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Pell and Associated Pell Braid Sequences as GCDs of Sums of k Consecutive Pell, Balancing, and Related Numbers

aBa Mbirika[1] and Janee Schrader
Department of Mathematics
University of Wisconsin-Eau Claire
Eau Claire, WI 54702
USA
mbirika@uwec.edu
schradjm5500@uwec.edu

Jürgen Spilker
Institute of Mathematics
University of Freiburg
79085 Freiburg im Breisgau
Germany
juergen.spilker@t-online.de

Abstract

We consider the greatest common divisor (GCD) of all sums of k consecutive terms of a sequence $(S_n)_{n\geq 0}$ where the terms S_n come from exactly one of following six well-known sequences' terms: Pell P_n, associated Pell Q_n, balancing B_n, Lucas-balancing C_n, cobalancing b_n, and Lucas-cobalancing c_n numbers. For each of the six GCDs, we provide closed forms dependent on k. Moreover, each of these closed forms can be realized as braid sequences of Pell and associated Pell numbers in an intriguing manner. We end with partial results on GCDs of sums of squared terms and open questions.

[1]Corresponding author.

1 Introduction

In 2021, Guyer and Mbirika gave closed forms for the greatest common divisor (GCD) of all sums of k consecutive generalized Fibonacci numbers [4]. Further, in 2022, Mbirika and Spilker generalized those results to the setting of the GCD of all sums of k consecutive squares of generalized Fibonacci numbers [7]. In this current paper, we extend the results of 2021 to the following six well-known sequences: Pell $(P_n)_{n\geq 0}$, associated Pell $(Q_n)_{n\geq 0}$, balancing $(B_n)_{n\geq 0}$, Lucas-balancing $(C_n)_{n\geq 0}$, cobalancing $(b_n)_{n\geq 0}$, and Lucas-cobalancing $(c_n)_{n\geq 0}$. Moreover, these GCDs can be realized as braid sequences. A *braid sequence* arises when we intertwine two sequences. For example in Figure 1, we intertwine the sequence $(P_n)_{n\geq 1}$ in the top row with the associated Pell sequence $(Q_n)_{n\geq 1}$ in the bottom row.

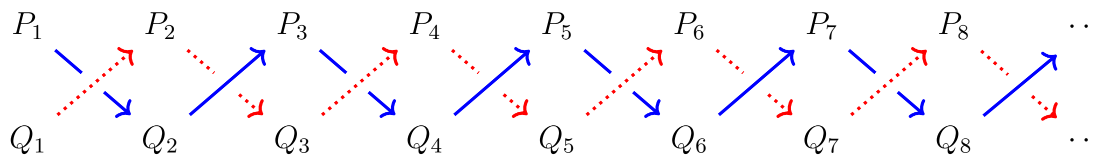


Figure 1: The braiding of $(P_n)_{n\geq 1}$ and $(Q_n)_{n\geq 1}$.

The red-dotted path is the sequence $\big(\gcd(B_n, b_n)\big)_{n\geq 1}$ and the blue-solid path is the sequence $\big(\gcd(B_n, b_n + 1)\big)_{n\geq 1}$. Both braid sequences easily follow from the identities

$$B_n = P_n Q_n, \quad b_n = \begin{cases} P_n Q_{n-1}, & \text{if } n \text{ is even;} \\ P_{n-1} Q_n, & \text{if } n \text{ is odd,} \end{cases} \qquad b_n + 1 = \begin{cases} P_{n-1} Q_n, & \text{if } n \text{ is even;} \\ P_n Q_{n-1}, & \text{if } n \text{ is odd,} \end{cases}$$

and the facts that $\gcd(P_{n-1}, P_n) = 1$ and $\gcd(Q_{n-1}, Q_n) = 1$.

In this paper we give closed forms for the braid sequence of GCD-values $\big(\mathcal{S}^m(k)\big)_{k\geq 1}$ when $m = 1$ for each of the six sequences and give partial results for when $m = 2$. We chose these six sequences, in particular, since the GCD-values $\big(\mathcal{S}(k)\big)_{k\geq 1}$ for each of the six sequences all involve Pell and associated Pell numbers in an intriguing manner. The breakdown of this paper is as follows. In Section 2, we provide definitions of the six sequences and some historical origins of the relatively newer sequences $(B_n)_{n\geq 0}$, $(C_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$, and $(c_n)_{n\geq 0}$. In Section 3, we give preliminary identities used to prove our main results, which are in Sections 4 and 5. Finally, in Section 6, we address our progress towards the $m = 2$ setting and provide some open questions.

Convention 1. When $(S_n)_{n\geq 0}$ is any of the six sequences $(P_n)_{n\geq 0}$, $(Q_n)_{n\geq 0}$, $(B_n)_{n\geq 0}$, $(C_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$, or $(c_n)_{n\geq 0}$, we establish the notation $\mathcal{S}^m(k)$ to denote the GCD of all sums of k consecutive m^{th} powers of sequence terms, in the respective six settings:

$$\mathscr{P}^m(k) = \gcd\left\{\left(\sum_{i=1}^{k} P_{n+i}^m\right)_{n\geq 0}\right\}, \qquad \mathbb{Q}^m(k) = \gcd\left\{\left(\sum_{i=1}^{k} Q_{n+i}^m\right)_{n\geq 0}\right\},$$

$$\mathscr{B}^m(k) = \gcd\left\{\left(\sum_{i=1}^{k} B_{n+i}^m\right)_{n\geq 0}\right\}, \qquad \mathscr{C}^m(k) = \gcd\left\{\left(\sum_{i=1}^{k} C_{n+i}^m\right)_{n\geq 0}\right\},$$

$$\mathscr{b}^m(k) = \gcd\left\{\left(\sum_{i=1}^{k} b_{n+i}^m\right)_{n\geq 0}\right\}, \qquad \mathscr{c}^m(k) = \gcd\left\{\left(\sum_{i=1}^{k} c_{n+i}^m\right)_{n\geq 0}\right\}.$$

Remark 2. When $m = 1$, we omit the superscript and simply write $\mathcal{S}(k)$.

1.1 Motivation: a new proof of an old result

At the 20^{th} *International Conference on Fibonacci Numbers and Their Applications* in Sarajevo in 2022, Mbirika presented his and collaborator Guyer's results on the GCD of all sums of k consecutive generalized Fibonacci numbers [4]. Conference participant Florian Luca communicated to Mbirika an observation that leads to a simple proof in the Fibonacci setting when k is even. In this setting, the Guyer-Mbirika result was the following:

$$\mathcal{F}(k) = \begin{cases} F_{k/2}, & \text{if } k \equiv 0 \pmod 4; \\ L_{k/2}, & \text{if } k \equiv 2 \pmod 4, \end{cases}$$

where $\mathcal{F}(k)$ denotes the GCD of all sums of k consecutive Fibonacci numbers. For ease of notation, set $\sigma_F(k,n)$ to be the sum $\sum_{i=1}^{k} F_{n+i}$. Luca's observations was the following:

$$\sigma_F(k,n) = F_{n+k+2} - F_{n+2} = \begin{cases} F_{k/2} L_{(k/2+2)+n}, & \text{if } k \equiv 0 \pmod 4; \\ L_{k/2} F_{(k/2+2)+n}, & \text{if } k \equiv 2 \pmod 4. \end{cases} \tag{1}$$

From this identity, the Guyer-Mbirika result is easily proven. For example for $k = 20$, using Identity (1) and the fact that consecutive Lucas numbers are relatively prime, we have

$$\begin{aligned} \mathcal{F}(20) = \gcd(\sigma_F(20,0), \sigma_F(20,1), \sigma_F(20,2), \ldots) &= \gcd(F_{10}L_{12}, F_{10}L_{13}, F_{10}L_{14}, \ldots) \\ &= F_{10} \cdot \gcd(L_{12}, L_{13}, L_{14}, \ldots) \\ &= F_{10}, \end{aligned}$$

as expected. For Identity (1), Luca noted that the first equality is easily shown if we utilize the fact that $\sum_{i=1}^{k} F_i = F_{k+2} - 1$, and the second equality follows by the known result

$$F_a - F_b = \begin{cases} F_{\frac{a-b}{2}} L_{\frac{a+b}{2}}, & \text{if } a - b \equiv 0 \pmod{4}; \\ L_{\frac{a-b}{2}} F_{\frac{a+b}{2}}, & \text{if } a - b \equiv 2 \pmod{4}, \end{cases} \tag{2}$$

if we set $a := n+k+2$ and $b := n+2$. Identity (2) follows directly from a 1963 result proven by Ruggles [10, p. 77]. In this current paper, we generalize Identity (2) into the Pell and associated Pell settings in Lemmas 12 and 13. Moreover, using the latter two lemmas we generalize Identity (1) to compute the sums $\sigma_S(k, n)$ for five (of our six) different sequences $(S_n)_{n\geq 0}$ in one of our main results given in Theorem 14.

2 Definitions of the six sequences and some remarks

We first recall the recursive definitions of the six sequences used in this paper, and then we follow with their well-known Binet forms.

