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On Tribonacci Numbers that are Products of Factorials

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Abstract

We determine all Tribonacci numbers of positive or negative indices that are products of factorials.

1 Introduction

The Tribonacci numbers $\{T_n\}_{n\geq 0}$ are given by $T_0 = 0,\ T_1 = T_2 = 1$ and $T_{n+3} = T_{n+2} + T_{n+1} + T_n$ for $n \geq 0$. In [5], Marques and Lengyel determined all Tribonacci numbers T_n that are factorials. They showed that if $T_n = m!$ then $n \in \{1, 2, 3, 7\}$. Here we take this a couple of steps further.

Theorem 1. *If $n \geq 1$ and*

$$T_n = m_1!\, m_2! \cdots m_k!$$

for some positive integers $m_1 \leq m_2 \leq \cdots \leq m_k$, then $n \in \{1, 2, 3, 4, 7\}$.

Next, one can extend the sequence of Tribonacci numbers in the negative direction using the recurrence relation. Namely, since $T_0 = 0,\ T_1 = T_2 = 1$, one computes that $T_{-1} = 0,\ T_{-2} = 1$ and for $n \geq 3$, we have

$$T_{-n} = -T_{-(n-1)} - T_{-(n-2)} + T_{-(n-3)}.$$

The Tribonacci numbers with negative indices change signs infinitely often. Nevertheless, we can still ask what about the equation $|T_n| = m_1! \cdots m_k!$. Here is our result.

Theorem 2. *If $n \in \mathbb{Z}$ and $|T_n| = m_1!\, m_2! \cdots m_k!$ then*

$$n \in \{-9, -8, -7, -5, -3, -2, 1, 2, 3, 4, 7\}.$$

Note the near miss $T_{-32} = -2^4 \cdot 3 \cdot 5^2 \cdot 7 = 7! \cdot (5/3)$.

2 Preliminaries

The main ingredient is an exact formula for the exponent of 2 in T_n. Let $\nu_p(m)$ be the exponent of the prime p in the factorization of the integer m with convention that $\nu_p(0) = \infty$. Here is Theorem 1 in [5].

Theorem 3. *For $n \geq 1$, we have*

$$\nu_2(T_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod 4; \\ 1, & \text{if } n \equiv 3, 11 \pmod{16}; \\ 2, & \text{if } n \equiv 4, 8 \pmod{16}; \\ \nu_2(n) - 1, & \text{if } n \equiv 0 \pmod{16}; \\ \nu_2(n+4) - 1 & \text{if } n \equiv 12 \pmod{16}; \\ \nu_2(n+17) + 1 & \text{if } n \equiv 15 \pmod{32}; \\ \nu_2(n+1) + 1 & \text{if } n \equiv 31 \pmod{32}. \end{cases}$$

A similar theorem is proved for the prime $p = 3$ in [1].

Theorem 4. *For $n \geq 1$, we have*

$$\nu_3(T_n) = \begin{cases} 0, & \text{if } n \equiv 1,2,3,4,5,6,8,10,11 \pmod{13}; \\ 1, & \text{if } n \equiv 7 \pmod{13}; \\ \nu_3(n)+2, & \text{if } n \equiv 0 \pmod{13}; \\ \nu_3(n+1)+2, & \text{if } n \equiv 12 \pmod{13}; \\ 4, & \text{if } n \equiv 9 \pmod{39}; \\ \nu_3(n+17)+4, & \text{if } n \equiv 22 \pmod{39}; \\ \nu_3(n+4)+4, & \text{if } n \equiv 35 \pmod{39}. \end{cases}$$

2.1 The proof of Theorem 1

Assume $k \geq 1$, $n \geq 3$ and

$$T_n = m_1! \cdots m_k!.$$

We may assume that $2 \leq m_1 \leq \cdots \leq m_k$. It is well-known that letting α be real root of $x^3 - x^2 - x - 1 = 0$, then $\alpha > 1.83$ and $T_n \geq \alpha^{n-2}$ holds for all $n \geq 1$. Thus,

$$\alpha^{n-2} \leq T_n = m_1! \cdots m_k! < m_1^{m_1} \cdots m_k^{m_k},$$

so

$$n < 2 + \frac{m_1 \log m_1 + \cdots + m_k \log m_k}{\log(1.83)}. \tag{1}$$

Note that

$$\nu_2(m!) = \lfloor m/2 \rfloor + \lfloor m/4 \rfloor + \cdots \geq m/2$$

holds for all $m \geq 2$, except when $m = 3$, when $\nu_2(3!) = (3-1)/2 \geq 3/3$. Hence, the inequality $\nu_2(m!) \geq m/3$ holds for all $m \geq 2$. Next, by Theorem 3, we see that

$$\begin{aligned} \nu_2(T_n) &\leq \max\{2, \nu_2(n) - 1, \nu_2(n+4) - 1, \nu_2(n+1) + 1, \nu_2(n+17) + 1\} \\ &\leq \frac{\log(n+17)}{\log 2} + 1. \end{aligned}$$

