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Patterns in Continued Fractions of Square Roots

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Abstract

We examine the structure of the periodic continued fractions of square roots of non-square positive integers given by an integer-valued quadratic polynomial $Q(n) = (an+b)^2 + (gn+h)$. The aim is to identify repeated blocks of partial quotients in the period. The quotients in the period form a palindrome, and when the period length is even, the period has a central term a_n. The paper focuses on periods with $a_n = a_0$ or $a_n = a_0 - 1$, where a_0 is the initial partial quotient. For $a_n = a_0$ we give an algorithm to obtain formulas involving repeated blocks comprising three or more elements, not all equal.

1 Introduction

1.1 Preliminaries

Irrational square roots entered into mathematics with the Pythagorean theorem which led to the discovery of the irrationality of $\sqrt{2}$. About 400 BCE, rational approximations $\frac{a}{b}$ to $\sqrt{2}$ appeared in India and Greece. Bombelli (*Algebra*, 1572) and Cataldi, who continued Bombelli's work, introduced continued fractions to approximate square roots. For an irrational number α, an expression of the form

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

with $a_i \in \mathbb{N}$ for $i \geq 0$, is called a *simple continued fraction* (scf) of α. It is generally denoted by a space-saving symbolism: $[a_0; a_1, a_2, a_3, \ldots]$. The integers a_i are called *partial quotients*. The rational number represented by the truncated continued fraction $[a_0; a_1, a_2, a_3, \ldots, a_n]$ is called the nth *convergent* (c_n) of α. If we define the sequences $\{p_k\}$, $\{q_k\}$:

$$\begin{aligned} p_{-2} &= 0, \quad p_{-1} = 1, \quad p_k = a_k\, p_{k-1} + p_{k-2} \text{ for } k \geq 0, \\ q_{-2} &= 1, \quad q_{-1} = 0, \quad q_k = a_k\, q_{k-1} + q_{k-2} \text{ for } k \geq 0, \end{aligned}$$

then $[a_0; a_1, \ldots, a_k] = \dfrac{p_k}{q_k}$. Note that $\begin{bmatrix} p_{-1} & p_{-2} \\ q_{-1} & q_{-2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity matrix.

If the quotients repeat from a point r onward, i.e., $a_{m\ell+r+k} = a_{r+k}$, $m \in \mathbb{N}$, $0 \leq k \leq \ell$, with period length ℓ, then the scf is said to be *periodic* and is written as

$$[a_0; a_1, a_2, \ldots a_{r-1}, \overline{a_r, a_{r+1}, a_{r+2}, \ldots, a_{r+\ell-1}}\,].$$

If the scf repeats from the start, $[\,\overline{a_0, a_1, \ldots, a_n}\,]$, it is called *purely* periodic.

Euler proved in 1737 that the value of every periodic scf is a quadratic irrational of the form $\frac{P+\sqrt{d}}{Q}$ with $P, Q \in \mathbb{Z}$, $Q \neq 0$ and $d \in \mathbb{Z}^+$ not a perfect square. In 1770, Lagrange proved the converse of Euler's theorem that each quadratic irrational has a periodic scf expansion.

The *continued fraction algorithm*, the algorithm for obtaining the scf of $\sqrt{d}$, is given in many books. See [24, § 8.4]. The period of the scf of $\sqrt{d}$ is symmetrical and for period length $\ell(d)$ its form is

$$\sqrt{d} = [a_0;\ \overline{a_1, a_2, a_3, \ldots, a_{\ell-3}, a_{\ell-2}, a_{\ell-1}, 2a_0}\,]$$

with $a_1 = a_{\ell-1}$, $a_2 = a_{\ell-2}, \ldots$; $a_1, a_2, \ldots, a_{\ell-1}$ form a palindrome that may or may not have a central term. When $\ell(d) = 2n$, the period is symmetric around a_n with $a_{n+1} = a_{n-1}$, $a_{n+2} = a_{n-2}$, and so on. If $\ell(d) = 2n+1$, then $a_{n+1} = a_n$, $a_{n+2} = a_{n-1}$, and so on. $\ell(d)$ is odd if and only if $d = a^2 + b^2$, $\gcd(a, b) = 1$ [21]. In the scf expansion of $\sqrt{d}$ with period length ℓ, each partial quotient a_k for $0 \leq k < \ell$ satisfies $a_k < \sqrt{d}$ [15, p. 245, 3(f)] .

1.2 Schinzel's criterion for bounded period length

Schinzel [23] gave two theorems on the period length ℓ in the scf of $\sqrt{f(n)}$.
(A) Let $f(x) = \sum_{k=0}^{p} c_k\, x^{p-k}$ be an integer-valued polynomial with $c_0 > 0$. If (i) p is odd or (ii) p is even and c_0 is not a rational square, then $\overline{\lim}\ \ell\left(\sqrt{f(n)}\right) = \infty$.
(B) Let $f(n) = a^2n^2 + bn + c,\ a, b, c \in \mathbb{Z},\ a > 0$. Then the inequality $\overline{\lim}\ \ell\left(\sqrt{f(n)}\right) < \infty$ holds if and only if $\Delta \mid 4\gcd(2a^2, b)^2$ where $\Delta = b^2 - 4a^2c \neq 0$.

These two theorems together fully solved the problem of when $\ell\left(\sqrt{f(n)}\right)$ can be bounded independently of n if $f(n)$ is an integer-valued quadratic polynomial [11, pp. 134–135].

In [20], van der Poorten and Williams obtained the scf of $\sqrt{A^2X^2 + 2BX + C}$ in terms of C's scf expansion. In this case Schinzel's condition becomes $B^2 - A^2C \mid 4\gcd(A^2, B)^2$.

Cheng et al. [6, 7] considered the polynomial $D(X) = A^2X^2 + 2BX + C$ that satisfies Schinzel's condition. They found that the scf of $\sqrt{D(X)}$ has the form

$$[q_0(X);\ \overline{\mathcal{S}_0, q_1(X), \mathcal{S}_1, q_2(X), \ldots, \mathcal{S}_{\kappa-1}, q_\kappa(X)}\,].$$

Here the period comprises κ segments, each consisting of a string $\mathcal{S}_i$ (an ordered set of natural numbers) followed by a linear function $q_{i+1}(X)$. Using the notation $\overleftarrow{\mathcal{S}_j}$ for the reverse of $\mathcal{S}_j$, the symmetry of the period shows that $\mathcal{S}_{\kappa-1} = \overleftarrow{\mathcal{S}_0}$, $\mathcal{S}_{\kappa-2} = \overleftarrow{\mathcal{S}_1}$, and so on. $q_0(X) = a_0$, $q_\kappa(X) = 2q_0(X)$. Furthermore, $q_{\kappa-1}(X) = q_1(X)$, $q_{\kappa-2}(X) = q_2(X)$, and so on.

We present two types of results (presumably new) obtained by us: (i) Scf expansions in individual formulas (numbered in the text) containing two parameters m, n; (ii) Theorems yielding polynomials that give *predictable patterns each consisting of a single string or block repeated k times*. Throughout the paper, d is a non-square positive integer generated by $Q(n) = (an + b)^2 + (gn + h)$, $a, b, g, h \in \mathbb{Z}$, $(gn + h) < 2(an + b)$. This form of polynomial makes a_0 evident. Our technique involves 'tweaking' the continued fraction of some smaller quadratic irrational whose quotients replicate themselves endlessly as a singleton, a pair, triple, etc. Tweaking inserts a_0 or $a_0 - 1$ in the middle of the pattern. We use the matrix method to validate our methodology.

1.3 Two useful continued fraction expansions

Consider the equation $x^2 - (2m + 1)x - 1 = 0$ with roots $\frac{2m+1\pm\sqrt{(2m+1)^2+4}}{2}$. The equation can be written as $x = (2m + 1) + \frac{1}{x}$. Replacing x by $(2m + 1) + \frac{1}{x}$ in the RHS repeatedly leads to the scf of the positive root of the preceding equation:

$$\frac{2m + 1 + \sqrt{(2m + 1)^2 + 4}}{2} = 2m + 1 + \cfrac{1}{2m + 1 + \cfrac{1}{2m + 1 + \cfrac{1}{2m + 1 + \cdots}}},$$

which is denoted by $[2m+1;\overline{2m+1}\,]$. The convergents of this scf are given by

$$\frac{u_2}{u_1},\ \frac{u_3}{u_2},\ldots,\frac{u_{n+1}}{u_n},\ldots.$$

where u_n ($n \in \mathbb{N}_0$) has the closed form

$$u_n = \frac{\left(2m+1+\sqrt{(2m+1)^2+4}\right)^n - \left(2m+1-\sqrt{(2m+1)^2+4}\right)^n}{2^n\ \sqrt{(2m+1)^2+4}}. \tag{1}$$

Note that u_n is even when $n = 3r$, and odd when $n = 3r+1,\ 3r+2$.

The value $m = 0$ yields a purely periodic scf for the *golden ratio* whose convergents are ratios of the consecutive Fibonacci numbers $\frac{F_{n+1}}{F_n}$. The Fibonacci sequence $(F_n)_{n\geq 0}$ is given by the recurrence relation

$$F_{n+1} = F_n + F_{n-1} \text{ for } n \geq 1 \text{ with } F_0 = 0,\ F_1 = 1.$$

In this case (1) reduces to the well-known Euler-Binet closed formula for F_n.

It is obvious that $x^2 - 2mx - k = 0$ ($m \in \mathbb{N}$, $1 \leq k \leq m$) yields the quadratic irrational $m \pm \sqrt{m^2+k}$.

The equation: $x^2 - 2x - 1 = 0$, having roots $1 \pm \sqrt{2}$, can be rewritten as $x^2 = 2x + 1$ or $x = 2 + \frac{1}{x}$. Substituting $2 + \frac{1}{x}$ for x repeatedly in the RHS yields the scf for the positive root:

$$1+\sqrt{2} = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2+\cdots}}} \Longrightarrow \sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2+\cdots}}},$$

where the left scf is purely periodic unlike the right one. In general, the scf of $\sqrt{d}$ is not purely periodic, while that of $\lfloor\sqrt{d}\rfloor + \sqrt{d}$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, is purely periodic.

The convergents of $\sqrt{2} = [1;\overline{2}\,]$ are given by $\frac{p_n}{q_n} = \frac{2p_{n-1}+p_{n-2}}{2q_{n-1}+q_{n-2}}$ for $n \geq 1$ with $p_0 = a_0 = 1, q_0 = 1, p_{-1} = 1, q_{-1} = 0$. So, for $n \geq 0$ we have

$$\frac{1}{1},\ \frac{3}{2},\ \frac{7}{5},\ \frac{17}{12},\ \frac{41}{29},\ \frac{99}{70},\ \frac{239}{169},\ldots.$$

$\langle \frac{p_{2n-1}-1}{2},\ \frac{p_{2n-1}+1}{2}, q_{2n-1}\rangle$ constitute Pythagorean triples. The numerators and denominators occur in A001333 and A000129 [25]. Their closed forms are

$$p_k = \frac{(1+\sqrt{2})^k + (1-\sqrt{2})^k}{2}; \quad q_k = \frac{(1+\sqrt{2})^k - (1-\sqrt{2})^k}{2\sqrt{2}}.$$

From the closed forms, we can deduce the following relations:

$$p_k = q_{k+1} - q_k = q_k + q_{k-1}, \tag{2a}$$

$$2q_k = p_{k+1} - p_k = p_k + p_{k-1}. \tag{2b}$$

2 Matrix method in continued fractions

2.1 Linear recurrences with constant coefficients

The general form of a second-order linear recurrence with constant integer coefficients is $u_{n+1} = a\,u_n + b\,u_{n-1}$ with $a, b \in \mathbb{Z}\,(b \neq 0)$. Its characteristic equation is $x^2 - ax - b = 0$. If α, β are its roots, then $\alpha+\beta = a,\ \alpha\cdot\beta = -b,\ \alpha-\beta = \sqrt{a^2+4b}$. If $a^2+4b > 0$, both roots are real with $\alpha \neq \beta$. In this case the general solution of the given recurrence is $u_n = \lambda\alpha^n + \mu\beta^n$ for $n = 0, 1, 2, \ldots$ for arbitrary numbers λ and μ. If two initial values u_0, u_1 are given, these two numbers are uniquely determined by $\lambda + \mu = u_0;\ \lambda\alpha + \mu\beta = u_1$ [16, p. 199, Th.4.10, eq(4.5)].