Definition 3. The *Pell sequence* $(P_n)_{n\geq 0}$ and the *associated Pell sequence* $(Q_n)_{n\geq 0}$ are defined by the recurrence relations $P_n = 2P_{n-1} + P_{n-2}$ and $Q_n = 2Q_{n-1} + Q_{n-2}$, respectively, with initial conditions $P_0 = 0$, $P_1 = 1$, $Q_0 = 1$, and $Q_1 = 1$. In the OEIS, these are sequences A000129 and A001333, respectively [12].

Remark 4. In the literature, there is unfortunately some discrepancy on the precise definition of the *Pell-Lucas sequence*. Though many sources attribute the OEIS sequence A002203 as the "companion Pell sequence" (or equivalently, the Pell-Lucas sequence), we choose to follow Koshy [6] and many others in the literature who define the Pell-Lucas sequence as we have done in Definition 3 and call $(Q_n)_{n\geq 0}$ the "associated Pell sequence".

Remark 5. The associated Pell (respectively, Pell) sequence is the sequence of numerators (respectively, denominators) of the rational convergents to $\sqrt{2}$; that is, $\lim\limits_{n\to\infty} \frac{Q_n}{P_n} = \sqrt{2}$.

Before we give the recursive definition of the remaining four sequences, we first discuss how these four sequences were originally defined. In 1999, Behera and Panda [1] defined an integer $n \in \mathbb{N}$ to be a balancing number if it is a solution to the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r), \tag{3}$$

where r is the balancer corresponding to n. The terms in the sequence of balancing numbers and their corresponding balancers are denoted B_n and R_n, respectively. For example, $B_2 = 6$ and $R_2 = 2$ since $1 + 2 + \cdots + 5 = 7 + 8$. Later in 2005, Panda and Ray [9] slightly modified Equation (3) to the Diophantine equation

$$1 + 2 + \cdots + n = (n+1) + (n+2) + \cdots + (n+r). \tag{4}$$

In this new setting, they called the value n a cobalancing number and the corresponding r a cobalancer. The terms in the sequence of cobalancing numbers and their corresponding cobalancers are denoted b_n and r_n, respectively. It turns out that every balancer is also a cobalancing number in the following sense: $R_n = b_n$. Moreover, every cobalancer is also a balancing number in the following sense: $r_{n+1} = B_n$. Hence, in this paper we consider the sequences $(B_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ and neither $(R_n)_{n\geq 0}$ nor $(r_n)_{n\geq 0}$.

Behera and Panda also showed that B_n is a balancing number if and only if $8B_n^2+1$ is a perfect square. So consider the sequence, denoted $(C_n)_{n\geq 0}$, of positive roots of $\sqrt{8B_n^2+1}$ for each $n \geq 0$. This sequence is called the Lucas-balancing sequence and is named so since the value C_n is associated to B_n in many manners similar to the relationship between L_n and F_n [8]. Lastly, it is known that b_n is a cobalancing number if and only if $8b_n^2+8b_n+1$ is a perfect square. So consider the sequence, denoted $(c_n)_{n\geq 0}$, of positive roots of $\sqrt{8b_n^2+8b_n+1}$ for each $n \geq 1$ and set $c_0 := -1$. This sequence is called the Lucas-cobalancing sequence.

We now give the recursive definitions of the four sequences $(B_n)_{n\geq 0}$, $(C_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$, and $(c_n)_{n\geq 0}$. Then in Table 1, we give the first eleven terms of each of the six sequences.

Definition 6. The *balancing sequence* $(B_n)_{n\geq 0}$ and the *Lucas-balancing sequence* $(C_n)_{n\geq 0}$ are defined by the recurrence relations $B_n = 6B_{n-1} - B_{n-2}$ and $C_n = 6C_{n-1} - C_{n-2}$, respectively, with initial conditions $B_0 = 0$, $B_1 = 1$, $C_0 = 1$, and $C_1 = 3$. In the OEIS, these are sequences A001109 and A001541, respectively [12].

Definition 7. The *cobalancing sequence* $(b_n)_{n\geq 0}$ and the *Lucas-cobalancing sequence* $(c_n)_{n\geq 0}$ are defined by the recurrence relations $b_n = 6b_{n-1} - b_{n-2} + 2$ and $c_n = 6c_{n-1} - c_{n-2}$, respectively, with initial conditions $b_0 = 0$, $b_1 = 0$, $c_0 = -1$, and $c_1 = 1$. In the OEIS, these are sequences A053141 and A002315, respectively [12].

Finally, let $\gamma = 1+\sqrt{2}$ and $\delta = 1-\sqrt{2}$. Then we have the following well-known Binet forms for the sequence terms P_n, Q_n, B_n, C_n, b_n, and c_n, respectively:

$$P_n = \tfrac{\gamma^n-\delta^n}{2\sqrt{2}}, \qquad B_n = \tfrac{\gamma^{2n}-\delta^{2n}}{4\sqrt{2}}, \qquad b_n = \tfrac{\gamma^{2n-1}-\delta^{2n-1}}{4\sqrt{2}} - \tfrac{1}{2},$$

$$Q_n = \tfrac{\gamma^n+\delta^n}{2}, \qquad C_n = \tfrac{\gamma^{2n}+\delta^{2n}}{2}, \qquad c_n = \tfrac{\gamma^{2n-1}+\delta^{2n-1}}{2}.$$

3 Some old and new identities

In this section, we provide the preliminary identities used to prove our main results in Sections 4 and 5. Some of these identities are well known, but most are new.

3.1 Sum identities, Cassini's identities, and GCD identities

The following lemma follows from results in Koshy's book [6], the Binet formulas given in Section 2, or Catarino et al. [2].

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
P_n	0	1	2	5	12	29	70	169	408	985	2378
Q_n	1	1	3	7	17	41	99	239	577	1393	3363
B_n	0	1	6	35	204	1189	6930	40391	235416	1372105	7997214
C_n	1	3	17	99	577	3363	19601	114243	665857	3880899	22619537
b_n	0	0	2	14	84	492	2870	16730	97512	568344	3312554
c_n	-1	1	7	41	239	1393	8119	47321	275807	1607521	9369319

Table 1: The first 11 Pell P_n, associated Pell Q_n, balancing B_n, cobalancing b_n, Lucas-balancing C_n, and Lucas-cobalancing c_n numbers.

Lemma 8. *For all $k \geq 1$, the following sum identities hold for the six sequences Pell $(P_n)_{n\geq 0}$, associated Pell $(Q_n)_{n\geq 0}$, balancing $(B_n)_{n\geq 0}$, Lucas-balancing $(C_n)_{n\geq 0}$, cobalancing $(b_n)_{n\geq 0}$, and Lucas-cobalancing $(c_n)_{n\geq 0}$:*

$$\sum_{i=1}^{k} P_i = \frac{1}{2}(Q_{k+1} - 1), \qquad (5)$$

$$\sum_{i=1}^{k} Q_i = P_{k+1} - 1, \qquad (6)$$

$$\sum_{i=1}^{k} B_i = \frac{1}{4}(P_{2k+1} - 1), \qquad (7)$$

$$\sum_{i=1}^{k} C_i = \frac{1}{2}(Q_{2k+1} - 1), \qquad (8)$$

$$\sum_{i=1}^{k} b_i = \frac{1}{4}(b_{k+1} - b_k - 2k), \qquad (9)$$

$$\sum_{i=1}^{k} c_i = \frac{1}{2}(Q_{2k} - 1). \qquad (10)$$

Proof of Identity (5). This is well known (see [6, Identity (10.1)]). □

Proof of Identity (6). This is well known (see [6, Identity (10.2)]). □

Proof of Identity (7). By the Binet formulas, we have $B_i = \frac{\gamma^{2i}-\delta^{2i}}{4\sqrt{2}} = \frac{1}{2} \cdot \frac{\gamma^{2i}-\delta^{2i}}{2\sqrt{2}} = \frac{1}{2}P_{2i}$. It follows that $\sum_{i=1}^{k} B_i = \frac{1}{2}\sum_{i=1}^{k} P_{2i} = \frac{1}{2}\left(\frac{1}{2}P_{2k+1} - \frac{1}{2}\right) = \frac{1}{4}(P_{2k+1} - 1)$, where the second equality holds by a well-known identity for the sum of the first k even-indexed Pell numbers (see [6, Identity (10.4)]). □

Proof of Identity (8). By the Binet formulas, we have $C_i = \frac{\gamma^{2i}-\delta^{2i}}{2} = Q_{2i}$. It follows that $\sum_{i=1}^{k} C_i = \sum_{i=1}^{k} Q_{2i} = \frac{1}{2}(Q_{2k+1}-1)$, where the second equality holds by a well-known identity for the sum of the first k even-indexed associated Pell numbers (see [6, Identity (10.5)]). □

Proof of Identity (9). This is well known (see [2, Proposition 3.6]). □

Proof of Identity (10). By the Binet formulas, we have $c_i = \frac{\gamma^{2i-1}-\delta^{2i-1}}{2} = Q_{2i-1}$. It follows that $\sum_{i=1}^{k} c_i = \sum_{i=1}^{k} Q_{2i-1} = \frac{1}{2}(Q_{2k} - 1)$, where the second equality holds by a well-known identity for the sum of the first k odd-indexed associated Pell numbers (see [6, Identity (10.6)]). □

Cassini's identity for the Fibonacci numbers has an analogue in both the Pell and associated Pell settings. We use the following Cassini's identities in the next two lemmas to prove the closed forms of $\mathscr{P}(k)$ and $\mathbb{Q}(k)$ given in Theorems 15 and 16, respectively.

