Corrigendum to Article 23.3.5
There are no Collatz-m-Cycles with $m \leq 91$

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The derivation of the inequalities $T(n_{m-m_2+1+\ell}) < \frac{3}{(2^v-1)^\delta}$ in the proof of Theorem 21 at the bottom of page 14 contains an error. To show the statement

$$T(n_{m-m_2+1}) + \sum_{i=m-m_2+2}^{m} T(n_i) < \frac{3}{2^v-1} + \frac{3\cdot(m_2-1)}{(2^v-1)^\delta},$$

which is used later in the proof, we can proceed as follows.

For a local minimum n_i in the hypothetically existing Collatz cycle, define

$$x_i := \frac{\log(n_i+1)}{\log 2}.$$

Then, by Lemma 8, we know that $1 \leq k_i \leq x_i$, and by Lemma 20 that $n_{i+1} < n_i^\delta$, which implies $x_{i+1} < \delta x_i$. Using the same argument as on page 13 in the original proof of Theorem 21, we know that there are m_2 consecutive local minima $n_{m-m_2+1}, \ldots, n_m$ such that $x_{m-m_2+1} + \cdots + x_m \geq k_{m-m_2+1} + \cdots + k_m \geq \frac{m_2}{m}\cdot K$. Without loss of generality, let the cycle be entered in such a way that this sum is maximal. Then $x_{m-m_2} \leq x_m$; otherwise, we would have chosen the shifted interval of local minima. Since $x_{m-m_2+1} < \delta x_{m-m_2}$, this implies $x_{m-m_2+1} < \delta x_m$.

By Remark 7, we have $T(n_i) < \frac{3}{n_i}$. Thus,

$$\sum_{i=m-m_2+1}^{m} T(n_i) < \sum_{i=m-m_2+1}^{m} \frac{3}{2^{x_i}-1} =: S(x_{m-m_2+1}, \ldots, x_m).$$

Since all x_i are positive, the function S is convex and attains its maximum value at a vertex of the polytope determined by the constraints $x_i \geq 1$, $x_{i+1} \leq \delta x_i$, $x_{m-m_2+1} < \delta x_m$, and $x_{m-m_2+1} + \cdots + x_m \geq \frac{m_2}{m} \cdot K$. Thus, at least m_2 of these constraints must be active, i.e., must hold with equality.

Suppose there is a value x_i such that $x_i = 1$. Then $x_{i+1} \leq \delta, \ldots, x_{i-1} \leq \delta^{m_2-1}$. Hence,

$$x_{m-m_2+1} + \cdots + x_m \leq \frac{\delta^{m_2} - 1}{\delta - 1} < \frac{m_2}{m} \cdot K,$$

contradicting the sum constraint. The last inequality follows from the definition of m_2 in the statement of Theorem 21.

Thus, m_2 of the remaining $m_2 + 1$ constraints must be active. If all constraints $x_{i+1} \leq \delta x_i$, $x_{m-m_2+1} < \delta x_m$ were active simultaneously, this would lead to the contradiction $x_m = \delta^{m_2} x_m$, and hence $\delta = 1$, contradicting $\delta = \frac{\log 3}{\log 2}$. Therefore, the sum condition must be active, and we have

$$x_{m-m_2+1} + \cdots + x_m = \frac{m_2}{m} \cdot K,$$

together with all but one of the equations $x_{i+1} = \delta x_i$, and $x_{m-m_2+1} = \delta x_m$. Consequently, S attains its maximum at a point whose coordinates are a cyclic permutation of $x_{m-m_2+1}, \delta x_{m-m_2+1}, \ldots, \delta^{m_2-1} x_{m-m_2+1}$, and

$$\frac{m_2}{m} \cdot K = x_{m-m_2+1} + \delta x_{m-m_2+1} + \cdots + \delta^{m_2-1} x_{m-m_2+1} = x_{m-m_2+1} \cdot \frac{\delta^{m_2} - 1}{\delta - 1}.$$

Hence,

$$x_{m-m_2+1} = \frac{m_2}{m} \cdot K \cdot \frac{\delta - 1}{\delta^{m_2} - 1} = v,$$

and we obtain

$$\begin{aligned}
T(n_{m-m_2+1}) + \sum_{i=m-m_2+2}^{m} T(n_i) &< S(x_{m-m_2+1}, \ldots, x_m) \\
&\leq S(v, \delta v, \ldots, \delta^{m_2-1} v) \leq S(v, \delta v, \ldots, \delta v) \\
&= \frac{3}{2^v - 1} + \frac{3 \cdot (m_2 - 1)}{2^{\delta v} - 1} < \frac{3}{2^v - 1} + \frac{3 \cdot (m_2 - 1)}{(2^v - 1)^\delta},
\end{aligned}$$

which is exactly what we wanted to prove.

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