Lenstra and Shallit [14] proved that the numerators and denominators of the convergents to an irrational number θ satisfy a (sometimes higher order) linear recurrence with constant coefficients if and only if θ is a quadratic irrational.

2.2 Correspondence between matrices and convergents

Given a sequence $a_0, a_1, a_2, \ldots$, we have

$$\begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_2 & 1 \\ 1 & 0 \end{bmatrix}\cdots\begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} \quad \text{for } n = 0, 1, 2, \ldots,$$

if and only if $\dfrac{p_n}{q_n} = [a_0; a_1, \ldots, a_n]$ for $n = 0, 1, 2, \ldots$. This sets up a correspondence between certain products of 2×2 matrices and continued fractions ([2, p. 45], [4, p. 142], [5, p. 28], [9], [10, p. 244], [18, p. 104], [19, p. 87]).

If $\alpha = [0; a_1, a_2, a_3, \ldots]$ and hence $\dfrac{p_n}{q_n} = [0; a_1, a_2, \ldots, a_n]$, then the convergents get inverted:

$$M_n := \begin{bmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_2 & 1 \\ 1 & 0 \end{bmatrix}\cdots\begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix}$$

and the matrix M_n is symmetrical if and only if $a_1, a_2, \ldots, a_n$ is a palindrome and so $p_n = q_{n-1}$. See proof of Theorem 2.1 in [1]. Note that in this case we have for all $n \geq 1$ that

$$\begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_2 & 1 \\ 1 & 0 \end{bmatrix}\cdots\begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} q_n & q_{n-1} \\ q_{n-1} & q_{n-2} \end{bmatrix}.$$

Writing $\alpha = a_0 + \frac{1}{\alpha_1},\ \alpha_1 = a_1 + \frac{1}{\alpha_2}, \ldots, \alpha_n = a_n + \frac{1}{\alpha_{n+1}}\ (\alpha_i > 1)$, i.e., $\alpha = [a_0, a_1, \ldots, a_n, \alpha_{n+1}]$ with $\alpha_{n+1} = [a_{n+1}, a_{n+2}, \ldots]$, we have the following formula expressing α in terms of the *complete quotient* α_{n+1} and two neighbouring convergents ([8, p. 80, eq(14)], [2, p. 45], [4, p. 148, 2.6]):

$$\alpha = \frac{\alpha_{n+1}p_n + p_{n-1}}{\alpha_{n+1}q_n + q_{n-1}}. \tag{3}$$

Given a continued fraction for a number of the form $\sqrt{d}$ (with d a non-square integer):

$$\sqrt{d} = [a_0;\ \overline{a_1, a_2, a_3, \ldots, a_{\ell-3}, a_{\ell-2}, a_{\ell-1}, 2a_0}\,]$$

with period length ℓ, Rippon and Taylor [21] deduced using (3) the following lemma:

Lemma 1. *If*

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_2 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_{\ell-1} & 1 \\ 1 & 0 \end{bmatrix}$$

then $\sqrt{d} = \sqrt{a^2 + b}$ *with*

$$a = a_0; \qquad b = \frac{2a_0 B + C}{A} \tag{4}$$

Proof. As in [21], if $\beta = a_0 + \alpha = [\overline{2a_0, a_1, a_2, \ldots, a_{\ell-1}}\,]$, the convergents of β are found from the columns of

$$\begin{bmatrix} 2a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix},$$

so by periodicity and (3),

$$\beta = \frac{(2a_0 A + B)\beta + (2a_0 B + C)}{A\beta + B}.$$

Solving for β leads to the desired result. □

2.3 Power of the matrix associated with convergents of $\sqrt{2}$

As Khovanskii [12, p. 292, Ex. 20] states, the matrix

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} p_0 & 2q_0 \\ q_0 & p_0 \end{bmatrix}$$

leads to $\sqrt{2} = [1; \overline{2}\,]$.

Using the values from Subsection 1.3, we deduce the relation

Lemma 2.

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^k = \begin{bmatrix} p_{k-1} & 2q_{k-1} \\ q_{k-1} & p_{k-1} \end{bmatrix}.$$

Proof. We use induction. The statement is obviously true for $k = 1$. Assume it to be true for $k = m$ so that

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^m = \begin{bmatrix} p_{m-1} & 2q_{m-1} \\ q_{m-1} & p_{m-1} \end{bmatrix}.$$

Now

$$\begin{bmatrix} p_{m-1} & 2q_{m-1} \\ q_{m-1} & p_{m-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} p_{m-1} + 2q_{m-1} & 2p_{m-1} + 2q_{m-1} \\ p_{m-1} + q_{m-1} & p_{m-1} + 2q_{m-1} \end{bmatrix} = \begin{bmatrix} p_m & 2q_m \\ q_m & p_m \end{bmatrix},$$

by using (2a) and (2b). Thus we get

$$\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}^{m+1} = \begin{bmatrix}p_m & 2q_m\\ q_m & p_m\end{bmatrix}.$$

The statement is thus true for $k = m + 1$ also. Hence it is true for all k. □

Next, we record a useful formula:

$$\begin{bmatrix}2 & 1\\ 1 & 0\end{bmatrix}^{k} = \begin{bmatrix}q_k & q_{k-1}\\ q_{k-1} & q_{k-2}\end{bmatrix}$$

provable by induction. We then have two relevant products, also provable by induction:

$$\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix} \cdot \begin{bmatrix}2 & 1\\ 1 & 0\end{bmatrix}^{k} = \begin{bmatrix}p_k & p_{k-1}\\ q_k & q_{k-1}\end{bmatrix}; \quad \begin{bmatrix}2 & 1\\ 1 & 0\end{bmatrix}^{k} \cdot \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}p_k & q_k\\ p_{k-1} & q_{k-1}\end{bmatrix}. \tag{5}$$

2.4 Matrix associated with $\sqrt{3}$

The convergents of $\sqrt{3} = [1; \overline{1, 2}\,]$ are $\frac{1}{1}, \frac{2}{1}, \frac{5}{3}, \frac{7}{4}, \frac{19}{11}, \frac{26}{15}, \frac{71}{41}, \frac{97}{56}, \frac{265}{153}, \ldots$, whose numerators occur in A002531 and denominators in A002530 and have these closed forms:

$$p_{2n-1} = \frac{(1+\sqrt{3})^{2n} + (1-\sqrt{3})^{2n}}{2^{n+1}}; \; q_{2n-1} = \frac{(1+\sqrt{3})^{2n} - (1-\sqrt{3})^{2n}}{2^{n+1}\sqrt{3}},$$
$$p_{2n} = \frac{(1+\sqrt{3})^{2n+1} + (1-\sqrt{3})^{2n+1}}{2^{n+1}}; \; q_{2n} = \frac{(1+\sqrt{3})^{2n+1} - (1-\sqrt{3})^{2n+1}}{2^{n+1}\sqrt{3}}.$$

From the closed forms, we deduce the following relations:

$$p_{2n-1} = q_{2n} - q_{2n-1}; \; p_{2n+1} = q_{2n} + q_{2n+1}; \tag{6}$$
$$3q_{2n} + 2q_{2n-1} = q_{2n+2}; \; 2q_{2n-1} + q_{2n-2} = q_{2n}. \tag{7}$$

By using the relations (6) and (7), we can prove (for $k \in \mathbb{N}$) by induction:

Lemma 3.

$$\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}^{k} = \begin{bmatrix}q_{2k} & 2q_{2k-1}\\ q_{2k-1} & q_{2k-2}\end{bmatrix}. \tag{8}$$

2.5 Power of a general matrix

We introduce a matrix to be used later.

Lemma 4.

$$\begin{bmatrix}2m+1 & 1\\ 1 & 0\end{bmatrix}^{n} = \begin{bmatrix}u_{n+1} & u_n\\ u_n & u_{n-1}\end{bmatrix},$$

where u_n satisfies the recurrence relation

$$u_{n+1} = (2m+1)\; u_n + u_{n-1} \text{ with } u_1 = 1,\; u_0 = 0.$$

Proof. The statement is obviously true for $n = 1$ in view of (4). Assume the statement to be true for $n = r$ so that

$$\begin{bmatrix} 2m+1 & 1 \\ 1 & 0 \end{bmatrix}^r = \begin{bmatrix} u_{r+1} & u_r \\ u_r & u_{r-1} \end{bmatrix}.$$

Then

$$\begin{aligned} \begin{bmatrix} 2m+1 & 1 \\ 1 & 0 \end{bmatrix}^{r+1} &= \begin{bmatrix} u_{r+1} & u_r \\ u_r & u_{r-1} \end{bmatrix} \begin{bmatrix} 2m+1 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} (2m+1)u_{r+1} + u_r & u_{r+1} \\ (2m+1)u_r + u_{r-1} & u_r \end{bmatrix} = \begin{bmatrix} u_{r+2} & u_{r+1} \\ u_{r+1} & u_r \end{bmatrix} \end{aligned}$$

by using (4). So the statement is true for $n = r + 1$ also. Hence it is true for all n. □

3 Formulas without central term or central term $< a_0$

3.1 Selected formulas without central term

3.1.1 Formulas with $\ell(d) = 2$

These two are the only expansions ($m \geq 1$) of length 2 [3, Th. 3].

$$\sqrt{(mn)^2 + n} = [mn; \overline{2m, 2mn}\,], n > 1; \ \sqrt{(mn)^2 + 2n} = [mn; \overline{m, 2mn}\,], n \geq 1. \tag{9}$$

3.1.2 Formulas with $\ell(d) = 3$

This formula of Perron [17, p. 100] gives the only possible scf:

$$\sqrt{((4m^2+1)n + m)^2 + 4mn + 1} = [(4m^2+1)n + m; \overline{2m, 2m, 2((4m^2+1)n + m)}\,].$$

3.1.3 Formula with $\ell(d) = 5$

This formula will be used in a later section:

$$\sqrt{(2n+1)^2 + 4} = [2n+1; \ \overline{n, 1, 1, n, 4n+2}\,]. \tag{10}$$

Proof. Writing $\sqrt{d} = \sqrt{a^2 + b} = a + y$, and $y = [0; \frac{a-1}{2}, 1, 1, \frac{a-1}{2}, 2a + y]$ gives the equation $y^2 + 2ay - 4 = 0$ with roots $y = -a \pm \sqrt{a^2 + 4}$. For $\frac{a-1}{2}$ to be integer, a must be $2n + 1$. □

3.2 Formulas with central term $< a_0 - 1$

3.2.1 Formula with $\ell(d) = 6$

We can easily establish the following formula ($n \in \mathbb{N}$) by the continued fraction algorithm:

$$\sqrt{(2n+2)^2 + (4n+1)} = [2n+2; \ \overline{1, n, 2, n, 1, 2(2n+2)}\,].$$

3.2.2 Formula with $\ell(d) = 8$

This formula ($n \in \mathbb{N}$) can be proved by the continued fraction algorithm:

$$\sqrt{(4n+5)^2 + (8n+3)} = [4n+5;\ \overline{1, n, 2, 2n+2, 2, n, 1, 2(4n+5)}\,].$$

3.2.3 Formula with $\ell(d) = 12$

This formula ($n = 0, 1, 2, \dots$) can be proved by the continued fraction algorithm:

$$\sqrt{(209n+48)^2 + (264n+61)} = [209n+48; \overline{1, 1, 1, 2, 2, 38n+8, 2, 2, 1, 1, 1, 2(209n+48)}\,].$$

3.3 Formulas with central term $a_0 - 1$

3.3.1 Formula with $\ell(d) = 10$

We discovered this formula for scf expansions for $m, n \in \mathbb{N}$:

$$\begin{aligned}\sqrt{((8m^2+1)n + 2m(4m-1))^2 + ((4m-1)^2+1)n + (4m-1)^2 - 2(2m-1)} = \\ [(8m^2+1)n + 2m(4m-1); \overline{1, 2m-1, 2m+1, 1, (8m^2+1)n + 2m(4m-1) - 1,} \\ \overline{1, 2m+1, 2m-1, 1, 2((8m^2+1)n + 2m(4m-1))}\,].\end{aligned}$$

The coefficients occur in A081585 and A080856. The formula can be proved by the continued fraction algorithm.

