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The Sixth Moment of Random Determinants

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Abstract

We determine the sixth moment of the determinant of an asymmetric $n \times n$ random matrix where the entries are drawn independently from an arbitrary distribution Ω over $\mathbb{R}$ with mean 0. Furthermore, we derive the asymptotic behavior of the sixth moment of the determinant as the size of the matrix tends to infinity.

1 Introduction

The behavior of the determinant of a random matrix has been extensively studied. One line of work analyzed the kth moment of a random determinant, i.e., the expected value of the kth power of the determinant of a random matrix. Turán observed that the second moment of the determinant of an $n \times n$ matrix (where the entries have mean 0 and variance 1) is $n!$. The fourth moment of the determinant was determined by Nyquist, Rice, and Riordan [14]. For the special case when the entries of the random matrix are Gaussian, several works [7, 14] showed that the kth moment of the determinant is $\prod_{j=0}^{\frac{k}{2}-1} \frac{(n+2j)!}{(2j)!}$ for even k.

Several variations of this question have also been analyzed. Dembo [4] generalized these results for $p \times n$ matrices (where we consider $E[\det(MM^\top)^{\frac{k}{2}}]$ rather than $E[\det(M)^k]$). Very recently, Beck [5] determined the fourth moment of the determinant of an $n \times n$ random matrix with independent entries from an arbitrary distribution (which may not have mean 0). For symmetric random matrices, the second moment of the determinant was analyzed by Zhurbenko [20].

A second line of work analyzed the distribution of the determinant of a random matrix. Girko [9, 8] showed that under some assumptions on the entries of the random matrix, the logarithm of the determinant obeys a central limit theorem, but his proofs are quite difficult. Nguyen and Vu [13] gave a simpler proof of a stronger version of this central limit theorem.

A third line of work analyzed the probability that a random $n \times n$ matrix with ± 1 entries is singular. Komlós [11, 12] was the first to prove that this probability is $o(1)$. Kahn, Komlós, and Szemerédi [10] proved that this probability is at most $.999^n$, which was the first exponential upper bound. A series of works [16, 17, 2] improved this upper bound culminating in the work of Tikhomirov [18] who proved an upper bound of $(\frac{1}{2}+o(1))^n$, which is tight. For the symmetric case, Costello, Tao, and Vu [3] showed that the probability that a random symmetric matrix with ± 1 entries is singular is $o(1)$ and further work has improved this bound.

For a recent survey of results on random matrices, see [19]. In this paper, we extend the first line of work by analyzing the sixth moment of the determinant of a random matrix.

1.1 The fourth moment of random determinants

Before stating our main result, we describe the results of Nyquist, Rice, and Riordan [14] on the fourth moment of the determinant of a random matrix.

Definition 1. Given a distribution Ω over $\mathbb{R}$, we define $\mathcal{M}_{n\times n}(\Omega)$ to be the distribution of $n \times n$ matrices where each entry is drawn independently from Ω.

Definition 2. Given a distribution Ω over $\mathbb{R}$, we define m_k to be the kth moment of Ω, i.e.,

$$m_k = E_{x\sim\Omega}[x^k].$$

Definition 3. We define $f_k(n) = E_{M\sim\mathcal{M}_{n\times n}(\Omega)}[\det(M)^k]$ to be the expected value of the k-th power of the determinant of a random $n \times n$ matrix. Similarly, we define $p_k(n)$ to be the expected value of the k-th power of the permanent of a random $n \times n$ matrix.

Remark 4. Both $f_k(n)$ and $p_k(n)$ depend on the moments of Ω, but we write $f_k(n)$ and $p_k(n)$ rather than $f_{k,\Omega}(n)$ and $p_{k,\Omega}(n)$ for brevity.

Nyquist, Rice, and Riordan [14] showed that $f_4(n) = n!\, y_n$ where y_n obeys the recurrence relation

$$y_n = (n + m_4 - 1)y_{n-1} + (3 - m_4)(n - 1)y_{n-2}.$$

where $y_0 = 1$ and $y_1 = m_4$. They further observed that if we take the generating function $Y(t) = \sum_{t=0}^{\infty} \frac{y_n t^n}{n!}$ then $Y(t) = (1-t)^{-3} e^{(m_4-3)t}$. From this generating function, they found the equation

$$f_4(n) = n!\, y_n = \frac{(n!)^2}{2} \sum_{k=0}^{n} \frac{(n-k+1)(n-k+2)}{k!} (m_4 - 3)^k.$$

To prove their results, Nyquist, Rice, and Riordan [14] counted $4 \times n$ tables with certain properties. As we describe in Section 2, we use the same general approach though our analysis is considerably more intricate.

1.2 Our results

Our main results are as follows.

Theorem 5. *For any distribution Ω over $\mathbb{R}$ such that $m_1 = m_3 = 0$ and $m_2 = 1$, the formal generating function $F_6(t) = \sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} f_6(n)$ for $f_6(n)$ is*

$$F_6(t) = \frac{e^{t(m_6 - 15m_4 + 30)}}{48(1 + 3t - m_4 t)^{15}} \sum_{i=0}^{\infty} \frac{(1+i)(2+i)(4+i)!\, t^i}{(1 + 3t - m_4 t)^{3i}}.$$

Performing Taylor expansion on this generating function, we get a formula for computing the sixth moment of random determinants, namely:

Corollary 6. *For any distribution Ω over $\mathbb{R}$ such that $m_1 = m_3 = 0$ and $m_2 = 1$, we have that $f_6(n)$ equals*

$$n!^2 \sum_{j=0}^{n} \sum_{i=0}^{j} \frac{(1+i)(2+i)(4+i)!}{48(n-j)!} \binom{14 + j + 2i}{j - i} (m_6 - 15m_4 + 30)^{n-j} (m_4 - 3)^{j-i}.$$

Remark 7. If $m_2 \neq 1$ then we can scale the distribution Ω by $\frac{1}{\sqrt{m_2}}$ (which changes the determinant of matrices in $\mathcal{M}_{n \times n}(\Omega)$ by a factor of $(\frac{1}{\sqrt{m_2}})^n$) and then apply the result in Corollary 6.

Remark 8. If $\Omega = N(0,1)$ then $m_4 = 3$ and $m_6 = 15$ so $f_6(n) = P_n = \frac{n!(n+2)!(n+4)!}{48}$, which is a special case of the result that $f_k(n) = \prod_{j=0}^{\frac{k}{2}-1} \frac{(n+2j)!}{(2j)!}$ when $\Omega = N(0,1)$ and k is even.

Another generalization is when $m_3 \neq 0$.

Theorem 9. *For any distribution Ω over $\mathbb{R}$ such that $m_1 = 0$ and $m_2 = 1$, the formal generating function $F_6(t) = \sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} f_6(n)$ for $f_6(n)$ is*

$$F_6(t) = (1 + m_3^2 t)^{10} \frac{e^{t(m_6 - 10m_3^2 - 15m_4 + 30)}}{48(1 + 3t - m_4 t)^{15}} \sum_{i=0}^{\infty} \frac{(1+i)(2+i)(4+i)!\, t^i}{(1 + 3t - m_4 t)^{3i}}.$$

Corollary 10. *For any distribution Ω over $\mathbb{R}$ such that $m_1 = 0$ and $m_2 = 1$,*

$$f_6(n) = n!^2 \sum_{j=0}^{n} \sum_{i=0}^{j} \sum_{k=0}^{n-j} \frac{(1+i)(2+i)(4+i)!}{48(n-j-k)!} \binom{10}{k} \binom{14+j+2i}{j-i} q_6^{n-j-k} q_4^{j-i} q_3^k,$$

where $q_6 = m_6 - 10m_3^2 - 15m_4 + 30$, $q_4 = m_4 - 3$, and $q_3 = m_3^2$.

Below, we show the values of $f_k(n)$ and $p_k(n)$ when $\Omega = \{-1, 1\}$ for small values of k and n. When $\Omega = \{-1, 1\}$, we note that $f_4(n)$ is the integer sequence A052127 in the On-Line Encyclopedia of Integer Sequences [15, A052127]. In the entry for this integer sequence, it is noted that $f_4(n) \sim (n!)^2 \frac{(n^2+7n+10)}{(2e^2)}$ as $n \to \infty$.

n	$f_2(n)$, A000142	$f_4(n)$, A052127	$f_6(n)$, A357571	$p_2(n)$	$p_4(n)$	$p_6(n)$
1	1	1	1	1	1	1
2	2	8	32	2	8	32
3	6	96	1536	6	96	2976
4	24	2112	282624	24	2112	513024
5	120	68160	66846720	120	68160	157854720

Table 1: Values of $f_k(n)$ and $p_k(n)$ for $\Omega = \{-1, 1\}$, $k \le 6$, and $n \le 5$.

Remark 11. Note that for all $n \in \mathbb{N}$, $p_2(n) = f_2(n)$ and $p_4(n) = f_4(n)$. However, for $n \ge 3$, $p_6(n) > f_6(n)$.

We can describe the asymptotic behavior of f_6 using the following asymptotic expansion.

Theorem 12. *For all $R \in \mathbb{N} \cup \{0\}$,*

$$f_6(n) = \frac{e^{3q_4}(n!)^2}{48} \left(\sum_{k=0}^{R} c_k (n+6-k)! \right) \pm O((n!)^2 (n+6-R-1)!),$$

where the coefficients c_k are the Taylor expansion coefficients of the function $C(t) = \sum_{k \ge 0} c_k t^k$,

$$C(t) = e^{(q_6 - 3q_4^2)t + q_4^3 t^2} (1 + q_3 t)^{10} \Big(1 - 2(3q_4 + 4)t + 3(5q_4^2 + 8q_4 + 4)t^2$$
$$- 4(q_4^2(5q_4 + 6))t^3 + q_4^3(15q_4 + 8)t^4 - 6q_4^5 t^5 + q_4^6 t^6 \Big).$$

Remark 13. For the first terms in the expansion, we have

$$f_6(n) \sim \frac{e^{3m_4 - 9}}{48} n!^3 \Big(n^6 + (m_6 - 3m_4^2 - 3m_4 + 34)n^5 + \frac{1}{2} \left(m_6^2 - 10m_3^4 + 9m_4^4\right.$$
$$\left. + 20m_4^3 - 183m_4^2 - 126m_4 - 6m_4^2 m_6 - 6m_4 m_6 + 56m_6 + 905\right) n^4 + \cdots \Big).$$

Remark 14. Note that when $\Omega = \{-1, 1\}$, as $n \to \infty$,

$$f_6(n) \sim \frac{(n!)^3}{48e^6}\left(n^6 + 29n^5 + 335n^4 + \frac{5861n^3}{3} + \frac{17944n^2}{3} + \frac{44036n}{5} + \frac{167536}{45} - \frac{210176}{63n}\right).$$

2 Preliminaries

To prove Theorem 5, we need a few definitions and a key lemma. For this section and the next section, we assume that $m_1 = m_3 = 0$.