Lemma 9 (Cassini's identity for $(P_n)_{n\geq 0}$)**.** *For all $k \geq 1$, we have $P_{k-1}P_{k+1} = P_k^2 + (-1)^k$.*

Proof. See Horadam [5, Identity (30)]. □

Koshy mentions Cassini's identity in the associated Pell setting [6, Identity (35)]; however, he provides no proof. As we could not find a proof in the literature of this identity, we provide our own proof using the Binet formula for Q_n.

Lemma 10 (Cassini's identity for $(Q_n)_{n\geq 0}$)**.** *For all $k \geq 1$, we have*

$$Q_{k-1}Q_{k+1} = Q_k^2 + 2(-1)^{k-1}.$$

Proof. By the Binet formula for the associated Pell sequence, we have

$$\begin{aligned}
Q_{k+1}Q_{k-1} - Q_k^2 &= \left(\frac{\gamma^{k+1}+\delta^{k+1}}{2}\right)\left(\frac{\gamma^{k-1}+\delta^{k-1}}{2}\right) - \left(\frac{\gamma^{k}+\delta^{k}}{2}\right)^2 \\
&= \frac{\gamma^{2k} + \gamma^{k+1}\delta^{k-1} + \gamma^{k-1}\delta^{k+1} + \delta^{2k}}{4} - \frac{\gamma^{2k} + 2(\gamma\delta)^k + \delta^{2k}}{4} \\
&= \frac{\gamma^2(\gamma\delta)^{k-1} + \delta^2(\gamma\delta)^{k-1} - 2(\gamma\delta)^k}{4} \\
&= \frac{(\gamma^2+\delta^2)(-1)^{k-1} - 2(-1)^k}{4} \qquad \text{(since } \gamma\delta = -1\text{)} \\
&= \frac{6(-1)^{k-1} + 2(-1)^{k-1}}{4} \\
&= 2(-1)^{k-1},
\end{aligned}$$

where the fifth equality holds since $3 = Q_2 = \frac{\gamma^2+\delta^2}{2}$ implies $\gamma^2 + \delta^2 = 6$. □

Lemma 11. *For all $n \geq 1$, we have the following five identities:*

$$\gcd(P_n, P_{n+1}) = 1, \quad (11) \qquad \gcd(P_{2n-1}, P_{2n+1}) = 1, \quad (14)$$
$$\gcd(Q_n, Q_{n+1}) = 1, \quad (12) \qquad \gcd(Q_n, Q_{n+2}) = 1. \quad (15)$$
$$\gcd(P_{2n}, P_{2n+2}) = 2, \quad (13)$$

Proof. Identity (11) follows from Lemma 9, while Identity (12) follows from Lemma 10 and the fact that associated Pell numbers are always odd. Identities (13) and (14) hold by Flórez et al. [3] in Proposition 2 part (2) if we set $x := 1$, and Identity (15) holds by Flórez et al. [3] in Proposition 2 part (1) if we set $x := 1$. □

3.2 New identities used to prove our main results

Lemma 12. *For all $s, r \geq 1$ where s is even, the following identity holds:*

$$P_{s+r} - P_r = \begin{cases} 2P_{s/2}Q_{s/2+r}, & \text{if } s \equiv 0 \pmod 4; \\ 2Q_{s/2}P_{s/2+r}, & \text{if } s \equiv 2 \pmod 4. \end{cases}$$

Proof. Let $s, r \geq 1$ be given where s is even.

Case I. Suppose $s \equiv 0 \pmod 4$. Then $\frac{s}{2}$ is even and hence $(\gamma\delta)^{s/2} = 1$. Observe that

$$\begin{aligned} P_{s+r} - P_r &= \frac{\gamma^{s+r} - \delta^{s+r}}{2\sqrt{2}} - \frac{\gamma^r - \delta^r}{2\sqrt{2}} \\ &= \frac{1}{2\sqrt{2}} \left(\gamma^{s+r} - \delta^{s+r} - (\gamma\delta)^{s/2}(\gamma^r - \delta^r)\right) \qquad \text{(since } (\gamma\delta)^{s/2} = 1) \\ &= \frac{1}{2\sqrt{2}} \left(\gamma^{s/2} - \delta^{s/2}\right) \left(\gamma^{s/2+r} + \delta^{s/2+r}\right) \\ &= 2 \cdot \frac{\gamma^{s/2} - \delta^{s/2}}{2\sqrt{2}} \cdot \frac{\gamma^{s/2+r} + \delta^{s/2+r}}{2} \\ &= 2P_{s/2}Q_{s/2+r}. \end{aligned}$$

Case II. Suppose $s \equiv 2 \pmod 4$. Then $\frac{s}{2}$ is odd and hence $(\gamma\delta)^{s/2} = -1$. Observe that

$$\begin{aligned} P_{s+r} - P_r &= \frac{\gamma^{s+r} - \delta^{s+r}}{2\sqrt{2}} - \frac{\gamma^r - \delta^r}{2\sqrt{2}} \\ &= \frac{1}{2\sqrt{2}} \left(\gamma^{s+r} - \delta^{s+r} + (\gamma\delta)^{s/2}(\gamma^r - \delta^r)\right) \qquad \text{(since } (\gamma\delta)^{s/2} = -1) \\ &= \frac{1}{2\sqrt{2}} \left(\gamma^{s/2} + \delta^{s/2}\right) \left(\gamma^{s/2+r} - \delta^{s/2+r}\right) \\ &= 2 \cdot \frac{\gamma^{s/2} + \delta^{s/2}}{2} \cdot \frac{\gamma^{s/2+r} - \delta^{s/2+r}}{2\sqrt{2}} \\ &= 2Q_{s/2}P_{s/2+r}. \end{aligned}$$

□

Lemma 13. *For all $s, r \geq 1$ where s is even, the following identity holds:*

$$Q_{s+r} - Q_r = \begin{cases} 4P_{s/2}P_{s/2+r}, & \text{if } s \equiv 0 \pmod 4; \\ 2Q_{s/2}Q_{s/2+r}, & \text{if } s \equiv 2 \pmod 4. \end{cases}$$

Proof. Let $s, r \geq 1$ be given where s is even.

Case I. Suppose $s \equiv 0 \pmod 4$. Then $\frac{s}{2}$ is even and hence $(\gamma\delta)^{s/2} = 1$. Observe that

$$\begin{aligned} Q_{s+r} - Q_r &= \frac{\gamma^{s+r} + \delta^{s+r}}{2} - \frac{\gamma^r + \delta^r}{2} \\ &= \frac{1}{2}\left(\gamma^{s+r} + \delta^{s+r} - (\gamma\delta)^{s/2}(\gamma^r + \delta^r)\right) \qquad \text{(since } (\gamma\delta)^{s/2} = 1\text{)} \\ &= \frac{1}{2}\left(\gamma^{s/2} - \delta^{s/2}\right)\left(\gamma^{s/2+r} - \delta^{s/2+r}\right) \\ &= 4 \cdot \frac{\gamma^{s/2} - \delta^{s/2}}{2\sqrt{2}} \cdot \frac{\gamma^{s/2+r} - \delta^{s/2+r}}{2\sqrt{2}} \\ &= 4P_{s/2}P_{s/2+r}. \end{aligned}$$

Case II. Suppose $s \equiv 2 \pmod 4$. Then $\frac{s}{2}$ is odd and hence $(\gamma\delta)^{s/2} = -1$. Observe that

$$\begin{aligned} Q_{s+r} - Q_r &= \frac{\gamma^{s+r} + \delta^{s+r}}{2} - \frac{\gamma^r + \delta^r}{2} \\ &= \frac{1}{2}\left(\gamma^{s+r} + \delta^{s+r} + (\gamma\delta)^{s/2}(\gamma^r + \delta^r)\right) \qquad \text{(since } (\gamma\delta)^{s/2} = -1\text{)} \\ &= \frac{1}{2}\left(\gamma^{s/2} + \delta^{s/2}\right)\left(\gamma^{s/2+r} + \delta^{s/2+r}\right) \\ &= 2 \cdot \frac{\gamma^{s/2} + \delta^{s/2}}{2} \cdot \frac{\gamma^{s/2+r} + \delta^{s/2+r}}{2} \\ &= 2Q_{s/2}Q_{s/2+r}. \end{aligned}$$

□

Using the latter Lemmas 12 and 13, we are now ready to prove our main sum identities in the following theorem, which we use to prove our main results in Section 4.

