Journal of Integer Sequences, Vol. 26 (2023),
Article 23.7.5

Sequences Derived from the Symmetric Powers of $\{1, 2, \ldots, k\}$

Po-Yi Huang and Wen-Fong Ke
Department of Mathematics
National Cheng Kung University
Tainan 701
Taiwan
pyhuang@ncku.edu.tw
wfke@mail.ncku.edu.tw

Abstract

For a fixed integer k, we define a sequence $A_k = (a_k(n))_{n\geq 0}$ and a corresponding sparse subsequence S_k using the cardinality of the n-th symmetric power of the set $\{1, 2, \ldots, k\}$. For $k \in \{2, \ldots, 8\}$, we find recursive formulas for S_k, and show that the values $a_k(0)$, $a_k(1)$, and $a_k(3)$ are sufficient for constructing A_k.

1 Introduction

Let C and D be finite subsets of $\mathbb{N}$. We recall that the *symmetric difference* of C and D, denoted by $C \triangle D$, is the union of $C \setminus D$ and $D \setminus C$. If $C = \{c_1, \ldots, c_s\}$ and $D = \{d_1, \ldots, d_t\}$, both nonempty, we set $C \nabla D = c_1D \triangle c_2D \triangle \cdots \triangle c_sD$, where $c_iD = \{c_id_1, \ldots, c_id_t\}$ for all i. For example,

$$\{1, 2, 3\} \nabla \{2, 4\} = \{2, 4\} \triangle \{4, 8\} \triangle \{6, 12\} = \{2, 6, 8, 12\}.$$

When either C or D is empty, we put $C \nabla D = \varnothing$. This binary operation ∇ on the finite subsets of $\mathbb{N}$, called the *symmetric product,* is associative, commutative and distributive over $\triangle$. We let F denote the set of all finite subsets of $\mathbb{N}$. Consider the map that takes the i-th prime p_i to the i-th variable x_i of $X = \{x_1, x_2, \ldots\}$. We can extend it in a natural way to a map that takes the natural number $t = 2^{s_1}3^{s_2}\cdots$ ($s_i \geq 0$ with finitely many nonzero s_i) to the monomial $m_t = x_1^{s_1}x_2^{s_2}\cdots$, then further to a map f that takes the nonempty finite subset $T = \{t_1, \ldots, t_\ell\}$ of $\mathbb{N}$ to the multivariate polynomial $f(T) = m_{t_1} + \cdots + m_{t_\ell}$. Set $f(\varnothing) = 0$. Then $f : F \to \mathbb{Z}_2[X]$ is a ring isomorphism from $(F, \triangle, \nabla)$ to the unique factorization domain $(\mathbb{Z}_2[X], +, \cdot)$ of multivariate polynomials on $X = \{x_i \mid i \in \mathbb{N}\}$ over the Galois field of order two.

For $k \in \mathbb{N}$, set $H_k = \{1, \ldots, k\}$ and define $H_k^{\nabla 0} = \{1\}$, $H_k^{\nabla 1} = H_k$, $H_k^{\nabla 2} = H_k \nabla H_k$, and $H_k^{\nabla n} = H_k^{\nabla(n-1)} \nabla H_k$ for $n > 2$. We call $H_k^{\nabla n}$ the *n-th symmetric power* of H_k. It is clear that $H_k^{\nabla 2} = \{1^2, \ldots, k^2\}$. In general, we have $H_k^{\nabla 2^t} = \left\{1^{2^t}, \ldots, k^{2^t}\right\}$ for $t \in \mathbb{N}$. One easily sees that $H_k^{\nabla(2n)} = (H_k^{\nabla n})^{\nabla 2} = \left\{x^2 \mid x \in H_k^{\nabla n}\right\}$ for all $n \in \mathbb{N}$.

In this paper, we consider the sequences $A_k = (a_k(n))_{n\geq 0}$, where $a_k(n) = |H_k^{\nabla n}|$, the cardinality of the n-th symmetric power of H_k. Thus, for all $k, t \in \mathbb{N}$, we have $a_k(2^t) = |H_k^{\nabla 2^t}| = k$. These sequences arise during the investigation of the minimal distances of certain linear codes (cf. [1, 5, 7, 8]) about which we will not go into detail.

The first few of these sequences can be found in the database *The On-Line Encyclopedia of Integer Sequences (OEIS)* [10]:

- $A_1 = [1, 1, 1, \ldots]$ is A000012;
- $A_2 = [1, 2, 2, 4, 2, 4, 4, 8, 2, 4, 4, 8, 4, 8, 8, 16, 2, \ldots]$ is A001316;
- $A_3 = [1, 3, 3, 9, 3, 9, 9, 27, 3, 9, 9, 27, 9, 27, 27, 81, 3, \ldots]$ is A048883;
- $A_4 = [1, 4, 4, 12, 4, 16, 12, 40, 4, 16, 16, 48, 12, 48, 40, 128, 4, \ldots]$ is A253064.

These appear as the numbers of ON cells at the n-th generation when certain "odd-rule" cellular automata, with rules 001, 003, 013 and 017, on the square grid are started in generation 0 with a single ON cell at the origin (see [9]).

The sequences A_k, $k \geq 5$, are not present in the OEIS database. While it is easy to imagine that $a_k(n)$ could get as large as possible with n when $k \geq 2$, the situation that $a_k(2^t) = k$ for all $t \geq 0$ makes the sequences A_k interesting.

We discuss in this article how one can compute A_k when $2 \leq k \leq 8$. We show that $a_k(n)$ is closely related to the binary expansion of n. Here, if $n = \epsilon_0 + 2\epsilon_1 + \cdots + 2^\ell\epsilon_\ell$, where $\ell \geq 0$, and each $\epsilon_i \in \{0, 1\}$ with $\epsilon_\ell = 1$, then the binary expansion of n is $[\epsilon_\ell\epsilon_{\ell-1}\cdots\epsilon_1\epsilon_0]$, and we write $(n)_2 = [\epsilon_\ell\epsilon_{\ell-1}\cdots\epsilon_1\epsilon_0]$.

In the next section, we make an observation that allows us to restrict our attention to the *sparse subsequences* $S_k = (a_k(2^n - 1))_{n\geq 0}$ when $k \leq 7$. In Section 3, we focus on determining the recursive formula for S_k when $2 \leq k \leq 8$. In the last section, we return to the sequence A_8.

2 An observation

Note that every number in $H_k^{\nabla n}$, $2 \leq k \leq 8$ and $0 \leq n$, is a product of some powers of 2, 3, 5, and 7.

Lemma 1. *Let $k, n \in \mathbb{N}$ with $1 \leq k \leq 7$. If $n = \alpha + \beta \cdot 2^{s+1}$ for some $\alpha, \beta, s \in \mathbb{N}$ with $\alpha < 2^s$, then $|H_k^{\nabla n}| = |H_k^{\nabla \alpha}| \cdot |H_k^{\nabla \beta}|$.*

Proof. We have $H_k^{\nabla n} = H_k^{\nabla \alpha} \nabla H_k^{\nabla(\beta \cdot 2^{s+1})}$. Let $u \in H_k^{\nabla \alpha}$ and $v \in H_k^{\nabla(\beta \cdot 2^{s+1})}$. Then there are $e_i, f_i \in \mathbb{N} \cup \{0\}$, $1 \leq i \leq 4$, such that $u = 2^{e_1}3^{e_2}5^{e_3}7^{e_4}$ and $v = 2^{f_1}3^{f_2}5^{f_3}7^{f_4}$. Since $k \leq 7$ and $\alpha < 2^s$, we have $e_i \leq 2\alpha < 2^{s+1}$ for each i. On the other hand, we have $2^{s+1} \mid f_i$ for each i. Hence, we have $H_k^{\nabla \alpha} \cap H_k^{\nabla(\beta \cdot 2^{s+1})} = \varnothing$. Suppose further that $u' = 2^{e_1'}3^{e_2'}5^{e_3'}7^{e_4'} \in H_k^{\nabla \alpha}$ and $v' = 2^{f_1'}3^{f_2'}5^{f_3'}7^{f_4'} \in H_k^{\nabla(\beta \cdot 2^{s+1})}$ are such that $uv = u'v'$. Then, for each i, we have $e_i + f_i = e_i' + f_i'$, which is equivalent to $f_i' - f_i = e_i - e_i'$. Since $2^{s+1} \mid (f_i' - f_i)$ and $|e_i - e_i'| < 2^{s+1}$, we get $f_i' - f_i = 0$. It follows that $e_i = e_i'$, and so $u = u'$ and $v = v'$. Therefore, we have $|H_k^{\nabla n}| = |H_k^{\nabla \alpha}| \cdot |H_k^{\nabla \beta}|$ as claimed. □

