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Binomial Fibonacci Sums from Chebyshev Polynomials

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Abstract

We explore new types of binomial sums with Fibonacci and Lucas numbers. The binomial coefficients under consideration are $\frac{n}{n+k}\binom{n+k}{n-k}$ and $\frac{k}{n+k}\binom{n+k}{n-k}$. We derive the identities by relating the underlying sums to Chebyshev polynomials. Finally, we study some combinatorial sums and derive a connection with a recent paper by Chu and Guo from 2022.

1 Preliminaries

As usual, the Fibonacci numbers F_n and the Lucas numbers L_n are defined, for $n \in \mathbb{Z}$, through the recurrence relations $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$, with initial values $F_0 = 0$, $F_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ with $L_0 = 2$, $L_1 = 1$. For negative subscripts we have $F_{-n} = (-1)^{n-1}F_n$ and $L_{-n} = (-1)^n L_n$. They possess the explicit formulas (Binet forms)

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n, \quad n \in \mathbb{Z}, \tag{1}$$

with $\alpha = (1+\sqrt{5})/2$ being the golden ratio and $\beta = -1/\alpha$. The sequences $(F_n)_{n\geq 0}$ and $(L_n)_{n\geq 0}$ are indexed in the On-Line Encyclopedia of Integer Sequences [19] as entries A000045 and A000032, respectively. For more information we refer to Koshy [15] and Vajda [21], who have written excellent books dealing with Fibonacci and Lucas numbers.

For an integer $n \geq 0$, the Chebyshev polynomials $\big(T_n(x)\big)_{n\geq 0}$ of the first kind are defined by the second-order recurrence relation [18]

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),$$

while the Chebyshev polynomials $\big(U_n(x)\big)_{n\geq 0}$ of the second kind are defined by

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x).$$

The Chebyshev polynomials possess the representations

$$T_n(x) = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n}{2k} (x^2-1)^k x^{n-2k}, \tag{2}$$

$$U_n(x) = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n+1}{2k+1} (x^2-1)^k x^{n-2k}. \tag{3}$$

Also, the sequences $T_n(x)$ and $U_n(x)$ have the Binet-like formulas

$$T_n(x) = \frac{1}{2}\Big((x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n\Big), \tag{4}$$

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}\Big((x+\sqrt{x^2-1})^{n+1} - (x-\sqrt{x^2-1})^{n+1}\Big). \tag{5}$$

The Chebyshev polynomials of the first and second kind are connected by the equations

$$\begin{aligned} T_{n+1}(x) &= xT_n(x) - (1-x^2)U_{n-1}(x), \\ U_{n+1}(x) &= xU_n(x) + T_{n+1}(x), \end{aligned} \tag{6}$$

from which we also get

$$T_n(x) = \frac{1}{2}\big(U_n(x) - U_{n-2}(x)\big).$$

The properties of Chebyshev polynomials have been studied extensively in the literature. In the recent papers [1, 7, 8, 9, 13, 14, 16, 17, 22] the reader can find additional information about them, especially about their products, convolutions, and power sums, as well as their connections with Fibonacci numbers and polynomials.

There exists a countless number of binomial sum identities involving Fibonacci and Lucas numbers. For some new papers in this field, we refer to [2, 3, 4, 5]. Motivated by these studies, the goal of this paper is to extend existing results to other types of binomial coefficients. More precisely, in this paper we deal with sums of the following forms

$$\sum_{k=0}^{n} \frac{n}{n+k}\binom{n+k}{n-k} x_k \qquad \text{and} \qquad \sum_{k=0}^{n} \frac{k}{n+k}\binom{n+k}{n-k} x_k,$$

where the x_k are some weighted Fibonacci (Lucas) entries. Among the huge number of Fibonacci (Lucas) sums in the literature, we could not find references treating these forms, although the binomial coefficients remind us of coefficients of Girard–Waring type [10], Jennings [11], and also Kilic and Ioanescu [12]. It may be also of interest that we can write the sums under consideration equivalently involving three binomial coefficients. For instance, the first type of sum equals

$$\sum_{k=0}^{n} \frac{\binom{n+k-1}{k}\binom{n}{k}}{\binom{2k}{k}} x_k,$$

and the second is similar.

