

If we substitute $\lambda = 1$ in Eq. (25), we obtain the degenerate Stirling numbers of the second kind (6), while for $\alpha \to 0$, we get the λ-Stirling numbers of the second kind (5). Moreover, the (α,λ)-Stirling numbers of the second kind $S_2(n,k;\lambda|\alpha)$ can be reduced to some other well-known special numbers. Let us recall that in this context, the generalized Stirling numbers $S(n,k;\alpha,\beta,r)$, with $\alpha\beta \neq 0$, were introduced by Hsu and Shiue [11], and that they satisfy the following identity [11, Thm. 2]:

$$(1+\alpha t)^{\frac{r}{\alpha}}\left(\frac{(1+\alpha t)^{\frac{\beta}{\alpha}}-1}{\beta}\right) = k!\sum_{n\geq 0} S(n,k;\alpha,\beta,r)\frac{t^n}{n!}. \tag{26}$$

Letting $r = 0$ and $\beta = 1$ in (26) gives

$$S(n,k;\alpha,1,0) = S_2(n,k;1|\alpha).$$

The degenerate weighted Stirling numbers of the second kind $S(n,k;\lambda|\theta)$, introduced by Howard [10], by means of the generating function

$$(1+\theta t)^{\frac{\lambda}{\theta}}\left((1+\theta t)^{\frac{1}{\theta}}-1\right)^k = k!\sum_{n\geq k} S(n,k;\lambda|\theta)\frac{t^n}{n!},$$

can be reduced to the (α, λ)-Stirling numbers $S_2(n, k; \lambda|\alpha)$ as follows:

$$S(n, k; 0|\alpha) = S_2(n, k; 1|\alpha).$$

In addition, Howard [9] introduced the degenerate associated Stirling numbers of the second kind $S_r(n, k|\lambda)$ as follows:

$$k! \sum_{n \geq (r+1)k} S_r(n, k|\lambda) \frac{t^n}{n!} = \left((1 + \lambda t)^{\frac{1}{\lambda}} - 1 - G(r, \lambda)\right)^k, \tag{27}$$

where $G(r, \lambda) = \sum_{i=1}^{r} (1 - \lambda)(1 - 2\lambda) \cdots (1 - (i - 1)\lambda) \frac{x^i}{i!}$. The case $r = 0$ has been investigated by Carlitz [3]. Therefore, we have the following relation between the degenerate Stirling numbers $S_r(n, k|\lambda)$ and the (α, λ)-Stirling numbers $S_2(n, k; \lambda|\alpha)$:

$$S_0(n, k|\alpha) = S_2(n, k; 1|\alpha).$$

Theorem 7. *The (α, λ)-Stirling numbers of the second kind $S_2(n, k; \lambda|\alpha)$ can be expressed by the following formulas:*

$$S_2(n, k; \lambda|\alpha) = \frac{1}{k!} \sum_{m=0}^{n} \sum_{l=0}^{k} \lambda^l \binom{k}{l} l^m \alpha^{n-m} (-1)^{k-l} s(n, m) \tag{28}$$

or

$$S_2(n, k; \lambda|\alpha) = \frac{1}{k!} \sum_{l=0}^{k} \lambda^l \binom{k}{l} \left(\frac{l}{\alpha}\right)_n \alpha^n (-1)^{k-l}. \tag{29}$$

Proof. According to Eq. (25), we have

$$\sum_{n \geq 0} S_2(n, k; \lambda|\alpha) \frac{t^n}{n!} = \frac{1}{k!} \sum_{l=0}^{k} \lambda^l e^{\frac{l \log(1+\alpha t)}{\alpha}} (-1)^{k-l} \binom{k}{l} \tag{30}$$

$$= \frac{1}{k!} \sum_{l=0}^{k} \lambda^l \sum_{m \geq 0} \frac{l^m (\log(1 + \alpha t))^m}{\alpha^m m!} (-1)^{k-l} \binom{k}{l}. \tag{31}$$

Using (3) in (31), we get

$$\sum_{n \geq 0} S_2(n, k; \lambda|\alpha) \frac{t^n}{n!} = \frac{1}{k!} \sum_{n \geq 0} \sum_{m=0}^{n} \sum_{l=0}^{k} \lambda^l \binom{k}{l} l^m \alpha^{n-m} (-1)^{k-l} s(n, m) \frac{t^n}{n!}.$$

Hence, Eq. (28) follows by comparing the coefficient of $\frac{t^n}{n!}$. □

In the following theorem, we provide a relation between the (α, λ)-Stirling numbers of the second kind $S_2(n, k; \lambda|\alpha)$ and the degenerate Simsek numbers $y_1(n, k; \lambda|\alpha)$.