Definition 15. Given natural numbers k and n where k is even, we define an even $k \times n$ table to be a $k \times n$ table where each row is a permutation of $[n]$ and each column contains each number that appears in the column an even number of times. We define $T_{k,n}$ to be the set of all even $k \times n$ tables.

Definition 16. Given an even table t of size $k \times n$, we define its sign $\operatorname{sgn}(t)$ to be the product of the signs of its rows, which are permutations of $[n]$.

Definition 17. Given a column c where each element is in $[n]$, we define its weight $w(c)$ to be

$$w(c) = \prod_{j=1}^{n} m_{\#\text{ of times } j \text{ appears in column c}}.$$

For even $6 \times n$ tables, we say that a column is a 6-column if it contains some number 6 times, a 4-column if it contains one number four times and another number two times, and a 2-column if it contains three different numbers two times. Observe that the weight of a 6-column is m_6, the weight of a 4-column is m_4m_2, and the weight of a 2-column is m_2^3.

Definition 18. Given an even $k \times n$ table t, we define its weight $w(t)$ to be the product of the weights of its columns.

We can use the following proposition to reduce the problem of finding the sixth moment of a random determinant to a combinatorial problem.

Proposition 19. *For all even $k \in \mathbb{N}$, $f_k(n) = \sum_{t \in T_{k,n}} \operatorname{sgn}(t)w(t)$ and $p_k(n) = \sum_{t \in T_{k,n}} w(t)$.*

Proof. We observe that

$$f_k(n) = E_{A \sim \mathcal{M}_{n\times n}(\Omega)}\left[\sum_{\pi_1, \pi_2, \ldots, \pi_k \in S_n}\left(\prod_{i=1}^{k} \operatorname{sgn}(\pi_i)\right)\prod_{p=1}^{n}\left(\prod_{q=1}^{k} A_{p,\pi_q(p)}\right)\right]$$

and

$$p_k(n) = E_{A \sim \mathcal{M}_{n\times n}(\Omega)}\left[\sum_{\pi_1, \pi_2, \ldots, \pi_k \in S_n}\prod_{p=1}^{n}\left(\prod_{q=1}^{k} A_{p,\pi_q(p)}\right)\right].$$

For each $p \in [n]$, we have that $E_{A\sim\mathcal{M}_{n\times n}(\Omega)}[\prod_{q=1}^{k} A_{p,\pi_q(p)}] = w(p)$ (i.e., the weight of column p), so $f_k(n) = \sum_{t\in T_{k,n}} \mathrm{sgn}(t)w(t)$ and $p_k(n) = \sum_{t\in T_{k,n}} w(t)$, as needed.

□

Thus, computing the kth moment of a random determinant is equivalent to summing the signed weights of all even tables of size $k \times n$.

Corollary 20. *If Ω is the uniform Bernoulli distribution (i.e., the uniform distribution on $\{-1, 1\}$) then $f_k(n) = \sum_{t\in T_{k,n}} \mathrm{sgn}(t)$ and $p_k(n) = |T_{k,n}|$.*

Corollary 21. *If $k = 2$, $k = 4$, or $n \leq 2$ then $p_k(n) = f_k(n)$. If $n \geq 3$, $k \geq 6$, and k is even then $p_k(n) > f_k(n)$.*

To analyze $f_6(n)$, it is useful to consider tables together with pairings of identical elements in each column.

Definition 22. Given an even $k \times n$ table t, we define a *pairing* P on t to be a set of matchings $\{M_i : i \in [n]\}$, one for each column, where each matching M_i pairs up identical elements of column i. We define $\mathcal{P}(t)$ to be the set of all pairings on t.

Example 23. The table on the left below is an even 6×4 table with 27 possible *pairings*. The table on the right shows one of the 27 possible pairings.

1	2	4	3
1	2	4	3
1	3	4	2
1	3	4	2
2	4	1	3
2	4	1	3

1	2	4	3
1	2	4	3
1	3	4	2
1	3	4	2
2	4	1	3
2	4	1	3

Proposition 24. *For each even $6 \times n$ table t,*

$$|\mathcal{P}(t)| = 15^{\#\text{ of }6\text{-columns in t}} 3^{\#\text{ of }4\text{-columns in t}}.$$

Definition 25. We define $P_n = \sum_{t\in T_{k,n}} \mathrm{sgn}(t)|\mathcal{P}(t)|$.

It turns out that P_n can be easily computed and this is crucial for our results.

Lemma 26. *For all $n \in \mathbb{N}$, $P_n = n(n+2)(n+4)P_{n-1}$ where $P_0 = 1$.*

This lemma follows from the fact that when $\Omega = N(0, 1)$ and k is even, the kth moment of the determinant is $\prod_{j=0}^{\frac{k}{2}-1} \frac{(n+2j)!}{(2j)!}$. We give a direct proof of this lemma in Appendix A.

Note that $P_n = \sum_{t\in T_{k,n}} \mathrm{sgn}(t)15^{\#\text{ of 6-columns in t}}3^{\#\text{ of 4-columns in t}}$ while $f_6(n) = \sum_{t\in T_{k,n}} \mathrm{sgn}(t){m_6}^{\#\text{ of 6-columns in t}}{m_4}^{\#\text{ of 4-columns in t}}$. If $\Omega = N(0, 1)$ (or we at least have that $m_1 = m_3 = 0$, $m_2 = 1$, $m_4 = 3$, and $m_6 = 15$) then $f_6(n) = P_n$.

In the next section, we show how to handle other distributions Ω using inclusion/exclusion.

We finish our preliminaries section with the following generation function result on derangements:

Lemma 27. *Let S_n be the set of all permutations of order n on $[n] = \{1, 2, 3, \ldots, n\}$ and let D_n be the set of all* derangements *of the same order on $[n]$, i.e., the permutations in S_n which have no fixed points. If we let $C(\pi)$ denote the number of cycles in a permutation π and take $C_n(u) = \sum_{\pi \in D_n} u^{C(\pi)}$, then*

$$C_n(u) = (n-1)(C_{n-1}(u) + u\, C_{n-2}(u))$$

and

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} C_n(u) = \frac{e^{-ux}}{(1-x)^u}.$$

Proof. See the chapter on Bivariate generating functions in the textbook "Analytic Combinatorics" by Flajolet and Sedgewick [6]. For completeness, we present our own derivation. We proceed recursively based on the position of the node n. We can create a derangement $\pi \in D_n$ by either:

1. Adding the node n to one of the cycles of a derangement $\pi' \in D_{n-1}$. That is, if $i \to \pi(i)$, then we insert n as $i \to n \to \pi(i)$. Since there are $n-1$ nodes in π', there are $n-1$ different $\pi \in D_n$ we can create. In this case, the number of cycles is unchanged, i.e., $C(\pi) = C(\pi')$.

2. Adding a cycle $(n, n-1)$ of length two to $\pi'' \in D_{n-2}$. We can then replace $n-1$ by any $i \in \pi''$. This gives $n-1$ new derangements $\pi \in D_n$ created from π'', all of them having $C(\pi) = C(\pi'') + 1$.

We can obtain all derangements D_n in this way. These two possibilities are shown in the figures below.

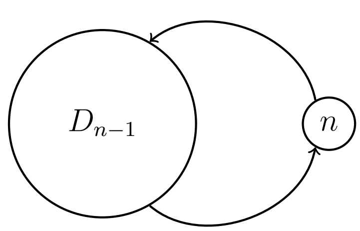

Figure 1: $D_{n-1} \to D_n$.

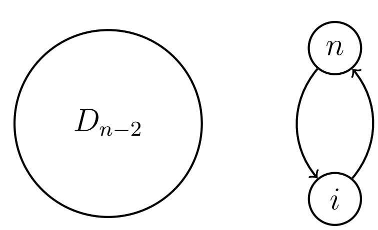

Figure 2: $D_{n-2} \to D_n$.

In terms of $C_n(u)$, we get the desired recurrence relation

$$\begin{aligned} C_n(u) = \sum_{\pi \in D_n} u^{C(\pi)} &= (n-1) \sum_{\pi' \in D_n} u^{C(\pi')} + (n-1) \sum_{\pi'' \in D_{n-2}} u^{C(\pi'')+1} \\ &= (n-1)(C_{n-1}(u) + u\, C_{n-2}(u)), \end{aligned}$$

from which one can deduce its generating function easily. □

3 Proof of Theorem 5

Before preceding to the proof of Theorem 5, we first prove the following result on the sixth moment of random determinants.