Theorem 14. *For all $k \geq 1$, set $\sigma_S(k,n) := \sum_{i=1}^{k} S_{n+i}$ where $(S_n)_{n\geq 0}$ is any sequence. Then the following identities hold for the five sequences Pell $(P_n)_{n\geq 0}$, associated Pell $(Q_n)_{n\geq 0}$, balancing $(B_n)_{n\geq 0}$, Lucas-balancing $(C_n)_{n\geq 0}$, and Lucas-cobalancing $(c_n)_{n\geq 0}$:*

$$\sigma_P(k,n) = \frac{1}{2}(Q_{n+k+1} - Q_{n+1}) = \begin{cases} 2P_{k/2}P_{k/2+n+1}, & \text{if } k \equiv 0 \pmod 4; \\ Q_{k/2}Q_{k/2+n+1}, & \text{if } k \equiv 2 \pmod 4. \end{cases} \tag{16}$$

$$\sigma_Q(k,n) = P_{n+k+1} - P_{n+1} = \begin{cases} 2P_{k/2}Q_{k/2+n+1}, & \text{if } k \equiv 0 \pmod 4; \\ 2Q_{k/2}P_{k/2+n+1}, & \text{if } k \equiv 2 \pmod 4. \end{cases} \tag{17}$$

$$\sigma_B(k,n) = \frac{1}{4}\left(P_{2k+2n+1} - P_{2n+1}\right) = \begin{cases} \frac{1}{2}P_kQ_{k+2n+1}, & \text{if } k \text{ is even;} \\ \frac{1}{2}Q_kP_{k+2n+1}, & \text{if } k \text{ is odd.} \end{cases} \tag{18}$$

$$\sigma_C(k,n) = \frac{1}{2}\left(Q_{2k+2n+1} - Q_{2n+1}\right) = \begin{cases} 2P_k P_{k+2n+1}, & \textit{if } k \textit{ is even;} \\ Q_k Q_{k+2n+1}, & \textit{if } k \textit{ is odd.} \end{cases} \tag{19}$$

$$\sigma_c(k,n) = \frac{1}{2}\left(Q_{2k+2n} - Q_{2n}\right) = \begin{cases} 2P_k P_{k+2n}, & \textit{if } k \textit{ is even;} \\ Q_k Q_{k+2n}, & \textit{if } k \textit{ is odd.} \end{cases} \tag{20}$$

Proof of Identity (16). Let $k \geq 2$ be given where k is even. Observe that

$$\begin{aligned} \sigma_P(k,n) := \sum_{i=1}^{k} P_{n+i} &= \sum_{i=1}^{k+n} P_i - \sum_{i=1}^{n} P_i \\ &= \frac{1}{2}(Q_{k+n+1} - Q_{n+1}), \end{aligned}$$

where the last equality holds by Identity (5) of Lemma 8. By Lemma 13, if we set $s := k$ and $r := n+1$, then we have

$$\sigma_P(k,n) = \begin{cases} \frac{1}{2}\left(4P_{k/2}P_{k/2+n+1}\right) = 2P_{k/2}P_{k/2+n+1}, & \text{if } k \equiv 0 \pmod 4; \\ \frac{1}{2}\left(2Q_{k/2}Q_{k/2+n+1}\right) = Q_{k/2}Q_{k/2+n+1}, & \text{if } k \equiv 2 \pmod 4. \end{cases}$$

□

Proof of Identity (17). Let $k \geq 2$ be given where k is even. Observe that

$$\begin{aligned} \sigma_Q(k,n) := \sum_{i=1}^{k} Q_{n+i} &= \sum_{i=1}^{k+n} Q_i - \sum_{i=1}^{n} Q_i \\ &= P_{k+n+1} - P_{n+1}, \end{aligned}$$

where the last equality holds by Identity (6) of Lemma 8. By Lemma 12, if we set $s := k$ and $r := n+1$, then we have

$$\sigma_Q(k,n) = \begin{cases} 2P_{k/2}Q_{k/2+n+1}, & \text{if } k \equiv 0 \pmod 4; \\ 2Q_{k/2}P_{k/2+n+1}, & \text{if } k \equiv 2 \pmod 4. \end{cases}$$

□

Proof of Identity (18). Let $k \geq 1$ be given. Observe that

$$\sigma_B(k,n) := \sum_{i=1}^{k} B_{n+i} = \sum_{i=1}^{k+n} B_i - \sum_{i=1}^{n} B_i$$

$$= \frac{1}{4}\left(P_{2k+2n+1} - P_{2n+1}\right),$$

where the last equality holds by Identity (7) of Lemma 8. By Lemma 12, if we set $s := 2k$ and $r := 2n+1$, then we have

$$\sigma_B(k,n) = \begin{cases} \frac{1}{4}\left(2P_{2k/2}Q_{2k/2+2n+1}\right) = \frac{1}{2}P_kQ_{k+2n+1}, & \text{if } k \text{ is even;} \\ \frac{1}{4}\left(2Q_{2k/2}P_{2k/2+2n+1}\right) = \frac{1}{2}Q_kP_{k+2n+1}, & \text{if } k \text{ is odd,} \end{cases}$$

since $s \equiv 0 \pmod 4$ if and only if k is even, and $s \equiv 2 \pmod 4$ if and only if k is odd. □

Proof of Identity (19). Let $k \geq 1$ be given. Observe that

$$\sigma_C(k,n) := \sum_{i=1}^{k} C_{n+i} = \sum_{i=1}^{k+n} C_i - \sum_{i=1}^{n} C_i$$
$$= \frac{1}{2}\left(Q_{2k+2n+1} - Q_{2n+1}\right),$$

where the last equality holds by Identity (8) of Lemma 8. By Lemma 13, if we set $s := 2k$ and $r := 2n+1$, then we have

$$\sigma_C(k,n) = \begin{cases} \frac{1}{2}\left(4P_{2k/2}P_{2k/2+2n+1}\right) = 2P_kP_{k+2n+1}, & \text{if } k \text{ is even;} \\ \frac{1}{2}\left(2Q_{2k/2}Q_{2k/2+2n+1}\right) = Q_kQ_{k+2n+1}, & \text{if } k \text{ is odd,} \end{cases}$$

since $s \equiv 0 \pmod 4$ if and only if k is even, and $s \equiv 2 \pmod 4$ if and only if k is odd. □

Proof of Identity (20). Let $k \geq 1$ be given. Observe that

$$\sigma_c(k,n) := \sum_{i=1}^{k} c_{n+i} = \sum_{i=1}^{k+n} c_i - \sum_{i=1}^{n} c_i$$
$$= \frac{1}{2}\left(Q_{2k+2n} - Q_{2n}\right),$$

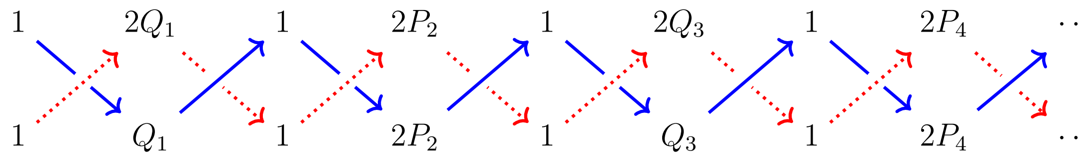
where the last equality holds by Identity (10) of Lemma 8. By Lemma 13, if we set $s := 2k$ and $r := 2n$, then we have

$$\sigma_c(k,n) = \begin{cases} \frac{1}{2}\left(4P_{2k/2}P_{2k/2+2n}\right) = 2P_kP_{k+2n}, & \text{if } k \text{ is even;} \\ \frac{1}{2}\left(2Q_{2k/2}Q_{2k/2+2n}\right) = Q_kQ_{k+2n}, & \text{if } k \text{ is odd,} \end{cases}$$

since $s \equiv 0 \pmod 4$ if and only if k is even, and $s \equiv 2 \pmod 4$ if and only if k is odd. □

4 Main results for $\mathscr{P}(k)$, $\mathbb{Q}(k)$, $\mathscr{B}(k)$, $\mathscr{C}(k)$, and $c(k)$

These are the braids for braid sequences $(\mathscr{P}(k))_{k\geq 1}$ in solid blue and $(\mathbb{Q}(k))_{k\geq 1}$ in dotted red. In Theorems 15 and 16, respectively, we give the proofs of these two braids.



k	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
$\mathscr{P}(k)$	1	1	1	4	1	7	1	24	1	41	1	140	1	239
$\mathbb{Q}(k)$	1	2	1	4	1	14	1	24	1	82	1	140	1	478

Table 2: The first 14 terms of the sequences $(\mathscr{P}(k))_{k\geq 1}$ and $(\mathbb{Q}(k))_{k\geq 1}$.