3.3.2 Formula with $\ell(d) = 12$

We further discovered this formula for $m = 2, 3, 4, \dots$, and $n \in \mathbb{N}$:

$$\begin{aligned}\sqrt{((2m^2+4m+1)n + m)^2 + 2(2m^2+2m-1)n + (2m-1)} = \\ [(2m^2+4m+1)n + m; \overline{1, m-1, 1, 2m, 1, (2m^2+4m+1)n + m - 1,} \\ \overline{1, 2m, 1, m-1, 12((2m^2+4m+1)n + m)}\,].\end{aligned}$$

The coefficients occur in A056220 and A142463. It can be proved similarly.

Cheng et al. give Example 3.1 in [6] with $D(X) = 119^2X^2 + 2(2205)X + 343$ where 7^2 divides A, B, C. As $\Delta = B^2 - A^2C = 2 \cdot 7^4$, Schinzel's condition is satisfied. For each r in $X \equiv r \pmod 7$ with $r = 0, 1, 2, \dots, 6$, the structure of the period is similar. Period length being even for every r, every period has the central or middle term. It is expedient to rewrite $D(X)$ as $(119X + 18)^2 + (126X + 19)$ to make a_0 evident. We find that $a_m = a_0 - 1$ only when $r = 0, 1, 3, 4$. So we get these four expansions (of lengths 28, 32, 76, 80) valid for $n \in \mathbb{N}_0$:

$$\begin{aligned}\sqrt{(833n+375)^2 + 882n + 397} = [833n+375; \overline{1, 1, 8, 17n+7, 1, 1, 4, 1, 10, 34n+15, 4, 3, 1,} \\ \overline{833n+374, 1, 3, 4, 34n+15, 10, 1, 4, 1, 1, 17n+7, 8, 1, 1, 2(833n+375)}\,].\end{aligned}$$

$$\sqrt{(833n+137)^2+882n+145} = [833n+137;\overline{1,1,8,17n+2,1,2,4,1,1,1,2,34n+5,4,3,1,}$$
$$\overline{833n+136,1,3,4,34n+5,2,1,1,1,4,2,1,17n+2,8,1,1,2(833n+137)}\,].$$

$$\sqrt{(833n+851)^2+882n+901} = [833n+851;\overline{1,1,8,17n+17,3,1,5,4,1,34n+33,}$$
$$\overline{1,18,1,4,1,17n+16,1,1,4,1,10,34n+34,1,1,1,38,1,17n+16,2,2,11,1,1,34n+34,}$$
$$\overline{4,3,1,833n+850,1,3,4,34n+34,1,1,11,2,2,17n+16,1,38,1,1,1,34n+34,}$$
$$\overline{10,1,4,1,1,17n+16,1,4,1,18,1,34n+33,1,4,5,1,3,17n+17,8,1,1,2(833n+851)}\,].$$

$$\sqrt{(833n+494)^2+882n+523} = [833n+494;\overline{1,1,8,17n+9,1,38,1,1,1,34n+19,}$$
$$\overline{1,1,11,2,2,17n+9,1,2,4,1,1,1,2,34n+19,1,4,5,1,3,17n+9,1,4,1,18,1,34n+19,}$$
$$\overline{4,3,1,833n+493,1,3,4,34n+19,1,18,1,4,1,17n+9,3,1,5,4,1,34n+19,}$$
$$\overline{2,1,1,1,4,2,1,17n+9,2,2,11,1,1,34n+19,1,1,1,38,1,17n+9,8,1,1,2(833n+494)}\,].$$

What we are interested in are expansions involving a *single string* so that we can find expansions wherein the same string repeats k times. For example, we found these three expansions of length 28 each of which can be proved by the continued fraction algorithm:

$$\sqrt{(23689n+26)^2+(38574n+43)} = [23689n+26;$$
$$\overline{1,4,2,1,1,1,1,1,4,3,1,9,1,23689n+25,1,9,1,3,4,1,1,1,1,1,2,4,1,2(23689n+26)}\,],$$
$$\sqrt{(203009n+24)^2+(192646n+23)} = [203009n+24;$$
$$\overline{2,9,3,2,1,1,3,1,6,4,1,2,1,203009n+23,1,2,1,4,6,1,3,1,1,2,3,9,2,2(203009n+24)}\,],$$
$$\sqrt{(326471n+22)^2+(44450n+3)} = [326471n+22;$$
$$\overline{14,1,2,4,1,1,3,2,5,1,6,1,1,326471n+21,1,1,6,1,5,2,3,1,1,4,2,1,14,2(326471n+22)}\,].$$

The next scf having the period length of 30 can be proved similarly:

$$\sqrt{(3690313n+33)^2+(558498n+5)} = [3690313n+33;\overline{13,4,1,1,1,5,2,1,2,}$$
$$\overline{3,9,6,1,1,3690313n+32,1,1,6,9,3,2,1,2,5,1,1,1,4,13,2(3690313n+33)}\,].$$

3.3.3 Formula with repeated 2's

Perron gives this formula [17, p. 114] with $\ell(d)=6$:

$$\sqrt{(3n+1)^2+(2n+1)} = [3n+1;\ \overline{2,1,3n,1,2,6n+2}\,].$$

Kraitchik [13, p. 47] gives this formula with $\ell(d)=8$:

$$\sqrt{(7n+1)^2+(6n+1)} = [7n+1;\ \overline{2,2,1,7n,1,2,2,14n+2}\,].$$

These formulas are special cases of the following theorem:

Theorem 5. *Let p_k be the numerator in the k-th convergent of $\sqrt{2}$. Then for $k = 1, 2, \dots$*

$$\sqrt{(p_k\ n+1)^2 + (2p_{k-1}\ n+1)} = [p_k n + 1; \overline{\underbrace{2, 2, \dots, 2}_{k}, 1, p_k n, 1, \underbrace{2, 2, \dots, 2}_{k}, 2(p_k n + 1)}\ ].$$

Proof. Let $\sqrt{d} := a + y = [a; \overline{2, 1, a-1, 1, 2, 2a}\,]$. Then

$$y = [0; 2, 1, a-1, 1, 2, y + 2a],$$

which leads to the equation $3y^2 + 6ay - (2a+1) = 0$. Solving the quadratic equation, we get the positive root $y = -a + \sqrt{a^2 + \frac{2a+1}{3}}$. This implies that $b = \frac{2a+1}{3}$, which must be an integer if d is to be an integer. The solution $a = 3n + 1$ gives $b = 2n + 1$.

When $\sqrt{d} = [a; \overline{2, 2, 1, a-1, 1, 2, 2, 2a}\,]$, the same procedure gives the equation $7y^2 + 14ay - (6a+1) = 0$ whose positive root yields $b = \frac{6a+1}{7}$ and for d to be an integer we have $a = 7n + 1$ and $b = 6n + 1$.

With $\sqrt{d} = [a; \overline{2, 2, 2, 1, a-1, 1, 2, 2, 2, 2a}\,]$ the procedure gives the equation $17y^2 + 34ay - (14a + 3) = 0$ whose positive root yields $b = \frac{14a+3}{17}$ and for d to be an integer we have $a = 17n + 1$ and $b = 14n + 1$.

And if 2 is repeated k times in the period, we get the equation $p_{k+1}y^2 + 2p_{k+1}ay - (2p_k a + p_{k-1}) = 0$ whose positive root yields $b = \frac{2p_k a + p_{k-1}}{p_{k+1}}$ and for d to be an integer we have $a = p_{k+1}n + 1$ and $b = 2p_k n + 1$ where p_k is the numerator of c_k of $\sqrt{2}$. To apply (4) to the general case, we note that in this case

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^{k-1} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} p_k n & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^{k-1}.$$

Using (5), we get

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^{k-1} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} p_k & q_k \\ p_{k-1} & q_{k-1} \end{bmatrix}$$

we can rewrite this matrix as follows:

$$\begin{aligned} \begin{bmatrix} A & B \\ B & C \end{bmatrix} &= \begin{bmatrix} p_k & q_k \\ p_{k-1} & q_{k-1} \end{bmatrix} \cdot \begin{bmatrix} p_k n & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{bmatrix} \\ &= \begin{bmatrix} p_k^3 n + 2p_k\ q_k & p_k^2\ p_{k-1}n + p_k\ q_{k-1} + p_{k-1}\ q_k \\ p_k^2\ p_{k-1}n + p_k\ q_{k-1} + p_{k-1}\ q_k & p_k\ p_{k-1}^2 n + 2p_{k-1}\ q_{k-1} \end{bmatrix}. \end{aligned}$$

We now use Lemma 1.

Since in Theorem 5 $a_0 = p_k n + 1$, we have that $a = a_0 = p_k n + 1$ and

$$\begin{aligned} b &= \frac{2a_0 B + C}{A} \\ &= \frac{2(p_k n + 1)(p_k^2\ p_{k-1}n + p_k\ q_{k-1} + p_{k-1}\ q_k) + p_k\ p_{k-1}^2 n + 2p_{k-1}\ q_{k-1}}{p_k^3 n + 2p_k\ q_k}. \end{aligned}$$

What we have to show is that $b = 2p_{k-1}n + 1$, or that

$$\begin{aligned}&2(p_k n + 1)(p_k^2\ p_{k-1}n + p_k\ q_{k-1} + p_{k-1}\ q_k) + p_k\ p_{k-1}^2 n + 2p_{k-1}\ q_{k-1}\\ &= (2p_{k-1}n + 1)(p_k^3 n + 2p_k\ q_k).\end{aligned}$$

After canceling equal terms involving n^2 on both sides, and subtracting $2p_k\ p_{k-1}\ q_k n$ from both sides, we are left with

$$\text{LHS} = \left(2p_k^2\ q_{k-1} + 2p_k^2\ p_{k-1} + p_k\ p_{k-1}^2\right) n + 2p_k\ q_{k-1} + 2p_{k-1}\ q_k + 2p_{k-1}\ q_{k-1}.$$

$$\text{RHS} = \left(p_k^3 + 2p_k\ p_{k-1}\ q_k\right) n + 2p_k\ q_k.$$

Using the relation $p_{k-1} + q_{k-1} = q_k$, as deduced in (2b), we have

$$\text{LHS} = \left(2p_k^2\ q_k + p_k\ p_{k-1}^2\right) n + 2(p_k\ q_{k-1} + p_{k-1}\ q_k + p_{k-1}\ q_{k-1}).$$

Comparing LHS and RHS, we have to show that

$$2p_k\ q_k + p_{k-1}^2 = p_k^2 + 2p_{k-1}\ q_k \text{ and } p_k\ q_{k-1} + p_{k-1}(q_k + q_{k-1}) = p_k\ q_k. \tag{11}$$

The equation at the left is a consequence of (2b): we need to show that

$$2q_k = p_k + p_{k-1} \Rightarrow 2q_k(p_k - p_{k-1}) = p_k^2 - p_{k-1}^2 \Rightarrow 2p_k\ q_k + p_{k-1}^2 = p_k^2 + 2p_{k-1}\ q_k.$$

Using the relation (2a): $p_k = q_k + q_{k-1}$, the equation at the right in (11) reduces to

$$p_k\ q_{k-1} + p_{k-1}\ p_k = p_k\ q_k,$$

which follows on using the relation $p_{k-1} + q_{k-1} = q_k$ again. □

Our investigation with various blocks other than the singleton (2), which becomes (2, 2), (2, 2, 2), etc., suggests the conjecture that *no other string can repeat in the period with central term* $= a_0 - 1$.

4 Formulas with central term a_0

4.1 Formulas with $\ell(d) = 6$

For any fixed $m \in \mathbb{N}$ and $n = 1, 2, 3, \ldots$, we have

$$\sqrt{((2m^2+1)n + m)^2 + 2(2mn+1)} = [(2m+1)n + m; \overline{m, 2m, (2m^2+1)n + m, 2m, m, 2((2m+1)n + m)}].$$

The formula can be proved by means of the continued fraction algorithm.