In terms of binary expansions, the numbers α, β, and n in Lemma 1 can be presented as $(\alpha)_2 = [x \cdots y]$, $(\beta)_2 = [b \cdots d]$, and $(n)_2 = [(b \cdots d)(0 \cdots 0)(x \cdots y)]$, where at least one 0 is present in $(0 \cdots 0)$. Thus, in the cases that $k \leq 7$, the lemma implies that if we can compute the sparse subsequence $S_k = (a_k(2^n - 1))_{n \geq 0}$, then we can compute every term $a_k(n)$ of A_k according to the binary expansion of n. For example, if $(n)_2 = [10110111011111]$, that is $n = 11727$, then $|H_k^{\nabla n}| = |H_k^{\nabla 1}| \cdot |H_k^{\nabla 3}| \cdot |H_k^{\nabla 7}| \cdot |H_k^{\nabla 15}|$ for $k \in \{1, 2, \ldots, 7\}$.

In general, Lemma 1 does not hold for $k \geq 8$. For example, we have $\left|H_8^{\nabla 3}\right| = 48$, $|H_8^{\nabla 8}| = |H_8| = 8$, and $|H_8^{\nabla 11}| = 368 \neq 8 \cdot 48$. We will deal with A_8 separately in the last section.

3 The sparse subsequences

In this section, we consider the sparse subsequences $S_k = (a_k(2^n - 1))_{n \geq 0}$ where $2 \leq k \leq 8$. For simplicity, with k fixed, we write $\theta_n = a_k(2^n - 1) = |H_k^{\nabla(2^n - 1)}|$ for $n \geq 0$. We have the following recursive formulas for S_k.

Proposition 2. *For each k, we have $\theta_0 = 1$ and $\theta_1 = k$.*

(1) *If $k = 2$, then $\theta_{n+1} = 2\theta_n$ for $n \geq 0$.*

(2) *If $k = 3$, then $\theta_{n+1} = 3\theta_n$ for $n \geq 0$.*

(3) *If $k = 4$, then $\theta_{n+2} = 2\theta_{n+1} + 4\theta_n$ for $n \geq 0$.*

(4) *If $k = 5$, then $\theta_{n+2} = 3\theta_{n+1} + 6\theta_n$ for $n \geq 0$.*

(5) *If $k = 6$, then $\theta_{n+1} = 5\theta_n$ for $n \geq 1$.*

(6) *If $k = 7$, then $\theta_{n+2} = 6\theta_{n+1} + \theta_n$ for $n \geq 0$.*

(7) *If $k = 8$, then $\theta_{n+3} = 7\theta_{n+2} - 2\theta_{n+1} - 24\theta_n$ for $n \geq 0$ with $\theta_2 = 48$.*

Together with Lemma 1 and the reduction rules to be derived in Section 4 for $k = 8$, this proposition gives us the following corollary.

Corollary 3. *Let $n, k \in \mathbb{N}$.*

(1) *If $2 \le k \le 7$, then $a_k(n)$ can be computed from $a_k(0) = 1$ and $a_k(1) = k$.*

(2) *If $k = 8$, then $a_k(n)$ can be computed from $a_k(0) = 1$, $a_k(1) = 8$, and $a_k(3) = 48$.*

The rest of the section is devoted to showing the validity of the formulas in Proposition 2. We start with S_8 as it provides the best illustration of our approach.

3.1 The case $k = 8$

For $n \ge 0$, the set $H_8^{\nabla(2^n-1)}$ is the disjoint union of sets of "chains". These chains are classified as types A, B, and C. A chain of type A is of the form $\{x, 2x, 4x, \ldots\}$; a chain of type B is of the form $\{x, 4x, 16x, \ldots\}$; and a chain of type C is just a singleton that is not contained in any chain of type A or type B. These chains are uniquely determined via the following procedure. First, collect all possible chains of type A. After removing the elements of these collected chains from $H_8^{\nabla(2^n-1)}$, collect all possible chains of type B from the remaining elements. After removing the elements of the type B chains thus collected, each of the remaining elements is itself a chain of type C. Let $\mathcal{C}_n$ be the collection of the chains of $H_8^{\nabla(2^n-1)}$.

For $n = 0$, we are looking at $H_8^{\nabla(2^0-1)} = H_8^{\nabla 0} = \{1\}$. The chains in $\mathcal{C}_0$ are

- type A: none;
- type B: none;
- type C: $\{1\}$.

For $n = 1$, we are looking at $H_8^{\nabla(2^1-1)} = H_8$. The chains in $\mathcal{C}_1$ are

- type A: $\{1, 2, 4, 8\}$ and $\{3, 6\}$;
- type B: none;
- type C: $\{5\}$ and $\{7\}$.

For $n = 2$, we are looking at

$$H_8^{\nabla(2^2-1)} = H_8^{\nabla 3} = H_8^{\nabla 2} \nabla H_8 = \{1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2\} \nabla \{1, 2, 3, 4, 5, 6, 7, 8\}.$$

The chains in $\mathcal{C}_2$ are

- type A:

$$\begin{array}{ll} \{1,2\}, & \{3,6,12,24,48,96,192,384\}, \\ \{9,18\}, & \{25,50,100,200\}, \\ \{27,54,108,216\}, & \{49,98,196,392\}, \\ \{75,150\}, & \{144,288\}, \\ \{147,294\}, & \{256,512\}; \end{array}$$

- type B:

$$\begin{array}{ll} \{5,20,80,320\}, & \{7,28,112,448\}, \\ \{45,180\}, & \{63,252\}; \end{array}$$

- type C:

$$\{125\},\{175\},\{245\},\{343\}.$$

Before we continue, let us define some numbers. For a fixed $n \geq 0$, we look at $\mathcal{C}_n$, and set the numbers b_n, c_n, u_n, v_n, and r_n as follows:

- b_n is the total number of the elements in the chains of type A;
- c_n is the number of chains of type A;
- u_n is the total number of the elements in the chains of type B;
- v_n is the number of chains of type B;
- r_n is the total number of elements in the chains of type C.

Thus,

$$\begin{aligned} &b_0 = c_0 = u_0 = v_0 = 0 \text{ and } r_0 = 1; \\ &b_1 = 6, c_1 = 2, u_1 = v_1 = 0 \text{ and } r_1 = 2; \\ &b_2 = 32, c_2 = 10, u_2 = 12, v_2 = 4 \text{ and } r_2 = 4. \end{aligned}$$

We refer to $V_n = (b_n, c_n, u_n, v_n, r_n)^{\mathrm{t}}$, a column vector, as the *structural vector* of $H_8^{\nabla(2^n-1)}$. They can be obtained inductively as we will do next. Note that we have

$$\theta_n = |H_8^{\nabla(2^n-1)}| = b_n + u_n + r_n.$$

Assume that $n \geq 2$ and that we have obtained the collection of chains $\mathcal{C}_n$ and the structural vector $(b_n, c_n, u_n, v_n, r_n)^{\mathrm{t}}$ of $H_8^{\nabla(2^n-1)}$. We want to get $\mathcal{C}_{n+1}$ and the structural vector $(b_{n+1}, c_{n+1}, u_{n+1}, v_{n+1}, r_{n+1})^{\mathrm{t}}$ of $H_8^{\nabla(2^{n+1}-1)}$. Realizing that

$$H_8^{\nabla(2^{n+1}-1)} = (H_8^{\nabla(2^n-1)})^{\nabla 2} \nabla H_8 = \triangle_{C\in\mathcal{C}_n, D\in\mathcal{C}_1}(C^{\nabla 2} \nabla D),$$

we compute all $C^{\nabla 2} \nabla D$ for $C \in \mathcal{C}_n$ and $D \in \mathcal{C}_1$.