Theorem 15. *For all $k \geq 1$, the GCD of all sums of k consecutive Pell numbers is*

$$\mathscr{P}(k) = \begin{cases} 2P_{k/2}, & \text{if } k \equiv 0 \pmod 4; \\ Q_{k/2}, & \text{if } k \equiv 2 \pmod 4; \\ 1, & \text{if } k \equiv 1,3 \pmod 4. \end{cases}$$

Proof. Let $k \geq 1$ be given. Recall $\sigma_P(k,n) = \sum_{i=1}^{k} P_{n+i}$. So by definition of $\mathscr{P}(k)$, we have

$$\mathscr{P}(k) = \gcd\left(\sigma_P(k,0), \sigma_P(k,1), \sigma_P(k,2), \ldots\right). \tag{21}$$

Case I. Suppose $k \equiv 0 \pmod 4$. By Identity (16) of Theorem 14, it follows that

$$\begin{aligned} \mathscr{P}(k) &= \gcd(2P_{k/2}P_{k/2+1}, 2P_{k/2}P_{k/2+2}, 2P_{k/2}P_{k/2+3}, \ldots) \\ &= 2P_{k/2} \cdot \gcd(P_{k/2+1}, P_{k/2+2}, P_{k/2+3}, \ldots) \\ &= 2P_{k/2}, \end{aligned}$$

where the last equality holds by Identity (11) of Lemma 11.

Case II. Suppose $k \equiv 2 \pmod 4$. By Identity (16) of Theorem 14, it follows that

$$\begin{aligned} \mathscr{P}(k) &= \gcd(Q_{k/2}Q_{k/2+1}, Q_{k/2}Q_{k/2+2}, Q_{k/2}Q_{k/2+3}, \ldots) \\ &= Q_{k/2} \cdot \gcd(Q_{k/2+1}, Q_{k/2+2}, Q_{k/2+3}, \ldots) \\ &= Q_{k/2}, \end{aligned}$$

where the last equality holds by Identity (12) of Lemma 11.

Case III. Suppose k is odd. Assume by way of contradiction that p^j divides $\mathcal{P}(k)$ for some prime p with $j \geq 1$. By Equation (21), it follows that p^j divides $\sigma_P(k, n)$ for all $n \geq 0$. Hence p^j divides both $\sigma_P(k, 1) - \sigma_P(k, 0)$ and $\sigma_P(k, 2) - \sigma_P(k, 1)$. Observe that

$$\sigma_P(k, 1) - \sigma_P(k, 0) = (P_2 + P_3 + \cdots + P_{k+1}) - (P_1 + P_2 + \cdots + P_k) = P_{k+1} - 1, \text{and}$$
$$\sigma_P(k, 2) - \sigma_P(k, 1) = (P_3 + P_4 + \cdots + P_{k+2}) - (P_2 + P_3 + \cdots + P_{k+1}) = P_{k+2} - 2.$$

Hence p^j divides both $P_{k+1} - 1$ and $P_{k+2} - 2$. Since $P_k + 2P_{k+1} = P_{k+2}$, we have

$$\begin{aligned} P_k &= P_{k+2} - 2P_{k+1} \\ &= P_{k+2} - 2 - 2P_{k+1} + 2 \\ &= (P_{k+2} - 2) - 2(P_{k+1} - 1), \end{aligned}$$

and therefore p^j divides P_k. Thus $P_k P_{k+2} \equiv 0 \pmod{p^j}$. Moreover, since p^j divides $P_{k+1} - 1$, we have p^j divides $P_{k+1}^2 - 1$, and thus $P_{k+1}^2 + 1 \equiv 2 \pmod{p^j}$. By the Pell Cassini Identity, Lemma 9, we have $P_k P_{k+2} = P_{k+1}^2 + 1$ since k is odd. It follows that $2 \equiv 0 \pmod{p^j}$ and thus p^j divides 2, forcing $p = 2$ and $j = 1$. Since p^j divides $P_{k+1} - 1$ and we know $p^j = 2$, this implies $P_{k+1} - 1$ is even. However, k being odd implies P_{k+1} is even and hence $P_{k+1} - 1$ is also odd, which yields a contradiction. Thus there exists no prime that divides $\mathcal{P}(k)$ when k is odd, and hence $\mathcal{P}(k) = 1$ for all odd k. □

Theorem 16. *For all $k \geq 1$, the GCD of all sums of k consecutive associated Pell numbers is*

$$\mathbb{Q}(k) = \begin{cases} 2P_{k/2}, & \text{if } k \equiv 0 \pmod{4}; \\ 2Q_{k/2}, & \text{if } k \equiv 2 \pmod{4}; \\ 1, & \text{if } k \equiv 1, 3 \pmod{4}. \end{cases}$$

Proof. Let $k \geq 1$ be given. Recall $\sigma_Q(k, n) = \sum_{i=1}^{k} Q_{n+i}$. So by definition of $\mathbb{Q}(k)$, we have

$$\mathbb{Q}(k) = \gcd\left(\sigma_Q(k, 0), \sigma_Q(k, 1), \sigma_Q(k, 2), \ldots\right). \tag{22}$$

Case I. Suppose $k \equiv 0 \pmod{4}$. By Identity (17) of Theorem 14, it follows that

$$\begin{aligned} \mathbb{Q}(k) &= \gcd(2P_{k/2}Q_{k/2+1}, 2P_{k/2}Q_{k/2+2}, 2P_{k/2}Q_{k/2+3}, \ldots) \\ &= 2P_{k/2} \cdot \gcd(Q_{k/2+1}, Q_{k/2+2}, Q_{k/2+3}, \ldots) \\ &= 2P_{k/2}, \end{aligned}$$

where the last equality holds by Identity (12) of Lemma 11.

Case II. Suppose $k \equiv 2 \pmod 4$. By Identity (17) of Theorem 14, it follows that

$$\begin{aligned}\mathbb{Q}(k) &= \gcd(2Q_{k/2}P_{k/2+1}, 2Q_{k/2}P_{k/2+2}, 2Q_{k/2}P_{k/2+3}, \ldots)\\ &= 2Q_{k/2} \cdot \gcd(P_{k/2+1}, P_{k/2+2}, P_{k/2+3}, \ldots)\\ &= 2Q_{k/2},\end{aligned}$$

where the last equality holds by Identity (11) of Lemma 11.

Case III. Suppose k is odd. Assume by way of contradiction that p^j divides $\mathbb{Q}(k)$ for some prime p with $j \geq 1$. By Equation (22), it follows that p^j divides $\sigma_Q(k,n)$ for all $n \geq 0$. Hence p^j divides both $\sigma_Q(k,1) - \sigma_Q(k,0)$ and $\sigma_Q(k,2) - \sigma_Q(k,1)$. Observe that

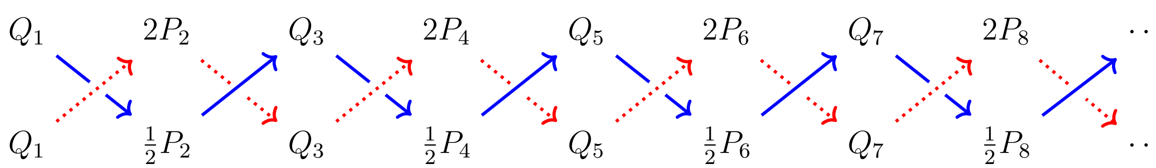
$\sigma_Q(k,1) - \sigma_Q(k,0) = (Q_2 + Q_3 + \cdots + Q_{k+1}) - (Q_1 + Q_2 + \cdots + Q_k) = Q_{k+1} - 1$, and
$\sigma_Q(k,2) - \sigma_Q(k,1) = (Q_3 + Q_4 + \cdots + Q_{k+2}) - (Q_2 + Q_3 + \cdots + Q_{k+1}) = Q_{k+2} - 3$.

Hence p^j divides both $Q_{k+1} - 1$ and $Q_{k+2} - 3$. Since $Q_k + 2Q_{k+1} = Q_{k+2}$, we have

$$\begin{aligned}Q_k &= Q_{k+2} - 2Q_{k+1}\\ &= Q_{k+2} - 3 - 2Q_{k+1} + 3\\ &= (Q_{k+2} - 3) - 2(Q_{k+1} - 1) + 1,\end{aligned}$$

and therefore p^j divides $Q_k - 1$, and thus $Q_k \equiv 1 \pmod{p^j}$. Also, since p^j divides $Q_{k+2} - 3$, we have $Q_{k+2} \equiv 3 \pmod{p^j}$. Therefore, $Q_kQ_{k+2} \equiv 3 \pmod{p^j}$. Moreover, since p^j divides $Q_{k+1} - 1$, we have p^j divides $Q_{k+1}^2 - 1$, and thus $Q_{k+1}^2 - 2 \equiv -1 \pmod{p^j}$. By the associated Pell Cassini Identity, Lemma 10, we have $Q_kQ_{k+2} = Q_{k+1}^2 - 2$ since k is odd. It follows that $3 \equiv -1 \pmod{p^j}$, and hence p^j divides 4, so $p = 2$ is forced. Since p^j divides $Q_{k+2} - 3$ and we know $p = 2$, this implies $Q_{k+2} - 3$ is even. However, for any k we have Q_{k+2} being odd and hence $Q_{k+2} - 3$ is also odd, which yields a contradiction. Thus there exists no prime that divides $\mathbb{Q}(k)$ when k is odd, and hence $\mathbb{Q}(k) = 1$ for all odd k. □

These are the braids for braid sequences $\left(\mathscr{B}(k)\right)_{k\geq 1}$ in solid blue and $\left(\mathscr{C}(k)\right)_{k\geq 1}$ in dotted red. In Theorems 17 and 18, respectively, we give the proofs of these two braids.



k	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
$\mathscr{B}(k)$	1	1	7	6	41	35	239	204	1393	1189	8119	6930	47321
$\mathscr{C}(k)$	1	4	7	24	41	140	239	816	1393	4756	8119	27720	47321

Table 3: The first 13 terms of the sequences $\left(\mathscr{B}(k)\right)_{k\geq 1}$ and $\left(\mathscr{C}(k)\right)_{k\geq 1}$.