2 Binomial Fibonacci and Lucas sums from identities involving $T_n(x)$

We start by deriving two identities involving $T_n(x)$, which we prove for the readers' convenience.

Theorem 1. *For all $x \in \mathbb{C}$ and non-negative integers n and m, we have the following identities:*

$$\sum_{k=0}^{n} (-2)^k \frac{n}{n+k}\binom{n+k}{n-k}(1 \mp x)^k = (\pm 1)^n T_n(x), \tag{7}$$

$$\sum_{k=0}^{n} (-2)^k \frac{n}{n+k}\binom{n+k}{n-k}\big(1 \mp T_m(x)\big)^k = (\pm 1)^n T_{nm}(x). \tag{8}$$

Proof. It suffices to prove (7). Let $P_n^{(a,b)}(x)$, $a, b > -1$, be the Jacobi polynomial. Then $P_n^{(a,b)}(x)$ has the truncated series representation

$$P_n^{(a,b)}(x) = \frac{\Gamma(a+n+1)}{n!\Gamma(a+b+n+1)} \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(a+b+n+k+1)}{2^k\Gamma(a+k+1)}(x-1)^k.$$

Now we use the fact that $T_n(x) = \frac{4^n(n!)^2}{(2n)!}P_n^{(-1/2,-1/2)}(x)$. The statement follows after some steps of simplifications using $\Gamma\big(n+\frac{1}{2}\big) = \frac{(2n)!\sqrt{\pi}}{4^n n!}$.

Identity (8) follows immediately if $x \to T_m(x)$, because $T_n\big(T_m(x)\big) = T_{nm}(x)$. $\square$

We note that the relation $T_n(2x^2-1) = T_{2n}(x)$ in conjunction with (7) yields

$$\sum_{k=0}^{n} 4^k \frac{n}{n+k}\binom{n+k}{n-k}(x^2-1)^k = T_{2n}(x) \tag{9}$$

and

$$\sum_{k=0}^{n} (-4)^k \frac{n}{n+k}\binom{n+k}{n-k}x^{2k} = (-1)^n T_{2n}(x). \tag{10}$$

Example 2. Setting $x = 0$ and $x = -1$ in (7) we get

$$\sum_{k=0}^{n} (-2)^k \frac{n}{n+k}\binom{n+k}{n-k} = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ (-1)^{n/2}, & \text{otherwise;} \end{cases}$$

$$\sum_{k=0}^{n} (-4)^k \frac{n}{n+k}\binom{n+k}{n-k} = (-1)^n.$$

Example 3. Directly from (7) we have the following interesting sums:

$$\sum_{k=0}^{n} (-2)^k \frac{n}{n+k}\binom{n+k}{n-k}L_k = T_n(\alpha) + T_n(\beta),$$

$$\sum_{k=0}^{n} (-2)^k \frac{n}{n+k}\binom{n+k}{n-k}F_k = \frac{T_n(\alpha) - T_n(\beta)}{-\sqrt{5}}.$$

It follows from these formulas that $T_n(\alpha) + T_n(\beta)$ and $\frac{1}{\sqrt{5}}\big(T_n(\alpha) - T_n(\beta)\big)$ are integers. The sequence $\big(\frac{1}{\sqrt{5}}\big(T_n(\alpha) - T_n(\beta)\big)\big)_{n\geq 0} = \big\{0, 1, 2, 5, 16, 45, 130, 377, 1088, 3145, \dots\big\}$ is sequence A138573 in the OEIS [19]. But more is true as we show in the next theorem.

Theorem 4. *For an integer s, the sequences $\big(T_n(\alpha^s) + T_n(\beta^s)\big)_{n\geq 0}$ and $\big(\frac{T_n(\alpha^s) - T_n(\beta^s)}{\sqrt{5}}\big)_{n\geq 0}$ are integers. Moreover, the sequences $\big(U_n(\alpha^s) + U_n(\beta^s)\big)_{n\geq 0}$ and $\big(\frac{U_n(\alpha^s) - U_n(\beta^s)}{\sqrt{5}}\big)_{n\geq 0}$ are also integer sequences for each s.*