Lemma 28. *For any distribution Ω over $\mathbb{R}$ such that $m_1 = m_3 = 0$ and $m_2 = 1$,*

$$f_6(n) = \sum_{j=0}^{n} \sum_{a=0}^{j} \binom{n}{j} \binom{j}{a} (m_6 - 15)^a (m_4 - 3)^{(j-a)} D_{n,a,j-a},$$

where

1. *We have that $P_n = n(n+2)(n+4)P_{n-1}$ where $P_0 = 1$. Equivalently, $P_n = \frac{n!(n+2)!(n+4)!}{2!4!}$.*
2. *We take $C_n = (n-1)(C_{n-1} + 15C_{n-2})$ where $C_0 = 1$ and $C_1 = 0$.*
3. *We take $H_{n,j,a,b} = \sum_{x=1}^{j} \frac{j!}{x!} \binom{j-1}{x-1} \prod_{y=0}^{x-1} (3(n-a-b) - y)$.*
4. *We take $D_{n,a,b} = \left(\prod_{j=0}^{a+b-1} (n-j) \right) \left(\sum_{i=0}^{b} \binom{b}{i} C_i H_{n,b-i,a,b} \right) P_{n-a-b}$.*

Proof. The idea behind the proof is as follows. We consider the tables where we know that some set $A \subseteq [n]$ of elements appear six times in a 6-column and another set $B \subseteq [n] \setminus A$ of elements appear four times in a 4-column and two times in a different column. We do not know whether the elements in $[n] \setminus (A \cup B)$ appear six times in a 6-column, appear four times in a 4-column and two times in a different column, or appear two times in three different columns. We consider these tables together with pairings for the columns which are unaccounted for by A and B (i.e., the columns which don't contain six of the same element in A or four of the same element in B), which we call the center columns.

To obtain $f_6(n)$, we compute the contribution from each A and B and then take an appropriate linear combination of these contributions so that the contribution from each individual table t is $\operatorname{sgn}(t)w(t)$.

Definition 29. Given $A \subseteq [n]$ and $B \subseteq [n] \setminus A$, we define $D_{n,A,B}$ to be the set of tables in $T_{6,n}$ such that the elements in A appear six times in a 6-column and the elements in B appear four times in a 4-column and two times in a different column.

Definition 30. Given $t \in T_{6,n}$, $A \subseteq [n]$, and $B \subseteq [n] \setminus A$ such that $t \in D_{n,A,B}$, we define the *center columns* of t to be the columns which do not contain six of the same element of A and which do not contain four of the same element of B. We define $\mathcal{P}_{\text{center}}(t)$ to be the set of pairings on the center columns of t.

Example 31. The following table is a 6×9 table $t \in D_{9,\{7\},\{1,5,6,8,9\}}$ together with a pairing $P \in \mathcal{P}_{\text{center}}(t)$ on the center columns.

7	8	5	2	4	3	6	1	9
7	9	5	2	4	3	6	1	8
7	8	3	5	4	2	6	1	9
7	8	3	5	4	2	6	1	9
7	9	5	2	6	3	1	4	8
7	8	5	2	6	3	1	4	9

By symmetry, the contribution from each $D_{n,A,B}$ only depends on $|A|$ and $|B|$.

Definition 32. Given $n, a, b \in \mathbb{N} \cup \{0\}$ such that $a + b \leq n$, we define $D_{n,a,b}$ to be

$$D_{n,a,b} = \sum_{t \in D_{n,A,B}} \text{sgn}(t) |\mathcal{P}_{\text{center}}(t)|$$

where $A \subseteq [n]$, $B \subseteq [n] \setminus A$, $|A| = a$, and $|B| = b$.

Lemma 33. *For all $n \in \mathbb{N} \cup \{0\}$,*

$$f_6(n) = \sum_{j=0}^{n} \sum_{a=0}^{j} \binom{n}{j} \binom{j}{a} (m_6 - 15)^a (m_4 - 3)^{(j-a)} D_{n,a,j-a}.$$

Proof. Observe that

$$\begin{aligned}
&\sum_{j=0}^{n} \sum_{a=0}^{j} \binom{n}{j} \binom{j}{a} (m_6 - 15)^a (m_4 - 3)^{(j-a)} D_{n,a,j-a} \\
&= \sum_{A \subseteq [n]} \sum_{B \subseteq [n] \setminus A} \sum_{t \in D_{n,A,B}} (m_6 - 15)^{|A|} (m_4 - 3)^{|B|} \text{sgn}(t) |\mathcal{P}_{\text{center}}(t)|.
\end{aligned}$$

Given a table $t \in T_{6,n}$, let A' be the set of elements in $[n]$ which appear six times in a 6-column of t and let B' be the set of elements which appear four times in a 4-column of t. Now consider the contribution from t in

$$\sum_{A \subseteq [n]} \sum_{B \subseteq [n] \setminus A} \sum_{t \in D_{n,A,B}} (m_6 - 15)^{|A|} (m_4 - 3)^{|B|} \text{sgn}(t) |\mathcal{P}_{\text{center}}(t)|.$$

We have that whenever $A \subseteq A'$ and $B \subseteq B'$, $t \in D_{n,A,B}$ and $|\mathcal{P}_{\text{center}}(t)|= 15^{|A'\setminus A|}3^{|B'\setminus B|}$. Thus, the contribution from t is

$$\begin{aligned}&\sum_{A\subseteq A'}\sum_{B\subseteq B'}(m_6-15)^{|A|}(m_4-3)^{|B|}15^{|A'\setminus A|}3^{|B'\setminus B|}\operatorname{sgn}(t)\\&= {m_6}^{|A'|}{m_4}^{|B'|}\operatorname{sgn}(t) = \operatorname{sgn}(t)w(t).\end{aligned}$$

This implies that

$$\sum_{j=0}^{n}\sum_{a=0}^{j}\binom{n}{j}\binom{j}{a}(m_6-15)^a(m_4-3)^{(j-a)}D_{n,a,j-a} = \sum_{t\in T_{6,n}}\operatorname{sgn}(t)w(t) = f_6(n),$$

as needed. □

We now compute $D_{n,a,b}$.

Lemma 34. *For all $n, a, b \in \mathbb{N}\cup\{0\}$ such that $a+b\le n$,*

$$D_{n,a,b} = \left(\prod_{j=0}^{a+b-1}(n-j)\right)\left(\sum_{i=0}^{b}\binom{b}{i}C_iH_{n,b-i,a,b}\right)P_{n-a-b}$$

where C_n is given by the recurrence relation $C_n = (n-1)(C_{n-1}+15C_{n-2})$, $C_0 = 1$, $C_1 = 0$ and

$$H_{n,j,a,b} = \sum_{x=1}^{j}\frac{j!}{x!}\binom{j-1}{x-1}\prod_{y=0}^{x-1}(3(n-a-b)-y).$$

Proof. To prove this lemma, we group the tables in $D_{n,A,B}$ based on the structure of the 4-columns containing four of the same element of B.

Definition 35. Given a table $t \in D_{n,A,B}$, we define the directed graph $G(t)$ to be the graph with vertices $V(G(t)) = B\cup\{v_{\text{center}}\}$ and the following edges. For each $j \in B$,

1. If there is a $j' \in B\setminus\{b\}$ such that there are two j in the 4-column containing four j' then we add an edge from j to j'.

2. If there is no such j' then we add an edge from j to v_{center}.

For each vertex $v \in V(G(t))$, we define the outdegree $\deg^+(v)$ of v to be the number of edges going from v to another vertex and we define the indegree $\deg^-(v)$ of v to be the number of edges going from another vertex to v.

Proposition 36. *For all $t \in D_{n,A,B}$, $G(t)$ has the following properties.*

1. *For all $j \in B$, $\deg^+(j) = 1$ and $\deg^-(j) \le 1$.*

2. $\deg^+(v_{\text{center}}) = 0$.

Corollary 37. *For all $t \in D_{n,A,B}$, $G(t)$ consists of directed cycles and paths which end at v_{center}, all of which are disjoint except for their common endpoint.*

Example 38. The figure on the left below shows a table t together with a pairing $P \in \mathcal{P}_{\text{center}}(t)$ on the center columns of t. The figure on the right shows the resulting graph $G(t)$.

7	8	5	2	4	3	6	1	9
7	9	5	2	4	3	6	1	8
7	8	3	5	4	2	6	1	9
7	8	3	5	4	2	6	1	9
7	9	5	2	6	3	1	4	8
7	8	5	2	6	3	1	4	9

Figure 3: A 6×9 table $t \in D_{9,\{7\},\{1,5,6,8,9\}}$.

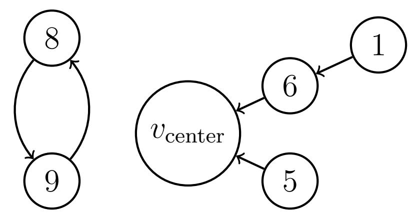

Figure 4: The associated $G(t)$.

Note that as the cycles and paths are disjoint, we can split any table $t \in D_{n,A,B}$ into 6-columns corresponding to A, 4-columns corresponding to B_{cycles}, 4-columns corresponding to B_{paths}, and center columns, respectively.

Example 39. The figures below show the split of table $t \in D_{9,\{7\},\{1,5,6,8,9\}}$ from Example 31.

7
7
7
7
7
7

Figure 5: A.

8	9
9	8
8	9
8	9
9	8
8	9

Figure 6: B_{cycles}.

5	6	1
5	6	1
3	6	1
3	6	1
5	1	4
5	1	4

Figure 7: B_{paths}.

2	4	3
2	4	3
5	4	2
5	4	2
2	6	3
2	6	3

Figure 8: center.

We now consider how many ways there are to start with a table $t_{\text{center}} \in T_{6,n-a-b}$ together with a pairing $P \in \mathcal{P}(t_{\text{center}})$ and construct a table $t \in D_{n,A,B}$ (we will automatically have that $\text{sgn}(t) = \text{sgn}(t_{\text{center}})$ and $\mathcal{P}_{\text{center}}(t) = P$). Before giving the entire analysis, we describe the parts of the analysis corresponding to the cycles and paths of $G(t)$ as these are the trickiest parts of the analysis.

To handle the cycles of $G(t)$, we observe that the columns of t corresponding to these cycles are independent from the rest of t. This means that once the locations and elements of these columns are chosen, it is sufficient to count the number of possible tables for these columns. This can be done as follows.

Definition 40. Define C_n to be the number of tables $t_{\text{cycles}} \in D_{n,\emptyset,[n]}$ such that $G(t_{\text{cycles}})$ consists of directed cycles and for each $i \in [n]$, column i contains four i.