Theorem 17. *For all $k \geq 1$, the GCD of all sums of k consecutive balancing numbers is*

$$\mathscr{B}(k) = \begin{cases} \frac{1}{2}P_k, & \text{if } k \text{ is even;} \\ Q_k, & \text{if } k \text{ is odd.} \end{cases}$$

Proof. Let $k \geq 1$ be given. Recall $\sigma_B(k, n) = \sum_{i=1}^{k} B_{n+i}$. So by definition of $\mathscr{B}(k)$, we have

$$\mathscr{B}(k) = \gcd\left(\sigma_B(k, 0), \sigma_B(k, 1), \sigma_B(k, 2), \ldots\right). \tag{23}$$

Case I. Suppose k is even. By Identity (18) of Theorem 14, it follows that

$$\begin{aligned} \mathscr{B}(k) &= \gcd\left(\frac{1}{2}P_kQ_{k+1}, \frac{1}{2}P_kQ_{k+3}, \frac{1}{2}P_kQ_{k+5}, \ldots\right) \\ &= \frac{1}{2}P_k \cdot \gcd\left(Q_{k+1}, Q_{k+3}, Q_{k+5}, \ldots\right) \\ &= \frac{1}{2}P_k, \end{aligned}$$

where the last equality holds by Identity (15) of Theorem 11.

Case II. Suppose k is odd. By Identity (18) of Theorem 14, it follows that

$$\begin{aligned} \mathscr{B}(k) &= \gcd\left(\frac{1}{2}Q_kP_{k+1}, \frac{1}{2}Q_kP_{k+3}, \frac{1}{2}Q_kP_{k+5}, \ldots\right) \\ &= \frac{1}{2}Q_k \cdot \gcd\left(P_{k+1}, P_{k+3}, P_{k+5}, \ldots\right) \\ &= Q_k, \end{aligned}$$

where the last equality holds by Identity (13) of Theorem 11. □

Theorem 18. *For all $k \geq 1$, the GCD of all sums of k consecutive Lucas-balancing numbers is*

$$\mathscr{C}(k) = \begin{cases} 2P_k, & \text{if } k \text{ is even;} \\ Q_k, & \text{if } k \text{ is odd.} \end{cases}$$

Proof. Let $k \geq 1$ be given. Recall $\sigma_C(k, n) = \sum_{i=1}^{k} C_{n+i}$. So by definition of $\mathscr{C}(k)$, we have

$$\mathscr{C}(k) = \gcd\left(\sigma_C(k, 0), \sigma_C(k, 1), \sigma_C(k, 2), \ldots\right). \tag{24}$$

Case I. Suppose k is even. By Identity (19) of Theorem 14, it follows that

$$\begin{aligned}\mathscr{C}(k) &= \gcd\left(2P_kP_{k+1}, 2P_kP_{k+3}, 2P_kP_{k+5}, \ldots\right)\\ &= 2P_k \cdot \gcd(P_{k+1}, P_{k+3}, P_{k+5}, \ldots)\\ &= 2P_k,\end{aligned}$$

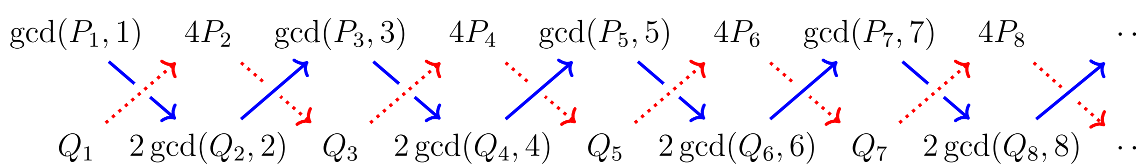
where the last equality holds by Identity (14) of Theorem 11.

Case II. Suppose k is odd. By Identity (19) of Theorem 14, it follows that

$$\begin{aligned}\mathscr{C}(k) &= \gcd\left(Q_kQ_{k+1}, Q_kQ_{k+3}, Q_kQ_{k+5}, \ldots\right)\\ &= Q_k \cdot \gcd(Q_{k+1}, Q_{k+3}, Q_{k+5}, \ldots)\\ &= Q_k,\end{aligned}$$

where the last equality holds by Identity (15) of Theorem 11. □

These are the braids for braid sequences $\left(\mathscr{b}(k)\right)_{k\geq 1}$ in solid blue and $\left(\mathscr{c}(k)\right)_{k\geq 1}$ in dotted red. In Theorem 19, we give the proof of the $\left(\mathscr{c}(k)\right)_{k\geq 1}$ braid. However, we leave the proof of the $\left(\mathscr{b}(k)\right)_{k\geq 1}$ braid to Section 5.



k	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
$\mathscr{b}(k)$	2	2	2	4	2	2	2	8	2	2	2	12	2
$\mathscr{c}(k)$	1	8	7	48	41	280	239	1632	1393	9512	8119	55440	47321

Table 4: The first 13 terms of the sequences $\left(\mathscr{b}(k)\right)_{k\geq 1}$ and $\left(\mathscr{c}(k)\right)_{k\geq 1}$.

Theorem 19. *For all $k \geq 1$, the GCD of all sums of k consecutive Lucas-cobalancing numbers is*

$$\mathscr{c}(k) = \begin{cases} 4P_k, & \text{if } k \text{ is even;}\\ Q_k, & \text{if } k \text{ is odd.}\end{cases}$$

Proof. Let $k \geq 1$ be given. Recall $\sigma_c(k,n) = \sum_{i=1}^{k} c_{n+i}$. So by definition of $\mathscr{c}(k)$, we have

$$\mathscr{c}(k) = \gcd\left(\sigma_c(k,0), \sigma_c(k,1), \sigma_c(k,2), \ldots\right). \tag{25}$$

Case I. Suppose k is even. By Identity (20) of Theorem 14, it follows that

$$\begin{aligned}\mathcal{c}(k) &= \gcd(2P_kP_k, 2P_kP_{k+2}, 2P_kP_{k+4}, \ldots) \\ &= 2P_k \cdot \gcd(P_k, P_{k+2}, P_{k+4}, \ldots) \\ &= 4P_k,\end{aligned}$$

where the last equality holds by Identity (13) of Theorem 11.

Case II. Suppose k is odd. By Identity (20) of Theorem 14, it follows that

$$\begin{aligned}\mathcal{c}(k) &= \gcd(Q_kQ_k, Q_kQ_{k+2}, Q_kQ_{k+4}, \ldots) \\ &= Q_k \cdot \gcd(Q_k, Q_{k+2}, Q_{k+4}, \ldots) \\ &= Q_k,\end{aligned}$$

where the last equality holds by Identity (15) of Theorem 11. □

5 Main results for $\mathcal{b}(k)$

For the five sequences Pell $(P_n)_{n\geq 0}$, associated Pell $(Q_n)_{n\geq 0}$, balancing $(B_n)_{n\geq 0}$, Lucas-balancing $(C_n)_{n\geq 0}$, and Lucas-cobalancing $(c_n)_{n\geq 0}$, the closed forms of the GCD of all sums of k consecutive terms involved braids of Pell and associate Pell numbers. However, in the setting of the cobalancing numbers, something much different occurs, namely we have the following closed form for the GCD of all sums of k consecutive cobalancing numbers:

$$\mathcal{b}(k) = \begin{cases} \gcd(P_k, k), & \text{if } k \text{ is even;} \\ 2\gcd(Q_k, k), & \text{if } k \text{ is odd.} \end{cases} \tag{26}$$

To prove this, we first derive an intermediary form of $\mathcal{b}(k)$ in Theorem 23 of Subsection 5.1. Then we prove our main result, Identity (26), in Theorem 28 of Subsection 5.2.

5.1 An intermediary result for $\mathcal{b}(k)$

In the proof of Theorem 23, we use the following easily-derived GCD result (see [7, Lemma 3.1]).