(1) Suppose that $x_i \cdot \{1, 2, 4, \dots, 2^{\ell_i - 1}\}$, $i = 1, 2, \dots, c_n$, are the chains of type A in $\mathcal{C}_n$. Take $x = x_i$ and $\ell = \ell_i$ $(1 \le i \le c_n)$. Then

$$\{x, 2x, 4x, \dots, 2^{\ell-1}x\}^{\nabla 2} = x^2 \cdot \{1, 4, 16, 64, \dots, 2^{2(\ell-1)}\}.$$

(a) We have

$$\begin{aligned} &x^2 \cdot \{1, 4, 16, 64, \dots, 2^{2(\ell-1)}\} \nabla \{1, 2, 4, 8\} \\ &\quad = x^2 \cdot \{1, 2^{2\ell}, 2, 2^{2\ell+1}\} = \{x^2, 2x^2\} \cup \{2^{2\ell}x^2, 2 \cdot 2^{2\ell}x^2\}. \end{aligned}$$

Since $\ell \ge 2$, this yields two distinct chains of type A with length 2. Thus, these symmetric products contribute $4c_n$ to b_{n+1}, and $2c_n$ to c_{n+1}.

(b) We have

$$\begin{aligned} x^2 \cdot \{1, 2, 4, \dots, 2^{\ell-1}\}^{\nabla 2} &\nabla \{3, 6\} \\ &= 3x^2 \cdot \{1, 4, 16, \dots, 2^{2(\ell-1)}\} \nabla \{1, 2\} \\ &= 3x^2 \cdot \{1, 2, 4, 8, \dots, 2^{2\ell-1}\}, \end{aligned}$$

a chain of type A with length 2ℓ. Thus, these contribute $2b_n$ to b_{n+1}, and c_n to c_{n+1}.

(c) We have

$$x^2 \cdot \{1, 2, 4, \dots, 2^{\ell-1}\}^{\nabla 2} \nabla \{5\} = 5x^2 \cdot \{1, 4, 16, \dots, 2^{2(\ell-1)}\}$$

and

$$x^2 \cdot \{1, 2, 4, \dots, 2^{\ell-1}\}^{\nabla 2} \nabla \{7\} = 7x^2 \cdot \{1, 4, 16, \dots, 2^{2(\ell-1)}\}.$$

Each of them is a chain of type B with length ℓ. Such symmetric products contribute $2b_n$ to u_{n+1}, and $2c_n$ to v_{n+1}.

(2) Suppose that $y_i \cdot \{1, 4, 16, \dots, 4^{\ell'_i - 1}\}$, $i = 1, 2, \dots, v_n$, are the chains of type B in $\mathcal{C}_n$. Take $y = y_i$ and $\ell' = \ell'_i$ $(1 \le i \le v_n)$. Then

$$\{y, 4y, 16y, \dots, 4^{\ell'-1}y\}^{\nabla 2} = y^2 \cdot \{1, 16, 256, \dots, 4^{2(\ell'-1)}\}.$$

(a) As

$$\begin{aligned} \{y, 4y, 16y, \dots, 4^{\ell'-1}y\}^{\nabla 2} \nabla \{1, 2, 4, 8\} &= cup_{i=0}^{\ell'-1} 4^{2i} y^2 \cdot \{1, 2, 4, 8\} \\ &= y^2 \cdot \{1, 2, 4, 8, 16, 32, \dots, 2^{4\ell'-1}\}, \end{aligned}$$

which is a chain of type A of length $4\ell'$, these contribute $4u_n$ to b_{n+1}, and v_n to c_{n+1}.

(b) As

$$\{y, 4y, 16y, \ldots, 4^{\ell'-1}y\}^2 \nabla \{3, 6\} = cup_{i=0}^{\ell'-1} 3 \cdot 4^{2i} y^2 \cdot \{1, 2\},$$

which yields ℓ' chains of type A each of length 2, these contribute $2u_n$ to b_{n+1}, and u_n to c_{n+1}.

(c) Each

$$\{y, 4y, 16y, \ldots, 4^{\ell'-1}y\}^2 \nabla \{5\} = cup_{i=0}^{\ell'-1} 5 \cdot 4^{2i} y^2 \cdot \{1\}$$

and

$$\{y, 4y, 16y, \ldots, 4^{\ell'-1}y\}^2 \nabla \{7\} = cup_{i=0}^{\ell'-1} 7 \cdot 4^{2i} y^2 \cdot \{1\}$$

yields $2\ell'$ chains of type C. Thus, these contribute $2u_n$ to r_{n+1}.

(3) Suppose that $\{z_i\}$, $i = 1, 2, \ldots, r_n$, are the chains of type C in $\mathcal{C}_n$. Take $z = z_i$ $(1 \leq i \leq r_n)$.

(a) Each $\{z\}^{\nabla 2} \nabla \{1, 2, 4, 8\} = \{z^2, 2z^2, 4z^2, 8z^2\}$ is a chain of type A with length 4. Thus, these contribute $4r_n$ to b_{n+1}, and r_n to c_{n+1}.

(b) Each $\{z\}^{\nabla 2} \nabla \{3, 6\} = \{3z^2, 6z^2\}$ gives a chain of type A with length 2. Thus, these contribute $2r_n$ to b_{n+1}, and r_n to c_{n+1}.

(c) Each of $\{z\}^{\nabla 2} \nabla \{5\} = \{5z^2\}$ and $\{z\}^{\nabla 2} \nabla \{7\} = \{7z^2\}$ gives a chain of type C. Thus, these contribute $2r_n$ to r_{n+1}.

Summarizing, we get the following system of recursive linear equations expressing b_{n+1}, c_{n+1}, u_{n+1}, v_{n+1}, and r_{n+1} in terms of b_n, c_n, u_n, v_n, and r_n:

$$\begin{cases} b_{n+1} = 2b_n + 4c_n + 6u_n + 6r_n, \\ c_{n+1} = 3c_n + u_n + v_n + 2r_n, \\ u_{n+1} = 2b_n, \\ v_{n+1} = 2c_n, \\ r_{n+1} = 2u_n + 2r_n. \end{cases}$$

Now, we need to clarify that the symmetric products of chains as described above do indeed give us the correct structural vector V_{n+1} of $H_8^{\nabla(2^{n+1}-1)}$ from that of $H_8^{\nabla(2^n-1)}$. Specifically, we need to demonstrate the following. Let C_1 and C_2 be two chains in $H_8^{\nabla(2^n-1)}$, and let D_1 and D_2 be two chains in H_8. If either $C_1 \neq C_2$ or $D_1 \neq D_2$, then

(1) the chains obtained in the symmetric products $C = C_1^{\nabla 2} \nabla D_1$ and $C' = C_2^{\nabla 2} \nabla D_2$ are distinct chains in $H_8^{\nabla(2^{n+1}-1)}$, and

(2) no longer chains can be formed from these chains.

Since $H_8^{\nabla(2^0-1)} = H_8^{\nabla 0} = \{1\}$, it is clear that the chains in $H_8^{\nabla(2^1-1)} = H_8$ are exactly produced in this way. Also, one can easily check that the chains in $H_8^{\nabla(2^2-1)} = H^{\nabla 3}$ can be produced exactly in this way from those in H_8.

Now, if we can show that $C \cap C'$ is empty, then (1) is true. Further, if we can show that $C \cap 2C'$, $C' \cap 2C$, $C \cap 4C'$, and $C' \cap 4C$ are all empty, then no concatenating chains to form longer chains are possible, and so (2) holds.

We will show that these are indeed true after we make the following general observation.

Lemma 4. *Let C_1 and C_2 be two nonempty finite subsets of $\mathbb{N}$, and $\ell \geq 0$. Then the following statements are equivalent.*

(1) *For $j \in \{0, \ldots, \ell\}$, the intersections $C_1 \cap (2^j C_2)$ and $C_2 \cap (2^j C_1)$ are empty.*

(2) *If $2^{g_1} v_1 \in C_1$ and $2^{g_2} v_2 \in C_2$, where $g_i \geq 0$ and $2 \nmid v_i$ for $i = 1$ and 2, then either $v_1 \neq v_2$ or $|g_1 - g_2| \geq \ell + 1$.*

Proof. Let $2^{g_1} v_1 \in C_1$ and $2^{g_2} v_2 \in C_2$, where $g_i \geq 0$ and $2 \nmid v_i$ for $i = 1$ and 2.