4.2 Formulas with $\ell(d) = 8$

We find the following result in Kraitchik's book [13, p. 47]:

$$\sqrt{(7n+5)^2 + 2(4n+3)} = [7n+5;\ \overline{1,1,3,7n+5,3,1,1,14n+10}\,],\ n \in \mathbb{N}_0.$$

It is a special case ($m = 2$) of the following *general formula* for any fixed $m \in \mathbb{N} \setminus \{1\}$:

$$\begin{aligned}\sqrt{((2m^2-1)n+2m^2-m-1)^2+2(2mn+2m-1)} = [&(2m^2-1)n+2m^2-m-1;\\ &\overline{m-1,1,2m-1,(2m^2-1)n+2m^2-m-1,2m-1,}\\ &\overline{1,m-1,2((2m^2-1)n+2m^2-m-1)}\,],\end{aligned}$$

which can be proved by using the continued fraction algorithm. The sequences appearing here occur in A056220 and A014106.

4.3 Formula with $\ell(d) = 10$

We have this pair of expansions valid for $n = 1, 2, \ldots,$

$$\begin{aligned}\sqrt{(9n+3)^2+18} &= [9n+3;\ \overline{n,2,1,2n,9n+3,2n,1,2,n,2(9n+3)}\,],\\ \sqrt{(9n+6)^2+18} &= [9n+6;\ \overline{n,1,2,2n+1,9n+6,2n+1,2,1,n,2(9n+6)}\,].\end{aligned}$$

Both can be proved easily by the continued fraction algorithm.

5 Formulas with replicating pair $(m,\, 2m)$

5.1 Formula with repeated pair $(1, 2)$

Theorem 6. *Let q_k denote the denominator of the convergent to $\sqrt{3}$. Then for $k \in \mathbb{N}$*

$$\begin{aligned}\sqrt{(q_{2k}\ n+1)^2+2(2q_{2k-1}\ n+1)} = [&q_{2k}\ n+1;\\ &\overline{\underbrace{1,2,1,2,\ldots,1,2}_{k},(q_{2k}\ n+1),\underbrace{2,1,2,1,\ldots,2,1}_{k},2(q_{2k}\ n+1)}\ ].\end{aligned}$$

Proof. As noted in [14], we have $q_{n+4} = 4q_{n+2} - q_n$ for all $n \geq 0$.

The sequences defined above give $\dfrac{q_{2k}}{q_{2k-1}}$ for $k = 1, 2, 3, \ldots$

$$\frac{3}{1}, \frac{11}{4}, \frac{41}{15}, \frac{153}{56}, \frac{571}{209}, \ldots.$$

From Lemma 2 we have

$$\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}^k = \begin{bmatrix} q_{2k} & 2q_{2k-1} \\ q_{2k-1} & q_{2k-2} \end{bmatrix}.$$

Hence,

$$\begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}^2 = \begin{bmatrix}11 & 8\\4 & 3\end{bmatrix}; \quad \begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}^3 = \begin{bmatrix}41 & 30\\15 & 11\end{bmatrix}; \quad \begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}^4 = \begin{bmatrix}153 & 112\\56 & 41\end{bmatrix}.$$

Applying the same procedure as in the proof of Theorem 5, we have $\sqrt{d} = \sqrt{a^2+b} = [a; \overline{1, 2, a, 2, 1, 2a}\,]$ and

$$y = [0; 1, 2, a, 2, 1, y + 2a],$$

which after some long calculation leads to the equation $3y^2 + 6ay - (4a + 2) = 0$. Solving the quadratic equation, we get as positive root $y = -a + \sqrt{a^2 + \frac{4a+2}{3}}$. For d to be an integer we have to take $a = 3n + 1$ giving $b = 4n + 2$.

When $\sqrt{d} = [a; \overline{1, 2, 1, 2, a, 2, 1, 2, 1, 2a}\,]$, the same procedure gives the equation $11y^2 + 22ay - (16a + 6) = 0$ whose positive root yields $b = \frac{16a+6}{11}$ and for d to be an integer we take $a = 11n + 1$ and $b = 16n + 2$.

With $\sqrt{d} = [a; \overline{1, 2, 1, 2, 1, 2, a, 2, 1, 2, 1, 2, 1, 2a}\,]$, the procedure gives the equation $41y^2 + 82ay - (60a + 22) = 0$ and hence for d to be an integer we take $a = 41n + 1$ and $b = 60n + 2$.

And for k times the pairs $(1, 2)$ and $(2, 1)$, the equation obtained is $q_{2k}\ y^2 + 2q_{2k}a\ y - 2(2q_{2k-1}a + q_{2k-2}) = 0$, which gives $a = q_{2k}n + 1,\ b = 2(2q_{2k-1}n + 1)$.

□

The first three formulas follow.

$$\sqrt{(3n+1)^2 + 2(2n+1)} = [3n+1;\ \overline{1, 2, 3n+1, 2, 1, 2(3n+1)}\,].$$
$$\sqrt{(11n+1)^2 + 2(8n+1)} = [11n+1;\ \overline{1, 2, 1, 2, 11n+1, 2, 1, 2, 1, 2(11n+1)}\,].$$
$$\sqrt{(41n+1)^2 + 2(30n+1)} = [41n+1; \overline{1, 2, 1, 2, 1, 2, 41n+1, 2, 1, 2, 1, 2, 1, 2(41n+1)}\,].$$

5.2 Generalization of Theorem 6

Let $\sqrt{m^2+2} = [m; \overline{m, 2m}\,]$, and let q_{2k}, q_{2k-1} be the denominators of its convergents. We have $q_{2k+1} = mq_{2k} + q_{2k-1}$ and the recurrence with gap 2 $x_{i+2} = 2(m^2 + 1)x_i - x_{i-2}$, which gives

$$q_{2k+1} = 2(m^2+1)q_{2k-1} - q_{2k-3},\ q_1 = 1,\ q_{-1} = 1,$$
$$q_{2k+2} = 2(m^2+1)q_{2k} - q_{2k-2},\ q_0 = 1,\ q_2 = m.$$

We then have the following generalization of Theorem 6:

Theorem 7. *Let the numbers be as defined above. Then for $k \in \mathbb{N}$ we have*

$$\sqrt{(q_{2k}\ n+m)^2 + 2(2q_{2k-1}\ n+1)} = [\ q_{2k}\ n+m; \overline{\underbrace{m, 2m, m, 2m, \ldots, m, 2m}_{k},}$$
$$\overline{(q_{2k}\ n+m), \underbrace{2m, m, 2m, m, \ldots, 2m, m}_{k}, 2(q_{2k}\ n+m)}\ ].$$

Proof. The proof of the general case is similar to the proof of Theorem 5. We prove the case $m = 1$:

$$\left(\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix} \cdot \begin{bmatrix}2 & 1\\1 & 0\end{bmatrix}\right)^k \cdot \begin{bmatrix}q_{2k}n+1 & 1\\1 & 0\end{bmatrix} \cdot \left(\begin{bmatrix}2 & 1\\1 & 0\end{bmatrix} \cdot \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\right)^k = \begin{bmatrix}A & B\\B & C\end{bmatrix}$$

or, using (8):

$$\begin{bmatrix}q_{2k} & q_{2k-1}\\2q_{2k-1} & q_{2k-2}\end{bmatrix} \cdot \begin{bmatrix}q_{2k}n+1 & 1\\1 & 0\end{bmatrix} \begin{bmatrix}q_{2k} & 2q_{2k-1}\\q_{2k-1} & q_{2k-2}\end{bmatrix} = \begin{bmatrix}A & B\\B & C\end{bmatrix}.$$

Here $a = a_0 = q_{2k}n + 1$, so applying (4) to the matrix noted above gives

$$b = \frac{2aB + C}{A} = 2(2q_{2k-1}n + 1)$$

using the relation $2q_{2k-1} + q_{2k-2} = q_{2k}$ from (7). □

5.2.1 Formulas for cases $m = 2, 3$

The convergents of $\sqrt{6} = [2; \overline{2, 4}\,]$ for $k \geq 0$, with denominators occurring in A041006 and A041007, are

$$\frac{2}{1}, \frac{5}{2}, \frac{22}{9}, \frac{49}{20}, \frac{218}{89}, \frac{485}{198}, \frac{2158}{881}, \frac{48271}{1960}, \frac{21362}{8721}, \ldots.$$

They yield these following formulas:

$$\sqrt{(9n+2)^2 + 2(2 \cdot 2n + 1)} = [9n + 2; \overline{2, 4, 9n + 2, 4, 2, 2(9n + 2)}\,].$$
$$\sqrt{(89n+2)^2 + 2(2 \cdot 20n + 1)} = [89n + 2; \overline{2, 4, 2, 4, 89n + 2, 4, 2, 4, 2, 2(89n + 2)}\,].$$
$$\sqrt{(881n+2)^2 + 2(2 \cdot 198n + 1)} = [881n + 2; \overline{2, 4, 2, 4, 2, 4, 41n + 1, 4, 2, 4, 2, 4, 2, 2(881n + 2)}\,].$$

The convergents of $\sqrt{11} = [3; \overline{3, 6}\,]$ from for $k \geq 0$ with denominators occurring in A041014 and A041015, are

$$\frac{3}{1}, \frac{10}{3}, \frac{63}{19}, \frac{199}{60}, \frac{1257}{379}, \frac{3970}{1197}, \frac{25077}{7561}, \frac{79201}{23880}, \frac{500283}{150841}, \ldots.$$

leading to the following formulas:

$$\sqrt{(19n+3)^2 + 2(2 \cdot 3n + 1)} = [19n + 3; \overline{3, 6, 19n + 3, 6, 3, 2(19n + 3)}\,].$$
$$\sqrt{(379n+3)^2 + 2(2 \cdot 60n + 1)}$$
$$= [379n + 3; \overline{3, 6, 3, 6, 379n + 3, 6, 3, 6, 3, 2(379n + 3)}\,].$$
$$\sqrt{(7561n+3)^2 + 2(2 \cdot 1197n + 1)}$$
$$= [7561n + 3; \overline{3, 6, 3, 6, 3, 6, 7561n + 3, 6, 3, 6, 3, 6, 3, 2(7561n + 3)}\,].$$

6 Formulas with repeated triple $\langle 1,1,3\rangle$

First, we establish a lemma to be used for proving the next theorem. We set $\alpha = 4+\sqrt{17}$, $\beta = 4-\sqrt{17}$ and so $\alpha+\beta = 8,\ \alpha-\beta = 2\sqrt{17}$ and get the values

$$(\alpha-1)\alpha-(\beta-1)\beta = (\alpha-\beta)(\alpha+\beta-1) = 14\sqrt{17},\quad (\alpha-1)\beta-(\beta-1)\alpha = 2\sqrt{17},$$

leading to

$$\frac{(\alpha-1)\alpha-(\beta-1)\beta}{\alpha-\beta} = 7,\quad \frac{(\alpha-1)\beta-(\beta-1)\alpha}{\alpha-\beta} = 1.$$

We can then form the following unimodular matrix:

$$\begin{bmatrix} 7 & 2 \\ 4 & 1 \end{bmatrix} = \frac{1}{\alpha-\beta}\begin{bmatrix} (\alpha-1)\alpha-(\beta-1)\beta & 2(\alpha-\beta) \\ 4(\alpha-\beta) & (\alpha-1)\beta-(\beta-1)\alpha \end{bmatrix}.$$

Lemma 8. *We have*

$$\begin{bmatrix} 7 & 2 \\ 4 & 1 \end{bmatrix}^k = \frac{1}{\alpha-\beta}\begin{bmatrix} (\alpha-1)\alpha^k-(\beta-1)\beta^k & 2(\alpha^k-\beta^k) \\ 4(\alpha^k-\beta^k) & (\alpha-1)\beta^k-(\beta-1)\alpha^k \end{bmatrix}.$$

Proof. We use induction.