Lemma 20. *Let $(a_i)_{i\geq 0}$ be a sequence of integers. Then the following identity holds:*

$$\gcd(a_0, a_1, a_2, a_3 \ldots) = \gcd(a_0, a_1 - a_0, a_2 - a_1, a_3 - a_2, \ldots).$$

Theorem 21. *For all $k \geq 1$, set $\sigma_b(k, n) := \sum_{i=1}^{k} b_{n+i}$. Then the following identity holds:*

$$\sigma_b(k, n) = \frac{1}{2}(B_{k+n} - B_n - k). \tag{27}$$

Proof. Let $k \geq 1$ be given. Observe that

$$\begin{aligned}
\sigma_b(k,n) := \sum_{i=1}^{k} b_{n+i} &= \sum_{i=1}^{k+n} b_i - \sum_{i=1}^{n} b_i \\
&= \frac{1}{4}\left(b_{k+n+1} - b_{k+n} - 2(k+n)\right) - \frac{1}{4}\left(b_{n+1} - b_n - 2n\right) \\
&= \frac{1}{4}\left(2B_{k+n} - 2k - 2n\right) - \frac{1}{4}\left(2B_n - 2n\right) \\
&= \frac{1}{4}\left(2B_{k+n} - 2B_n - 2k\right) \\
&= \frac{1}{2}\left(B_{k+n} - B_n - k\right),
\end{aligned}$$

where the second equality holds by Identity (9) of Lemma 8, and the third equality holds from the identity $2B_r = b_{r+1} - b_r$ for all $r \geq 1$ by Panda and Ray [9, Corollary 4.2]. □

Lemma 22. *For all $i, k \geq 1$, the following identity holds:*

$$\frac{1}{2}(P_{2k+2i} - P_{2i}) - \frac{1}{2}(P_{2k+2i-2} - P_{2i-2}) = \begin{cases} 2P_kQ_{k+2i-1}, & \text{if } k \text{ is even;} \\ 2Q_kP_{k+2i-1}, & \text{if } k \text{ is odd.} \end{cases}$$

Proof. For ease of notation, set $t_i := \frac{1}{2}(P_{2k+2i} - P_{2i}) - \frac{1}{2}(P_{2k+2i-2} - P_{2i-2})$. Then we have

$$\begin{aligned}
t_i &= \frac{1}{2}\big((P_{2k+2i} - P_{2k+2i-2}) - (P_{2i} - P_{2i-2})\big) \\
&= \frac{1}{2}(2P_{2k+2i-1} - 2P_{2i-1}) && \text{(by the Pell recurrence)} \\
&= P_{2k+2i-1} - P_{2i-1} \\
&= \begin{cases} 2P_kQ_{k+2i}, & \text{if } k \text{ is even;} \\ 2Q_kP_{k+2i}, & \text{if } k \text{ is odd,} \end{cases}
\end{aligned}$$

where the last equality holds by Lemma 12, if we set $s := 2k$ and $r := 2i - 1$ in the third equality. Then observe that $s \equiv 0 \pmod 4$ if and only if k is even, and $s \equiv 2 \pmod 4$ if and only if k is odd. □

Theorem 23. *For all $k \geq 1$, the GCD of all sums of k consecutive cobalancing numbers is*

$$\mathscr{b}(k) = \begin{cases} \gcd\left(\frac{1}{2}(B_k - k), P_k\right), & \text{if } k \text{ is even;} \\ \gcd\left(\frac{1}{2}(B_k - k), 2Q_k\right), & \text{if } k \text{ is odd.} \end{cases}$$

Proof. Let $k \geq 1$ be given. Recall $\sigma_b(k,n) = \sum_{i=1}^{k} b_{n+i}$. So by definition of $\mathscr{b}(k)$, we have

$$\mathscr{b}(k) = \gcd\left(\sigma_b(k,0), \sigma_b(k,1), \sigma_b(k,2), \ldots\right)$$

$$= \gcd\left(\frac{1}{2}(B_k - k), \frac{1}{2}(B_{k+1} - B_1 - k), \frac{1}{2}(B_{k+2} - B_2 - k), \dots\right),$$

where the second equality holds by Theorem 21. For ease of notation, set $r_i := B_{k+i} - B_i - k$ and $s_i := B_{k+i} - B_i - (B_{k+i-1} - B_{i-1})$ and $t_i := \frac{1}{2}(P_{2k+2i} - P_{2i}) - \frac{1}{2}(P_{2k+2i-2} - P_{2i-2})$. It is clear that $s_i = r_i - r_{i-1}$ for all $i \geq 1$. Moreover by the Binet formulas, $B_i = \frac{P_{2i}}{2}$ holds, so we have $s_i = t_i$ for all $i \geq 1$. Observe that

$$\begin{aligned}
\mathscr{b}(k) &= \frac{1}{2}\gcd(r_0, r_1, r_2, \dots) \\
&= \frac{1}{2}\gcd(B_k - k, s_1, s_2, \dots) && \text{(by Lemma 20)} \\
&= \frac{1}{2}\gcd(B_k - k, \gcd(s_1, s_2, \dots)) \\
&= \frac{1}{2}\gcd(B_k - k, \gcd(t_1, t_2, \dots)).
\end{aligned}$$

Case I. Suppose k is even. Then $t_i = 2P_kQ_{k+2i-1}$ for all $i \geq 1$ by Lemma 22. It follows that

$$\begin{aligned}
\mathscr{b}(k) &= \frac{1}{2}\gcd(B_k - k, \gcd(2P_kQ_{k+1}, 2P_kQ_{k+3}, 2P_kQ_{k+5}, \dots) \\
&= \frac{1}{2}\gcd(B_k - k, 2P_k \cdot \gcd(Q_{k+1}, Q_{k+3}, Q_{k+5}, \dots)) \\
&= \frac{1}{2}\gcd(B_k - k, 2P_k) \\
&= \gcd\left(\frac{1}{2}(B_k - k), P_k\right),
\end{aligned}$$

where the third equality holds by Identity (15) of Theorem 11.

Case II. Suppose k is odd. Then $t_i = 2Q_kP_{k+2i-1}$ for all $i \geq 1$ by Lemma 22. It follows that

$$\begin{aligned}
\mathscr{b}(k) &= \frac{1}{2}\gcd(B_k - k, \gcd(2Q_kP_{k+1}, 2Q_kP_{k+3}, 2Q_kP_{k+5}, \dots)) \\
&= \frac{1}{2}\gcd(B_k - k, 2Q_k \cdot \gcd(P_{k+1}, P_{k+3}, P_{k+5}, \dots)) \\
&= \frac{1}{2}\gcd(B_k - k, 4Q_k) \\
&= \gcd\left(\frac{1}{2}(B_k - k), 2Q_k\right),
\end{aligned}$$

where the third equality holds by Identity (13) of Theorem 11. □

5.2 Our main result for $\mathcal{C}\!\!\!\!b(k)$ involving $\gcd(P_k, k)$ and $\gcd(Q_k, k)$

To prove the results in this subsection, we use the p-adic valuation function and some of its well-known properties in Lemma 25 whose proofs we omit. In this subsection, we sometimes apply Lemma 25 without reference.

Definition 24. For each $n \geq 1$ and p a prime, the *p-adic valuation* of n, denoted $\nu_p(n)$, is the smallest nonnegative integer k such that p^k divides n.

Lemma 25. *For all $a, b \in \mathbb{Z}$ and p a prime, the following identities hold:*

$$\nu_p(\gcd(a,b)) = \min(\nu_p(a), \nu_p(b)), \tag{28}$$

$$\nu_p(a \cdot b) = \nu_p(a) + \nu_p(b). \tag{29}$$

Lemma 26. *For all $k \geq 1$, we have $\nu_2(P_k) = \nu_2(k)$.*

Proof. This follows from a more general result by Sanna (see [11, Theorem 1.5]). □

Lemma 27. *For all $k \geq 2$, we have $\nu_2(P_kQ_k - k) \geq 2$.*

Proof. We first claim that if $\ell \geq 4$ is even, then $\nu_2(P_\ell - \ell) \geq 3$ holds. Observe that

$$P_\ell = \sum_{i=1}^{\ell/2} \binom{\ell}{2i-1} 2^{i-1} \equiv \ell + \binom{\ell}{3} 2 + \binom{\ell}{5} 2^2 \pmod 8, \tag{30}$$

where the first equality holds by [6, Identity (9.10)]. For $\ell \geq 4$, we have $\binom{\ell}{3} = \frac{\ell\cdot(\ell-1)\cdot(\ell-2)}{2\cdot 3}$ and thus $\nu_2\left(\binom{\ell}{3}\right) \geq 2$, so $\nu_2\left(\binom{\ell}{3}2\right) = \nu_2\left(\binom{\ell}{3}\right) + \nu_2(2) \geq 2+1 = 3$ holds. Moreover for $\ell \geq 6$, we have $\binom{\ell}{5} = \frac{\ell\cdot(\ell-1)\cdot(\ell-2)\cdot(\ell-3)\cdot(\ell-4)}{2^3\cdot 15}$ and thus $\nu_2\left(\binom{\ell}{5}\right) \geq 1$, so $\nu_2\left(\binom{\ell}{5}2^2\right) = \nu_2\left(\binom{\ell}{5}\right) + \nu_2(2^2) \geq 1 + 2 = 3$ holds. Hence 8 divides both $\binom{\ell}{3}2$ and $\binom{\ell}{5}2^2$, and so Identity (30) implies $P_\ell - \ell$ is also divisible by 8, and thus $\nu_2(P_\ell - \ell) \geq 3$, as desired. Now let $k \geq 2$ be given. Observe that

$$\nu_2(P_kQ_k - k) = \nu_2(2P_kQ_k - 2k) - 1 = \nu_2(P_{2k} - 2k) - 1 \geq 3 - 1 = 2,$$

where the second equality holds since $P_{2k} = 2P_kQ_k$. □

Theorem 28. *For all $k \geq 1$, the GCD of all sums of k consecutive cobalancing numbers is*

$$\mathcal{C}\!\!\!\!b(k) = \begin{cases} \gcd(P_k, k), & \text{if } k \text{ is even;} \\ 2\gcd(Q_k, k), & \text{if } k \text{ is odd.} \end{cases}$$

Proof. By Theorem 23, it suffices to show that $\gcd\left(\frac{1}{2}(B_k - k), P_k\right) = \gcd(P_k, k)$ when k is even, and that $\gcd\left(\frac{1}{2}(B_k - k), 2Q_k\right) = 2\gcd(Q_k, k)$ when k is odd.