Theorem 8. *Let n and k be non-negative integer. Let λ and α be complex number such that $\alpha \neq 0$. Then*

$$S_2\big(n,k;\lambda^2|\frac{\alpha}{2}\big) = \frac{k!}{2^n}\sum_{j=0}^{n}\binom{n}{j}S_2(j,k;\lambda|\alpha)y_1(n-j,k;\lambda|\alpha). \tag{32}$$

Proof. From Eqs. (16) and (25), we get the following functional relation between the generating functions of the (α,λ)-Stirling numbers of the second kind $S_2(n,k;\lambda|\alpha)$ and the degenerate Simsek numbers $y_1(n,k;\lambda|\alpha)$:

$$\frac{(\lambda^2 e^{\frac{2}{\alpha}\log(1+\alpha t)}-1)^k}{k!} = k!G_{S_2}(t,k;\lambda|\alpha)F_{y_1}(t,k;\lambda|\alpha). \tag{33}$$

Thus,

$$\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}\lambda^{2l}e^{\frac{2l}{\alpha}\log(1+\alpha t)}(-1)^{k-l} = k!\sum_{n\geq 0}S_{2,\alpha}(n,k;\lambda)\frac{t^n}{n!}\sum_{n\geq 0}y_1(n,k;\lambda|\alpha)\frac{t^n}{n!}.$$

Therefore,

$$\begin{aligned}\frac{1}{k!}\sum_{n\geq 0}\sum_{m=0}^{n}\sum_{l=0}^{k}\binom{k}{l}\lambda^{2l}(-1)^{k-l}2^m l^m\alpha^{n-m}s(n,m)\frac{t^n}{n!} &= k!\sum_{n\geq 0}\sum_{j=0}^{n}\binom{n}{j}S_2(j,k;\lambda|\alpha)\\ &\quad\times y_1(n-j,k;\lambda|\alpha)\frac{t^n}{n!}.\end{aligned}$$

Equating the coefficient of $\frac{t^n}{n!}$ in both sides of the above identity yields

$$\frac{1}{k!}\sum_{m=0}^{n}\sum_{l=0}^{k}\binom{k}{l}\lambda^{2l}(-1)^{k-l}2^m l^m\alpha^{n-m}s(n,m) = k!\sum_{j=0}^{n}\binom{n}{j}S_2(n,k;\lambda|\alpha)y_1(n,k;\lambda|\alpha).$$

Then, using (28), the left-hand side of the above identity becomes

$$2^n S_2\big(n,k;\lambda^2|\frac{\alpha}{2}\big),$$

and the desired result follows. □

Remark 9. Similarly, one can prove Theorem 8 using Eqs. (29) and (33), as well as the fact that

$$\begin{aligned}\frac{(\lambda^2 e^{\frac{2}{\alpha}\log(1+\alpha t)}-1)^k}{k!} &= \frac{1}{k!}(\lambda^2(1+\alpha t)^{\frac{2}{\alpha}}-1)^k\\ &= \frac{1}{k!}\sum_{n\geq 0}\sum_{l=0}^{k}\binom{k}{l}\lambda^{2l}(-1)^{k-l}\Big(\frac{2l}{\alpha}\Big)_n\alpha^n\frac{t^n}{n!}.\end{aligned}$$

It follows that, by letting $\alpha \to 0$, Eq. (32) reduces to the following formula proved by Simsek [28] for λ-Stirling numbers $S_2(n,k;\lambda)$ and Simsek numbers $y_1(n,k;\lambda)$:

$$S_2(n,k;\lambda^2) = \frac{k!}{2^n}\sum_{j=0}^{n}\binom{n}{j}S_2(j,k;\lambda)y_1(n-j,k;\lambda). \tag{34}$$