The statement is obviously true for $k = 1$. Now assume it to be true for an integer $m > 1$,

$$\begin{bmatrix} 7 & 2 \\ 4 & 1 \end{bmatrix}^m = \frac{1}{\alpha-\beta}\begin{bmatrix} (\alpha-1)\alpha^m-(\beta-1)\beta^m & 2(\alpha^m-\beta^m) \\ 4(\alpha^m-\beta^m) & (\alpha-1)\beta^m-(\beta-1)\alpha^m \end{bmatrix}.$$

Then

$$\begin{aligned}\begin{bmatrix} 7 & 2 \\ 4 & 1 \end{bmatrix}^{m+1} &= \frac{1}{(\alpha-\beta)^2}\begin{bmatrix} (\alpha-1)\alpha^m-(\beta-1)\beta^m & 2(\alpha^m-\beta^m) \\ 4(\alpha^m-\beta^m) & (\alpha-1)\beta^m-(\beta-1)\alpha^m \end{bmatrix} \\ \times &\begin{bmatrix} (\alpha-1)\alpha-(\beta-1)\beta & 2(\alpha-\beta) \\ 4(\alpha-\beta) & (\alpha-1)\beta-(\beta-1)\alpha \end{bmatrix} = \frac{1}{(\alpha-\beta)^2}\begin{bmatrix} a & b \\ c & d \end{bmatrix}.\end{aligned}$$

After multiplication, we obtain the expression

$$\begin{aligned} a =&(\alpha-\beta)(\alpha+\beta-1)[(\alpha-1)\alpha^m-(\beta-1)\beta^m]+8(\alpha-\beta)(\alpha^m-\beta^m) \\ &= (\alpha-\beta)[(\alpha-1)\alpha^{m+1}-(\beta-1)\beta^{m+1}+(\alpha-1)(\beta-1)(\alpha^m-\beta^m)+8(\alpha^m-\beta^m)] \\ &= (\alpha-\beta)[(\alpha-1)\alpha^{m+1}-(\beta-1)\beta^{m+1}+(\alpha^m-\beta^m)((\alpha-1)(\beta-1)+8)] \\ &= (\alpha-\beta)[(\alpha-1)\alpha^{m+1}-(\beta-1)\beta^{m+1}], \end{aligned}$$

the last term in the penultimate line becoming 0 as $(\alpha-1)(\beta-1) = (3+\sqrt{17})(3-\sqrt{17}) = -8$. $b = 2(\alpha-\beta)[(\alpha-1)\alpha^m-(\beta-1)\beta^m]+2(\alpha^m-\beta^m)(\alpha-\beta) = 2(\alpha-\beta)(\alpha^{m+1}-\beta^{m+1})$. Further,

$$c = 4(\alpha^m-\beta^m)(\alpha-\beta)(\alpha+\beta-1)+4(\alpha-\beta)[(\alpha-1)\beta^m-(\beta-1)\alpha^m] = 4(\alpha-\beta)(\alpha^{m+1}-\beta^{m+1}).$$

$$\begin{aligned} d &= 8(\alpha-\beta)(\alpha^m-\beta^m) + (\alpha-\beta)[(\alpha-1)\beta^m-(\beta-1)\alpha^m] \\ &= (\alpha-\beta)[8\alpha^m - 8\beta^m + (\alpha-1)\beta^m - (\beta-1)\alpha^m] \\ &= (\alpha-\beta)[\beta^m(-5+\sqrt{17}) + \alpha^m(5+\sqrt{17})] \\ &= (\alpha-\beta)[(\alpha-1)\beta^{m+1} - (\beta-1)\alpha^{m+1}]. \end{aligned}$$

The values of a, b, c, d show that the statement is true for $m+1$ also. Thus it is true for all k. □

Methodology. Let $y = [1; 1, 3, 1, 1, 3, 1, 1, 3, \ldots]$. We write it as $y = [1, 1, 3, y]$ or $y = \frac{7y+2}{4y+1}$. Hence the associated quadratic equation is $2y^2-3y-1 = 0$ giving the positive root $y = \frac{3+\sqrt{17}}{4}$. Now $\sqrt{17} = [4; \overline{8}\,] \Rightarrow 3+\sqrt{17} = [7; \overline{8}\,]$. The convergents P'_k/Q'_k $(k \geq 0)$ of $3+\sqrt{17}$ are

$$\frac{7}{1}, \frac{57}{8}, \frac{463}{65}, \frac{3761}{528}, \frac{30551}{4289}, \frac{248169}{34840}, \ldots.$$

The numerators and denominators satisfy the recurrence $x_{k+1} = 8x_k + x_{k-1}$.

Let $\dfrac{p_k}{q_k}$ be the k-th convergent to y. Define $P_k = p_{3k-1}$, $Q_k = q_{3k-1}$, Then the recurrence sequences $\{P_k\}$, $\{Q_k\}$ are

$$\begin{aligned} P_k &= 8\ P_{k-1} + P_{k-2}; \quad P_0 = 7, P_{-1} = 1, \\ Q_k &= 8\ Q_{k-1} + Q_{k-2}; \quad Q_0 = 4, Q_{-1} = 0. \end{aligned}$$

The specially defined convergents, $P_k = P'_k$, $Q_k = 4Q'_k$ (for $k \geq 0$) are

$$\frac{7}{4}, \frac{57}{32}, \frac{463}{260}, \frac{3761}{2112}, \frac{30551}{17156}, \frac{248169}{139360}, \ldots.$$

Theorem 9. *Let P_k, Q_k be the numbers as defined above. Then for $k \in \mathbb{N}_0$ we have*

$$\sqrt{\left(\frac{P_k(2n+1)+3}{2}\right)^2 + Q_k(2n+1)+2} = \Bigg[\frac{P_k(2n+1)+3}{2};$$
$$\overline{\underbrace{1,1,3,1,1,3,\ldots,1,1,3}_{k+1}, \frac{P_k(2n+1)+3}{2}, \underbrace{3,1,1,3,1,1,\ldots,3,1,1}_{k+1},\ P_k(2n+1)+3}\Bigg].$$

Proof. One may verify that $Q_{k+1}-Q_k = 4P_k$, $2Q_k = P_k + P_{k-1}$, $2(P_k - P_{k-1}) = 3Q_k + Q_{k-1}$. A few powers of the associated matrix are

$$\begin{bmatrix} 7 & 2 \\ 4 & 1 \end{bmatrix}^2 = \begin{bmatrix} 57 & 16 \\ 32 & 9 \end{bmatrix}; \quad \begin{bmatrix} 7 & 2 \\ 4 & 1 \end{bmatrix}^3 = \begin{bmatrix} 463 & 130 \\ 260 & 73 \end{bmatrix}; \quad \begin{bmatrix} 7 & 2 \\ 4 & 1 \end{bmatrix}^4 = \begin{bmatrix} 3761 & 1056 \\ 2112 & 593 \end{bmatrix}.$$

Using Lemma 8 we have

$$\begin{bmatrix}7 & 2\\4 & 1\end{bmatrix}^k = \frac{1}{\alpha-\beta}\begin{bmatrix}(\alpha-1)\alpha^k-(\beta-1)\beta^k & 2(\alpha^k-\beta^k)\\ 4(\alpha^k-\beta^k) & (\alpha-1)\beta^k-(\beta-1)\alpha^k\end{bmatrix} := \begin{bmatrix}P_k & Q_k/2\\ Q_k & \frac{2P_{k-1}+Q_{k-1}}{2}\end{bmatrix}$$

and from this we deduce the matrix defined in Lemma 1:

$$\begin{bmatrix}A & B\\ B & C\end{bmatrix} = \begin{bmatrix}P_k & Q_k/2\\ Q_k & \frac{2P_{k-1}+Q_{k-1}}{2}\end{bmatrix}\begin{bmatrix}\frac{P_k(2n+1)+3}{2} & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}P_k & Q_k\\ Q_k/2 & \frac{2P_{k-1}+Q_{k-1}}{2}\end{bmatrix}.$$

Applying Lemma 1 with $a = a_0 = \frac{P_k(2n+1)+3}{2}$,

$$b = \frac{(P_k(2n+1)+3)Q_k + 2P_{k-1} + Q_{k-1}}{P_k} = Q_k(2n+1)+2,$$

using the relation $2(P_k - P_{k-1}) = 3Q_k + Q_{k-1}$, which can easily be proved by comparing the initial values of the sequences at the LHS and the RHS.

□

The first three formulas yielded by Theorem 9 follow.

$$\sqrt{(7n+5)^2+2(4n+3)} = [7n+5;\ \overline{1,1,3,7n+5,3,1,1,2(7n+5)}\,].$$
$$\sqrt{(57n+30)^2+2(32n+17)} = [57n+30;$$
$$\overline{1,1,3,1,1,3,57n+30,3,1,1,3,1,1,2(57n+30)}\,].$$
$$\sqrt{(463n+233)^2+2(260n+131)} = [463n+233;$$
$$\overline{1,1,3,1,1,3,1,1,3,463n+233,3,1,1,3,1,1,3,1,1,2(463n+233)}\,].$$

7 Repeated odd partial quotients

7.1 General formula for $\ell-1$ quotient $(2m+1)$, $m \geq 0$

7.1.1 Methodology

As derived earlier, $\frac{(2m+1)+\sqrt{(2m+1)^2+4}}{2} = [2m+1;\overline{2m+1}\,]$. Let $(2m+1)$ be the repeated partial quotient in the period. We will use the scf for $-(2m+1)+\sqrt{(2m+1)^2+4} = [0;m,1,1,m,4m+2]$ given in (10). Now if we calculate the inverse of its convergent at each quotient we find that the denominator is odd at the quotient a_1, at a_2 and at a_4, while it is even at the quotient a_3 and at a_5. That is, only in the truncated fractions $[a_1,a_2,a_3]$ and $[a_1,a_2,a_3,a_4,2a_0]$ the denominator is even. Let us define $C_k := c_{5k-2}$ and $P_k := p_{5k-2}, Q_k := q_{5k-2}$ and $C'_k := c_{5k}$ and $P'_k := p_{5k}, Q'_k =: q_{5k}$, in terms of the regular convergents.

Computing $c_3 = \dfrac{p_3}{q_3} = [m,1,1]$, and denoting it by $C_1 = \frac{P_1}{Q_1}$, we have

$$C_1 = \frac{P_1}{Q_1} = \frac{2m+1}{2} = \frac{2m+1}{2},$$

and its successor $c_8 = \dfrac{p_8}{q_8} = [m,1,1,m,4m+2,m,1,1]$, denoted by $C_2 = \frac{P_2}{Q_2}$, is given by

$$C_2 = \frac{P_2}{Q_2} = \frac{(2m+1)^4 + 3(2m+1)^2 + 1}{2(2m+1)^3 + 4(2m+1)}.$$

Next, we compute $c_5 = \dfrac{p_5}{q_5} = [m,1,1,m,4m+2]$ and denote it by $C_1' = \frac{P_1'}{Q_1'}$:

$$C_1' = \frac{P_1'}{Q_1'} = [m,1,1,m,4m+2] = \frac{(2m+1)[(2m+1)^2+2]}{2[(2m+1)^2+1]},$$

and its successor with two full periods $[m,1,1,m,4m+2,m,1,1,m,4m+2]$, the convergent $c_{10} = \dfrac{p_{10}}{q_{10}}$, denoting it by $C_2' = \frac{P_2'}{Q_2'}$:

$$C_2' = \frac{P_2'}{Q_2'} = \frac{(2m+1)^6 + 5(2m+1)^4 + 6(2m+1)^2 + 1}{2[(2m+1)^5 + 4(2m+1)^3 + 3(2m+1)}.$$

Then

$$\begin{aligned} P_{k+1} &= MP_k + P_{k-1};\ Q_{k+1} = MQ_k + Q_{k-1}, \quad k \geq 2, \\ P_{k+1}' &= MP_k' + P_{k-1}'; \quad Q_{k+1}' = MQ_k' + Q_{k-1}', \quad k \geq 2, \end{aligned}$$

with $M := (2m+1)((2m+1)^2+3)$ [14, p. 352].