Case I. Suppose k is even. We claim that $\nu_p\left(\gcd(\frac{1}{2}(B_k-k),P_k)\right)=\nu_p(\gcd(P_k,k))$ for all primes p. Suppose $p\neq 2$ and $j\geq 1$. Then the following sequence of biconditionals holds:

$$\begin{aligned} p^j \text{ divides } \gcd\left(\frac{1}{2}(B_k-k),P_k\right) &\iff p^j \text{ divides } P_k \text{ and } B_k-k \\ &\iff p^j \text{ divides } P_k \text{ and } k \\ &\iff p^j \text{ divides } \gcd(P_k,k), \end{aligned}$$

where the first biconditional holds since p is an odd prime, and the second one holds since $B_k=P_kQ_k$. Thus $\nu_p\left(\gcd(\frac{1}{2}(B_k-k),P_k)\right)=\nu_p(\gcd(P_k,k))$.

On the other hand, suppose $p=2$. Since Q_k is odd, k is even, and $\nu_2(P_k)=\nu_2(k)$ by Lemma 26, we have $\nu_2(P_kQ_k-k)>\nu_2(P_k)$. It follows that $\nu_2(P_kQ_k-k)\geq\nu_2(P_k)+1$, and so $\nu_2(P_kQ_k-k)-1\geq\nu_2(P_k)$. Since $B_k=P_kQ_k$, then $\nu_2(B_k-k)=\nu_2(P_kQ_k-k)$, and thus $\nu_2\left(\frac{1}{2}(B_k-k)\right)=\nu_2(P_kQ_k-k)-1\geq\nu_2(P_k)$. By Identity (28) of Lemma 25, we have

$$\begin{aligned} \nu_2\left(\gcd\left(\frac{1}{2}(B_k-k),P_k\right)\right) &= \min\left(\nu_2\left(\frac{1}{2}(B_k-k)\right),\nu_2(P_k)\right) \\ &= \nu_2(P_k) \\ &= \min\left(\nu_2(P_k),\nu_2(k)\right) \\ &= \nu_2(\gcd(P_k,k)). \end{aligned}$$

Case II. Suppose k is odd. We claim that $\nu_p\left(\gcd(\frac{1}{2}(B_k-k),2Q_k)\right)=\nu_p(2\gcd(Q_k,k))$ for all primes p. Suppose $p\neq 2$ and $j\geq 1$. Then the following sequence of biconditionals holds:

$$\begin{aligned} p^j \text{ divides } \gcd\left(\frac{1}{2}(B_k-k),2Q_k\right) &\iff p^j \text{ divides } Q_k \text{ and } B_k-k \\ &\iff p^j \text{ divides } Q_k \text{ and } k \\ &\iff p^j \text{ divides } \gcd(Q_k,k) \\ &\iff p^j \text{ divides } 2\gcd(Q_k,k), \end{aligned}$$

where the first and fourth biconditionals hold since p is an odd prime, and the second one holds since $B_k=P_kQ_k$. Thus $\nu_p\left(\gcd(\frac{1}{2}(B_k-k),2Q_k)\right)=\nu_p(2\gcd(Q_k,k))$.

On the other hand, suppose $p=2$. Since Q_k and k are odd, we have $\gcd(Q_k,k)$ is odd and hence $\nu_2(2\gcd(Q_k,k))=1$. Since $B_k=P_kQ_k$, Lemma 27 implies that $\nu_2(B_k-k)\geq 2$, and so $\nu_2\left(\frac{1}{2}(B_k-k)\right)\geq 1$. By Identity (28) of Lemma 25, we have

$$\nu_2\left(\gcd\left(\frac{1}{2}(B_k-k),2Q_k\right)\right)=\min\left(\nu_2\left(\frac{1}{2}(B_k-k)\right),\nu_2(2Q_k)\right)=1,$$

where the last equality holds since $\nu_2\left(\frac{1}{2}(B_k-k)\right)\geq 1$, and Q_k being odd implies $\nu_2(2Q_k)=1$. It follows that $\nu_2\left(\gcd(\frac{1}{2}(B_k-k),2Q_k)\right)=1=\nu_2(2\gcd(Q_k,k))$. $\square$

6 Further directions and open questions

6.1 GCD of sums of k consecutive squares

Recall in Convention 1, we use the notation $\mathcal{S}^m(k)$ to denote the GCD of all sums of k consecutive m^{th} powers of sequence terms. Based on evidence collected from the software `Mathematica`, we have the following closed forms for the GCD of all sums of k consecutive squares of Pell and associated Pell numbers. We present this proof in a future paper.

Partial Result 29. *For all $k \geq 1$, the GCDs of all sums of k consecutive squares of Pell and associated Pell numbers, respectively, are*

$$\mathscr{P}^2(k) = \begin{cases} \frac{1}{2}P_k, & \text{if } k \text{ is even;} \\ 1, & \text{if } k \text{ is odd,} \end{cases} \quad \text{and} \quad \mathbb{Q}^2(k) = \begin{cases} P_k, & \text{if } k \text{ is even;} \\ 1, & \text{if } k \text{ is odd.} \end{cases}$$

Question 30. *Can we find nice closed forms for $\mathscr{B}^2(k)$, $\mathscr{C}^2(k)$, $\mathscr{b}^2(k)$, and $\mathscr{c}^2(k)$? Thus far from evidence collected from `Mathematica`, we do have the following observations:*

- *For all $k \geq 1$, we have* $\mathscr{B}^2(k) = \begin{cases} \frac{1}{2}\mathscr{C}^2(k), & \text{if } k \text{ is even;} \\ \mathscr{C}^2(k), & \text{if } k \text{ is odd.} \end{cases}$
- *For all $k \geq 1$, we have* $\mathscr{C}^2(k) = \begin{cases} \mathscr{c}^2(k), & \text{if } k \not\equiv 3 \pmod 6; \\ \frac{1}{3}\mathscr{c}^2(k), & \text{if } k \equiv 3 \pmod 6. \end{cases}$
- *For all $k \geq 1$, we have $\mathscr{b}^2(k) \equiv 0 \pmod 4$.*

6.2 The GCD of a sequence term and its index

Recall that by Theorem 28, we found that

$$\mathscr{b}(k) = \begin{cases} \gcd(P_k, k), & \text{if } k \text{ is even;} \\ 2\gcd(Q_k, k), & \text{if } k \text{ is odd.} \end{cases}$$

Unlike the other five GCD closed forms, $\mathscr{P}(k)$, $\mathbb{Q}(k)$, $\mathscr{B}(k)$, $\mathscr{C}(k)$, and $\mathscr{c}(k)$, the formula for $\mathscr{b}(k)$ is a not necessarily a closed form in the same sense, since we still need to compute both $\gcd(P_k, k)$ and $\gcd(Q_k, k)$. This leads to the following question.

Question 31. *When $(S_n)_{n\geq 0}$ is any of the six sequences $(P_n)_{n\geq 0}$, $(Q_n)_{n\geq 0}$, $(B_n)_{n\geq 0}$, $(C_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$, or $(c_n)_{n\geq 0}$, can we find a closed form for the values $\gcd(S_k, k)$ for all $k \geq 1$?*

When $(S_n)_{n\geq 0}$ is the Pell or associated Pell sequence, we have partial results towards closed forms for $\gcd(S_k, k)$ that involve the *entry point* (or *rank of apparition*), $e_S(p)$, which is the smallest index $r > 0$ such that p divides S_r where p is a prime.

Conjecture 32. We claim that $\gcd(Q_k, k) > 1$ if and only if there exists a prime p such that p divides k and the rank of apparition (or entry point), $e_Q(p)$ divides k. For example, $\gcd(Q_{21}, 21) = 7$ and for the prime $p = 7$, we have p divides 21 and $e_Q(p) = 3$ divides 21.

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