Proof. For a complex variable x, from (2) we have

$$T_n(\alpha^s x) = \sum_{k=0}^{\lfloor n/2\rfloor}\sum_{j=0}^{k}(-1)^j\binom{n}{2k}\binom{k}{j}(\alpha^s x)^{n-2j},$$

$$T_n(\beta^s x) = \sum_{k=0}^{\lfloor n/2\rfloor}\sum_{j=0}^{k}(-1)^j\binom{n}{2k}\binom{k}{j}(\beta^s x)^{n-2j}.$$

In particular,

$$T_n(\alpha^s) = \begin{cases} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} L_s^k \alpha^{s(n-k)}, & \text{if } s \text{ is odd;} \\ \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (\sqrt{5}F_s)^k \alpha^{s(n-k)}, & \text{otherwise,} \end{cases}$$

$$T_n(\beta^s) = \begin{cases} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} L_s^k \beta^{s(n-k)}, & \text{if } s \text{ is odd;} \\ \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (-\sqrt{5}F_s)^k \beta^{s(n-k)}, & \text{otherwise.} \end{cases}$$

Thus,

$$T_n(\alpha^s) + T_n(\beta^s) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} L_s^k L_{s(n-k)}, \qquad s \text{ odd,}$$

$$\frac{T_n(\alpha^s) - T_n(\beta^s)}{\sqrt{5}} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} L_s^k F_{s(n-k)}, \qquad s \text{ odd,}$$

and

$$T_n(\alpha^s) + T_n(\beta^s) = \sum_{k=0}^{\lfloor n/4 \rfloor} 5^k F_s^{2k} \left(\binom{n}{4k} L_{s(n-2k)} + 5\binom{n}{4k+2} F_s F_{s(n-2k-1)} \right), \qquad s \text{ even,}$$

$$\frac{T_n(\alpha^s) - T_n(\beta^s)}{\sqrt{5}} = \sum_{k=0}^{\lfloor n/4 \rfloor} 5^k F_s^{2k} \left(\binom{n}{4k} F_{s(n-2k)} + \binom{n}{4k+2} F_s L_{s(n-2k-1)} \right), \qquad s \text{ even.}$$

For Chebyshev polynomials of the second kind similar formulas hold. From (3) using Binet's formulas (1) we have

$$U_n(\alpha^s) + U_n(\beta^s) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2k+1} L_s^k L_{s(n-k)}, \qquad s \text{ odd,}$$

$$\frac{U_n(\alpha^s) - U_n(\beta^s)}{\sqrt{5}} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2k+1} L_s^k F_{s(n-k)}, \qquad s \text{ odd,}$$

and

$$U_n(\alpha^s) + U_n(\beta^s) = \sum_{k=0}^{\lfloor n/4 \rfloor} 5^k F_s^{2k} \left(\binom{n+1}{4k+1} L_{s(n-2k)} + 5\binom{n+1}{4k+3} F_s F_{s(n-2k-1)} \right), \qquad s \text{ even,}$$

$$\frac{U_n(\alpha^s) - U_n(\beta^s)}{\sqrt{5}} = \sum_{k=0}^{\lfloor n/4 \rfloor} 5^k F_s^{2k} \left(\binom{n+1}{4k+1} F_{s(n-2k)} + \binom{n+1}{4k+3} F_s L_{s(n-2k-1)} \right), \qquad s \text{ even.}$$

□

Example 5. Here we proceed with additional interesting sums of the same kind as in Example 3:

$$\sum_{k=0}^{n}(-2)^k\frac{n}{n+k}\binom{n+k}{n-k}L_{2k}=(-1)^n\big(T_n(\alpha)+T_n(\beta)\big),$$

$$\sum_{k=0}^{n}(-2)^k\frac{n}{n+k}\binom{n+k}{n-k}F_{2k}=\frac{(-1)^n}{\sqrt{5}}\big(T_n(\alpha)-T_n(\beta)\big),$$

$$\sum_{k=0}^{n}(-2)^k\frac{n}{n+k}\binom{n+k}{n-k}L_{3k}=(-1)^n\big(T_n(2\alpha)+T_n(2\beta)\big),$$

$$\sum_{k=0}^{n}(-2)^k\frac{n}{n+k}\binom{n+k}{n-k}F_{3k}=\frac{(-1)^n}{\sqrt{5}}\big(T_n(2\alpha)-T_n(2\beta)\big),$$

