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Arithmetic Functions that Remain Constant on Runs of Consecutive Integers

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Abstract

We bound from above the length of the longest sequence of consecutive numbers less than or equal to x with the same number of divisors. We also bound the length of the longest sequence of consecutive numbers less than or equal to x for which the number of divisors is decreasing. Finally, we consider variants of this problem such as the corresponding sequences for the sum-of-proper-divisors function and the Carmichael function. In particular, we show that it is impossible for the sum-of-proper-divisors function to be equal on six consecutive integers.

1 Introduction

Let $d(n)$ be the number of divisors of n. In 1952, Erdős and Mirsky [4] defined the function $F(x)$ as the largest number k for which there exists some $n \leq x - k$ such that $d(n+1) = d(n+2) = \cdots = d(n+k)$. Unfortunately, they did not obtain any non-trivial bounds.

Over the years, multiple people have investigated whether $d(n) = d(n+k)$ for infinitely many values of n for a given k. Spiro [18] proved this statement for $k = 5040$. Heath-Brown [11] then solved the $k = 1$ case. Finally, Pinner [15] showed that $d(n) = d(n+k)$ has infinitely many solutions for all k. (For further discussion of this problem, see Guy [9, §B18]. For bounds on the number of solutions to the equation $d(n) = d(n+1)$ with $n \leq x$, see Erdős, Pomerance, and Sárközy [5] and Hildebrand [12].)

Schinzel's hypothesis H implies that $d(n+1) = d(n+2) = \cdots = d(n+k)$ has infinitely many solutions for all k. However, there is no unconditional proof of this result. Last year, Letsko found a solution to this equation with $k = 20$, which implies that $F(x) \geq 20$ for all sufficiently large x. (For examples of runs of integers with the same number of divisors, see A006558.)

Spătaru [17] recently made the first substantial improvement on Erdős and Mirsky's original question. (From here on, $\log_k$ is the kth iterate of the logarithm.)

Theorem 1. *As $x \to \infty$, we have*

$$F(x) = \exp(O(\sqrt[3]{\log x \log_2 x})).$$

In this note, we derive an alternate proof of Theorem 1, which we independently obtained before Spătaru [17] appeared in print. Later on, we derive a substantially stronger conditional bound.

We also consider a related function. Let $G(x)$ be the largest k for which the inequality $d(n+1) \geq d(n+2) \geq \cdots \geq d(n+k)$ holds for some $n \leq x - k$. (The bounds we obtain also hold for increasing, decreasing, and non-decreasing sequences.) We modify Spătaru's proof to bound $G(x)$ as well.

Theorem 2. *We have*

$$G(x) = \exp(O(\sqrt{\log x \log_2 x})).$$

We can generalize these functions. For a given arithmetic function f, let $F_f(x)$ (resp., $G_f(x)$) be the largest k for which $f(n+1), f(n+2), \ldots, f(n+k)$ is constant (resp., decreasing). Note that $F_f(x) \leq G_f(x)$ for all f, x. Pollack, Pomerance, and Treviño [16] showed that $F_\varphi(x) \sim \log_3 x / \log_6 x$, where φ is the totient function. (For recent research on the equation $\varphi(n) = \varphi(n+1)$, see Bayless and Kinlaw [2] or Kinlaw, Kobayashi, and Pomerance [13]. Though Erdős [3] conjectured that $\varphi(n+1) = \varphi(n+2) = \cdots = \varphi(n+k)$ has infinitely many solutions for all k, the only known solution to $\varphi(n+1) = \varphi(n+2) = \varphi(n+3)$ is $n = 5185$ [14]. For the non-consecutive case, see Tao [19].)

Let $\omega(n)$ (resp., $\Omega(n)$) be the number of distinct (resp., not necessarily distinct) prime factors of n. Erdős, Pomerance, and Sárközy [6, Thm. 5] proved that

$$F_\omega(x) \leq \exp((1/\sqrt{2} + o(1))\sqrt{\log x \log_2 x})$$

and

$$F_\Omega(x) \leq \exp((\sqrt{\log 2} + o(1))\sqrt{\log x})$$

using a variation of the arguments that we use here. They also bounded the largest k for which there exists a number $n \leq x - k$ such that $\omega(n+1), , \ldots, \omega(n+k)$ (resp., $\Omega(n+1), \ldots, \Omega(n+k)$) are all distinct.