This gives

$$\frac{\log(n+17)}{\log 2} + 1 \geq \nu_2(T_n) = \nu_2(m_1!) + \cdots + \nu_2(m_k!).$$

Thus, putting

$$x := \nu_2(m_1!) + \cdots + \nu_2(m_k!),$$

we have

$$n > 2^{x-1} - 17, \tag{2}$$

and since $m_i \le 3\nu_2(m_i!)$ for $i = 1, \ldots, k$, we have

$$\begin{aligned} m_1 \log m_1 + \cdots + m_k \log m_k &< (m_1 + \cdots + m_k) \log(m_1 + \cdots + m_k) \\ &\le 3x \log(3x), \end{aligned}$$

so (1) yields

$$n < 2 + \frac{3x \log(3x)}{\log(1.83)}. \tag{3}$$

Combining (2) and (3), we get

$$2^{x-1} - 17 < 2 + \frac{3x \log(3x)}{\log(1.83)},$$

giving $x \le 8$. Since $\nu_2(11!) = 8$, we conclude that either $T_n = 11!$ and $k = 1$, or $m_k \le 10$, in which case $P(T_n) \le 7$, where $P(m)$ is the largest prime factor of m. Since $T_n = 11!$ is not possible, we conclude that $P(T_n) \le 7$. All Tribonacci numbers whose largest prime factor is at most 7 have been determined in [2] and they are $T_1 = T_2 = 1,\ T_2 = 2,\ T_3 = 4,\ T_5 = 7,\ T_7 = 2^3 \cdot 3,\ T_9 = 3^4,\ T_{12} = 2^3 \cdot 3^2 \cdot 7,\ T_{15} = 2^6 \cdot 7^2$, and the only numbers from the previous list that are products of factorials correspond to $n \in \{1, 2, 3, 4, 7\}$.

3 The proof of Theorem 2

This is trickier, since the lower bounds on $|T_{-n}|$ are quite weak. However, we can follow the arguments from [3] and [4]. Namely, we put $\Lambda := \{\alpha, \beta, \gamma\}$ for the set of roots of $P(X) = X^3 - X^2 - X - 1$. We assume $\beta = \alpha^{-1/2} e^{i\theta}$ where $\theta \in (0, \pi)$ and $\gamma = \alpha^{-1/2} e^{-i\theta}$. So, β is the complex nonreal root of $P(X)$ in the upper half-plane. Then

$$T_n = c_\alpha \alpha^n + c_\beta \beta^n + c_\gamma \gamma^n \qquad \text{holds for all} \quad n \in \mathbb{Z}.$$

Here

$$c_\lambda = \frac{\lambda}{P'(\lambda)} \quad \text{for} \quad \lambda \in \Lambda.$$

The Tribonacci sequence $\{T_n\}_{n \in \mathbb{Z}}$ is periodic modulo 2^k with period 2^{k+2} for all $k \ge 1$. To see this, it is easier to work with

$$22T_n = (22c_\alpha)\alpha^n + (22c_\beta)\beta^n + (22c_\gamma)\gamma^n.$$

The numbers c_λ are not algebraic integers, but $22c_\lambda$ are algebraic integers of minimal polynomial $X^3 - 22X - 242$ for $\lambda \in \Lambda$. Since $X^3 - X^2 - X - 1$ divides $X^4 - 2X + 1$, it follows that $\lambda^4 \equiv 1 \pmod 2$ for $\lambda \in \Lambda$. Here, for algebraic integers γ, δ and an integer $m \ge 1$, we write $\gamma \equiv \delta \pmod m$ if $(\gamma - \delta)/m$ is an algebraic integer. By induction we get $\lambda^{2^{k+1}} \equiv 1 \pmod{2^k}$ for all $k \ge 1$. In particular, for all $n \in \mathbb{Z}$, we have

$$22T_{n+2^{k+1}} = \sum_{\lambda \in \Lambda} (22c_\lambda)\lambda^{n+2^{k+1}} \equiv \sum_{\lambda \in \Lambda} (22c_\lambda)\lambda^n \equiv 22T_n \pmod{2^k}.$$

The above congruence implies that $T_{n+2^{k+1}} \equiv T_n \pmod{2^{k-1}}$ holds for all $k \geq 2$. Hence, $\{T_n\}_{n\in\mathbb{Z}}$ is periodic modulo 2^k with period 2^{k+2} for all $k \geq 1$. In particular, the Marques–Lengyel formulas from Theorem 3 hold for all integers n, not only for the positive ones. Let us see why.