Theorem 10. *The degenerate Simsek numbers $y_1(n,k;\lambda|\alpha)$ satisfy the following recursive formula:*

$$y_1(n+1,k;\lambda|\alpha) = (k-\alpha n)y_1(n,k;\lambda|\alpha) - y_1(n,k-1;\lambda|\alpha). \tag{35}$$

Proof. From (16), we have the following partial differential equations:

$$\frac{\partial}{\partial t}F_{y_1}(t,k;\lambda|\alpha) = \frac{k}{(1+\alpha t)}F_{y_1}(t,k;\lambda|\alpha) - \frac{1}{(1+\alpha t)}F_{y_1}(t,k-1;\lambda|\alpha) \tag{36}$$

and

$$\frac{\partial}{\partial t}\Big((1+\alpha t)F_{y_1}(t,k;\lambda|\alpha)\Big) = \sum_{n\geq 0} y_1(n+1,k;\lambda|\alpha)\frac{t^n}{n!} + \alpha\sum_{n\geq 0}(n+1)y_1(n,k;\lambda|\alpha)\frac{t^n}{n!}. \tag{37}$$

On the other hand, we have

$$\begin{aligned}\frac{\partial}{\partial t}\Big((1+\alpha t)F_{y_1}(t,k;\lambda|\alpha)\Big) &= \alpha F_{y_1}(t,k;\lambda|\alpha) + (1+\alpha t)\frac{\partial}{\partial t}F_{y_1}(t,k;\lambda|\alpha)\\ &= \alpha\sum_{n\geq 0} y_1(n,k;\lambda|\alpha)\frac{t^n}{n!} + k\sum_{n\geq 0} y_1(n,k;\lambda|\alpha)\frac{t^n}{n!}\\ &\quad - \sum_{n\geq 0} y_1(n,k-1;\lambda|\alpha)\frac{t^n}{n!}.\end{aligned} \tag{38}$$

Comparing the coefficient of $\frac{t^n}{n!}$ in both right-hand sides of (37) and (38) completes the proof. □

Let us define the numbers $B(n,k;\lambda|\alpha)$ by the following identity:

$$B(n,k;\lambda|\alpha) := k!y_1(n,k;\lambda|\alpha), \tag{39}$$

i.e., by means of the generating function

$$(\lambda e^{\frac{\log(1+\alpha t)}{\alpha}} + 1)^k = \sum_{n\geq 0} B(n,k;\lambda|\alpha)\frac{t^n}{n!}. \tag{40}$$

Observe that the first terms of $B(n,k;1|\alpha)$ for fixed n are of the form $c_k(\alpha)2^k$, for some function $c_k(\alpha)$:

$$B(0,k;1|\alpha) = 2^k,$$
$$B(1,k;1|\alpha) = k2^{k-1},$$
$$B(2,k;1|\alpha) = k(k+1-2\alpha)2^{k-2}.$$

Remark 11. Note that for $\alpha \to 0$ in Eq. (39), we obtain the numbers $B_0(n,k;\lambda) := B(n,k;\lambda) = k!y_1(n,k;\lambda)$, considered by Xu [32] and Goubi [7]. If moreover we set $\lambda = 1$, then we obtain Eq. (11).

Theorem 12. *Let n and k be non-negative integer. Then*

$$B(n,k;\lambda|\alpha) = \lambda^k \sum_{m=0}^{n} \sum_{j=0}^{m} A_{k,m-j}\left(\frac{1}{\lambda}\right) \binom{m}{j} k^j s(n,m) \alpha^{n-m}, \tag{41}$$

where $A_{k,m}(a) = \sum_{i=0}^{k} \binom{k}{i} a^i (-i)^m$.

Proof. We have

$$\begin{aligned}
\left(\lambda e^{\frac{\log(1+\alpha t)}{\alpha}} + 1\right)^k &= \sum_{i=0}^{k} \binom{k}{i} \lambda^{k-i} e^{\frac{k-i}{\alpha}\log(1+\alpha t)} \\
&= \sum_{i=0}^{k} \binom{k}{i} \lambda^{k-i} \left(\sum_{m\geq 0} \frac{k^m}{\alpha^m} \frac{(\log(1+\alpha t))^m}{m!} \right) \\
&\qquad \times \left(\sum_{m\geq 0} \frac{(-i)^m}{\alpha^m} \frac{(\log(1+\alpha t))^m}{m!} \right) \\
&= \sum_{n\geq 0} \sum_{m=0}^{n} \sum_{j=0}^{m} \sum_{i=0}^{k} \binom{k}{i} \binom{m}{j} \lambda^{k-i} \alpha^{n-m} k^j (-i)^{m-j} s(n,m) \frac{t^n}{n!}.
\end{aligned}$$