Lemma 41. *For all $n \geq 2$, $C_n = (n-1)(C_{n-1} + 15C_{n-2})$ where $C_0 = 1$ and $C_1 = 0$ and as a consequence,*

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} C_n = \frac{e^{-15x}}{(1-x)^{15}}.$$

Proof. Notice that $G(t_{\text{cycles}})$ is a derangement of order n. On the other hand, given a derangement $\pi \in D_n$, we determine the number of tables $t_{\text{cycles}} \in D_{n,\emptyset,[n]}$ with the property that $G(t_{\text{cycles}}) = \pi$ and for each $i \in [n]$, column i contains four i. Note that for each i and j in a given cycle of π, the four rows of column i which contain i are the same as the four rows of column j which contain j. Thus, for each cycle C, to determine the entries of the part of t_{cycles} corresponding to C, it is sufficient to take an arbitrary i in C and choose which four rows of column i contain i. There are $\binom{6}{4} = 15$ choices for this, so there are $15^{C(\pi)}$ distinct tables t_{cycles} which we can construct from π. Thus,

$$C_n = \sum_{\pi \in D_n} 15^{C(\pi)} = C_n(15).$$

By Lemma 27, we get the recurrence relation and the generating function. □

Example 42. The figures below show a table t_{cycles} and the corresponding derangement $G(t_{\text{cycles}})$.

1	2	3	4	5
1	2	3	4	5
1	2	5	3	4
1	2	5	3	4
2	1	3	4	5
2	1	3	4	5

Figure 9: A 6×5 table $t_{cylces} \in D_{5,\emptyset,[5]}$.

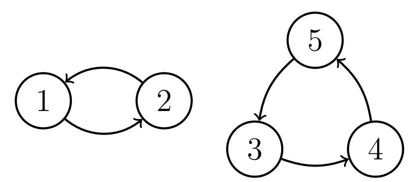

Figure 10: The associated $G(t_{\text{cycles}})$.

To handle the paths of $G(t)$, we observe that if we want to construct t so that $G(t)$ has a path $b_1 \to b_2 \to \cdots \to b_l \to v_{\text{center}}$, we can do this as follows:

1. We start with a table t_{center} for the center columns and a pairing $P \in \mathcal{P}(t_{\text{center}})$.

2. For each $j \in [l]$, we add a column containing six b_j.

3. We choose a pair of entries of t_{center} which are paired together by P. Let $z \in [n]$ be the element in these entries and let r_1 and r_2 be the two rows of these entries.

4. For each $j \in \{0, 1, \ldots, l-1\}$, we swap this pair of z with the two b_{l-j} in rows r_1 and r_2 and then move on to the next j. When we are done,

    1. The column which started with six b_1 now has two z in rows r_1 and r_2.

2. For each $j \in [l-1]$, the column which started with six b_{j+1} now has two b_j in rows r_1 and r_2.
3. The original pair of z are replaced by a pair of b_l.

Example 43. The figures below show how to extend a table t_{center} and a pairing $P \in \mathcal{P}(t_{\text{center}})$ based on a path to v_{center} and an endpoint for this path in t_{center} (i.e., a pair of entries in t_{center} which are paired up by P).

3	2	6	4	1	5
3	2	5	6	1	4
3	4	2	6	1	**5**
3	4	6	2	1	5
3	1	5	2	6	4
3	1	2	4	6	**5**

Figure 11: A table t_{center} with a pairing P.

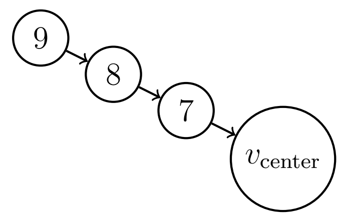

Figure 12: A path to v_{center}.

9	8	7	3	2	6	4	1	5
9	8	7	3	2	5	6	1	4
5	9	8	3	4	2	6	1	7
9	8	7	3	4	6	2	1	5
9	8	7	3	1	5	2	6	4
5	9	8	3	1	2	4	6	7

Figure 13: A 6×9 table t where $G(t)$ has the given path to v_{center}.

Following this logic, we can handle the paths in $G(t)$ as follows:

1. Choose the paths in $G(t)$.
2. For each path in $G(t)$, choose a pair of entries in t_{center} which are paired up by P to act as an endpoint for the path. Note that no two paths can have the same pair of entries as an endpoint.

Remark 44. This argument for analyzing the paths of $G(t)$ is the reason why it is important to consider the table t_{center} for the center columns together with a pairing $P \in \mathcal{P}(t_{\text{center}})$.

To count the number of ways to choose the paths in $G(t')$ and their endpoints, we use the following lemma which counts the number of possible graphs G_{paths} with a given number of paths to v_{center} and no cycles.

Lemma 45. *Let $B' \subseteq [n]$ and take $j = |B'|$. For all $x \in [j]$, there are $\frac{j!}{x!}\binom{j-1}{x-1}$ possible graphs G_{paths} on the vertices $B' \cup \{v_{\text{center}}\}$ which consist of x disjoint paths to v_{center} and no cycles.*

Proof. We can specify each such graph as follows:

1. Choose an ordering for the elements of B'. There are $j!$ possibilities for this ordering.
2. Choose the x paths by putting $x-1$ dividing lines among the elements of B'. Between each neighbouring elements of B', there must be at most one dividing line, since each path must have at least one vertex in B'. Hence, there are $\binom{j-1}{x-1}$ possibilities for this.

However, if we do this, each graph G_{paths} is counted $x!$ times, one for each possible ordering of the x paths. Thus, the number of possible graphs G_{paths} on the vertices $B' \cup \{v_{\text{center}}\}$ which consist of x disjoint paths to v_{center} and no cycles is $\frac{j!}{x!}\binom{j-1}{x-1}$, as needed. $\square$

Proposition 46. *Letting x be the number of paths in G_{paths}, there are $\prod_{y=0}^{x-1}(3(n-a-b)-y)$ ways to choose the endpoints for these paths.*

We now give the entire analysis for $D_{n,a,b}$. Given $A \subseteq [n]$ and $B \subseteq [n] \setminus A$ such that $|A|= a$ and $|B|= b$, we can compute our $D_{n,a,b} = \sum_{t\in D_{n,A,B}} \operatorname{sgn}(t)|\mathcal{P}_{\text{center}}(t)|$ as follows.

1. For each $i \in A$, we choose which column contains six copies of i. Similarly, for each $j \in B$, we choose which column contains four j. The number of choices for this is $\prod_{j=0}^{a+b-1}(n-j)$.

2. After choosing these columns, we choose a table $t_{\text{center}} \in T_{6,n-a-b}$ and a pairing $P \in \mathcal{P}(t_{\text{center}})$ to fill in the remaining columns. The number of choices for this is P_{n-a-b}.

3. We split into cases based on the number of vertices i in $G(t)$ which are contained in cycles. For each i, we choose which $\binom{b}{i}$ of the elements in B are contained in cycles. By Lemma 41, once these elements are chosen there are C_i possibilities for the columns containing these elements.

4. There are now $j = b - i$ elements of B which are contained in paths. We further split into cases based on the number x of paths in $G(t)$. By Lemma 45, there are $\frac{j!}{x!}\binom{j-1}{x-1}$ possibilities for what these paths are in $G(t)$.

   As discussed above, for each of the x paths, we need to choose a different pair of entries which are paired up by P to be an endpoint for the path and the number of choices for these pairs is $\prod_{y=0}^{x-1}(3(n-a-b)-y)$. Summing all of these possibilities up gives a factor of
$$H_{n,j,a,b} = \sum_{x=1}^{j} \frac{j!}{x!}\binom{j-1}{x-1}\prod_{y=0}^{x-1}(3(n-a-b)-y).$$

Putting everything together, we have that
$$D_{n,a,b} = \left(\prod_{j=0}^{a+b-1}(n-j)\right)\left(\sum_{i=0}^{b}\binom{b}{i}C_i H_{n,b-i,a,b}\right)P_{n-a-b},$$
as needed.

Since this argument is intricate and is a central argument for our results, we make this argument formal with the following bijection.

Lemma 47. *Given $A = \{i_1, \ldots, i_a\} \subseteq [n]$ and $B = \{j_1, \ldots, j_b\} \subseteq [n] \setminus A$, there is a bijection between the following sets of data:*

1. *A table $t \in D_{n,A,B}$ together with a pairing $P \in \mathcal{P}_{\text{center}}(t)$ on the center columns.*

2. *A tuple $(C_A, C_B, t_{\text{center}}, P, i, B_{\text{cycles}}, t_{\text{cycles}}, x, G_{\text{paths}}, E)$ where*

(a) *The indices C_A specify the locations of the* 6*-columns corresponding to the elements in A. More precisely, $C_A = (c_{i_1}, c_{i_2}, \dots, c_{i_a})$ where for each $k \in [a]$, column c_{i_k} contains six i_k.*

(b) *The indices C_B specify the locations of the* 4*-columns corresponding to the elements in B. More precisely, $C_B = (c_{j_1}, c_{j_2}, \dots, c_{j_b})$ where for each $k \in [b]$, column c_{j_k} contains four j_k.*

(c) *We let $t_{\text{center}} \in T_{6,n-|A|-|B|}$ and $P \in \mathcal{P}(t_{\text{center}})$ denote a table for the center columns and a pairing on this table.*

(d) *We let $i \in [0, |B|]$ denote the number of elements of B which are contained in cycles in $G(t)$.*

(e) *We let $B_{\text{cycles}} \subseteq B$ denote the i elements of B which are contained in cycles in $G(t)$.*

(f) *We let $t_{\text{cycles}} \in D_{i,\emptyset,[i]}$ denote a table such that $G(t_{\text{cycles}})$ contains only cycles.*

(g) *We let $x \in [0, |B|-i]$ denote the number of paths in $G(t)$.*

(h) *Letting $B_{\text{paths}} = B \setminus B_{\text{cycles}}$, we let G_{paths} denote a graph on the vertices $B_{\text{paths}} \cup v_{\text{center}}$ which consists of x disjoint paths to v_{center}.*

(i) *We let $E = (p_1, \dots, p_x)$ denote the pairs in P which are the endpoints for the x paths in G_{paths}.*

Proof. Given a table $t \in D_{n,A,B}$ together with a pairing $P \in \mathcal{P}_{\text{center}}(t)$ on the center columns, it is easy to find C_A, C_B, i, B_{cycles}, x, and G_{paths}. We obtain t_{center}, t_{cycles}, and E as follows.