Assume $n \neq 0, 1, 4, 17$, since for these values of b we have $T_{-n} = 0$, so the formula from Theorem 3 holds with both sides equal ∞. Let k be large ($k \geq n + 10$). Then $T_{-n} \equiv T_{2^k-n} \pmod{2^{k-2}}$ and $2^k - n$ is positive. Unless $-n \equiv 0, 1, 4, 17 \pmod{16}$, the formulas from Theorem 3 show that $\nu_2(2^k-n) \in \{0, 1, 2\}$ and $16 \mid 2^k$. It then follows that $\nu_2(T_{2^k-n}) = 0, 1, 2$ and since $T_{2^k-n} \equiv T_{-n} \pmod{2^{k-2}}$, it follows that $\nu_2(T_{-n}) = \nu_2(T_{2^k-n})$. Hence, the formula from Theorem 3 holds for the residue classes for $-n$ modulo 16 that are not one of $0, 1, 4, 17$. For the rest of the residue classes, $\nu_2(T_{2^k-n}) = \nu_2(2^k - n + a) + b$ for some $a \in \{0, 1, 4, 17\}$ and $b \in \{-1, 1\}$. But

$$\nu_2(2^k - n + a)) = \nu_2(2^k - (n - a)) = \nu_2(n - a) \quad \text{for large} \quad k \quad \text{since} \quad n - a \neq 0.$$

Since $T_{-n} \equiv T_{2^k-n} \pmod{2^{k-2}}$ and $\nu_2(T_{-n}) = \nu_2(T_{2^k-n}) = \nu_2(-n + a) + b$, the formula from Theorem 3 holds for $-n$ that are congruent to one of $\{0, 1, 4, 17\}$ modulo 16. Similar (and in fact easier) considerations modulo $13 \cdot 3^k$ show that the sequence $\{T_n\}_{n\in\mathbb{Z}}$ is periodic modulo 3^{k+1} with period $13 \cdot 3^k$ and that the formula from Theorem 4 holds for all integers n not only for the positive ones.

We next assume that $n \geq 18$. Then [3, Inequality (2.3)] (also see [4, Inequality (4.2)]) shows that

$$T_{-n} > \alpha^{n/2-1.2\times 10^{16}\log n}.$$

The condition $n \geq 18$ is needed since $T_{-17} = 0$. Thus, the analogue of inequality (1) when $|T_{-n}| = m_1! \cdots m_k!$ for $n \geq 18$ is

$$n/2 - 1.2 \times 10^{16} \log n < \frac{m_1 \log m_1 + \cdots + m_k \log m_k}{\log(1.83)}. \tag{4}$$

Further, using that Theorem 3 holds for $n \in \mathbb{Z}$, we get that

$$\begin{aligned}\nu_2(T_{-n}) &\leq \max\{2, \nu_2(n) - 1,\ \nu_2(n - 4) - 1, \nu_2(n - 1) + 3, \nu_2(n - 17) + 3\}\\ &\leq \frac{\log(n - 1)}{\log 2} + 3,\end{aligned}$$

so the analogue of inequality (2) is

$$n \geq 2^{x-3} + 1. \tag{5}$$

The function $y \mapsto y/2 - 1.2 \times 10^{16} \log y$ is increasing for $y > 2.4 \times 10^{16}$, so if $x \geq 58$, then $2^{x-3} + 1 \geq 2^{55} + 1 > 2.4 \times 10^{16}$, so

$$n/2 - 1.2 \times 10^{16} \log n \geq 2^{x-4} + 0.5 - 1.2 \times 10^{16}(x - 3) \log 2. \tag{6}$$

We thus get

$$2^{x-4} + 0.5 - 1.2 \times 10^{16}(x-3)\log 2 < \frac{3x\log(3x)}{\log(1.83)},$$

which gives $x \le 62$. Hence, $x \le 62$, so $m_1 + m_2 + \cdots + m_k \le 3x \le 186$. Thus,

$$m_1!\, m_2! \, \cdots \, m_k! \mid (m_1 + \cdots + m_k)! \mid 186!.$$

Further, from (4) we get

$$n/2 - 1.2 \times 10^{16} \log n < \frac{3x\log(3x)}{\log(1.83)} < \frac{186 \log 186}{\log(1.83)},$$

so $n < 10^{18}$. We are ready to do some Baker-Davenport reduction. This has been explained and used in many places. For our application, we refer the reader to [4, Lemma 5.1] and [3, Eqs. (3.4), (3.5)]. We write

$$c_\beta \beta^{-n} + c_\gamma \gamma^{-n} = \pm m_1! \cdots m_k! - c_\alpha \alpha^{-n}.$$