7.1.2 General formula

Theorem 10. *(i) Let P_k and Q_k be the numbers as defined above. Then for $k \in \mathbb{N}$*

$$\sqrt{\left(\frac{P_k(2n-1)+(2m+1)}{2}\right)^2 + \frac{Q_k(2n-1)}{2} + 1}$$
$$= \left[\frac{P_k(2n-1)+(2m+1)}{2}; \overline{\underbrace{2m+1, 2m+1, \ldots, 2m+1}_{3k-2},\ P_k(2n-1)+(2m+1)}\right].$$

(ii) Let P_k and Q_k be the numbers as defined above. Then for $k \in \mathbb{N}$

$$\sqrt{\left(\frac{P_k'(2n-1)+(2m+1)}{2}\right)^2 + \frac{Q_k'(2n-1)}{2} + 1}$$
$$= \left[\frac{P_k'(2n-1)+(2m+1)}{2}; \overline{\underbrace{2m+1, 2m+1, \ldots, 2m+1}_{3k},\ P_k'(2n-1)+(2m+1)}\right].$$

Proof. Part (i). Here the partial quotient $(2m+1)$ is repeated $(3k-2)$ times. In the 2×2 matrix approach, this just corresponds to the matrix

$$\begin{bmatrix} 2m+1 & 1 \\ 1 & 0 \end{bmatrix}^{3k-2} = \begin{bmatrix} u_{3k-1} & u_{3k-2} \\ u_{3k-2} & u_{3k-3} \end{bmatrix},$$

where u_i is the solution of the recurrence $u_{i+1} = (2m+1)\ u_i + u_{i-1}$ with initial values $u_0 = 0,\ u_1 = 1$ (Lemma 4 in Subsection 2.5). Now the general solution of this recurrence is

$$y_n = \lambda\alpha^n + \mu\beta^n$$

with $\alpha = \frac{2m+1+\sqrt{(2m+1)^2+4}}{2}$ and $\beta = \frac{2m+1-\sqrt{(2m+1)^2+4}}{2}$, for arbitrary constants λ and μ. Clearly $\alpha = -\frac{1}{\beta}$. We can show that y_n satisfies the relation

$$y_{n+6} = (2m+1)\left((2m+1)^2+3\right) y_{n+3} + y_n. \tag{12}$$

To do this we calculate

$$\begin{aligned} y_{n+6} - y_n &= \lambda\cdot\left(\alpha^{n+6} - \alpha^n\right) + \mu\cdot\left(\beta^{n+6} - \beta^n\right) \\ &= \lambda\cdot\alpha^{n+3}\left(\alpha^3 - \alpha^{-3}\right) + \mu\cdot\beta^{n+3}\left(\beta^3 - \beta^{-3}\right) \\ &= \lambda\cdot\alpha^{n+3}\left(\alpha^3 + \beta^3\right) + \mu\cdot\beta^{n+3}\left(\beta^3 + \alpha^3\right) \\ &= \left(\alpha^3 + \beta^3\right) y_{n+3}. \end{aligned}$$

Note that $\alpha^3 + \beta^3 = (2m+1)\{(2m+1)^2 + 3$ has been denoted by M above.

The solution we need, u_n, is given by y_n with $\lambda = -\mu = \frac{2}{\alpha-\beta}$:

$$u_n = \frac{\alpha^n - \beta^n}{\sqrt{(2m+1)^2+4}}$$

with

$$\begin{array}{lll} u_{-3} = (2m+1)^2+1, & u_0 = 0, & u_3 = (2m+1)^2+1, \\ u_{-2} = -(2m+1), & u_1 = 1, & u_4 = (2m+1)((2m+1)^2+2), \\ u_{-1} = 1, & u_2 = 2m+1, & u_5 = (2m+1)^4 + 3(2m+1)^2 + 1, \\ \vdots & \vdots & \vdots \end{array}$$

As one can see, the sequence $u_{-1},\ u_2,\ u_5, \ldots,$ that satisfies (12) is precisely the sequence P_k defined in 7.1.1; more specifically, $u_{3k-1} = P_k$. Furthermore, the sequence $u_{-2},\ u_1,\ u_4, \ldots,$ which also satisfies (12), is precisely the sequence $Q_k/2$; more specifically, $u_{3k-2} = Q_k/2$.

Note that for the sequence $u_{-3},\ u_0,\ u_3,\ \ldots,$ we have

$$u_{3k-3} = -(2m+1)u_{3k-2} + u_{3k-1}, \text{ or } u_{3k-3} = P_k - \tfrac{2m+1}{2}Q_k.$$

So the conclusion is

$$\begin{bmatrix} 2m+1 & 1 \\ 1 & 0 \end{bmatrix}^{3k-2} = \begin{bmatrix} u_{3k-1} & u_{3k-2} \\ u_{3k-2} & u_{3k-3} \end{bmatrix} = \begin{bmatrix} P_k & \frac{Q_k}{2} \\ \frac{Q_k}{2} & P_k - \frac{2m+1}{2}Q_k \end{bmatrix} = \begin{bmatrix} A & B \\ B & C \end{bmatrix}$$

the matrix at the RHS being the one defined in (4). We now apply (4). In (i) we have that

$$a = a_0 = \frac{P_k(2n-1) + (2m+1)}{2}$$

and as a consequence of what we did above, we get

$$b = \frac{2a_0 B + C}{A} = \frac{(P_k(2n-1) + (2m+1))\,\frac{Q_k}{2} + P_k - \frac{2m+1}{2}Q_k}{P_k} = \frac{Q_k(2n-1)}{2} + 1,$$

which is exactly what we had to prove.

To prove part (ii), we take $3k$-th power of the matrix used previously. Note that $P'_1 = u_4$, $Q'_1 = 2u_3$ and more generally $P'_k = u_{3k+1}, Q'_k = 2u_{3k}$ and $u_{3k-1} = u_{3k+1} - (2m+1)u_{3k}$. Hence $u_{3k+1} = P'_k$, $u_{3k} = \frac{Q'_k}{2}$ and $u_{3k-1} = P'_k - \frac{2m+1}{2}Q'_k$. The rest of the proof proceeds along the same lines as the proof of (i). □

7.2 Period with $\ell - 1$ units, case $m = 0$

Setting $m = 0$ in Theorem 10 yields the following corollary:

Corollary 11. *(i) Let F_{3k-1} be the $(3k-1)$-th Fibonacci number. Then for $k \in \mathbb{N}$ and $n = 1, 2, \ldots$*

$$\sqrt{\left(\frac{F_{3k-1}\,(2n-1)+1}{2}\right)^2 + F_{3k-2}\,(2n-1)+1} = \left[\frac{F_{3k-1}\,(2n-1)+1}{2}; \overline{\underbrace{1,1,\ldots,1}_{3k-2},\ F_{3k-1}\,(2n-1)+1}\right].$$

(ii) Let F_{3k+1} be the $(3k+1)$-th Fibonacci number. Then

$$\sqrt{\left(\frac{F_{3k+1}\,(2n-1)+1}{2}\right)^2 + F_{3k}\,(2n-1)+1} = \left[\frac{F_{3k+1}\,(2n-1)+1}{2}; \overline{\underbrace{1,1,\ldots,1}_{3k},\ F_{3k+1}\,(2n-1)+1}\right].$$

We have these recurrence relations (with $M = (2m+1)((2m+1)^2+3) = 4$):

$$F_{3k+4} = 4F_{3k+1} + F_{3k-2};\ F_{3k+3} = 4F_{3k} + F_{3k-3};\ F_{3k+2} = 4F_{3k-1} + F_{3k-4}.$$

For example, $F_{13} = 233$ and $F_{12} = 144$ together yield ($n = 1$) a dozen units:

$$\sqrt{13834} = [117; \overline{1,1,1,1,1,1,1,1,1,1,1,1,234}].$$

7.3 Period with $\ell - 1$ threes

Taking $m = 1$, we have

$$\sqrt{13} = [3; \overline{1, 1, 1, 1, 6}\,];\ -3 + \sqrt{13} = [0; \overline{1, 1, 1, 1, 6}\,].$$

We also have

$$\begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 33 & 10 \\ 10 & 3 \end{bmatrix}; \quad \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}^6 = \begin{bmatrix} 1189 & 360 \\ 360 & 109 \end{bmatrix};$$
$$\begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}^4 = \begin{bmatrix} 109 & 33 \\ 33 & 10 \end{bmatrix}; \quad \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}^7 = \begin{bmatrix} 3927 & 1189 \\ 1189 & 360 \end{bmatrix}.$$

To elucidate, we give the associated terminating continued fractions. Note that the element a_{21} (2nd row, 1st column) of each matrix is $Q/2$.

Example 12. (i) We have $\frac{P_1}{Q_1} = (1, 1, 1) = \frac{3}{2}$. Hence

$$\sqrt{(3n)^2 + 2n} = [3n; \overline{3, 6n}\,].$$

(ii) Subsequently we have that $\frac{P_1'}{Q_1'} = (1, 1, 1, 1, 6) = \frac{33}{20}$. Hence

$$\sqrt{(33n - 15)^2 + (20n - 9)} = [33n - 15; \overline{3, 3, 3, 2(33n - 15)}\,].$$

Example 13. (i) We have $\frac{P_2}{Q_2} = (1, 1, 1, 1, 6, 1, 1, 1) = \frac{109}{66}$. Hence

$$\sqrt{(109n - 53)^2 + (66n - 32)} = [109n - 53; \overline{3, 3, 3, 3, 2(109n - 53)}\,].$$

(ii) Subsequently we have $\frac{P_2'}{Q_2'} = (1, 1, 1, 1, 6, 1, 1, 1, 1, 6) = \frac{1189}{720}$. Hence

$$\sqrt{(1189n - 593)^2 + (720n - 359)} = [1189n - 593; \overline{3, 3, 3, 3, 3, 3, 2(1189n - 593)}\,].$$

Example 14. (i) We have $\frac{P_3}{Q_3} = (1, 1, 1, 1, 6, 1, 1, 1, 1, 6, 1, 1, 1) = \frac{36 \cdot 109 + 3}{36 \cdot 66 + 2} = \frac{3927}{2378}$ and

$$\sqrt{(3927n - 1962)^2 + (2378n - 1188)} = [3927n - 1962; \overline{3, 3, 3, 3, 3, 3, 3, 2(3927n - 1962)}\,].$$

(ii) Subsequently we have $\frac{P_3'}{Q_3'} = (1, 1, 1, 1, 6, 1, 1, 1, 1, 6, 1, 1, 1, 1, 6) = \frac{36 \cdot 1189 + 33}{36 \cdot 720 + 20} = \frac{42837}{25940}$ and

$$\begin{aligned} &\sqrt{(42837n - 21417)^2 + (25940n - 12969)} \\ &\quad = [42837n - 21417; \overline{3, 3, 3, 3, 3, 3, 3, 3, 3, 2(42837n - 21417)}\,]. \end{aligned}$$

7.4 Period with $\ell - 1$ fives

The choice $m = 2$ gives $-5 + \sqrt{29} = [0; \overline{2, 1, 1, 2, 29}\,]$. We list a few formulas here.

$$\sqrt{(5n)^2 + 2n} = [5n; \overline{5, 10n)}\,].$$
$$\sqrt{(701n - 348)^2 + (270n - 134)} = [701n - 348; \overline{5, 5, 5, 5, 2(701n - 348)}\,].$$
$$\sqrt{(98145n - 49070)^2 + (37802n - 18900)}$$
$$= [98145n - 49070; \overline{5, 5, 5, 5, 5, 5, 5, 5, 2(98145n - 49070)}\,].$$

$$\sqrt{(135n - 65)^2 + (52n - 25)} = [135n - 65; \overline{5, 5, 5, 2(135n - 65)}\,].$$
$$\sqrt{(18901n - 9448)^2 + (7280n - 3639)}$$
$$= [18901n - 9448; \overline{5, 5, 5, 5, 5, 5, 2(18901n - 9448)}\,].$$
$$\sqrt{(2646275n - 1323135)^2 + (1019252n - 509625)}$$
$$= [2646275n - 1323135; \overline{5, 5, 5, 5, 5, 5, 5, 5, 5, 2(2646275n - 1323135)}\,].$$

8 Repeated even partial quotients

We first prove a simple lemma to be used in the proof of the next theorem.

Lemma 15. *Let $\alpha := m + \sqrt{m^2 + 1}$, $\beta := m - \sqrt{m^2 + 1}$ so that $\alpha + \beta = 2m$, $\alpha - \beta = 2\sqrt{m^2 + 1}$ and $\alpha\beta = -1$. Then*

$$\begin{bmatrix} 2m & 1 \\ 1 & 0 \end{bmatrix}^k = \frac{1}{\alpha - \beta} \begin{bmatrix} \alpha^{k+1} - \beta^{k+1} & \alpha^k - \beta^k \\ \alpha^k - \beta^k & \alpha^{k-1} - \beta^{k-1} \end{bmatrix}.$$

Proof. We prove the lemma by induction.