Assume (1) and $v_1 = v_2$. If $j \geq 0$, then $g_1 = g_2 + j$ implies $2^{g_1} v_1 = 2^j \cdot (2^{g_2} v_2) \in C_1 \cap (2^j C_2)$, and $g_2 = g_1 + j$ implies $2^{g_2} v_2 = 2^j \cdot (2^{g_1} v_1) \in C_2 \cap (2^j C_1)$. By the assumption, these cannot happen for $j \leq \ell$. Hence $|g_1 - g_2| \geq \ell + 1$, and (2) is true.

Conversely, assume (2). If $2^{g_1} v_1 = 2^{g_2 + j} v_2 \in C_1 \cap (2^j C_2)$ or $2^{g_1 + j} v_1 = 2^{g_2} v_2 \in C_2 \cap (2^j C_1)$, where $j \geq 0$, then $v_1 = v_2$. In such cases, we have $j = |g_1 - g_2| \geq \ell + 1$. Therefore, we get $C_1 \cap (2^j C_2) = C_2 \cap (2^j C_1) = \varnothing$ whenever $j \in \{0, 1, \ldots, \ell\}$. This is (1). □

We now state and prove the desired statement.

Proposition 5. *Let $C_1, C_2 \in \mathcal{C}_n$, and $D_1, D_2 \in \mathcal{C}_1$. Put $C = C_1^{\nabla 2} \nabla D_1$ and $C' = C_2^{\nabla 2} \nabla D_2$. If either $C_1 \neq C_2$ or $D_1 \neq D_2$, then $C \cap (2^i C') = C' \cap (2^i C) = \varnothing$ for $i = 0, 1, 2$.*

Proof. Suppose first that $D_1 \neq D_2$. Then $(D_1 \setminus D_2) \cup (D_2 \setminus D_1)$ is not empty, and an odd prime can be found there. Without loss of generality, we may assume that $3 \in (D_1 \setminus D_2)$. If $2^{e_1} 3^{e_2} 5^{e_3} 7^{e_4} \in C = C_1^{\nabla 2} \nabla D_1$, then e_2 is odd. On the other hand, for $i \in \{0, 1, 2\}$, if $2^{e_1'} 3^{e_2'} 5^{e_3'} 7^{e_4'} \in 2^i C' = 2^i (C_2^{\nabla 2} \nabla D_2)$, then e_2' is even. Hence $C \cap (2^i C') = \varnothing$. Similarly, we have $C' \cap (2^i C) = \varnothing$ for $i \in \{0, 1, 2\}$.

Suppose that $D_1 = D_2 = D$ and $C_1 \neq C_2$. Consider $u_1^2 d_1 \in C_1^{\nabla 2} \nabla D$ and $u_2^2 d_2 \in C_2^{\nabla 2} \nabla D$, where $d_1, d_2 \in D$, $u_1 = 2^{g_1} v_1 \in C_1$, and $u_2 = 2^{g_2} v_2 \in C_2$ with $g_1 \geq 0$, $g_2 \geq 0$, and v_1 and v_2 being odd. As distinct chains of $H_8^{\nabla(2^n-1)}$, C_1 and C_2 satisfy

$$C_1 \cap (2^i C_2) = \varnothing \text{ and } C_2 \cap (2^i C_1) = \varnothing \text{ for } i \in \{0, 1, 2\}. \tag{1}$$

Write $d_1 = 2^w v$, where $w \geq 0$ and $2 \nmid v$. Since D is a chain in H_8 we have $d_2 = 2^{\pm\lambda} d_1 = 2^{\pm\lambda + w} v$ for some λ with $0 \leq \lambda \leq 3$. Therefore,

$$u_1^2 d_1 = 2^{2g_1 + w} v_1^2 v \text{ and } u_2^2 d_2 = 2^{2g_2 \pm \lambda + w} v_2^2 v.$$

Now, if $v_1^2 v = v_2^2 v$, then $v_1 = v_2$, and so $|g_1 - g_2| \geq 3$ by Lemma 4. Hence

$$|(2g_1 + w) - (2g_2 \pm \lambda + w)| \geq 2 \cdot |g_1 - g_2| - |\lambda| \geq 3.$$

Again, by Lemma 4, we have $C \cap (2^i C') = C' \cap (2^i C) = \varnothing$ for $i = 0$, 1, and 2. □

Expressing the linear recursive system (3.1) in matrix form, we have $V_{n+1} = MV_n$, where

$$M = \begin{pmatrix} 2 & 4 & 6 & 0 & 6 \\ 0 & 3 & 1 & 1 & 2 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 2 \end{pmatrix} \text{ and } V_n = \begin{pmatrix} b_n \\ c_n \\ u_n \\ v_n \\ r_n \end{pmatrix}, \quad n = 0, 1, 2, \ldots, \text{ with } V_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2}$$

Setting $U = (1, 0, 1, 0, 1)$, we have

$$\theta_{n+1} = b_{n+1} + u_{n+1} + r_{n+1} = UV_{n+1} = UMV_n.$$

Now, the minimal polynomial of M is $X^2(X^3 - 7X^2 + 2X + 24)$, and so we have

$$U(M^3 - 7M^2 + 2M + 24I)M^2 = 0.$$

A direct check shows that

$$\begin{aligned} UM &= (4, 4, 8, 0, 8)\,, \\ UM^2 &= (24, 28, 44, 4, 48)\,, \\ UM^3 &= (136, 188, 268, 28, 296)\,, \end{aligned}$$

and that

$$UM^3 - 7UM^2 + 2UM + 24U = 0. \tag{3}$$

Therefore, for $n \geq 0$, we have

$$\theta_{n+3} = UM^3 V_n = (7UM^2 - 2UM - 24U)V_n = 7\theta_{n+2} - 2\theta_{n+1} - 24\theta_n. \tag{4}$$

Remark 6. We can tabulate the above computations:

		$\{1,2,4,8\}$	$\{3,6\}$	$\{5\},\{7\}$	b_{n+1}	c_{n+1}	u_{n+1}	v_{n+1}	r_{n+1}
(b_n, c_n)	A_ℓ	$2\mathrm{A}_2$	$\mathrm{A}_{2\ell}$	$2\mathrm{B}_\ell$	$4c_n + 2b_n$	$3c_n$	$2b_n$	$2c_n$	
(u_n, v_n)	B_ℓ	$\mathrm{A}_{4\ell}$	$\ell\mathrm{A}_2$	$2\ell\mathrm{C}$	$4u_n + 2u_n$	$v_n + u_n$			$2u_n$
(r_n)	C	A_4	A_2	2C	$6r_n$	$2r_n$			$2r_n$

.

Here, in the table, we use A_ℓ and B_ℓ to indicate a chain of type A and type B of length ℓ, respectively; and we use C to indicate a chain of a singleton. Also, we use $2\mathrm{A}_\ell$ to indicate two disjoint chains of type A with length ℓ, etc. The (b_n, c_n), (u_n, v_n), and (r_n) in the first column are to remind the variables used for counting in the respective rows, and the break-downs are shown in the columns to the right.

For $D \in \{\{1,2,4,8\},\{3,6\},\{5\},\{7\}\}$, the first row records the pattern for the symmetric products $\mathrm{A}_\ell^{\nabla 2} \nabla D$, the second row records the pattern for the symmetric products $\mathrm{B}_\ell^{\nabla 2} \nabla D$, and the last row records the pattern for the symmetric products $\mathrm{C}^{\nabla 2} \nabla D$. The columns of $b_{n+1}, c_{n+1}, u_{n+1}, v_{n+1}, r_{n+1}$ show us the recursive equations we have seen above. For example, the column of b_{n+1} reads $b_{n+1} = 4c_n + 2b_n + 4u_n + 2u_n + 6r_n = 2b_n + 4c_n + 6u_n + 6r_n$. The corresponding matrix M of the system of linear equations can then be written down easily.