1. We obtain t_{cycles} from the entries of the columns containing the elements in B_{cycles}. More precisely, let $c'_1, \dots, c'_i$ be the columns containing the elements of B_{cycles} and let j'_k be the element which appears four times in column c'_k (note that $j'_1, \dots, j'_i$ are not necessarily in increasing order). If the entry in row r and column c'_k is j'_y then we take the entry in row r and column k of t_{cycles} to be y.

2. In order to obtain t_{center} and E, we first modify t as follows. We order the paths in G_{paths} based on the order of their starting vertices. Let $j_{k1} \to j_{k2} \to \cdots \to j_{kl_k} \to v_{\text{center}}$ be the kth path in G_{paths}. Let r_{k1} and r_{k2} be the rows of column $c_{j_{k1}}$ which contain an element $z_k \notin A \cup B$ rather than j_{k1}. Finally, let c^*_k be the center column containing two j_{kl_k}. We now make the following modifications to t.

    1. We replace the two z_k in rows r_{k1} and r_{k2} of column $c_{j_{k1}}$ with j_{k1}.
    2. For all $i \in [l_k - 1]$, the two j_{ki} in column $c_{j_{k(i+1)}}$ must be in rows r_{k1} and r_{k2}. We replace these two j_{ki} with $j_{k(i+1)}$.
    3. The two j_{kl_k} in column c^*_k must be in rows r_{k1} and r_{k2}. We replace these j_{kl_k} with z_k. We take p_k to be this pair of elements.

Let t' be the table we obtain after making these modifications for each of the k paths. The center columns of t' only contain elements in $[n] \setminus (A \cup B)$, so we can obtain t_{center} from t'. More precisely, letting $c''_1, \ldots, c''_{n-a-b}$ be the center columns of t' and letting $i'_1, \ldots, i'_{n-a-b}$ be the elements of $[n] \setminus (A \cup B)$, if t' has an element i'_y in row r and column c'_j for some $r \in [6]$ and $j, y \in [n-a-b]$ then we put the element y in row r and column j of t_{center}. We translate $P \in \mathcal{P}_{\text{center}}(t')$ and $E = (p_1, \ldots, p_k)$ from the center columns of t' to t_{center} accordingly.

Conversely, given a tuple $(C_A, C_B, t_{\text{center}}, P, i, B_{\text{cycles}}, t_{\text{cycles}}, x, G_{\text{paths}}, E)$, we construct $t \in D_{n,A,B}$ as follows.

1. For each $i_k \in A$, We start by putting six i_k in column c_{i_k}. Similarly, for each $j_k \in B$, we start by putting six j_k in column c_{j_k}.

2. We use t_{center} to fill in the center columns. More precisely, letting $c''_1, \ldots, c''_{n-a-b}$ be the center columns and letting $i'_1, \ldots, i'_{n-a-b}$ be the elements of $[n] \setminus (A \cup B)$, if t_{center} has an element $y \in [n-a-b]$ in row r and column j then we put the element i'_y in row r and column c''_j of t.

   We translate the pairing $P \in \mathcal{P}(t_{\text{center}})$ and the endpoints $E = (p_1, \ldots, p_x)$ to the center columns accordingly.

3. We use t_{cycles} to replace the columns of t containing the i elements of B_{cycles}. More precisely, if $c''_1, \ldots, c''_i$ are the columns containing the elements of B_{cycles} and j'_k is the element in column c''_k (note that $j'_1, \ldots, j'_i$ are not necessarily in increasing order) then if t_{cylces} has an element $y \in [i]$ in row r and column k then we put the element j'_y in row r and column c''_k.

   We let t' be the table which we obtain after this replacement.

4. We order the paths in G_{paths} based on their starting vertex. For each $k \in [x]$, let $j_{k1} \to j_{k2} \to \cdots \to j_{kl_k} \to v_{\text{center}}$ be the kth path in G_{paths}. After translating E to the center columns, p_k is a pair of elements z_k in rows r_{k1} and r_{k2} of column c^*_k for some $z_k \in [n] \setminus (A \cup B)$, $c^*_k \in \{c''_1, \ldots, c''_{n-a-b}\}$, and $r_{k1}, r_{k2} \in [6]$. We obtain t by performing the following modifications to t' for each of the k paths.

   1. We replace the two z_k in rows r_{k1} and r_{k2} of column c^*_k with j_{kl_k}.
   2. For all $i \in [l_k - 1]$, we replace the two $j_{k(i+1)}$ in rows r_{k1} and r_{k2} of column $c_{j_{k(i+1)}}$ with j_{ki}.
   3. We replace the two j_{k1} in rows r_{k1} and r_{k2} of column $c_{j_{k1}}$ with z_k.

Example 48. The figures below illustrate the bijection when $A = \{7\}$, $B = \{1, 5, 6, 8, 9\}$, $C_A = (1)$, $C_B = (8, 4, 7, 2, 9)$ (as these are the 4-columns for $1, 5, 6, 8, 9$ respectively), $i = 2$, $B_{\text{cycles}} = \{8, 9\}$, $x = 2$, and t_{center} t_{cycles}, G_{paths}, and E are as shown below.

1	3	2
1	3	2
2	3	1
2	3	1
1	3	2
1	3	2

1	2
2	1
1	2
1	2
2	1
1	2

Figure 14: t_{center} and t_{cycles}.

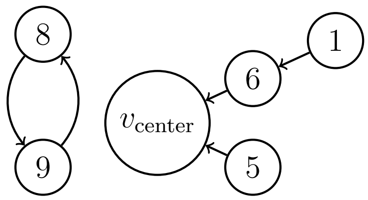

Figure 15: $G(t)$.

7	8	2	5	4	3	6	1	9
7	9	2	5	6	3	1	4	8
7	8	5	3	6	2	1	4	9
7	8	5	3	4	2	6	1	9
7	9	2	5	4	3	6	1	8
7	8	2	5	4	3	6	1	9

Figure 16: The resulting table t.

It is not hard to check that these maps are inverses of each other. $\square$

$\square$

This completes the proof of Lemma 34. Combining Lemmas 33 and 34 completes the proof of Lemma 28. $\square$

We now simplify the terms in Lemma 28.

Proposition 49.

$$H_{n,j,a,b} = \frac{(3(n-a-b)+j-1)!}{(3(n-a-b)-1)!}.$$

Proof. Originally,

$$H_{n,j,a,b} = \sum_{x=1}^{j} \frac{j!}{x!} \binom{j-1}{x-1} \prod_{y=0}^{x-1} (3(n-a-b)-y).$$

Let $z = 3(n-a-b)$. For the inner product, we can write

$$\prod_{y=0}^{x-1} (3(n-a-b)-y) = \frac{z!}{(z-x)!},$$

so

$$H_{n,j,a,b} = \sum_{x=1}^{j} \frac{j!}{x!} \binom{j-1}{x-1} \frac{z!}{(z-x)!} = j! \sum_{x=1}^{j} \binom{j-1}{x-1} \binom{z}{x} = j! \binom{z+j-1}{j}.$$

The last equality is a special case of the *Chu-Vandermonde Identity.* $\square$

With these simplifications, we can derive an expression for the generating function

$$F_6(t) = \sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} f_6(n).$$

By Lemma 28,

$$F_6(t) = \sum_{0\le a\le j\le n} \frac{t^n}{(n!\,)^2}\binom{n}{j}\binom{j}{a}(m_6-15)^a(m_4-3)^{(j-a)}D_{n,a,j-a}.$$

Summing with respect to $b = j-a$ instead of a and observing that

$$\begin{aligned} D_{n,a,b} &= \left(\prod_{k=0}^{a+b-1}(n-k)\right)\left(\sum_{i=0}^{b}\binom{b}{i}C_iH_{n,b-i,a,b}\right)P_{n-a-b} \\ &= \frac{n!}{(n-j)!}\left(\sum_{i=0}^{b}\binom{b}{i}C_iH_{n,b-i,j-b,b}\right)P_{n-j} \\ &= n!\left(\sum_{i=0}^{b}\binom{b}{i}C_iH_{n,b-i,j-b,b}\right)\frac{(n-j+2)!\,(n-j+4)!}{48}, \end{aligned}$$

we have that

$$\begin{aligned} F_6(t) = \sum_{0\le i\le b\le j\le n} &\frac{t^n}{n!}\binom{n}{j}\binom{j}{b}\binom{b}{i}(m_6-15)^{(j-b)}(m_4-3)^b \\ &\frac{(n-j+2)!\,(n-j+4)!}{48}H_{n,b-i,j-b,b}C_i. \end{aligned}$$

By Proposition 49, $H_{n,b-i,j-b,b} = (3n-3j+b-i-1)!\,/(3n-3j-1)!$. Using the reparametrization $b = i+s, j = b+r, n = j+q$, where s, r, q goes from 0 to ∞, we get

$$\begin{aligned} F_6(t) = \sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{i=0}^{\infty} &\frac{t^{i+s+r+q}}{q!\,r!\,s!\,i!}(m_6-15)^r(m_4-3)^{(i+s)} \\ &\frac{(q+2)!\,(q+4)!}{48}\frac{(3q+s-1)!}{(3q-1)!}C_i. \end{aligned}$$

Grouping the terms to separate the dependence on r, s, and i, we have that $F_6(t)$ equals

$$\left(\sum_{q=0}^{\infty}\frac{t^q}{q!}\frac{(q+2)!\,(q+4)!}{48}\right)\left(\sum_{r=0}^{\infty}\frac{t^r(m_6-15)^r}{r!}\right)$$
$$\left(\sum_{s=0}^{\infty}\frac{t^s}{s!}\frac{(3q+s-1)!}{(3q-1)!}(m_4-3)^s\right)\left(\sum_{i=0}^{\infty}\frac{t^i}{i!}(m_4-3)^iC_i\right).$$

Summing all the inner sums (the rightmost using Lemma 41),

$$F_6(t) = \sum_{q=0}^{\infty}\frac{t^q}{q!}\frac{(q+2)!\,(q+4)!}{48}e^{t(m_6-15)}\frac{1}{(1-t(m_4-3))^{3q}}\frac{e^{-15t(m_4-3)}}{(1-t(m_4-3))^{15}}.$$

4 Generalization for an arbitrary third moment

Restating Proposition 19, we can write

$$f_6(n) = \sum_{t \in T_{6,n}} \operatorname{sgn}(t) w(t),$$

where $T_{6,n}$ is the set of all permutation tables of length n with six rows whose columns fall in one of the following categories:

- 6-columns: six copies of a single number (weight m_6)
- 4-columns: four copies of one number and two copies of a distinct number (weight m_4)
- 2-columns: three pairs of distinct numbers (weight 1)

The weight $w(t)$ of a table t is simply the product of the weights of its columns.