The statement is obviously true for $k = 1$. Assume it to be true for an integer $m > 1$,

$$\begin{bmatrix} 2m & 1 \\ 1 & 0 \end{bmatrix}^m = \frac{1}{\alpha - \beta} \begin{bmatrix} \alpha^{m+1} - \beta^{m+1} & \alpha^m - \beta^m \\ \alpha^m - \beta^m & \alpha^{m-1} - \beta^{m-1} \end{bmatrix}.$$

Then

$$\begin{bmatrix} 2m & 1 \\ 1 & 0 \end{bmatrix}^{m+1} = \frac{1}{\alpha - \beta} \begin{bmatrix} \alpha^{m+1} - \beta^{m+1} & \alpha^m - \beta^m \\ \alpha^m - \beta^m & \alpha^{m-1} - \beta^{m-1} \end{bmatrix} \begin{bmatrix} \alpha + \beta & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with $a = (\alpha^{m+1} - \beta^{m+1})(\alpha + \beta) + \alpha^m - \beta^m = (\alpha^{m+2} - \beta^{m+2}) + (\alpha^m - \beta^m)(\alpha\beta + 1)$ and so $a = \alpha^{m+2} - \beta^{m+2}$ as the other term vanishes. Furthermore, $b = \alpha^{m+1} - \beta^{m+1}$. Similarly, $c = \alpha^{m+1} - \beta^{m+1}$ and $d = \alpha^m - \beta^m$. The values of a, b, c, d show that the statement is true for $m + 1$ also. Thus it is true for all k. □

8.1 Methodology

Let $2m$ $(m \geq 1)$ be the quotient that repeats in the period. We then use the scf for $\sqrt{(2m)^2+4}$, the special case $(n=2)$ of (9), for the convergents to be used. The convergents are computed from these recurrence relations $(k \geq 1)$:

$$p_{2k} = m\ p_{2k-1} + p_{2k-2};\ p_{2k+1} = 4m\ p_{2k} + p_{2k-1};\ p_0 = 1,\ p_1 = 2m,$$
$$q_{2k} = m\ q_{2k-1} + q_{2k-2};\ q_{2k+1} = 4m\ q_{2k} + q_{2k-1};\ q_0 = 0,\ q_1 = 1,$$

or from this one:

$$p_{k+2} = M'p_k - p_{k-2},\ p_{-1} = -2m, \quad q_{k+2} = M'q_k - q_{k-2},\ q_{-1} = 1$$

with $M' = 2(2m^2+1)$, and $c_2 = \dfrac{p_2}{q_2} = [2m; m] = \dfrac{2m^2+1}{m}$.

8.2 General formula for $\ell - 1$ partial quotients $2m,\ m \geq 1$

Theorem 16. *Let the numbers as defined above. Then for* $k \in \mathbb{N}$

(*i*) $\sqrt{(q_{2k}\ n\ + m)^2 + (q_{2k-1}\ n + 1)} = \left[\ q_{2k}\ n\ + m; \overline{\underbrace{2m, 2m, \ldots, 2m}_{2k-1}, 2(q_{2k}n + m)}\ \right]$,

(*ii*) $\sqrt{(q_{2k+1}\ n\ + m)^2 + (4q_{2k}\ n + 1)} = \left[\ q_{2k+1}\ n\ + m; \overline{\underbrace{2m, 2m, \ldots, 2m}_{2k}, 2(q_{2k+1}n + m)}\ \right]$.

Proof. We established above in Lemma 15:

$$\begin{bmatrix} 2m & 1 \\ 1 & 0 \end{bmatrix}^k = \frac{1}{\alpha - \beta} \begin{bmatrix} \alpha^{k+1} - \beta^{k+1} & \alpha^k - \beta^k \\ \alpha^k - \beta^k & \alpha^{k-1} - \beta^{k-1} \end{bmatrix},$$

where $\alpha := m + \sqrt{m^2+1},\ \beta := m - \sqrt{m^2+1}$. We thus have

$$\begin{bmatrix} 2m & 1 \\ 1 & 0 \end{bmatrix}^{2k-1} = \frac{1}{\alpha - \beta} \begin{bmatrix} \alpha^{2k} - \beta^{2k} & \alpha^{2k-1} - \beta^{2k-1} \\ \alpha^{2k-1} - \beta^{2k-1} & \alpha^{2k-2} - \beta^{2k-2} \end{bmatrix},$$
$$\begin{bmatrix} 2m & 1 \\ 1 & 0 \end{bmatrix}^{2k} = \frac{1}{\alpha - \beta} \begin{bmatrix} \alpha^{2k+1} - \beta^{2k+1} & \alpha^{2k} - \beta^{2k} \\ \alpha^{2k} - \beta^{2k} & \alpha^{2k-1} - \beta^{2k-1} \end{bmatrix}.$$

We find that (14) yields only denominators but not numerators:

$$\begin{bmatrix} 2m & 1 \\ 1 & 0 \end{bmatrix}^{2k-1} = \begin{bmatrix} 2q_{2k} & q_{2k-1} \\ q_{2k-1} & 2q_{2k-2} \end{bmatrix}; \quad \begin{bmatrix} 2m & 1 \\ 1 & 0 \end{bmatrix}^{2k} = \begin{bmatrix} q_{2k+1} & 2q_{2k} \\ 2q_{2k} & q_{2k-1} \end{bmatrix}.$$

These are the associated closed forms for the denominators:

$$q_{2k-1} = \frac{(m + \sqrt{m^2+1})^{2k-1} - (m - \sqrt{m^2+1})^{2k-1}}{2\sqrt{m^2+1}},$$
$$q_{2k} = \frac{(m + \sqrt{m^2+1})^{2k} - (m - \sqrt{m^2+1})^{2k}}{4\sqrt{m^2+1}},$$

while those for numerators are

$$p_{2k-1} = (m+\sqrt{m^2+1})^{2k-1} + (m-\sqrt{m^2+1})^{2k-1},$$
$$p_{2k} = \frac{(m+\sqrt{m^2+1})^{2k} + (m-\sqrt{m^2+1})^{2k}}{2}.$$

Applying (4) from Lemma 1 to the matrix proves both parts of Theorem 16.

For (i) we have $a_0 = a = q_{2k}n + m$ and the power $2k-1$ of the matrix giving

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix} = \begin{bmatrix} 2q_{2k} & q_{2k-1} \\ q_{2k-1} & 2q_{2k-2} \end{bmatrix}$$

so that

$$b = \frac{2(q_{2k}n+m)q_{2k-1} + 2q_{2k-2}}{2q_{2k}} = \frac{q_{2k}q_{2k-1}n + mq_{2k} + q_{2k-2}}{q_{2k}} = q_{2k-1}\ n + 1$$

using the relation $mq_{2k} + q_{2k-2} = q_{2k}$.

For (ii) we have $a_0 = a = q_{2k+1}n + m$ and the power $2k$ of the matrix giving

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix} = \begin{bmatrix} q_{2k+1} & 2q_{2k} \\ 2q_{2k} & q_{2k-1} \end{bmatrix}$$

so that

$$b = \frac{2(q_{2k+1}n+m)2q_{2k} + q_{2k-1}}{2q_{2k+1}} = \frac{4q_{2k}q_{2k+1}n + 4mq_{2k} + q_{2k-1}}{q_{2k+1}} = 4q_{2k}n + 1$$

using the relation $4mq_{2k} + q_{2k-1} = q_{2k+1}$.

□

8.3 Derivation of a combined formula for Theorem 16 (i) & (ii)

The two sequences of numbers in the formula are linked by

$$p_{2k} = 2mq_{2k} + q_{2k-1}; \quad p_{2k+1} = 2mq_{2k+1} + 4q_{2k}.$$

Expanding the square term under the root on the left of formula (i) gives us

$$q_{2k}^2 n + m^2 + (2mq_{2k} + q_{2k-1})n + 1 = q_{2k}^2 n \ + m^2 + p_{2k}n + 1. \tag{17}$$

Similarly, from (ii) we obtain

$$q_{2k+1}^2 n + m^2 + (2mq_{2k+1} + 4q_{2k})n + 1 = q_{2k+1}^2 n + m^2 + p_{2k+1}n + 1. \tag{18}$$

Similarity of the RHS of both (17) and (18) allows us to combine (i) and (ii):

$$\sqrt{((q_{k+1}n)^2 + m^2) + (p_{k+1}n + 1)} = [\, q_{k+1}n + m; \overline{\underbrace{2m, 2m, \ldots, 2m}_{k}, 2(q_{k+1}n + m)}\,]\,. \qquad (19)$$

We note that the even-numbered convergents are given by the matrix

$$\begin{bmatrix} 2m^2+1 & 4m(m^2+1) \\ m & 2m^2+1 \end{bmatrix}^k = \begin{bmatrix} p_{2k} & 4m(m^2+1)q_{2k} \\ q_{2k} & p_{2k} \end{bmatrix}$$

and the odd-numbered convergents by the product

$$\begin{bmatrix} 2m^2+1 & 4m(m^2+1) \\ m & 2m^2+1 \end{bmatrix}^k \cdot \begin{bmatrix} 2m & 4(m^2+1) \\ 1 & 2m \end{bmatrix} = \begin{bmatrix} p_{2k+1} & 4(m^2+1)q_{2k+1} \\ q_{2k+1} & p_{2k+1} \end{bmatrix}.$$

8.4 Period with $\ell - 1$ twos

We have $\sqrt{2^2+4} = [2; \overline{1, 4}\,]$. The recurrence relations for $k \geq 1$ are

$$p_{2k} = p_{2k-1} + p_{2k-2};\ p_{2k+1} = 4p_{2k} + p_{2k-1};\ p_0 = 1,\ p_1 = 2;$$
$$q_{2k} = q_{2k-1} + q_{2k-2};\ q_{2k+1} = 4q_{2k} + q_{2k-1};\ q_0 = 0,\ q_1 = 1.$$

The convergents with numerators/denominators in A041010 and A041011 are

$$\frac{2}{1}, \frac{3}{1}, \frac{14}{5}, \frac{17}{6}, \frac{82}{29}, \frac{99}{35}, \frac{478}{169}, \frac{577}{204}, \frac{2786}{985}, \ldots.$$

Formula (19) and four of the convergents give us

$$\sqrt{n^2 + 1^2 + (3n+1)} = [n+1; \overline{2, 2(n+1)}\,],$$
$$\sqrt{(5n)^2 + 1^2 + (14n+1)} = [5n+1; \overline{2, 2, 2(5n+1)}\,],$$
$$\sqrt{(6n)^2 + 1^2 + (17n+1)} = [6n+1; \overline{2, 2, 2, 2(6n+1)}\,],$$
$$\sqrt{(29n)^2 + 1^2 + (82n+1)} = [29n+1; \overline{2, 2, 2, 2, 2(29n+1)}\,].$$

8.5 Period with $\ell - 1$ fours

We have $\sqrt{4^2+4} = [4; \overline{2, 8}\,]$. The convergents with numerators/denominators in A041030 and A041031 are

$$\frac{4}{1}, \frac{9}{2}, \frac{76}{17}, \frac{161}{36}, \frac{1364}{305}, \frac{2889}{646}, \frac{24476}{5473}, \frac{51841}{11592}, \frac{439204}{98209}, \ldots.$$

Formula (19) and three of the convergents yield

$$\sqrt{(2n)^2 + 2^2 + (9n+1)} = [2n+2; \overline{4, 2(2n+2)}\,],$$
$$\sqrt{(17n)^2 + 2^2 + (76n+1)} = [17n+2; \overline{4, 4, 2(17n+2)}\,],$$
$$\sqrt{(36n)^2 + 2^2 + (161n+1)} = [36n+2; \overline{4, 4, 4, 2(36n+2)}\,].$$

8.6 Period with $\ell - 1$ sixes

We have $\sqrt{6^2+4} = [6; \overline{3, 12}\,]$. Its convergents with numerators/denominators in A041066 and A041067 are

$$\frac{6}{1}, \frac{19}{3}, \frac{234}{37}, \frac{721}{114}, \frac{8886}{1405}, \frac{27379}{4329}, \frac{337434}{53353}, \ldots.$$

Formula (19) and three of the convergents lead to

$$\sqrt{(3n)^2 + 3^2 + (19n+1)} = [3n+3; \overline{6, 2(3n+3)}\,],$$
$$\sqrt{(37n)^2 + 3^2 + (234n+1)} = [37n+3; \overline{6, 6, 2(37n+3)}\,],$$
$$\sqrt{(114n)^2 + 3^2 + (721n+1)} = [114n+3; \overline{6, 6, 6, 2(114n+3)}\,].$$

9 Algorithm involving five sequences

9.1 Methodology and general formula

Until now we dealt with formulas that involved only two sequences P_k, Q_k. We found empirically that repeated blocks having three (excepting $(1,1,3)$) or more numbers cannot be formulated using only these two sequences but need a third auxiliary sequence R_k related to P_k. Two more sequences S_k and T_k are computed further on. The computation of S_k and T_k involves the solution of certain linear Diophantine equations.