3.2 The case $k = 2, 3, 4, 5, 6, 7$

Recall that $H_k^{\nabla(2^{n+1}-1)} = (H_k^{\nabla(2^n-1)})^{\nabla 2} \nabla H_k = \triangle_{i=1}^k i(H_k^{\nabla(2^n-1)})^{\nabla 2}$. Since the elements in $(H_2^{\nabla(2^n-1)})^{\nabla 2}$ and in $(H_3^{\nabla(2^n-1)})^{\nabla 2}$ are squares for all $n \geq 1$, and those in $2(H_2^{\nabla(2^n-1)})^{\nabla 2}$, $2(H_3^{\nabla(2^n-1)})^{\nabla 2}$, and $3(H_3^{\nabla(2^n-1)})^{\nabla 2}$ are not, we have

$$\begin{aligned}
(H_2^{\nabla(2^n-1)})^{\nabla 2} \cap (2(H_2^{\nabla(2^n-1)})^{\nabla 2}) &= \varnothing,\\
(H_3^{\nabla(2^n-1)})^{\nabla 2} \cap (2(H_3^{\nabla(2^n-1)})^{\nabla 2}) &= \varnothing,\\
(H_3^{\nabla(2^n-1)})^{\nabla 2} \cap (3(H_3^{\nabla(2^n-1)})^{\nabla 2}) &= \varnothing.
\end{aligned}$$

Also it is clear that

$$(2(H_3^{\nabla(2^n-1)})^{\nabla 2}) \cap (3(H_3^{\nabla(2^n-1)})^{\nabla 2}) = \varnothing.$$

Therefore, for $k = 2$ and 3, and $n \geq 1$, we have

$$\begin{aligned}
\theta_{n+1} = |H_k^{\nabla(2^{n+1}-1)}| &= |\triangle_{i=1}^k i(H_k^{\nabla(2^n-1)})^{\nabla 2}|\\
&= \sum_{i=1}^k |i(H_k^{\nabla(2^n-1)})^{\nabla 2}|\\
&= k \cdot |(H_k^{\nabla(2^n-1)})^{\nabla 2}| = |H_k^{\nabla(2^n-1)}| \cdot |H_k| = k\theta_n. \qquad (5)
\end{aligned}$$

As $\theta_0 = 1$, (5) holds for $n \geq 0$.

For each $k \in \{4,5,6,7\}$ we analyze H_k and compute the symmetric products $C^{\nabla 2} \nabla D$ of the chains C of $H_k^{\nabla(2^n-1)}$ and the chains D of H_k to produce a table just like we did for $H_8^{\nabla(2^{n+1}-1)}$ in Remark 6. It turns out that in these cases we only need to collect chains of types A and C. Namely, for each $H_k^{\nabla(2^n-1)}$, $k \in \{4,5,6,7\}$, and $n \in \mathbb{N}$, we would collect all possible chains of type A. After removing the elements of these collected chains, each of the remaining elements gives us a chain of type C. The structural vector of $H_k^{\nabla(2^n-1)}$ is then set to be $V_n = (b_n, c_n, r_n)^{\mathrm{t}}$, where b_n is the total number of the elements in the chains of type A, c_n is the number of chains of type A, and r_n is the total number of elements in the chains of type C.

We also need to justify that when the chains C_1 and C_2 of $H_k^{\nabla(2^n-1)}$ are distinct and/or the chains D_1 and D_2 of H_k are distinct, the chains produced in the symmetric product

$C_1^{\nabla 2} \nabla D_1$ and those produced in $C_2^{\nabla 2} \nabla D_2$ are disjoint and cannot be concatenated to form longer chains. The following proposition similar to Proposition 5 provides us the justification.

Proposition 7. *Let $k \in \{4,5,6,7\}$. Let C_1 and C_2 be chains of $H_k^{\nabla(2^n-1)}$, and D_1 and D_2 that of H_k. Put $C = C_1^{\nabla 2} \nabla D_1$ and $C' = C_2^{\nabla 2} \nabla D_2$. If either $C_1 \neq C_2$ or $D_1 \neq D_2$, then for $i = 0$ and 1, $C \cap (2^i C') = C' \cap (2^i C) = \varnothing$.*

To prove Proposition 7, one can use the proof of Proposition 5 as a template. The only changes needed are to adjust the range of λ from $0 \le \lambda \le 3$ to $0 \le \lambda \le 2$ (as 8 is not present in H_k), change the inequality $|g_1 - g_2| \ge 3$ to $|g_1 - g_2| \ge 2$, and apply Lemma 4 to obtain the desired conclusion from the inequality

$$|(2g_1 + w) - (2g_2 \pm \lambda + w)| \ge 2 \cdot |g_1 - g_2| - |\lambda| \ge 2.$$

Thus, for each $k = 4,5,6,7$, M is a 3×3 matrix. The size $\theta_{n+1} = |H_k^{\nabla(2^{n+1}-1)}|$ satisfies $\theta_{n+1} = UMV_n$, where $U = (1,0,1)$ and $V_n = (b_n, c_n, r_n)^{\mathrm{t}}$. Therefore, to get the recursive formula for θ_k, it suffices to write down the table of the symmetric products of the chains, the matrix M, the minimal polynomial of M and the polynomial P in M with $UP(M) = 0$.

3.2.1 $k = 4$

The table of the symmetric products of the chains reads

		$\{1,2,4\}$	$\{3\}$	b_{n+1}	c_{n+1}	r_{n+1}
(b_n, c_n)	A_ℓ	$2A_2, (\ell-2)\mathrm{C}$	$\ell\mathrm{C}$	$4c_n$	$2c_n$	$2b_n - 2c_n$
(r_n)	C	A_3	C	$3r_n$	r_n	r_n

.

The corresponding matrix $M = \begin{pmatrix} 0 & 4 & 3 \\ 0 & 2 & 1 \\ 2 & -2 & 1 \end{pmatrix}$ has the minimal polynomial

$$X^3 - 3X^2 - 2X + 4 = (X-1)(X^2 - 2X - 4).$$

We have $U(M^2 - 2M - 4I) = 0$, and so for $n \ge 0$,

$$\theta_{n+2} = 2\theta_{n+1} + 4\theta_n.$$

3.2.2 $k = 5$

The table of the symmetric products of the chains reads

		$\{1,2,4\}$	$\{3\},\{5\}$	b_{n+1}	c_{n+1}	r_{n+1}
(b_n, c_n)	A_ℓ	$2\mathrm{A}_2, (\ell-2)\mathrm{C}$	$2\ell\mathrm{C}$	$4c_n$	$2c_n$	$3b_n - 2c_n$
(r_n)	C	A_3	2C	$3r_n$	r_n	$2r_n$

.

The corresponding matrix $M = \begin{pmatrix} 0 & 4 & 3 \\ 0 & 2 & 1 \\ 3 & -2 & 2 \end{pmatrix}$ has the minimal polynomial

$$X^3 - 4X^2 - 3X + 6 = (X - 1)(X^2 - 3X - 6).$$

We have $U(M^2 - 3M - 6I) = 0$, and so for $n \geq 0$,

$$\theta_{n+2} = 3\theta_{n+1} + 6\theta_n.$$

3.2.3 $k = 6$

The table of the symmetric products of the chains reads

		$\{3,6\}$	$\{1,2,4\}$	$\{5\}$	b_{n+1}	c_{n+1}	r_{n+1}
(b_n, c_n)	A_ℓ	$\mathrm{A}_{2\ell}$	$2\mathrm{A}_2, (\ell-2)\mathrm{C}$	$\ell\mathrm{C}$	$2b_n + 4c_n$	$3c_n$	$2b_n - 2c_n$
(r_n)	C	A_2	A_3	C	$5r_n$	$2r_n$	r_n

The corresponding matrix $M = \begin{pmatrix} 2 & 4 & 5 \\ 0 & 3 & 2 \\ 2 & -2 & 1 \end{pmatrix}$ has the minimal polynomial

$$X^3 - 6X^2 + 5X = X(X - 1)(X - 5).$$

We have $U(M^2 - 5M) = 0$, and so for $n \geq 1$,

$$\theta_{n+1} = 5\theta_n.$$

3.2.4 $k = 7$

The table of the symmetric products of the chains reads

		$\{3,6\}$	$\{1,2,4\}$	$\{5\},\{7\}$	b_{n+1}	c_{n+1}	r_{n+1}
(b_n, c_n)	A_ℓ	$\mathrm{A}_{2\ell}$	$2\mathrm{A}_2, (\ell-2)\mathrm{C}$	$2\ell\mathrm{C}$	$2b_n + 4c_n$	$3c_n$	$3b_n - 2c_n$
(r_n)	C	A_2	A_3	2C	$5r_n$	$2r_n$	$2r_n$

The corresponding matrix $M = \begin{pmatrix} 2 & 4 & 5 \\ 0 & 3 & 2 \\ 3 & -2 & 2 \end{pmatrix}$ has the minimal polynomial

$$X^3 - 7X^2 + 5X + 1 = (X - 1)(X^2 - 6X - 1).$$

We have $U(M^2 - 6M - I) = 0$, and so for $n \geq 0$,

$$\theta_{n+2} = 6\theta_{n+1} + \theta_n.$$

4 The sequence A_8

Recall that for the sparse subsequence $S_8 = (\theta_n)_{n\geq 0}$, where $\theta_n = |H_8^{\nabla(2^n-1)}|$, we associate a structural vector $V_n = (b_n, c_n, u_n, v_n, r_n)^{\mathrm{t}}$ with each $H_8^{\nabla(2^n-1)}$, so that $\theta_n = UMV_n$ for all $n \geq 0$ with the vector $U = (1, 0, 1, 0, 1)$ and the 5×5 matrix M in (2). We can extend this idea to A_8 in order to get every $a_8(n) = |H_8^{\nabla n}|$, $n \geq 0$.