Therefore, using (40) and comparing the coefficient of $\frac{t^n}{n!}$, we arrive at (41). ◻

Theorem 13. *For any non-negative integers n and k, we have*

$$B(n,k;\lambda|\alpha) = (\lambda+1)^{k-n} \sum_{i=0}^{n} a_{n-i}(\alpha,\lambda) k^i, \tag{42}$$

where the coefficients $a_{n-i}(\alpha,\lambda)$ are given by the formula

$$a_{n-i}(\alpha,\lambda) = \sum_{j=i}^{n} \lambda^j (\lambda+1)^{n-j} s(j,i) S_2(n,j|\alpha), \text{ for } i = 0,\ldots,n.$$

Proof. It follows from Theorem 6 that

$$B(n,k;\lambda|\alpha) = \sum_{j=0}^{n} \binom{k}{j} \lambda^j j! (\lambda+1)^{k-j} S_2(n,j|\alpha). \tag{43}$$

Hence,

$$\begin{aligned}
B(n,k;\lambda|\alpha) &= \sum_{j=0}^{n}(k)_j\lambda^j(\lambda+1)^{k-j}S_2(n,j|\alpha)\\
&= \sum_{j=0}^{n}\sum_{i=0}^{j}s(j,i)\lambda^j k^i(\lambda+1)^{k-j}S_2(n,j|\alpha)\\
&= (\lambda+1)^{k-n}\sum_{i=0}^{n}k^i\sum_{j=i}^{n}\lambda^j(\lambda+1)^{n-j}S_2(n,j|\alpha)s(j,i)\\
&= (\lambda+1)^{k-n}\sum_{i=0}^{n}k^i a_{n-i}(\alpha,\lambda),
\end{aligned}$$

where in the second equality we use Eq. (2). □

Remark 14. Setting $\lambda = 1$ in (42) yields

$$B(n,k;1|\alpha) = 2^{k-n}\sum_{i=0}^{n}k^i a_{n-i}(\alpha,1),$$

where the coefficients $a_{n-i}(\alpha,1)$ can be expressed in terms of Stirling numbers of the first kind, and the degenerate Stirling numbers of the second kind, as follows:

$$a_{n-i}(\alpha,1) = \sum_{j=i}^{n}2^{n-j}s(j,i)S_2(n,j|\alpha).$$

Letting $\alpha \to 0$ in Eq. (42), we obtain

$$B(n,k;\lambda|0) = \big\{a_n(0,\lambda) + ka_{n-1}(0,\lambda) + \cdots + k^n a_0(0,\lambda)\big\}(\lambda+1)^{k-n}.$$

This result was given by Xu [32] as a general answer to the first Simsek open problem [28], regarding the coefficients $a_n(0,\lambda)$.

Let us define

$$H_n(x;\lambda|\alpha) := \sum_{k\geq 0}B(n,k;\lambda|\alpha)x^k \tag{44}$$

as a generating function of the numbers $B(n,k;\lambda|\alpha)$. Then we have the resulting theorem.

Theorem 15. *For any non-negative integer n, the generating function $H_n(x;\lambda|\alpha)$ is given as follows:*

$$H_n(x;\lambda|\alpha) = \begin{cases}\sum\limits_{j=1}^{n}\frac{j!\lambda^j S_2(n,j|\alpha)}{(1-x(\lambda+1))^{j+1}}x^j, & \text{for } n\geq 1;\\ \frac{1}{1-x(1+\lambda)}, & \text{for } n=0.\end{cases} \tag{45}$$

For proving the above theorem, we will use the following lemma, given by Goubi [6].