When $m_3 \neq 0$, we can have the following type of column in addition to 2-columns, 4-columns, and 6-columns:

- 3-columns: three copies of one number and three copies of a distinct number (weight m_3^2)

To handle this, we make the following definitions.

Definition 50. We define $f_6^*(n) = E_{M \sim \mathcal{M}_{n\times n}(\Omega)}[\det(M)^6]$ to be the expected value of the sixth power of the determinant of a matrix $M \sim \mathcal{M}_{n\times n}(\Omega)$ (i.e., an $n \times n$ random matrix with entries drawn from Ω) where Ω is a distribution such that $m_1 = 0$.

We define the formal generating function $F_6^*(t)$ for $f_6^*(n)$ to be $F_6^*(t) = \sum_{n=0}^{\infty} \frac{t^n}{(n!)^2} f_6^*(n)$.

Remark 51. We write $f_6^*(n)$ instead of $f_6(n)$ to distinguish this case from the case we previously analyzed where $m_1 = m_3 = 0$.

Definition 52. We define $T_{6,n}^*$ to be the set of all $6 \times n$ tables whose rows are permutations of $[n]$ and whose columns are 2-columns, 3-columns, 4-columns, or 6-columns.

Similar to before, we can write $f_6^*(n)$ as the sum over tables in $T_{6,n}^*$ of the contribution from each table.

Proposition 53. *For all $n \in \mathbb{N}$, $f_6^*(n) = \sum_{t \in T_{6,n}^*} \operatorname{sgn}(t) w(t)$.*

We can evaluate $f_6^*(n)$ using our expression for $f_6(n)$.

Definition 54. We define $f_6(n) = \sum_{t \in T_{6,n}} \operatorname{sgn}(t) w(t) = E_{M \sim \mathcal{M}_{n\times n}(\Omega')}[\det(M)^6]$ where Ω' is the distribution where we first sample x from Ω and then randomly choose x or $-x$ (so Ω and Ω' have the same even moments but the odd moments of Ω' are 0).

Lemma 55. *For all $n \in \mathbb{N}$,*

$$f_6^*(n) = \sum_{j=0}^{n} \binom{n}{j}^2 f_6(n-j)\, j!\, m_3^{2j} (-1)^j \sum_{\pi \in D_j} (-10)^{C(\pi)}.$$

Proof. We first break each table $t \in T_{6,n}^*$ into tables s and t', where s contains all of the 3-columns of t. The signs of these tables are related as

$$\operatorname{sgn}(t) = \operatorname{sgn}(s)\operatorname{sgn}(t').$$

Definition 56. Given $J \subset [n]$, letting $j = |J|$, we define $T_{6,J}$ to be the set of all $6 \times |J|$ tables whose rows are permutations of J and whose columns are all 2-columns, 4-columns, or 6-columns. We define $Q_{6,J}$ to be the set of all $6 \times |J|$ tables whose rows are permutations of J and whose columns are all 3-columns.

Since the selection of the subset $J \subseteq [n]$ does not depend on the positions of the columns containing the elements of J in table t, we can write our sum as

$$f_6^*(n) = \sum_{J \subset [n]} \binom{n}{j} \sum_{t' \in T_{6,[n]/J}} \operatorname{sgn}(t') w(t') \sum_{s \in Q_{6,J}} \operatorname{sgn}(s) w(s).$$

No matter which numbers J are selected, as long as we select the same amount of them, the contribution is the same. Hence,

$$f_6^*(n) = \sum_{j=0}^{n} \binom{n}{j}^2 \sum_{t' \in T_{6,n-j}} \operatorname{sgn}(t') w(t') \sum_{s \in Q_{6,j}} \operatorname{sgn}(s) w(s),$$

where $Q_{6,j} = Q_{6,[j]}$. The first inner sum is simply $f_6(n-j)$. For the second inner sum, by symmetry, we can fix the first permutation in s to be the identity. Since $w(s) = m_3^{2j}$, we get

$$\sum_{s \in Q_{6,j}} w(s) \operatorname{sgn}(s) = j!\, m_3^{2j} \sum_{\substack{s \in Q_{6,j} \\ s_1 = \mathrm{id}}} \operatorname{sgn}(s).$$

Similarly as before, we construct a graph $G(s)$ from the numbers in table s and then show $G(s)$ is a derangement. Let c be a number in the first row of a given column of table s. Since it is a 3-column, we denote the other number in the column as c'. We construct a graph $G(s)$ for a given table s whose edges are all of these pairs $c \to c'$.

Example 57. Example showing the correspondence between a table s where $s_1 = id$ and the corresponding derangement $G(s)$.

1	2	3	4	5	6	7	8	9
3	6	1	4	5	7	2	8	9
1	2	3	9	4	6	7	5	8
1	6	3	9	4	7	2	5	8
3	2	1	9	4	6	7	5	8
3	6	1	4	5	7	2	8	9

Figure 17: A 6×9 table $s \in Q_{6,9}$.

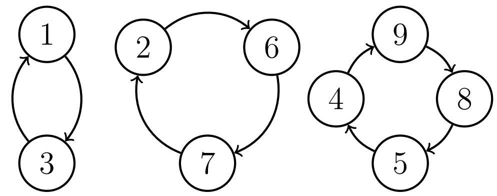

Figure 18: The associated $G(s)$.

For the signs, letting $\pi = G(s)$, we have

$$\operatorname{sgn}(s) = \operatorname{sgn}(\pi) = (-1)^{j - C(\pi)}.$$

Note that since c and c' are always different, the set of all $G(s)$ corresponds to the set D_j of all derangements. Since there are $\binom{5}{2} = 10$ possibilities how to arrange the leftover 5 numbers c and c' in each of the 3-columns corresponding to a given cycle of π, we get

$$\sum_{s \in Q_{6,j}} w(s) \operatorname{sgn}(s) = j!\, m_3^{2j} (-1)^j \sum_{\pi \in D_j} (-1)^{C(\pi)} 10^{C(\pi)}.$$

Putting everything together, we obtain that

$$f_6^*(n) = \sum_{j=0}^{n} \binom{n}{j}^2 f_6(n-j)\, j!\, m_3^{2j} (-1)^j \sum_{\pi \in D_j} (-10)^{C(\pi)}.$$

□

Corollary 58.

$$F_6^*(t) = (1 + m_3^2 t)^{10} e^{-10 m_3^2 t}\, F_6(t).$$

Proof. In terms of generating functions,

$$\begin{aligned} F_6^*(t) &= \sum_{n=0}^{\infty} \frac{t^n}{n!^2} f_6^*(n) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{t^{n-j}}{(n-j)!^2} f_6(n-j) \frac{(-m_3^2 t)^j}{j!} \sum_{\pi \in D_j} (-10)^{C(\pi)} \\ &= F_6(t) \sum_{j=0}^{\infty} \frac{(-m_3^2 t)^j}{j!} \sum_{\pi \in D_j} (-10)^{C(\pi)} = F_6(t) \frac{e^{-10 m_3^2 t}}{(1 + m_3^2 t)^{-10}}. \end{aligned}$$

The final equality is a special case of Lemma 27. Theorem 9 follows. □

5 Asymptotics

The proof relies directly on the calculus developed by Borinsky [1], enabling us to extract the asymptotic behaviour of coefficients from their factorially divergent generating function. We use the following result from Borinsky [1]:

Definition 59. We say a formal power series $f(t) = \sum_{n\geq 0} f_n t^n$ is factorially divergent of type (α, β), if $f_n \sim \sum_{k=0}^{R} c_k \alpha^{n+\beta-k}\Gamma(n+\beta-k)$ as $n \to \infty$ for any fixed R integer. We also define an operator $\mathcal{A}_\beta^\alpha$ acting of $f(t)$ such that $(\mathcal{A}_\beta^\alpha f)(t) = \sum_{k\geq 0} c_k t^k$. If moreover $f(t)$ is analytic at 0, then $(\mathcal{A}_\beta^\alpha f)(t) = 0$.

Lemma 60. *Let $f(t)$ and $g(t)$ be two factorially divergent power series of type (α, β), then*

$$(\mathcal{A}_\beta^\alpha(fg))(t) = (\mathcal{A}_\beta^\alpha f)(t)g(t) + f(t)(\mathcal{A}_\beta^\alpha g)(t), \text{ and}$$
$$(\mathcal{A}_\beta^\alpha(f\circ g))(t) = f'(g(t))(\mathcal{A}_\beta^\alpha g)(t) + \left(\frac{t}{g(t)}\right)^\beta e^{\frac{1}{\alpha}\left(\frac{1}{t}-\frac{1}{g(t)}\right)}(\mathcal{A}_\beta^\alpha f)(g(t)),$$

where the second equality holds when $g(t) = 1 + t + O(t^2)$.