For each n, we again collect the chains of types A, B and C in $H_8^{\nabla n}$, and define the structural vector Λ_n of $H_8^{\nabla n}$ to be $(b_n, c_n, u_n, v_n, r_n)^{\mathrm{t}}$. Note that the same variables b_n, c_n, u_n, v_n, and r_n are used here. Yet they are not the same in the indices as we have used them when dealing with the sparse subsequences S_k.

Now, the structural vector of $H_8^{\nabla 0} = \{1\}$ is $\Lambda_0 = (0, 0, 0, 0, 1)^{\mathrm{t}} = V_0$. Next, with $H_8^{\nabla 1} = \left(H_8^{\nabla(2^0-1)}\right)^2 \nabla H_8$, we apply the matrix M in (2) to Λ_0 to get Λ_1 (which is V_1 in this case). Thus, $\Lambda_1 = M\Lambda_0 = (6, 2, 0, 0, 2)^{\mathrm{t}}$. To get Λ_2, the structural vector of $H_8^{\nabla 2}$, we need another matrix for *symmetric squaring*.

Let $n \geq 1$. Suppose that we have obtained the collection of chains $\mathcal{C}$ of $H_8^{\nabla n}$ and the structural vector $\Lambda_n = (b_n, c_n, u_n, v_n, r_n)^{\mathrm{t}}$. For a type A chain of length ℓ, say $C = \{x, 2x, \ldots, 2^\ell x\}$, we have $C^{\nabla 2} = \{x^2, 4x^2, \ldots, 4^{\ell-1}x^2\}$, which is a type B chain of length ℓ in $H_8^{\nabla 2n}$. For a type B chain of length ℓ, say $C = \{x, 4x, \ldots, 4^{\ell-1}x\}$, we have $C^{\nabla 2} = \{x^2, 16x^2, \ldots, 16^{\ell-1}x^2\} = \cup_{i=0}^{\ell-1}\{16^i x^2\}$, which is the union of ℓ type C chains in $H_8^{\nabla 2n}$. Obviously, $\{x\}^{\nabla 2} = \{x^2\}$ for every type C chain $\{x\}$ in $\mathcal{C}$. This is a type C chain in $H_8^{\nabla 2n}$. Thus, the table for symmetric squaring is as follows:

	C	$C^{\nabla 2}$	b_{2n}	c_{2n}	u_{2n}	v_{2n}	r_{2n}
(b_n, c_n)	A_ℓ	B_ℓ			b_n	c_n	
(u_n, v_n)	B_ℓ	$\ell\mathrm{C}$					u_n
(r_n)	C	C					r_n

.

The corresponding matrix W is

$$W = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

We have $\Lambda_{2n} = W\Lambda_n$ and, in particular, $\Lambda_2 = W\Lambda_1 = (WM)\Lambda_0$.

Lemma 8. *Let $n \geq 0$ be arbitrary, and suppose that $(n)_2 = [\epsilon_\ell \epsilon_{\ell-1} \cdots \epsilon_1 \epsilon_0]$. Then*

$$\Lambda_n = E_0 E_1 \cdots E_{\ell-1} E_\ell \Lambda_0,$$

where, for each i, $E_i = M$ if $\epsilon_i = 1$, and $E_i = W$ if $\epsilon_i = 0$.

Proof. The result holds for $\ell = 0$ and $\ell = 1$ as we have just seen. Let $\ell \geq 2$. Take $m \in \mathbb{N}$ with $(m)_2 = [\epsilon_\ell \epsilon_{\ell-1} \cdots \epsilon_1]$. Assume that $\Lambda_m = E_1 \cdots E_{\ell-1} E_\ell \Lambda_0$ with $E_i = M$ if $\epsilon_i = 1$, and $E_i = W$ if $\epsilon_i = 0$.

If $\epsilon_0 = 0$, then $n = 2m$. In this case, we have $H_8^{\nabla n} = H_8^{\nabla(2m)} = \left(H_8^{\nabla m}\right)^{\nabla 2}$ and so $\Lambda_n = W\Lambda_m$.

If $\epsilon_0 = 1$, then $n = 2m + 1$. In this case, we have

$$H_8^{\nabla n} = H_8^{\nabla(2m+1)} = H_8^{\nabla(2m)} \nabla H_8 = \left(H_8^{\nabla m}\right)^{\nabla 2} \nabla H_8,$$

and so $\Lambda_n = M\Lambda_m$.

Therefore, by the induction hypothesis, we have

$$\Lambda_n = E_0 E_1 \cdots E_{\ell-1} E_\ell \Lambda_0$$

where $E_i = M$ if $\epsilon_i = 1$, and $E_i = W$ if $\epsilon_i = 0$. □

Recall that $U = (1, 0, 1, 0, 1)$, and $|H_8^{\nabla n}| = U\Lambda_n$ for all $n \geq 0$. We shall derive some reduction rules to help to compute every $|H_8^{\nabla n}|$, $n \in \mathbb{N}$. We already know that

$$|H_8^{\nabla 2m}| = |H_8^{\nabla m}| \text{ for all } m \in \mathbb{N}. \tag{6}$$

Notice that

$$W^2 = \Lambda_0 U,$$

and we have the following result similar to Lemma 1.

Lemma 9. *If $n = \alpha + \beta \cdot 2^{s+2}$ for some $\alpha, \beta, s \in \mathbb{N}$ with $\alpha < 2^s$, then*

$$|H_8^{\nabla n}| = |H_8^{\nabla \alpha}| \cdot |H_8^{\nabla \beta}|. \tag{7}$$

Proof. Let $(\alpha)_2 = [x \cdots z]$ and $(\beta)_2 = [b \cdots d]$. Then $(n)_2 = [(b \cdots d)(00 \cdots 0)(x \cdots z)]$, where there are at least two 0's between $(b \cdots d)$ and $(x \cdots z)$. Let C_1 and C_2 be the products of M's and W's such that $\Lambda_\alpha = C_1\Lambda_0$ and $\Lambda_\beta = C_2\Lambda_0$. Then for some $\ell \geq 0$, we have

$$\begin{aligned}
|H_8^{\nabla n}| = U\Lambda_n &= UC_1W^{\ell+2}C_2\Lambda_0 \\
&= UC_1\Lambda_0UW^{\ell}C_2\Lambda_0 \\
&= (U\Lambda_\alpha)(U\Lambda_{2^\ell\beta}) \\
&= |H_8^{\nabla\alpha}| \cdot |H_8^{\nabla(2^\ell\beta)}| \\
&= |H_8^{\nabla\alpha}| \cdot |(H_8^{\nabla\beta})^{\nabla 2^\ell}| = |H_8^{\nabla\alpha}| \cdot |H_8^{\nabla\beta}|.
\end{aligned}$$

□

Next, we collect some useful facts about M and W.

Lemma 10. *With M, W, U, and Λ_0 as above, we have*

(1) $U\mathit{\Lambda}_0 = 1$;

(2) $UW = U$;

(3) $UMW = 8U$;

(4) $UM^2W = UM + 40U$.

Consequently, if C is a product of M's and W's, then $UC = U(pM^2 + qM + rI)$ for some $p, q, r \in \mathbb{Z}$.