Lemma 16 ([6])**.** *For the formal generating function* $h(t) := \sum_{n\geq 0} a_n t^n$*, with* $a_0 \neq 0$ *and a complex number* $\beta \neq 0$*, we have*

$$h^{\beta}(t) = a_0^{\beta} + \sum_{n\geq 1}\sum_{k=1}^{n}(\beta)_k a_0^{\beta-k} B_{n,k}\Big(1!a_1, 2!a_2, \ldots, (n-k+1)!a_{n-k+1}\Big)\frac{t^n}{n!}, \tag{46}$$

where $B_{n,k}(x_1, \ldots, x_{n-k+1})$ *are exponential partial Bell polynomials* (8).

Proof of Theorem 15. It follows from Eqs. (40) and (44) that

$$\begin{aligned}\sum_{n\geq 0} H_n(x;\lambda|\alpha)\frac{t^n}{n!} &= \sum_{n\geq 0}\sum_{k\geq 0} B(n,k;\lambda|\alpha)x^k\frac{t^n}{n!}\\ &= \sum_{k\geq 0}(\lambda(1+\alpha t)^{\frac{1}{\alpha}}+1)^k x^k\\ &= \frac{1}{1-x(\lambda(1+\alpha t)^{\frac{1}{\alpha}}+1)}.\end{aligned}$$

Define $h(t) := (1-x(\lambda+1)) - \lambda x\sum_{n\geq 1}\left(\frac{1}{\alpha}\right)_n \alpha^n \frac{t^n}{n!}$ and set $\beta = -1$. Applying Lemma 16, we obtain

$$\begin{aligned}\sum_{n\geq 0} H_n(x;\lambda|\alpha)\frac{t^n}{n!} &= \frac{1}{1-x(\lambda+1)} + \Bigg(\sum_{n\geq 1}\sum_{k=1}^{n}(-1)^k k!(1-x(\lambda+1))^{-1-k}\\ &\qquad \times B_{n,k}\bigg(-\lambda x\alpha\Big(\frac{1}{\alpha}\Big)_1, -\lambda x\alpha^2\Big(\frac{1}{\alpha}\Big)_2, \ldots, -\lambda x\alpha^{n-k+1}\Big(\frac{1}{\alpha}\Big)_{n-k+1}\bigg)\Bigg)\frac{t^n}{n!}\\ &= \frac{1}{1-x(\lambda+1)} + \sum_{n\geq 1}\left(\sum_{k=1}^{n}(-1)^k k!(1-x(\lambda+1))^{-1-k}(-\lambda x)^k S_2(n,k|\alpha)\right)\frac{t^n}{n!},\end{aligned}$$

where in the second equality, we use Eq. (8). Hence, Eq. (45) follows. □

Remark 17. If we let $\alpha \to 0$ in Eq. (45), we get

$$H_n(x;\lambda|0) = \begin{cases} \sum_{j=1}^{n} \frac{j!\lambda^j S_2(n,j)}{(1-x(\lambda+1))^{j+1}}x^j, & \text{for } n \geq 1;\\ \frac{1}{1-x(1+\lambda)}, & \text{for } n = 0,\end{cases}$$

the generating function for $B(n,k;\lambda) := B(n,k;\lambda|0)$ given by Goubi [7, 8] and Xu [32] as a general answer to Simsek's second open question [28], regarding the generating function $H_n(x;\lambda|0) = \sum_{k\geq 0} B(n,k;\lambda|0)x^k$.

According to the formula

$$k!S_2(n,k|\alpha) = \sum_{j=1}^{k} \binom{k}{j} (-1)^{k-j} \left(\frac{j}{\alpha}\right)_n \alpha^n, \tag{47}$$

the generating function $H_n(x;\lambda|\alpha)$ can also be expressed as follows:

$$H_n(x;\lambda|\alpha) = \sum_{k=1}^{n} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \alpha^n \left(\frac{j}{\alpha}\right)_n \frac{\lambda^k x^k}{(1-x(\lambda+1))^{j+1}}. \tag{48}$$