Erdős [3] also conjectured that $F_\sigma(x) \to \infty$ as $x \to \infty$ as well, where σ is the sum-of-divisors function. A slight modification to Pollack et al.'s argument implies that $F_\sigma(x) \ll$

$\log_3 x/\log_6 x$. Weingartner [21] proved that $\sigma(n) = \sigma(n+k)$ has infinitely many solutions for all even k, conditional on Schinzel's hypothesis H. He also showed that if $k \leq 10^{10^7}$ is even, then $\sigma(n) = \sigma(n+k)$ has infinitely many solutions unconditionally. (For further discussion of the equation $\sigma(n) = \sigma(n+1)$, see Guy [9, §B13].) By modifying our proof of Theorem 1, we derive an alternate proof of the following result.

Theorem 3. *We have*

$$F_\sigma(x) = \exp(O(\sqrt{\log x \log_2 x})).$$

Let s be the sum-of-proper-divisors function. Surprisingly, we can prove a much stronger upper bound on $F_s(x)$.

Theorem 4. *For all x, we have $F_s(x) \leq 5$. In other words, the equation $s(n+1) = s(n+2) = \cdots = s(n+6)$ has no solutions.*

Though we are unaware of any papers on the equation $s(n) = s(n+1)$, there is a MathOverflow post [8] on this problem. Frank asked whether $n = 2$ is the only solution to $s(n) = s(n+1)$. Poonen then responded that there are no other solutions up to 10^{16}. (Note that $s(n) = s(n+1)$ is equivalent to the statement $\sigma(n) + \sigma(1) = \sigma(n+1)$. Guy [9, §B15] discusses the more general equation $\sigma(m) + \sigma(n) = \sigma(m+n)$.)

Finally, we prove the following upper bound for the Carmichael λ function. We define $\lambda(n)$ as the smallest integer m such that $a^m \equiv 1 \pmod{n}$ for all a coprime to n.

Theorem 5. *We have $F_\lambda(x) \ll (\log x \log_2 x)^2$.*

2 The divisor function

In this section, we prove our main result. From here on, we let $v_p(m)$ be the order of p in m, i.e., the largest a for which $p^a | m$. For a given x, let $k = F(x)$ and $K = \log k/\log 2$. By assumption, there exists some $n \leq x - k$ such that $d(n+1) = d(n+2) = \cdots = d(n+k)$. Let $D = d(n+i)$ for all $i \leq k$. Spătaru [17] proved the following result.

Lemma 6. *The number D is a multiple of every prime $\leq K$.*

Using the prime factorization of D, we can bound $F(x)$ from above.

Proof of Theorem 1. Recall that we wish to show that as $x \to \infty$, we have

$$F(x) = \exp(O(\sqrt[3]{\log x \log_2 x})).$$

We may assume that $k > \exp(C\sqrt[3]{\log x \log_2 x})$ for some positive constant C.

Fix $\epsilon < 1 - (\log 2/\log 3)$ and let $\mathcal{S}$ be the set of primes in $((1-\epsilon)K, K]$. Let q_1, q_2, and q_3 be three distinct elements of $\mathcal{S}$. For all i, we have $q_1 q_2 q_3 | d(n+i)$ by Lemma 6. Because $d(p_1^{a_1} \cdots p_r^{a_r}) = (a_1+1)\cdots(a_r+1)$, there are three possibilities:

1. We have $v_p(n+i) = aq_1q_2q_3 - 1$ for some prime p and positive a,
2. We have $v_{p_1}(n+i) = a_1q_1q_2 - 1$ and $v_{p_2}(n+i) = a_2q_3 - 1$ for some prime p_1, p_2 and positive a_1, a_2,
3. We have $v_{p_j}(n+i) = a_jq_j - 1$ for all $j \le 3$, where each p_j is prime and each a_j is positive.

Suppose $n+i$ satisfies Condition (1) for some $q_1, q_2, q_3 \in \mathcal{S}$. Then, we can bound p from above. In this case, we have

$$\begin{aligned} x &\ge p^{aq_1q_2q_3-1} \\ &\ge 2^{q_1q_2q_3-1} \\ &\ge 2^{(1-\epsilon)^3K^3-1} \\ &\ge 2^{((1-\epsilon)C/\log 2)^3 \log x \log_2 x - 1} \\ &\ge x^{(1+o(1))((1-\epsilon)C)^3/(\log 2)^2 \log_2 x}. \end{aligned}$$

However, this is impossible for x sufficiently large. Therefore, $n+i$ does not satisfy (1) for $q_1, q_2, q_3 \in \mathcal{S}$.