Proof. The four identities can be checked directly. Combining (2)–(4) and the fact that $UM^3 = 7UM^2 - 2UM - 24U$ (see (3)), we infer that $UC = U(pM^2 + qM + rI)$ for some $p, q, r \in \mathbb{Z}$ as claimed. □

Lemma 11. *If $(n)_2 = [(b \cdots d)01]$ and $(m)_2 = [b \cdots d]$, then*

$$|H_8^{\nabla n}| = 8 \cdot |H_8^{\nabla m}|. \tag{8}$$

Proof. As $\mathit{\Lambda}_n = MW\mathit{\Lambda}_m$, we have, by Lemma 10(3),

$$|H_8^{\nabla n}| = U\mathit{\Lambda}_n = UMW\mathit{\Lambda}_m = 8U\mathit{\Lambda}_m = 8 \cdot |H_8^{\nabla m}|.$$

□

Lemma 12. *If $(n)_2 = [(b \cdots d)011]$, $(m_1)_2 = [b \cdots d1]$ and $(m_2)_2 = [b \cdots d]$, then*

$$|H_8^{\nabla n}| = |H_8^{\nabla m_1}| + 40 \cdot |H_8^{\nabla m_2}|. \tag{9}$$

Proof. As $\mathit{\Lambda}_n = M^2W\mathit{\Lambda}_{m_2}$, we have, by Lemma 10(4),

$$\begin{aligned} |H_8^{\nabla n}| &= U\mathit{\Lambda}_n = UM^2W\mathit{\Lambda}_{m_2} = (UM + 40U)\mathit{\Lambda}_{m_2} \\ &= UM\mathit{\Lambda}_{m_2} + 40U\mathit{\Lambda}_{m_2} = U\mathit{\Lambda}_{m_1} + 40U\mathit{\Lambda}_{m_2} = |H_8^{\nabla m_1}| + 40 \cdot |H_8^{\nabla m_2}|. \end{aligned}$$

□

Lemma 13. *Let $(n)_2 = [(b \cdots d)111(x \cdots z)]$. With*

$$(m_1)_2 = [(b \cdots d)11(x \cdots z)],\ (m_2)_2 = [(b \cdots d)1(x \cdots z)],\ \text{and } (m_3)_2 = [(b \cdots d)(x \cdots z)],$$

we have

$$|H_8^{\nabla n}| = 7 \cdot |H_8^{\nabla m_1}| - 2 \cdot |H_8^{\nabla m_2}| - 24 \cdot |H_8^{\nabla m_3}|, \tag{10}$$

Proof. Let $\alpha \geq 0$ and $\beta \geq 0$ be such that $(\alpha)_2 = [x \cdots z]$ and $(\beta)_2 = [b \cdots d]$, and let C be the products of M's and W's such that $\Lambda_\alpha = C\Lambda_0$. Then $\Lambda_n = CM^3\Lambda_\beta$, $\Lambda_{m_1} = CM^2\Lambda_\beta$, $\Lambda_{m_2} = CM\Lambda_\beta$, and $\Lambda_{m_3} = C\Lambda_\beta$. By Lemma 10, we have $UC = U(pM^2 + qM + rI)$ for some $p, q, r \in \mathbb{Z}$. Let $D = pM^2 + qM + rI$. From (3), we get

$$\begin{aligned}
|H_8^{\nabla n}| = U\Lambda_n &= UCM^3\Lambda_\beta \\
&= UDM^3\Lambda_\beta \\
&= UM^3D\Lambda_\beta \\
&= U(7M^2 - 2M - 24I)D\Lambda_\beta \\
&= 7UM^2D\Lambda_\beta - 2UMD\Lambda_\beta - 24UD\Lambda_\beta \\
&= 7UDM^2\Lambda_\beta - 2UDM\Lambda_\beta - 24UD\Lambda_\beta \\
&= 7UCM^2\Lambda_\beta - 2UCM\Lambda_\beta - 24UC\Lambda_\beta \\
&= 7U\Lambda_{m_1} - 2U\Lambda_{m_2} - 24U\Lambda_{m_3} \\
&= 7 \cdot |H_8^{\nabla m_1}| - 2 \cdot |H_8^{\nabla m_2}| - 24 \cdot |H_8^{\nabla m_3}|.
\end{aligned}$$

□

We are in a position to show that we can get every $|H_8^{\nabla n}|$, $n \in \mathbb{N}$, with just $|H_8^{\nabla 0}|$, $|H_8^{\nabla 1}|$ and $|H_8^{\nabla 3}|$ by applying (6)–(9) repeatedly.

Let $\mathcal{S}$ be the set of all $n \geq 0$, such that none of the rules (6), (7), (8) and (9) applies. In particular, if $n \in \mathcal{S}$, then the binary expansion $(n)_2$, when read from the higher bits to the lower bits, does not have 00 or 111 anywhere, and does not end with 0 or 01.

- If $(n)_2$ has just one bit, then $(n)_2$ is either $(0)_2 = [0]$, or $(1)_2 = [1]$. We have $|H_8^{\nabla 0}| = 1$ and $|H_8^{\nabla 1}| = 8$.
- If $(n)_2$ has two bits, then $(n)_2$ has to be $(3)_2 = [11]$. We have $|H_8^{\nabla 3}| = 48$.

So we have $\{0, 1, 3\} \subseteq \mathcal{S}$.

Proposition 14. *The set* $\mathcal{S}$ *is exactly* $\{0, 1, 3\}$.

Proof. As $(4)_2 = [100]$, $(5)_2 = [101]$, $(6)_2 = [110]$, and $(7)_2 = [111]$, no integers with a 3 bit binary expansion belong to $\mathcal{S}$.

Let $n \in \mathcal{S}$. Suppose that $(n)_2$ has at least 4 bits. Then the last three bits of $(n)_2$ has to be 011. So $(n)_2 = [(b \cdots d)011]$. But then (9) applies, contradicting to that $n \in \mathcal{S}$. Therefore, $(n)_2$ has at most 3 bits, and we are done. □

Now Corollary 3(2) follows.

Example 15. (1) The identity (9) gives us

$$\left|H_8^{\nabla 11}\right| = \left|H_8^{\nabla 3}\right| + 40 \cdot \left|H_8^{\nabla 1}\right| = 368$$

since $(11)_2 = [1011]$, $(3)_2 = [11]$, and $(1)_2 = [1]$.

(2) Using (9) again, we get

$$\left|H_8^{\nabla 27}\right| = \left|H_8^{\nabla 7}\right| + 40 \cdot \left|H_8^{\nabla 3}\right| = \left|H_8^{\nabla 7}\right| + 1920$$

since $(27)_2 = [11011]$, $(7)_2 = [111]$, and $(3)_2 = [11]$. It follows from (10) that

$$\left|H_8^{\nabla 7}\right| = 7 \cdot \left|H_8^{\nabla 3}\right| - 2 \cdot \left|H_8^{\nabla 1}\right| - 24 \cdot \left|H_8^{\nabla 0}\right| = 296.$$

Hence $\left|H_8^{\nabla 27}\right| = 2216$.

We give two more reduction rules which can be useful for computing $|H_8^{\nabla n}|$. Here is the first one.