Let us observe that the generating function $H_n(x;\lambda|\alpha)$ can be written as a rational function:

$$H_n(x;\lambda|\alpha) = \frac{P_n(x;\lambda|\alpha)}{Q_{n+1}(x;\lambda)}, \tag{49}$$

where

$$P_n(x;\lambda|\alpha) = \begin{cases} \sum_{k=1}^{n} \lambda^k k! S_2(n,k|\alpha)(1-x(\lambda+1))^{n-k} x^k, & \text{for } n \geq 1; \\ 1, & \text{for } n = 0, \end{cases}$$

and

$$Q_n(x;\lambda) = \frac{1}{(1-x(\lambda+1))^n}.$$

The examples of $H_n(x;\lambda|\alpha)$ for some fixed values of n are given as follows:

$$H_0(x;\lambda|\alpha) = \frac{1}{1-x(1+\lambda)},$$

$$H_1(x;\lambda|\alpha) = \frac{\lambda x}{(1-x(1+\lambda))^2},$$

$$H_2(x;\lambda|\alpha) = \frac{\lambda(1+\alpha)x - (\lambda(\alpha+1) - \lambda^2(1-\alpha))x^2}{(1-x(1+\lambda))^3}.$$

Theorem 18. *Let n be a non-negative integer. The generating function $H_n(x;\lambda|\alpha)$ satisfies the following recursive formula:*

$$H_0(x;\lambda|\alpha) = \frac{1}{1-x(\lambda+1)} \tag{50}$$

and

$$H_n(x;\lambda|\alpha) = \frac{\lambda x}{1-x(\lambda+1)} \left(\sum_{k=0}^{n-1} \binom{n}{k} (1|\alpha)_{n-k} H_k(x;\lambda|\alpha) \right) \quad \text{for } n \geq 1, \tag{51}$$

where $(x|\alpha)_n := x(x-\alpha)(x-2\alpha)\cdots(x-(n-1)\alpha)$.

Proof. From the definition of $H_n(x;\lambda|\alpha)$, we can derive the following:

$$\begin{aligned}
\sum_{n\geq 0} H_n(x;\lambda|\alpha)\frac{t^n}{n!} &= \frac{1}{1-x(\lambda+1)} + \sum_{n\geq 1}\sum_{j=1}^{n} \frac{j!\lambda^j S_2(n,j|\alpha)x^j}{(1-x(\lambda+1))^{j+1}}\frac{t^n}{n!} \\
&= \frac{1}{1-x(\lambda+1)} + \sum_{j=1}^{\infty} j!\frac{(\lambda x)^j}{(1-x(\lambda+1))^{j+1}} \sum_{n=j}^{\infty} S_2(n,j|\alpha)\frac{t^n}{n!} \\
&= \frac{1}{1-x(\lambda+1)} + \sum_{j=1}^{\infty} \frac{(\lambda x)^j}{(1-x(\lambda+1))^{j+1}} (e^{\frac{\log(1+\alpha t)}{\alpha}} - 1)^j \\
&= \frac{1}{1-x(\lambda+1)} + \frac{1}{(1-x(\lambda+1))}\left(\frac{1}{1-\frac{\lambda x\left(e^{\frac{\log(1+\alpha t)}{\alpha}}-1\right)}{1-x(1+\lambda)}} - 1\right) \\
&= \frac{1}{1-x-\lambda e^{\frac{\log(1+\alpha t)}{\alpha}}},
\end{aligned}$$

where in the third equality, we use Eq. (6). By multiplying both sides of the above identity by $(1-x-x\lambda e^{\frac{\log(1+\alpha t)}{\alpha}})$ and comparing the coefficient of $\frac{t^n}{n!}$, we obtain the desired result. $\square$

Remark 19. Letting $\alpha \to 0$ in Eqs. (50) and (51), we obtain

$$H_0(x;\lambda|0) = \frac{1}{1-x(\lambda+1)}$$

and

$$H_n(x;\lambda|0) = \frac{\lambda x}{1-x(\lambda+1)}\left(\sum_{k=0}^{n-1}\binom{n}{k} H_{k,0}(x;\lambda)\right), \text{ for } n \geq 1.$$

This result has already been obtained by Goubi [7, 8] and Xu [32]. It answered the second Simsek problem [28], regarding the explicit formula for the generating function $H_n(x;1|0) = \sum_{k\geq 0} B(n,k;1|0)x^k$. For instance:

$$\begin{aligned}
H_0(x;1|0) &= \frac{1}{1-2x}, \\
H_1(x;1|0) &= \frac{x}{(1-2x)^2}, \\
H_2(x;1|0) &= \frac{x}{(1-2x)^3}.
\end{aligned}$$

2.2 Degenerate Apostol-type Euler numbers

In this section, we introduce the degenerate first and second kind Apostol-type Euler numbers, and establish a relation between these numbers and the degenerate Simsek numbers $y_1(n,k;\lambda|\alpha)$.