Let $r = \#\mathcal{S}$. Fix $i \le k$. Because $n+i$ does not satisfy (1), it is possible to partition $\mathcal{S}$ into two disjoint subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ with the following properties. Let $\mathcal{S}_1 = \{q_1, q_2, \ldots, q_{\#\mathcal{S}_1}\}$ and $\mathcal{S}_2 = \{q_{\#\mathcal{S}_1+1}, \ldots, q_r\}$. We may write

$$\prod_{j=1}^{\lfloor \#\mathcal{S}_1/2 \rfloor} p_j^{a_jq_{2j-1}q_{2j}-1} \prod_{j=\#\mathcal{S}_1+1}^{r} p_j^{a_jq_j-1} | n+i,$$

where each p_j is prime and each a_j is a positive integer. Note that each exponent is $\ge \lfloor (1-\epsilon)K \rfloor - 1$ and that there are at least $r/2$ distinct prime factors in the product above. Hence, $n+i \ge (p_{i,1}p_{i,2}\cdots p_{i,\lfloor r/2 \rfloor})^{\lfloor (1-\epsilon)K \rfloor - 1}$ for some primes $p_{i,1}, p_{i,2}, \ldots, p_{i,\lfloor r/2 \rfloor}$.

From here on, we assume that $n+i$ and $n+j$ are both odd. We can show that $p_{i,a} \ne p_{j,b}$ for all a and b. For every odd prime p, we have

$$p^{\lfloor (1-\epsilon)K \rfloor - 1} \ge 3^{(((1-\epsilon)/\log 2)\log k) - 2} = (1/9)k^{(1-\epsilon)(\log 3)/\log 2}.$$

However, this quantity is greater than k when k is sufficiently large. So, $p^{\lfloor (1-\epsilon)K \rfloor - 1}$ cannot divide both $n+i$ and $n+j$ for distinct i and j.

If we multiply all of these $n+i$ together, we obtain a multiple of

$$\left(\prod_{\substack{i: n+i \text{ is odd} \\ m \le r/2}} p_{i,m} \right)^{\lfloor (1-\epsilon)K \rfloor - 1}.$$

Let p_m be the mth prime and let

$$\theta(x) = \sum_{p \le x} \log p.$$

De la Vallée Poussin [20, p. 54] showed that $|\pi(x) - \mathrm{Li}(x)|$ and $|\theta(x) - x|$ are both $O(x \exp(-c\sqrt{\log x}))$ for some positive constant c. Therefore,

$$x^k > (n+1)(n+2)\cdots(n+k) \ge (p_1 p_2 \cdots p_{C_1 rk})^{\lfloor (1-\epsilon)K \rfloor - 1} = \exp\left((C_2 + o(1)) rkK \log(rk)\right)$$

for some positive constants C_1, C_2. So, we have

$$rkK \log(rk) \ll \log(x^k) = k \log x,$$

which implies that

$$rK \log(rk) \ll \log x.$$

Recall that $K \asymp \log k$ and $r \asymp \log k / \log\log k$. Therefore, we have

$$k = \exp(O(\sqrt[3]{\log x \log\log x})). \qquad \square$$

3 Decreasing sequences

Rather than considering sequences for which $d(n+i)$ is constant, we can consider sequences for which it is increasing or decreasing. Pollack et al. found a precise asymptotic formula for the analogous problem for the totient function.

Theorem 7 ([16, Thm. 1.5])**.** *The largest k for which there exists some $n \le x - k$ such that $\varphi(n+1) \ge \varphi(n+2) \ge \cdots \ge \varphi(n+k)$ is asymptotic to $\log_3 x / \log_6 x$.*

We modify their argument to obtain an upper bound for the divisor function. To do so, we make use of the following result of Spătaru.

Lemma 8 ([17, Lemma 4.2])**.** *Let n be an integer with smallest prime factor p. Then,*

$$\frac{\log n}{\log p} \ge \sum_q (q-1) v_q(d(n)).$$

Rather than using this lemma directly, we rewrite the righthand side using the following result.

Corollary 9. *For all m, we have*

$$\sum_q (q-1) v_q(m) \ge \frac{\log m}{\log 2}.$$

Proof. By definition, m is the product of $q^{v_q(m)}$ over all q. Therefore, we have

$$\log m = \sum_q (\log q) v_q(m),$$

which implies that

$$\frac{1}{\log m} \sum_q (q-1) v_q(m) = \left(\sum_q (q-1) v_q(m) \right) \left(\sum_q (\log q) v_q(m) \right)^{-1}.$$

Let

$$M = \min_{q|m} \left(\frac{q-1}{\log q} \right).$$

Then, we have

$$\sum_q (q-1) v_q(m) \geq \sum_q M (\log q) v_q(m) = M \sum_q (\log q) v_q(m),$$

giving us

$$\frac{1}{\log m} \sum_q (q-1) v_q(m) \geq M \geq \frac{1}{\log 2}.$$ □

Recall that $G(x)$ is the largest k for which there exists some $n \leq x-k$ such that $d(n+1) \geq d(n+2) \geq \cdots \geq d(n+k)$.