Lemma 16. *If* $(n)_2 = [(b \cdots d)011011]$, $(m_1)_2 = [(b \cdots d)011]$ *and* $(m_2)_2 = [b \cdots d]$, *then*

$$|H_8^{\nabla n}| = 47 \cdot |H_8^{\nabla m_1}| - 40 \cdot |H_8^{\nabla m_2}|. \tag{11}$$

Proof. We have

$$\begin{aligned}
|H_8^{\nabla n}| &= UM^2WM^2W\Lambda_{m_2} \\
&= (UM + 40U)M^2W\Lambda_{m_2} \\
&= UM^3W\Lambda_{m_2} + 40UM^2W\Lambda_{m_2} \\
&= (7UM^2 - 2UM - 24UI)W\Lambda_{m_2} + 40UM^2W\Lambda_{m_2} \\
&= 47UM^2W\Lambda_{m_2} - 2UMW\Lambda_{m_2} - 24UW\Lambda_{m_2} \\
&= 47U\Lambda_{m_1} - 16U\Lambda_{m_2} - 24U\Lambda_{m_2} \\
&= 47U\Lambda_{m_1} - 40U\Lambda_{m_2} \\
&= 47 \cdot |H_8^{\nabla m_1}| - 40 \cdot |H_8^{\nabla m_2}|.
\end{aligned}$$

□

The second one requires the following facts about WM, which can be checked directly. We note that the minimal polynomial of WM is $X^4 - 9X^3 + 8X^2$, and so

$$UWM((WM)^2 - 9WM + 8I) = 0. \tag{12}$$

Since $UW = U$, (12) is reduced to

$$UM((WM)^2 - 9WM + 8I) = 0. \tag{13}$$

Furthermore, it is also true that

$$UM^2((WM)^2 - 9WM + 8I) = 0. \tag{14}$$

Although $UM^3((WM)^2 - 9WM + 8I) \neq 0$, we still have

$$UM^3((WM)^2 - 9WM + 8I)\Lambda_0 = 0. \tag{15}$$

Lemma 17. *Let $n, m_1, m_2 \in \mathbb{N}$. If one of the following holds*

(1) $(n)_2 = [(b\cdots d)101011]$, $(m_1)_2 = [(b\cdots d)1011]$ *and* $(m_2)_2 = [(b\cdots d)11]$,

(2) $(n)_2 = [10101(x\cdots z)]$, $(m_1)_2 = [101(x\cdots z)]$, *and* $(m_2)_2 = [1(x\cdots z)]$,

then

$$|H_8^{\nabla n}| = 9\cdot|H_8^{\nabla m_1}| - 8\cdot|H_8^{\nabla m_2}|. \tag{16}$$

Proof.
(1) Let $\alpha \geq 0$ with $(\alpha)_2 = [b\cdots d]$. We have

$$|H_8^{\nabla m_2}| = UM^2\Lambda_\alpha,$$
$$|H_8^{\nabla m_1}| = UM^2WM\Lambda_\alpha,$$

and

$$|H_8^{\nabla n}| = UM^2(WM)^2\Lambda_\alpha.$$

Together with (14), this gives

$$|H_8^{\nabla n}| - 9\cdot|H_8^{\nabla m_1}| + 8\cdot|H_8^{\nabla m_2}| = UM^2((WM)^2 - 9WM + 8I))\Lambda_\alpha = 0.$$

(2) Let $\beta \geq 0$ with $(\beta)_2 = [x\cdots z]$. Let C be the product of M's and W's such that $\Lambda_\beta = C\Lambda_0$. We have $|H_8^{\nabla m_2}| = UCM\Lambda_0$, $|H_8^{\nabla m_1}| = UCMWM\Lambda_0$ and $|H_8^{\nabla n}| = UCM(WM)^2\Lambda_0$. Let $p, q, r \in \mathbb{Z}$ such that $UC = pUM^2 + qUM + rU$. Then

$$\begin{aligned}
&|H_8^{\nabla n}| - 9\cdot|H_8^{\nabla m_1}| + 8\cdot|H_8^{\nabla m_2}| \\
&\quad = UCM((WM)^2 - 9WM + 8I)\Lambda_0 \\
&\quad = U(pM^2 + qM + rI)M((WM)^2 - 9WM + 8I)\Lambda_0 \\
&\quad = pUM^3((WM)^2 - 9WM + 8I)\Lambda_0 \\
&\qquad + qUM^2((WM)^2 - 9WM + 8I)\Lambda_0 \\
&\qquad\quad + rUM((WM)^2 - 9WM + 8I)\Lambda_0.
\end{aligned}$$

According to (13)–(15), the last three terms are all zero. We are done. □

Here is an example of $|H_8^{\nabla n}|$ with a bit larger n.

Example 18. Let us compute $|H_8^{\nabla n}|$ where $(n)_2 = (11101011011)$, so $n = 1883$.

First of all, we apply (10) with $(m_1)_2 = [11101011]$ and $(m_2)_2 = [11101]$ to get

$$|H_8^{\nabla n}| = 47\cdot|H_8^{\nabla m_1}| - 40\cdot|H_8^{\nabla m_2}|,$$

Next, we put together (16) and (8) with $(\beta_1)_2 = [111011]$, $(\beta_2)_2 = [1111]$, and $(\gamma)_2 = [111]$, and we get

$$|H_8^{\nabla m_1}| = 9\cdot|H_8^{\nabla \beta_1}| - 8\cdot|H_8^{\nabla \beta_2}| \text{ and } |H_8^{\nabla m_2}| = 8\cdot|H_8^{\nabla \gamma}|.$$

Lastly, we infer from (10) that

$$|H_8^{\nabla\beta_1}| = 7 \cdot |H_8^{\nabla 27}| - 2 \cdot |H_8^{\nabla 11}| - 24 \cdot |H_8^{\nabla 3}| = 13624,$$
$$|H_8^{\nabla\beta_2}| = 7 \cdot |H_8^{\nabla 7}| - 2 \cdot |H_8^{\nabla 3}| - 24 \cdot |H_8^{\nabla 1}| = 1784,$$
$$|H_8^{\nabla\gamma}| = |H_8^{\nabla 7}| = 296,$$

and so $|H_8^{\nabla m_1}| = 108344$ and $|H_8^{\nabla m_2}| = 2368$. Therefore, $|H_8^{\nabla n}| = 4997448$.

When applying the above rules for computing $|H_8^{\nabla n}|$, there is no particular order that one have to follow. Some orders could be more efficient than others when applied in specific cases. Still, one can design certain general algorithm to compute $|H_8^{\nabla n}|$.

Finally, we remark that we do not have any formula for A_k and S_k when $k \geq 9$. The approach using chains like above fails since new types of chains need to be introduced, e.g., chains of the type $\{x, 3x, 9x, \ldots\}$, and they may not be disjoint from the chains of other types.

5 Acknowledgments

The authors like to thank the reviewer and the editor for their careful reading and comments to improve the clarity and the readability of the paper.

References

[1] A. E. Caicedo, T. A. C. Chartier, and P. P. Pach, Coloring the n-smooth numbers with n colors, *Electron. J. Combin.* **28** (2021), Paper 1.34.

[2] S. B. Ekhad, N. J. A. Sloane, and D. Zeilberger, Odd-rule cellular automata on the square grid, arxiv preprint arXiv:1503.04249 [math.CO], 2015. Available at https://arxiv.org/abs/1503.04249.

[3] S. B. Ekhad, N. J. A. Sloane, and D. Zeilberger, A meta-algorithm for creating fast algorithms for counting ON cells in odd-rule cellular automata, arxiv preprint arXiv:1503.01796 [math.CO], 2015. Available at https://arxiv.org/abs/1503.01796.

[4] S. B. Ekhad, N. J. A. Sloane, and D. Zeilberger, A meta-algorithm for creating fast algorithms for counting ON cells in odd-rule cellular automata (Maple package), 2015. Available at https://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/CAcount.html.

[5] P.-Y. Huang, W.-F. Ke, and G. F. Pilz, The cardinality of some symmetric differences, *Proc. Amer. Math. Soc.* **138** (2010), 787–797.

[6] T. Khovanova, There are no coincidences, arxiv preprint arXiv:1410.2193 [math.CO], 2014. Available at https://arxiv.org/abs/1410.2193.

[7] P. P. Pach and C. Szabó, On the minimal distance of a polynomial code, *Discrete Math. Theor. Comput. Sci.* **13** (2011), 33–43.

[8] G. F. Pilz, On polynomial near-ring codes, *Contributions to General Algebra* **8** (Linz, 1991), 233–238.

[9] N. J. A. Sloane, On the number of ON cells in cellular automata, in Steve Butler et al., eds., *Connections in Discrete Mathematics. A Celebration of the Work of Ron Graham*, Cambridge University Press, 2018, pp. 13–38. Also available at https://arxiv.org/abs/1503.01168.

[10] N. J. A. Sloane et al., *On-Line Encyclopedia of Integer Sequences*, electronically available at https://oeis.org, 2023.

2020 *Mathematics Subject Classification*: Primary 11B37; Secondary 11Y55.
Keywords: symmetric product, symmetric power, binary expansion, odd-rule cellular automaton.

(Concerned with sequences A000012, A001316, A048883, and A253064.)

Received June 19 2023; revised versions received June 25 2023; August 1 2023; August 4 2023. Published in *Journal of Integer Sequences*, August 5 2023.

Return to Journal of Integer Sequences home page.