For $k \in \mathbb{N}_0$, we define the degenerate second kind Apostol-type Euler polynomials $E_n^{*(k)}(x;\lambda|\alpha)$ of order k as the coefficients of the following Taylor power series:

$$\begin{aligned} F_P(t,x;k,\lambda|\alpha) :&= \left(\frac{2}{\lambda e^{\frac{\log(1+\alpha t)}{\alpha}} + \lambda^{-1} e^{\frac{\log(1-\alpha t)}{\alpha}}}\right)^k e^{\frac{x}{\alpha}\log(1+\alpha t)} \\ &= \sum_{n\geq 0} E_n^{*(k)}(x;\lambda|\alpha)\frac{t^n}{n!}. \end{aligned} \tag{52}$$

Substituting $x=0$ in Eq. (52), we obtain the degenerate second kind Apostol-Euler numbers of order k, denoted by $E_n^{*(k)}(\lambda|\alpha)$ and given by the means of the generating function:

$$F_{E^*}(t;k,\lambda|\alpha) := \left(\frac{2}{\lambda e^{\frac{\log(1+\alpha t)}{\alpha}} + \lambda^{-1} e^{\frac{\log(1-\alpha t)}{\alpha}}}\right)^k = \sum_{n\geq 0} E_n^{*(k)}(\lambda|\alpha)\frac{t^n}{n!}. \tag{53}$$

Moreover, we define the degenerate first kind Apostol-Euler numbers of order k, denoted by $E_n^{(k)}(\lambda|\alpha)$, as follows:

$$G_E(t;k,\lambda|\alpha) := \left(\frac{2}{\lambda e^{\frac{\log(1+\alpha t)}{\alpha}} + 1}\right)^k = \sum_{n\geq 0} E_n^{(k)}(\lambda|\alpha)\frac{t^n}{n!}. \tag{54}$$

Remark 20. It follows that for $\alpha \to 0$, the numbers $E_n^{(k)}(\lambda|\alpha)$ and $E_n^{*(k)}(\lambda|\alpha)$ reduce to first and second kind Apostol-Euler numbers, respectively (cf. [24, 28]).

The relation between the numbers $E_n^{(k)}(\lambda|\alpha)$ and $E_n^{*(k)}(\lambda|\alpha)$ is given by the following theorem:

Theorem 21. *Let k be a non-negative integer. Then*

$$\begin{aligned} E_n^{*(-k)}(\lambda|\alpha) = \sum_{j=0}^{k}\binom{k}{j}2^{j-k}(k-j)! \sum_{m=k-j}^{\infty} s(m,k-j)\alpha^{m+j-k}(n)_m \sum_{l=0}^{n-m}(-1)^{n-l} \\ \times \binom{n-m}{l} E_l^{(-j)}(\lambda|\alpha) B_{n-m-l}^{(-k+j)}(\lambda^{-1}|\alpha), \end{aligned} \tag{55}$$

where

$$H_B(t;k,\lambda|\alpha) := \left(\frac{\log(1+\alpha t)}{\alpha\left(\lambda e^{\frac{\log(1+\alpha t)}{\alpha}} - 1\right)}\right)^k = \sum_{n\geq 0} B_n^{(k)}(\lambda|\alpha)\frac{t^n}{n!},$$

and $B_n^{(k)}(\lambda|\alpha)$ can be considered as the degenerate Apostol-Bernoulli numbers of order k, which reduce, for $\alpha \to 0$, to the well-known Apostol-Bernoulli numbers $B_n^{(k)}(\lambda)$ of order k, defined by the generating function:

$$H_B(t;k,\lambda) := \left(\frac{t}{\lambda e^t - 1}\right)^k = \sum_{n\geq 0} B_n^{(k)}(\lambda)\frac{t^n}{n!},$$

(cf. [19, 25, 31]).