Proof of Theorem 2. A classic theorem [10, Thm. 317] states that there exists some $m \leq k/2$ such that

$$d(m) = 2^{(1+o(1)) \log(k/2)/\log_2(k/2)} = 2^{(1+o(1)) \log k/\log_2 k}.$$

Let $n+k = am+b$ with $0 < b < m$. Note that $k-b > k/2$ because $m \leq k/2$. Consider the subsequence $n+1, n+2, \ldots, n+(k-b)$. By assumption, $m | n+(k-b)$. So $d(m) \leq d(n+k-b)$, which implies that $d(m) \leq d(n+i)$ for all $i \leq k-b$. So,

$$d(n+i) \geq \exp\left((\log 2 + o(1)) \frac{\log k}{\log_2 k} \right)$$

for all $i \leq k-b$.

Spătaru [17, §4] also observed that in a sequence of $k-b$ consecutive numbers, at least one of those numbers consists entirely of prime factors p satisfying $\log p \gg \log(k-b) \gg \log k$. Select $i \leq k-b$ so that $n+i$ satisfies this property. The previous lemma and corollary imply that

$$\frac{\log x}{\log k} \geq \frac{\log(n+i)}{\log k} \gg \sum_q (q-1) v_q(d(n+i)) \gg \log d(n+i) \gg \frac{\log k}{\log_2 k}.$$

From this inequality, we obtain $(\log k)^2/\log_2 k \ll \log x$, giving us our desired bound. □

The function $G(x)$ has a much stronger conditional bound. Suppose the sequence $n+1, n+2, \ldots, n+k$ contains two primes $n+i, n+j > 3$. Then, $d(n+i) = d(n+j) = 2$, but $d(n+k) > 2$ for every $k \in (i,j)$ with $n+k$ composite. Therefore, $G(x)$ is at most as large as the largest gap between two consecutive primes $\leq x$. Cramér's conjecture states that this gap is $O((\log x)^2)$. Assuming this conjecture, $G(x) = O((\log x)^2)$ as well. (Unfortunately, the best unconditional upper bound on the gap between two consecutive primes $\leq x$ is $x^{0.525+o(1)}$ [1]. The best lower bound is $\gg \log x \log_2 x \log_4 x / \log_3 x$ [7].)

4 Sums of divisors

Let $F_\sigma(x)$ be the largest k for which there exists some $n \leq x-k$ such that $\sigma(n+1) = \sigma(n+2) = \cdots = \sigma(n+k)$. By modifying the techniques of Section 2, we bound $F_\sigma(x)$ from above. Let $T = \sigma(n+1) = \cdots = \sigma(n+k)$. Once again, we let $K = \log k / \log 2$.

Lemma 10. *We have*

$$T \geq \prod_{p \leq K} (2^p - 1).$$

Proof. Let $p \leq K$. Because $2^p \leq k$, there exists some $i \leq k$ such that $n+i \equiv 2^{p-1} \pmod{2^p}$. For this particular i, we have $\sigma(2^{p-1}) = 2^p - 1 | \sigma(n+i)$. Therefore, $2^p - 1 | T$ as well. In particular, $2^p - 1 | T$ for all $p \leq K$. However, if p and q are distinct primes, then $2^p - 1$ and $2^q - 1$ are relatively prime. Therefore, T is a multiple of

$$\prod_{p \leq K} (2^p - 1).$$

□

Proof of Theorem 3. By the previous lemma, there exists a constant C such that

$$T \geq \prod_{p \leq K} C^p = \exp\left((\log C) \sum_{p \leq K} p \right) = \exp\left((1+o(1)) \frac{\log C}{2} \frac{K^2}{\log K} \right).$$

However, Mertens' theorem implies that $T = \sigma(n+i) \ll x \log\log x$. Hence,

$$\frac{K^2}{\log K} \ll \log T \ll \log x,$$

which implies that

$$\log k \ll K \ll \sqrt{\log x \log_2 x}.$$

□

If we replace σ with the sum-of-proper-divisors function $s(n)$, we get a completely different result. To prove this result, we use the following fact about σ.