Recall Theorem 9, which states that

$$F_6(t) = (1+m_3^2 t)^{10}\,\frac{e^{t(m_6-10m_3^2-15m_4+30)}}{48(1+3t-m_4t)^{15}}\sum_{i=0}^{\infty}\frac{(1+i)(2+i)(4+i)!\,t^i}{(1+3t-m_4t)^{3i}}.$$

Hence, we can write $F_6(t) = h(t)f(g(t))$, where

$$\begin{aligned}
f(t) &= \sum_{i=0}^{\infty}(1+i)(2+i)(4+i)!\,t^i,\\
g(t) &= \frac{t}{(1+3t-m_4t)^3},\\
h(t) &= (1+m_3^2t)^{10}\,\frac{e^{t(m_6-10m_3^2-15m_4+30)}}{48(1+3t-m_4t)^{15}}
\end{aligned}$$

are factorially divergent of type $(1, 7)$ since

$$(1+i)(2+i)(4+i)! = \Gamma(i+7) - 8\Gamma(i+6) + 12\Gamma(i+5)$$

and $g(t)$ and $h(t)$ are analytic. Thus, by Lemma 60,

$$\begin{aligned}
(\mathcal{A}_7^1 F_6)(t) &= h(t)\left(\frac{t}{g(t)}\right)^7 e^{\frac{1}{t}-\frac{1}{g(t)}}(\mathcal{A}_7^1 f)(g(t))\\
&= h(t)\left(\frac{t}{g(t)}\right)^7 e^{\frac{1}{t}-\frac{1}{g(t)}}(1-8g(t)+12g^2(t)).
\end{aligned}$$

Apart from a factor $(n!\,)^2 e^{3(m_4-3)}/48$, this is our function $C(t)$ from the original statement of Theorem 12.

For $\Omega = \{-1, 1\}$, the asymptotic expression

$$f_6(n) \sim \frac{(n!)^3}{48e^6}\bigg(n^6 + 29n^5 + 335n^4 + \frac{5861n^3}{3} + \frac{17944n^2}{3} + \frac{44036n}{5} + \frac{167536}{45} - \frac{210176}{63n}\bigg)$$

gives an excellent approximation to $f_6(n)$ for $n \geq 10$. The following figure shows the ratio of this asymptotic expression to the actual value of $f_6(n)$ for n up to 20.

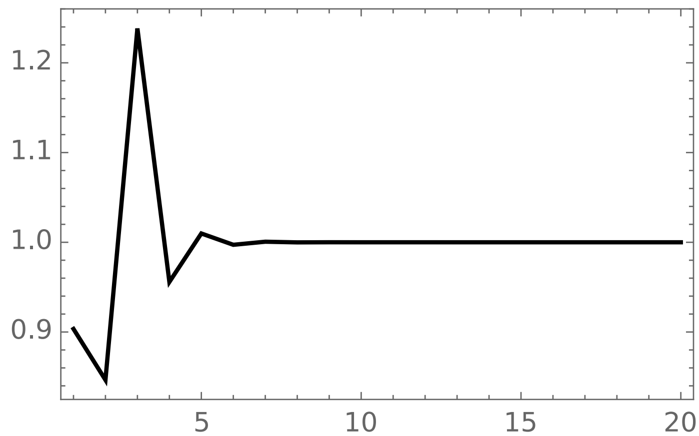

Figure 19: The ratio between the asymptotic expression and $f_6(n)$ for $\Omega = \{-1, 1\}$.

6 Acknowledgments

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A Direct proof of Lemma 26

Lemma. *(Restatement of Lemma 26). For all $n \in \mathbb{N}$, $P_n = n(n+2)(n+4)P_{n-1}$ where $P_0 = 1$.*

Proof. We recursively compute $P_n = \sum_{t \in T_{k,n}} \mathrm{sgn}(t)|\mathcal{P}(t)|$ based on where the six n are located in t.

We can count the cases where all of the n are in a 6-column as follows. Given a table $t \in T_{k,n-1}$ and a pairing $P \in \mathcal{P}(t)$, we can obtain a table $t' \in T_{k,n}$ and a pairing $P' \in \mathcal{P}(t)$ by choosing a location for the 6-column, choosing a pairing for this column, and using t and P to fill in the remainder of t' and P'. There are n possible places for the 6-column, it has 15 possible pairings, and $\mathrm{sgn}(t') = \mathrm{sgn}(t)$, so this gives a contribution of $15nP_{n-1}$.

We can count the cases where four of the n are in a 4-column and two of the n appear in a different column as follows. Given a table $t \in T_{k,n-1}$ and a pairing $P \in \mathcal{P}(t)$, we can obtain a table $t' \in T_{k,n}$ and a pairing $P' \in \mathcal{P}(t)$ with the following steps:

1. Choose which column will be the 4-column containing four of the n. We initially put all six n in this column.

2. Fill in the remaining columns using t and P.

3. Choose one of the $3(n-1)$ pairs in P and swap two of the n with this pair.

4. Choose a pairing for the remaining four n.

There are n possible places for the 4-column containing four of the n, there are $3(n-1)$ pairs in P which can be swapped with two of the n, there are 3 different pairings for the remaining four n, and $\operatorname{sgn}(t') = \operatorname{sgn}(t)$, so this gives a contribution of $3(3)n(n-1)P_{n-1} = 9n(n-1)P_{n-1}$.

The trickiest case to analyze is the case when the six n are split into three different columns. The idea for this case is that there is a correspondence between sets of two columns containing pairs of the elements a, b, c, d, e, f and sets of three columns containing pairs of the elements a, b, c, d, e, f where each column also contains a pair of n. This correspondence is highly non-trivial and relies on the signs of the permutations.

Definition 61. Let $S_1, S_2, S_3, S_4, S_5, S_6$ be six sets such that each set S_i contains two of the elements $\{a, b, c, d, e, f\}$ and each element in $\{a, b, c, d, e, f\}$ is contained in two of the sets $S_1, S_2, S_3, S_4, S_5, S_6$.

We define $T_2(S_1, S_2, S_3, S_4, S_5, S_6)$ to be the set of 6×2 tables t such that the ith row contains the elements in S_i and each element appears an even number of times in each column. Similarly, we define $T_3(S_1, S_2, S_3, S_4, S_5, S_6)$ to be the set of 6×3 tables t such that the ith row contains the elements in $S_i \cup \{n\}$ and each element appears an even number of times in each column.

For each $t \in T_2(S_1, S_2, S_3, S_4, S_5, S_6)$, we define $\operatorname{sgn}(t)$ to be the product of the signs of the rows of t where row i of t has sign 1 if the elements of S_i appear in order and sign -1 if the elements of S_i appear out of order. Similarly, for each $t \in T_3(S_1, S_2, S_3, S_4, S_5, S_6)$, we define $\operatorname{sgn}(t)$ to be the product of the signs of the rows of t where row i of t has sign 1 if it takes an even number of swaps to transform it into $S_i \cup \{n\}$ and -1 if it takes an odd number of swaps to transform it into $S_i \cup \{n\}$.

Lemma 62. *For all possible $S_1, S_2, S_3, S_4, S_5, S_6$,*

$$\sum_{t \in T_3(S_1,S_2,S_3,S_4,S_5,S_6)} \operatorname{sgn}(t) = 6 \sum_{t \in T_2(S_1,S_2,S_3,S_4,S_5,S_6)} \operatorname{sgn}(t).$$

Corollary 63. *For all $n \in \mathbb{N}$,*

$$\sum_{t \in T_{6,n} : n \text{ appears in 3 different columns}} \operatorname{sgn}(t)|\mathcal{P}(t)| = n(n-1)(n-2)P_{n-1}.$$

Proof. Recall that

$$\sum_{t \in T_{6,n-1}} \mathrm{sgn}(t)|\mathcal{P}(t)| = P_{n-1}.$$

We now apply Lemma 62 to the first two columns of the pairs (t, P) where $t \in T_{6,n-1}$ and $P \in \mathcal{P}(t)$. To do this, we use P to relabel the elements in the first two columns as a, b, c, d, e, f. One way to do this is as follows. We go through the rows one by one and assign the next unused label(s) to the element(s) which whose pair has not yet appeared. If there are two such elements, we assign the first unused label to the lower element and the next unused label to the higher element. If both elements are the same, we assign the first unused label to the column where the pair of this element appears first. If there is still a tie, we assign the same label to both elements and skip the next label. Lemma 62 still holds in this case as having $S_i = S_j = \{a, a\}$ instead of $S_i = S_j = \{a, b\}$ divides both sides by 2.

After doing this relabeling, for each $i \in [6]$, we take S_i to be the first two elements in row i. Applying Lemma 62, we obtain tables t' and pairings P' by taking P' to be the unique pairing for each column and inverting the labeling of the elements in the first two columns of t by $\{a, b, c, d, e, f\}$. This implies that whenever $n \geq 3$,

$$\sum_{t \in T_{6,n}: n \text{ appears in the first three columns}} \mathrm{sgn}(t)|\mathcal{P}(t)| = 6 \sum_{t \in T_{6,n-1}} \mathrm{sgn}(t)|\mathcal{P}(t)| = 6P_{n-1}.$$

There are $\binom{n}{3} = \frac{n(n-1)n-2)}{6}$ possibilities for which three columns contain n so we have that

$$\sum_{t \in T_{6,n}: n \text{ appears in 3 different columns}} \mathrm{sgn}(t)|\mathcal{P}(t)| = n(n-1)(n-2)P_{n-1},$$

as needed. □

Summing these three cases up, we have

$$\begin{aligned} P_n &= 15nP_{n-1} + 9n(n-1)P_{n-1} + n(n-1)(n-2)P_{n-1} \\ &= (n^3 + 6n^2 + 8n)P_{n-1} = n(n+2)(n+4)P_{n-1}. \end{aligned}$$

□

We now prove Lemma 62.

Proof of Lemma 62. Up to permutations of the rows and $\{a, b, c, d, e, f\}$, we have the following four cases for $S_1, S_2, S_3, S_4, S_5, S_6$:

1. $S_1 = S_2 = \{a, b\}$, $S_3 = S_4 = \{c, d\}$, and $S_5 = S_6 = \{e, f\}$.

2. $S_1 = S_2 = \{a, b\}$, $S_3 = \{c, d\}$, $S_4 = \{c, e\}$, $S_5 = \{d, f\}$, and $S_6 = \{e, f\}$.

3. $S_1 = \{a, b\}$, $S_2 = \{a, c\}$, $S_3 = \{b, d\}$, $S_4 = \{d, e\}$, $S_5 = \{c, f\}$, and $S_6 = \{e, f\}$.

4. $S_1 = \{a, b\}$, $S_2 = \{a, c\}$, $S_3 = \{b, c\}$, $S_4 = \{d, e\}$, $S_5 = \{d, f\}$, and $S_6 = \{e, f\}$.