Let $\mathcal{B} = (b_0, b_1, \ldots, b_j)$ with b_i positive integers not all equal. Let $\overleftarrow{\mathcal{B}} = (b_j, b_{j-1}, \ldots, b_0)$ be the reverse of $\mathcal{B}$. Let y denote the purely periodic continued fraction: $y = [\,\overline{b_0, b_1, \ldots, b_j}\,]$. We write it as $y = [b_0, b_1, \ldots, b_j, y]$, which leads to a quadratic equation of the form $cy^2 - by - a = 0$ with positive root $y = \frac{b+\sqrt{b^2+4ac}}{2c}$.

Empirical analysis revealed this *necessary condition* for the occurrence of block $\mathcal{B}$ in the next theorem: $\mathcal{B}$ can occur in the formula *only if* the coefficient of y^2 is twice the absolute value of the constant term in the equation.

We now define $\frac{P_0}{Q_0} = \mathrm{SCF}[b_0, b_1, \ldots, b_j]$ and $\frac{P_1}{Q_1} = \mathrm{SCF}[b_0, b_1, \ldots, b_j, b_0, b_1, \ldots, b_j]$ and so on. In general, P_k and Q_k are defined by

$$\begin{bmatrix} b_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_2 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} b_j & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} P_0 & V_0 \\ Q_0 & R_0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} P_k & V_k \\ Q_k & R_k \end{bmatrix} = \begin{bmatrix} P_0 & V_0 \\ Q_0 & R_0 \end{bmatrix}^{k+1}. \tag{20}$$

Note that using these notations the quadratic equation $cy^2 - by - a = 0$ equals $Q_0 y^2 - (P_0 - R_0)y - V_0 = 0$.

From now on we will assume that $Q_0 = 2V_0$ with as immediate consequence that $Q_k = 2V_k$. It follows from [14, p. 352] that P_k, Q_k, and R_k are solutions of the recurrence relation

$$y_{k+1} = My_k + (-1)^j y_{k-1}, k = 0, 1, 2, \ldots,$$

with initial values $P_{-1} = 1$, $Q_{-1} = 0$, $R_{-1} = 1$ and with P_0, Q_0 and R_0 defined by the product in (20). Furthermore, M is equal to the trace of

$$\begin{bmatrix} P_0 & \frac{1}{2}Q_0 \\ Q_0 & R_0 \end{bmatrix}.$$

Note that $M = \frac{Q_1}{Q_0}$.

A fourth sequence of numbers S_k is obtained defining S_k as the least positive integer such that $P_k \mid (Q_k \cdot S_k + R_k)$. From these four sequences, we deduce a fifth sequence of numbers T_k using the formula

$$T_k = \frac{Q_k \cdot S_k + R_k}{P_k}.$$

Theorem 17. *Let $P_k, Q_k, S_k, T_k, \mathcal{B}, \overleftarrow{\mathcal{B}}$ be the numbers/blocks as defined above. Then for $k, n \in \mathbb{N}_0$*

$$\sqrt{(P_k n + S_k)^2 + 2(Q_k n + T_k)} = [\, P_k n + S_k; \overline{\underbrace{\mathcal{B}, \mathcal{B}, \ldots, \mathcal{B}}_{k+1}, P_k n + S_k, \underbrace{\overleftarrow{\mathcal{B}}, \overleftarrow{\mathcal{B}}, \ldots, \overleftarrow{\mathcal{B}}}_{k+1},\ 2(P_k n + S_k)}\,].$$

Proof. The proof again uses (4). In this case

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix} = \begin{bmatrix} P_k & \frac{1}{2}Q_k \\ Q_k & R_k \end{bmatrix} \cdot \begin{bmatrix} P_k n + S_k & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} P_k & Q_k \\ \frac{1}{2}Q_k & R_k \end{bmatrix}$$

from which it follows that

$$\frac{2(P_k n + S_k)B + C}{A} = 2Q_k n + 2\frac{Q_k S_k + R_k}{P_k}.$$

For the second term to be an integer it is necessary that S_k is chosen in such a way that P_k divides $Q_k S_k + R_k$, or that the sequence T_k defined above consists only of integers. Hence we are looking for an integer solution $x = T_k$, $y = S_k$ with $x, y > 0$ of the linear equation

$$P_k \cdot x - Q_k \cdot y = R_k. \tag{21}$$

Now it is well known that such a solution exists if $(P_k, Q_k) = 1$ [8, p. 77]. That this condition is satisfied is a consequence of (20):

$$\det \begin{bmatrix} P_k & \frac{1}{2}Q_k \\ Q_k & R_k \end{bmatrix} = \det \begin{bmatrix} P_0 & \frac{1}{2}Q_0 \\ Q_0 & R_0 \end{bmatrix}^{k+1} = \left((-1)^{j+1}\right)^{k+1}. \tag{22}$$

A common factor of P_k and Q_k is a factor of this determinant and hence has to divide 1. Now it is easy to see from (22) (see also [8, p. 78]) that the integer solutions of the linear equation

$$P_k \cdot \hat{x} - Q_k \cdot \hat{y} = \left((-1)^{j+1}\right)^{k+1}$$

are determined by the denominator and the numerator of the penultimate convergent $\frac{\frac{1}{2}Q_k}{R_k}$ of the continued fraction of $\frac{P_k}{Q_k}$:

$$\hat{x} = R_k + mQ_k, \qquad \hat{y} = \frac{1}{2}Q_k + mP_k$$

(with m an arbitrary integer). Combining this with (21) we immediately see that

$$T_k \equiv \left((-1)^{j+1}\right)^{k+1} \cdot R_k^2 \pmod{Q_k}, \qquad S_k \equiv \left((-1)^{j+1}\right)^{k+1} \cdot \frac{1}{2}Q_kR_k \pmod{P_k}.$$

□

Remark 18. Let the positive root of the equation under review be denoted by $\frac{p+\sqrt{d}}{q}$. We found that

- $M = 2a_0$ and $P_0 = p + a_0$ if $\sqrt{d} = [a_0; \overline{2a_0}\,]$.
- $M = 2 + 2a_0$ and $P_0 = p + a_0 + 1$ if $\sqrt{d} = [a_0; \overline{1, 2a_0}\,]$.
- $M = 2 + 4a_0$ and and $P_0 = 2p + 2a_0 + 1$ if $\sqrt{d} = [a_0; \overline{2, 2a_0}\,]$.

9.2 Two illustrative examples

9.2.1 $\mathcal{B}$= (1, 1, 2, 3)

The characteristic equation is $10y^2 - 14y - 5 = 0$ with positive root $\frac{7+\sqrt{99}}{10}$ and scf expansion $\sqrt{99} = [9; \overline{1, 18}\,]$. SCF $[1, 1, 2, 3] = \frac{17}{10}$ and SCF $[1, 1, 2, 3, 1, 1, 2, 3] = \frac{339}{200}$ so that the multiplier is $M = 200/10 = 20 = 18 + 2$.

$$\frac{P_k}{Q_k} = \frac{17}{10}, \frac{339}{200}, \frac{6763}{3990}, \frac{134921}{79600}, \frac{2691657}{1588010}, \dots .$$

$$R_k = 3, 59, 1177, 23481, 468443, \dots .$$
$$S_k = 15, 137, 1354, 80954, 2153327, \dots .$$
$$T_k = 9, 81, 799, 47761, 1270409, \dots .$$

For $n = 0, 1, 2, \dots$, we have these first three formulas:

$$\sqrt{(17n + 15)^2 + 2(10n + 9)} = [17n + 15; \overline{1, 1, 2, 3, 17n + 15, 3, 2, 1, 1, 2(17n + 15)}\,],$$
$$\sqrt{(339n + 137)^2 + 2(200n + 81)} = [339n + 137; \overline{(1, 1, 2, 3)^2, 339n + 137, (3, 2, 1, 1)^2, 2(339n + 137)}\,],$$
$$\sqrt{(6763n + 1354)^2 + 2(3990n + 799)} = [6763n + 1354; \overline{(1, 1, 2, 3)^3,\ 6763n + 1354, (3, 2, 1, 1)^3,\ 2(6763n + 1354)}\,].$$

9.2.2 $\mathcal{B} = (1, 3, 1, 1, 2)$

The equation here is $18y^2 - 16y - 9 = 0$ with root $\frac{8+\sqrt{226}}{18}$ and expansion $\sqrt{226} = [15; \overline{30}]$. Now SCF$[1,3,1,1,2] = \frac{23}{18}$ and SCF $[1,3,1,1,2,1,3,1,1,2] = \frac{691}{540}$, which yield $M = 540/18 = 30$.

$$\frac{P_k}{Q_k} = \frac{23}{18}, \frac{691}{540}, \frac{20753}{16218}, \frac{623281}{487080}, \frac{18719183}{14628618}, \ldots.$$

$$R_k = 7, 211, 6337, 190321, 5710807, \ldots.$$
$$S_k = 6, 308, 18448, 484775, \ldots.$$
$$T_k = 5, 241, 14417, 378841, \ldots.$$

For $n = 0, 1, 2, \ldots$, we have

$$\begin{aligned}
\sqrt{(23n+6)^2 + 2(18n+5)} &= [23n+6; \\
&\quad \overline{1,3,1,1,2,23n+6,2,1,1,3,1,46n+12}\,], \\
\sqrt{(691n+308)^2 + 2(540n+241)} &= [691n+308; \\
&\quad \overline{(1,3,1,1,2)^2,\ 691n+308, (2,1,1,3,1)^2,\ 2(691n+308)}\,], \\
\sqrt{(20753n+18448)^2 + 2(16218n+14417)} &= [20753n+18448; \\
&\quad \overline{(1,3,1,1,2)^3,\ 20753n+18448, (2,1,1,3,1)^3,\ 2(20753n+18448)}\,].
\end{aligned}$$

9.3 On the composition of block $\mathcal{B}$

We found the blocks initially by inspecting all the scf expansions of $\sqrt{d}$ for $d \leq 1200$. However, one can take any arbitrary string of numbers and verify whether the equation obtained therefrom is of the form $2ay^2 - by - a = 0$. If yes, then that is a candidate, else not. Future work may lead to a more systematic way for producing these blocks. We give here some blocks of various lengths. They have been ordered on the basis of rising sums of the numbers in a block. The individual lists will grow as d rises. The length of the block will go on rising indefinitely.

9.3.1 Triples

We found that the only triple candidate is of the form $(n, 1, 2n+1),\ n \in \mathbb{N}$, whose characteristic equation is given by $y = [n, 1, 2n+1, y] \Rightarrow 2(n+1)y^2 - 2n(n+2)y - (n+1) = 0$.

9.3.2 Quadruples

(1, 1, 2, 3), (1, 4, 2, 2), (2, 2, 1, 4), (2, 1, 2, 5), (1, 6, 3, 2), (3, 2, 1, 6), (3, 1, 2, 7), (1, 10, 5, 2), (2, 8, 4, 4), (4, 4, 2, 8).

9.3.3 Quintuples

(1, 3, 1, 1, 2), (2, 3, 1, 1, 4), (1, 7, 1, 3, 2), (2, 5, 1, 2, 4), (3, 1, 2, 2, 7), (3, 7, 1, 3, 6).

9.3.4 Sextuples

(1, 3, 2, 1, 1, 2), (1, 1, 1, 1, 3, 3), (1, 5, 2, 1, 2, 2), (1, 1, 5, 2, 1, 3), (2, 1, 3, 1, 1, 5), (2, 1, 1, 1, 3, 5), (1, 6, 1, 2, 3, 2), (1, 1, 11, 5, 1, 3), (5, 2, 1, 2, 1, 10), (2, 2, 7, 14, 1, 4).

9.3.5 Septuples

(1, 3, 2, 2, 1, 1, 2), (1, 4, 1, 1, 3, 2, 2), (1, 1, 1, 2, 1, 6, 3), (2, 3, 2, 2, 1, 1, 4), (5, 4, 1, 1, 3, 2, 10).

9.3.6 Octuples

(1, 1, 2, 3, 1, 1, 2, 3), (1, 3, 1, 3, 7, 1, 1, 2), (2, 2, 1, 4, 2, 2, 1, 4).

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