Proof. According to Eqs. (53) and (54), we get

$$\begin{aligned} F_{E^*}(t;-k,\lambda|\alpha) = \sum_{j=0}^{k} \binom{k}{j} 2^{j-k}\alpha^{j-k}\big(\log(1-\alpha t)\big)^{k-j} G_E(t;-j,\lambda|\alpha) \\ \times H_B(-t;-k+j,\lambda^{-1}|\alpha). \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{n\geq 0} E_n^{*(-k)}(\lambda|\alpha)\frac{t^n}{n!} &= \sum_{j=0}^{k} 2^{j-k}\alpha^{j-k}\big(\log(1-\alpha t)\big)^{k-j}\left(\sum_{n\geq 0} E_n^{(-j)}(\lambda|\alpha)\frac{t^n}{n!}\right) \\ &\quad \times \left(\sum_{n\geq 0} B_n^{(-k+j)}(\lambda^{-1}|\alpha)\frac{(-t)^n}{n!}\right) \\ &= \sum_{j=0}^{k} 2^{j-k}\alpha^{j-k}\left((k-j)!\sum_{m=k-j}^{\infty} s(m,k-j)\frac{(-\alpha t)^m}{m!}\right) \\ &\quad \times \left(\sum_{n\geq 0} E_n^{(-j)}(\lambda|\alpha)\frac{t^n}{n!}\right)\left(\sum_{n\geq 0} B_n^{(-k+j)}(\lambda^{-1}|\alpha)\frac{(-t)^n}{n!}\right) \\ &= \sum_{n\geq 0}\sum_{j=0}^{k} 2^{j-k}(k-j)!\sum_{m\geq 0} s(m,k-j)\alpha^{m+j-k}(n)_m \\ &\quad \times \sum_{l=0}^{n-m}(-1)^{n-l}\binom{n-m}{l} E_l^{(-j)}(\lambda|\alpha) B_{n-m-l}^{(-k+j)}(\lambda^{-1}|\alpha)\frac{t^n}{n!}. \end{aligned}$$

Thus, by equating the coefficient of $\frac{t^n}{n!}$, we immediately arrive at Eq. (55). □

Using Eqs. (16) and (54), we obtain the following theorem.

Theorem 22. *Let k be a non-negative integer. Then we have the following relation between the degenerate first kind Apostol-Euler numbers $E_n^{(-k)}(\lambda|\alpha)$ and the degenerate Simsek numbers $y_1(n,k;\lambda|\alpha)$:*

$$k!y_1(n,k;\lambda|\alpha) = 2^k E_n^{(-k)}(\lambda|\alpha). \tag{56}$$

Example 23. According to Example 3, we have the following special cases of the degenerate first kind Apostol-Euler numbers $E_n^{(-k)}(\lambda|\alpha)$:

$$\begin{aligned} E_0^{(-k)}(\lambda|\alpha) &= 2^{-k}(\lambda+1)^k, \\ E_n^{(0)}(\lambda|\alpha) &= \delta_{n,0}, \\ E_n^{(-1)}(\lambda|\alpha) &= \frac{1}{2}\big(\delta_{n,0} + \lambda\big(\frac{1}{\alpha}\big)_n \alpha^n\big). \end{aligned}$$

Substituting $\lambda = 1$ and letting $\alpha \to 0$ in Eq. (56), yields

$$E_n^{(-k)} = 2^{-k}\sum_{j=0}^{k}\binom{k}{j} j^n,$$

the Euler numbers of the first kind of order $-k$, given by Simsek [28].

3 Conclusion

In this paper, we studied the degenerate Simsek numbers $y_1(n,k;\lambda|\alpha)$ and the (λ,α)-Stirling numbers of the second kind $S_2(n,k;\lambda|\alpha)$, based on generating functions. Some properties of $S_2(n,k;\lambda|\alpha)$ and $y_1(n,k;\lambda|\alpha)$, such as recurrence formulas, were presented. We also provided a relation between these numbers and with some kinds of special numbers. In particular, by specializing the parameters λ and α, we obtained the familiar Stirling numbers of the second kind, Simsek numbers [28], the λ-Stirling numbers of the second kind [19, 31], the degenerate Stirling numbers of the second kind [13, 14], the weighted Stirling numbers of the second kind [10], and the degenerate associated Stirling numbers of the second kind [9]. Finally, we introduced the degenerate first and second kind Apostol-Euler numbers of order k, and derived some related properties. A theorem which relates these numbers and $y_1(n,k;\lambda|\alpha)$ to each other was given.

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