Thus, assuming $n > 6000$, we get

$$\left| \left(-\frac{c_\gamma}{c_\beta}\right)\left(\frac{\beta}{\gamma}\right)^n - 1 \right| = \frac{|\pm m_1! \cdots m_k! - c_\alpha \alpha^{-n}|}{\alpha^{n/2}} < \frac{186! + 1}{\alpha^{n/2}} < \frac{1}{\alpha^{n/4}}.$$

The last inequality holds for $n > 6000$. Writing $\beta/\gamma = e^{2i\theta}$ and also $-c_\gamma/c_\beta = -e^{2i\omega}$ for some $\omega \in (0, 2\pi)$, we get

$$|e^{i(2n\theta+\pi-2\omega)} - 1| < \frac{1}{\alpha^{n/4}}.$$

The argument from [3] ((3.2)–(3.4)) shows that if $l := \lfloor (2n\theta + \pi - 2\omega)/\pi \rceil$, then the left-hand side above is at least

$$2\left| n\left(\frac{2\theta}{\pi}\right) - (l-1) - \frac{2\omega}{\pi} \right|.$$

Thus, we get

$$\left| n\left(\frac{2\theta}{\pi}\right) - (l-1) - \frac{2\omega}{\pi} \right| < \frac{2}{\alpha^{n/4}}.$$

The left-hand side above is $n\tau - m + \mu$, where $\tau = 2\theta/\pi$, $\mu = -2\omega/\pi$, $m = l - 1$ and the right-hand side above is of the form A/B^k, with the parameters $A = 2$, $B = \alpha$, $k = n/4$. The continued fraction of τ starts as follows:

$$[1, 2, 1, 1, 2, 6, 1, 5, 1, 1, 1, 11, 25, 2, 21, 1, 2, 1, 5, 4, 60, 8, 2, 1, 2, 8, 2, 1, 1, 60,$$
$$1, 5, 3, 1, 4, 29, 2, 24, 19, 1, \ldots]$$

We take $M := 10^{18}$, which is an upper bound on n. We take the 39th convergent p_{39}/q_{39} and with $q := q_{39}$, calculate $\varepsilon := \|q\mu\| - M\|q\tau\| > 0.4$, and we get by the Baker–Davenport reduction method that

$$n/4 = k < \frac{\log(Aq/\varepsilon)}{\log B} < \frac{\log(2q_{39}/0.4)}{\log(1.83)},$$

which gives $n \leq 361$, contradicting our assumption that $n > 6000$. So, $n \leq 6000$.

Now inequality (5) gives $x \leq 16$. Since $\nu_2(20!) = 17 > x$, it follows that $m_k \leq 19$. But we can do better. Assume that $m_k \geq 12$. Then $\nu_2(T_{-n}) \geq \nu_2(12!) = 11$ and $\nu_3(T_{-n}) \geq \nu_3(12!) \geq 5$. Theorems 3 and 4 show that n is congruent to one of $0, 1, 4, 17$ modulo 2^{10} and also modulo $13 \cdot 3$. Solving the above 4^2 possibilities with the Chinese remainder lemma we get that the only possibility for which $n < 6000$ is $n \equiv 4096 \pmod{2^{10} \cdot 3 \cdot 13}$, so $n = 4096$, but $\nu_3(T_{-4096}) = 4$ (not 5). This shows that $m_k \leq 11$.

Finally, let us note that $\nu_3(T_{-n}) \leq 9$. Indeed, if $\nu_3(T_{-n}) \geq 10$, then Theorem 4 shows that $n \equiv 0, 1, 4, 17 \pmod{13 \cdot 3^6}$ and $13 \cdot 3^6 > 6000$ (certainly n cannot be one of $0, 1, 4, 17$ since then $T_{-n} = 0$). So,

$$m_1! \cdots m_k! = 2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdot 11^e,$$

where $a \leq 16$, $b \leq 9$, $c \leq 5$, $d \leq 4$, $e \leq 2$. To see the upper bounds on the exponents of the primes larger than 3 above, note that if $c \geq 6$, then $5!^6 \mid m_1! \cdots m_k!$, which makes $\nu_2(T_{-n}) \geq 18$, a contradiction. The rest are proved in the same way. We created the list of all the numbers of the above form and intersected it with the list of absolute values $\{|T_{-n}|\}_{1 \leq n \leq 6000}$, obtaining some values in the intersection with the largest index $n = 33$ for which $|T_{-33}| = 2^6 \cdot 7^2$. From here, we recovered the solutions n for which $|T_{-n}|$ is a product of factorials namely $T_{-2} = 1$, $T_{-3} = -1$, $T_{-5} = 2$, $T_{-7} = 1$, $T_{-8} = 2^2$, $T_{-9} = -2^3$. This finishes the proof.

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