Lemma 11. *The quantity $\sigma(n)$ is odd if and only if $n = m^2$ or $n = 2m^2$ for some m.*

Proof. Suppose $\sigma(n)$ is odd. By definition,

$$\sigma(n) = \prod_p (1 + p + p^2 + \cdots + p^{v_p(n)}).$$

] If p and $v_p(n)$ are both odd, then $1 + p + p^2 + \cdots + p^{v_p(n)}$ is even because it is the sum of an even number of odd terms. Because $\sigma(n)$ is odd, $v_p(n)$ must be even for all odd p. So, $n = m^2$ or $2m^2$, depending on the parity of $v_2(n)$.

We now prove the converse. If $v_p(n)$ is even for all odd p, then

$$\prod_{p>2} (1 + p + \cdots + p^{v_p(n)})$$

is odd. In addition, $1+2+\cdots+2^{v_2(n)}$ is always odd because every term except 1 is even. □

Proof of Theorem 4. Suppose that $s(n+1) = s(n+2) = \cdots = s(n+6)$ for some integer n. Let $S = s(n+i)$ for all $i \leq 6$. Suppose S is even. Choose $i \leq 2$ so that $n+i$ is odd. Because $s(n+i)$ is even, we have that $\sigma(n+i)$ is odd, which implies that $n+i$ is a square. By a similar argument, the number $n+i+2$ is also a square, which is impossible. Therefore, S is odd.

The sequence $n+1, n+2, \ldots, n+6$ contains three even numbers m, $m+2$, and $m+4$. Because $s(m)$, $s(m+2)$, and $s(m+4)$ are odd, the numbers $\sigma(m)$, $\sigma(m+2)$, and $\sigma(m+4)$ must all be odd as well. There are two possibilities. Either two elements of $\{m, m+2, m+4\}$ are squares or two of them are double a square. However, the difference between two positive squares cannot be 2 or 4. If two elements of $\{m, m+2, m+4\}$ have the form $2r^2$ for some r, then two elements of $\{m/2, (m/2)+1, (m/2)+2\}$ are squares. But the difference between two positive squares cannot be 1 or 2, giving us a contradiction. □

5 The Carmichael function

Let λ be the Carmichael function. Though λ is an arithmetic function, it is neither additive nor multiplicative. Instead, for a given number $n = p_1^{a_1} \cdots p_r^{a_r}$, we have

$$\lambda(n) = \mathrm{lcm}(\lambda(p_1^{a_1}), \ldots, \lambda(p_r^{a_r})).$$

Let p^a be a prime power. Then,

$$\lambda(p^a) = \begin{cases} \varphi(p^a)/2, & \text{if } p = 2 \text{ and } a \geq 3; \\ \varphi(p^a), & \text{otherwise.} \end{cases}$$

We prove an upper bound on F_λ in a matter analogous to our previous functions.

Proof of Theorem 5. Let $L = \lambda(n+1) = \cdots = \lambda(n+k)$. We bound L from above. For each prime p, there exists an $i \le k$ such that $p^{\lfloor \log k/\log p \rfloor} | n+i$. For this i, we have

$$p^{\lfloor \log k/\log p \rfloor - 1} | \lambda(n+i) = L.$$

Hence, we have

$$L \ge \frac{1}{2} \prod_{p \le \sqrt{k}} p^{\lfloor \frac{\log k}{\log p} \rfloor - 1} \gg \prod_{p \le \sqrt{k}} p^{\frac{\log k}{\log p} - 2} = \prod_{p \le \sqrt{k}} \frac{k}{p^2}.$$

We now bound the numerator and denominator in our product. We have

$$\prod_{p \le \sqrt{k}} k = k^{\pi(\sqrt{k})} = \exp((\log k)\pi(\sqrt{k})) = \exp\left(2\sqrt{k} + (4 + o(1))\frac{\sqrt{k}}{\log k}\right)$$

and

$$\prod_{p \le \sqrt{k}} p^2 = \exp\left(2 \sum_{p \le \sqrt{k}} \log p\right) \le \exp\left(2\sqrt{k} + o\left(\frac{\sqrt{k}}{\log k}\right)\right).$$

Therefore, we have

$$L \ge \exp\left((4 + o(1))\frac{\sqrt{k}}{\log k}\right).$$

However, we have $L \le x$ because $\lambda(n+i) \le n+i$ for all i. So, $\sqrt{k}/\log k \ll \log x$, which implies that $k \ll (\log x \log_2 x)^2$. $\square$

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