We can see that these are the only possibilities as follows. If we construct a multi-graph where the vertices are $\{a, b, c, d, e, f\}$ and the edges are $\{S_1, S_2, S_3, S_4, S_5, S_6\}$ then in this multi-graph, every vertex will have degree 2.

1. If there is a cycle of length 2 then for the remaining 4 vertices, we will either have two more cycles of length 2 or a cycle of length 4. This gives cases 1 and 2.

2. If there is a cycle of length 3 then there must be another cycle of length 3 on the remaining vertices. This gives case 4.

3. If there are no cycles of length 2 or 3 then we must have a cycle of length 6. This gives case 3.

For the first three cases, we notice that $T_2(S_1, S_2, S_3, S_4, S_5, S_6)$ is nonempty as shown by the examples below. For the fourth case, we notice that $T_2(S_1, S_2, S_3, S_4, S_5, S_6)$ is empty.

$$\left\{\begin{matrix} a & b \\ a & b \\ c & d \\ c & d \\ e & f \\ e & f \end{matrix}\right\}, \left\{\begin{matrix} a & b \\ a & b \\ c & d \\ c & e \\ f & d \\ f & e \end{matrix}\right\}, \left\{\begin{matrix} a & b \\ a & c \\ d & b \\ d & e \\ f & c \\ f & e \end{matrix}\right\}$$

For all four cases, $T_3(S_1, S_2, S_3, S_4, S_5, S_6)$ is nonempty as shown by the examples below.

$$\left\{\begin{matrix} a & b & n \\ a & b & n \\ c & n & d \\ c & n & d \\ n & e & f \\ n & e & f \end{matrix}\right\}, \left\{\begin{matrix} a & b & n \\ a & b & n \\ c & n & d \\ c & n & e \\ n & f & d \\ n & f & e \end{matrix}\right\}, \left\{\begin{matrix} a & b & n \\ a & c & n \\ n & b & d \\ e & n & d \\ n & c & f \\ e & n & f \end{matrix}\right\}, \left\{\begin{matrix} a & b & n \\ a & n & c \\ n & b & c \\ d & e & n \\ d & n & f \\ n & e & f \end{matrix}\right\}$$

We now show that for each of the four cases,

$$\sum_{t \in T_3(S_1, S_2, S_3, S_4, S_5, S_6)} \operatorname{sgn}(t) = 6 \sum_{t \in T_2(S_1, S_2, S_3, S_4, S_5, S_6)} \operatorname{sgn}(t).$$

1. For the first case, $|T_2(S_1, S_2, S_3, S_4, S_5, S_6)| = 8$ as we can choose the order of $\{a, b\}$ in row 1, the order of $\{c, d\}$ in row 3, and the order of $\{e, f\}$ in row 5. All $t \in T_2(S_1, S_2, S_3, S_4, S_5, S_6)$ have positive sign as rows 2, 4, and 6 must be the same as rows 1, 3, and 5. Thus, $\sum_{t \in T_2(S_1, S_2, S_3, S_4, S_5, S_6)} \operatorname{sgn}(t) = 8$.

   To analyze $T_3(S_1, S_2, S_3, S_4, S_5, S_6)$, observe that there are 6 choices for the positions of the n in rows 1, 3, and 5 and we can again choose the order of $\{a, b\}$ in row 1, the order of $\{c, d\}$ in row 3, and the order of $\{e, f\}$ in row 5. Thus, $|T_3(S_1, S_2, S_3, S_4, S_5, S_6)| = 48$. All $t \in T_3(S_1, S_2, S_3, S_4, S_5, S_6)$ have positive sign as rows 2, 4, and 6 must be the same as rows 1, 3, and 5. Thus, $\sum_{t \in T_3(S_1,S_2,S_3,S_4,S_5,S_6)} \mathrm{sgn}(t) = 48$.

2. For the second case, $|T_2(S_1, S_2, S_3, S_4, S_5, S_6)| = 4$ as we can choose the order of $\{a, b\}$ in row 1 and the order of $\{c, d\}$ in row 3 and this uniquely determines the rest of the table. It can be checked that all $t \in T_2(S_1, S_2, S_3, S_4, S_5, S_6)$ have positive sign. Hence, we have that $\sum_{t \in T_2(S_1,S_2,S_3,S_4,S_5,S_6)} \mathrm{sgn}(t) = 4$.

   To analyze $T_3(S_1, S_2, S_3, S_4, S_5, S_6)$, observe that there are 6 choices for the order of $\{a, b, n\}$ in row 1. Once this order is chosen, there are two choices for the position of the n in row 3 and two choices for the order of $\{c, d\}$ in row 3. It can be checked that this uniquely determines the rest of the table and all $t \in T_3(S_1, S_2, S_3, S_4, S_5, S_6)$ have positive sign so we have that $\sum_{t \in T_3(S_1,S_2,S_3,S_4,S_5,S_6)} \mathrm{sgn}(t) = 24$.

3. For the third case, $|T_2(S_1, S_2, S_3, S_4, S_5, S_6)| = 2$ as we can choose the order of $\{a, b\}$ in row 1 and this uniquely determines the rest of the table. Both $t \in T_2(S_1, S_2, S_3, S_4, S_5, S_6)$ have negative sign so we have that $\sum_{t \in T_2(S_1,S_2,S_3,S_4,S_5,S_6)} \mathrm{sgn}(t) = -2$.

   For $T_3(S_1, S_2, S_3, S_4, S_5, S_6)$, there are 6 choices for the order of $\{a, b, n\}$ in row 1. When row 1 is a, b, n, we have the following four tables:

$$\left\{\begin{matrix} a & b & n \\ a & c & n \\ n & b & d \\ e & n & d \\ n & c & f \\ e & n & f \end{matrix}\right\}, \left\{\begin{matrix} a & b & n \\ a & n & c \\ d & b & n \\ d & n & e \\ n & f & c \\ n & f & e \end{matrix}\right\}, \left\{\begin{matrix} a & b & n \\ a & n & c \\ n & b & d \\ e & n & d \\ n & f & c \\ e & f & n \end{matrix}\right\}, \left\{\begin{matrix} a & b & n \\ a & n & c \\ n & b & d \\ n & e & d \\ f & n & c \\ f & e & n \end{matrix}\right\}$$

   Of these tables, the first, second, and fourth table have negative sign while the third table has positive sign so the net contribution is -2. Multiplying this by 6, we have that
$$\sum_{t \in T_3(S_1,S_2,S_3,S_4,S_5,S_6)} \mathrm{sgn}(t) = -12.$$

4. For the fourth case, $T_2(S_1, S_2, S_3, S_4, S_5, S_6)$ is empty because each column can only contain one of $\{a, b, c\}$ and one of $\{b, c, d\}$.

   To analyze $T_3(S_1, S_2, S_3, S_4, S_5, S_6)$, observe that we can choose the order of $\{a, b, n\}$ in row 1 and the order of $\{d, e, n\}$ in row 4 and this uniquely determines the rest of the table. The sign of each table will be the product of the sign for row 1 and the sign for row 4, so we have the same number of tables with positive and negative sign and thus $\sum_{t \in T_2(S_1,S_2,S_3,S_4,S_5,S_6)} \mathrm{sgn}(t) = \sum_{t \in T_3(S_1,S_2,S_3,S_4,S_5,S_6)} \mathrm{sgn}(t) = 0$.

$\square$

References

[1] M. Borinsky. Generating asymptotics for factorially divergent sequences. *Electron. J. Combin.* **25** (2018), 1–32.

[2] J. Bourgain, V. H. Vu, and P. M. Wood. Singularity of random Bernoulli matrices. *J. Funct. Anal.* **258** (2010), 559–603.

[3] K. P. Costello, T. Tao, and V. Vu. Random symmetric matrices are almost surely nonsingular. *Duke Math. J.* **135** (2006), 395–413.

[4] A. Dembo. On random determinants. *Quart. Appl. Math.* **47** (1989), 185–195.

[5] D. Beck. On the fourth moment of a random determinant. Arxiv preprint arXiv:2207.09311 [math.PR], 2022. Available at http://arxiv.org/abs/2207.09311.

[6] P. Flajolet and R. Sedgewick. *Analytic Combinatorics*. Cambridge University Press, 2009.

[7] G. E. Forsythe and J. W. Tukey. The extent of n-random unit vectors. *Bull. Amer. Math. Soc.* **58** (1952), 502–502.

[8] V. L. Girko. A refinement of the central limit theorem for random determinants. *Theory Probab. Appl.* **42** (1998), 121–129.

[9] V. L. Girko. The central limit theorem for random determinants. *Theory Probab. Appl.* **24** (1980), 729–740.

[10] J. Kahn, J. Komlós, and E. Szemerédi. On the probability that a random$\pm$1-matrix is singular. *J. Amer. Math. Soc.* **8** (1995), 223–240.

[11] J. Komlós. On the determinant of (0-1) matrices. *Studia Sci. Math. Hungar.* **2** (1967), 7–21.

[12] J. Komlós. On the determinant of random matrices. *Studia Sci. Math. Hungar.* **3** (1968), 387–399.

[13] H. H. Nguyen and V. Vu. Random matrices: law of the determinant. *Ann. Probab.* **42** (2014), 146–167.

[14] H. Nyquist, S. O. Rice, and J. Riordan. The distribution of random determinants. *Quart. Appl. Math.* **12** (1954), 97–104.

[15] N. J. A. Sloane et al. The On-line Encyclopedia of Integer Sequences, 2023. Available at https://oeis.org/.

[16] T. Tao and V. Vu. On random ± 1 matrices: singularity and determinant. *Random Structures Algorithms* **28** (2006), 1–23.

[17] T. Tao and V. Vu. On the singularity probability of random Bernoulli matrices. *J. Amer. Math. Soc.* **20** (2007), 603–628.

[18] K. Tikhomirov. Singularity of random Bernoulli matrices. *Ann. of Math.* **191** (2012), 593–634.

[19] V. Vu. Recent progress in combinatorial random matrix theory. Arxiv preprint arXiv:2005.02797 [math.co], available at http://arxiv.org/abs/2005.02797, 2020.

[20] I. G. Zhurbenko. Moments of random determinants. *Theory Probab. Appl.* **13** (1968), 682–686.

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