$$\sum_{k=0}^{n}\frac{n}{n+k}\binom{n+k}{n-k}L_{k}=T_n\Big(\frac{\sqrt{5}\alpha}{2}\Big)+T_n\Big(\frac{\sqrt{5}\beta}{2}\Big),$$

$$\sum_{k=0}^{n}\frac{n}{n+k}\binom{n+k}{n-k}F_{k}=\frac{1}{\sqrt{5}}\Big(T_n\Big(\frac{\sqrt{5}\alpha}{2}\Big)-T_n\Big(\frac{\sqrt{5}\beta}{2}\Big)\Big),$$

$$\sum_{k=0}^{n}4^k\frac{n}{n+k}\binom{n+k}{n-k}L_{k}=T_{2n}(\alpha)+T_{2n}(\beta),$$

$$\sum_{k=0}^{n}4^k\frac{n}{n+k}\binom{n+k}{n-k}F_{k}=\frac{1}{\sqrt{5}}\big(T_{2n}(\alpha)-T_{2n}(\beta)\big).$$

By substituting $x=L_p/2$ and $x=\sqrt{5}F_p/2$ in (4), in turn, we obtain the results stated in Lemma 6.

Lemma 6. *If p is an integer, then we have*

$$T_n\Big(\frac{L_p}{2}\Big)=\frac{1}{2}L_{pn},\qquad p\text{ even},\tag{11}$$

and

$$T_n\Big(\frac{\sqrt{5}F_p}{2}\Big)=\begin{cases}\frac{1}{2}L_{pn}, & \text{if } p \text{ is odd and } n \text{ is even;}\\ \frac{\sqrt{5}}{2}F_{pn}, & \text{if } p \text{ is odd and } n \text{ is odd.}\end{cases}\tag{12}$$

In particular, we have

$$T_n\Big(\frac{3}{2}\Big)=\frac{1}{2}L_{2n},$$

$$T_n\Big(\frac{\sqrt{5}}{2}\Big)=\begin{cases}\frac{1}{2}L_{n}, & \text{if } n \text{ is even;}\\ \frac{\sqrt{5}}{2}F_{n}, & \text{otherwise;}\end{cases}\tag{13}$$

$$T_n\big(\sqrt{5}\big)=\begin{cases}\frac{1}{2}L_{3n}, & \text{if } n \text{ is even;}\\ \frac{\sqrt{5}}{2}F_{3n}, & \text{otherwise.}\end{cases}\tag{14}$$

Theorem 7. *If n is a positive integer and p is an integer, then the following identities hold:*

$$\sum_{k=0}^{n}(-1)^{(p-1)(n-k)}\frac{n}{n+k}\binom{n+k}{n-k}L_p^{2k}=\frac{L_{2pn}}{2},$$
$$\sum_{k=0}^{n}(-1)^{p(n-k)}\frac{n}{n+k}\binom{n+k}{n-k}5^kF_p^{2k}=\frac{L_{2pn}}{2}.$$

Proof. Set $x=L_{2p}/2$ in (7), use (11) and the fact that

$$L_{2p}-2=\begin{cases}5F_p^2, & \text{if } p \text{ is even;}\\ L_p^2, & \text{otherwise;}\end{cases}\qquad L_{2p}+2=\begin{cases}5F_p^2, & \text{if } p \text{ is odd;}\\ L_p^2, & \text{otherwise;}\end{cases}\tag{15}$$

to get

$$\sum_{k=0}^{n}\frac{n}{n+k}\binom{n+k}{n-k}L_p^{2k}=\frac{1}{2}L_{2pn},\qquad p \text{ odd},$$
$$\sum_{k=0}^{n}\frac{n}{n+k}\binom{n+k}{n-k}(-1)^kL_p^{2k}=\frac{(-1)^n}{2}L_{2pn},\qquad p \text{ even},$$

and

$$\sum_{k=0}^{n}\frac{n}{n+k}\binom{n+k}{n-k}(-5)^kF_p^{2k}=\frac{(-1)^n}{2}L_{2pn},\qquad p \text{ odd},$$
$$\sum_{k=0}^{n}\frac{n}{n+k}\binom{n+k}{n-k}5^kF_p^{2k}=\frac{1}{2}L_{2pn},\qquad p \text{ even},$$

from which the stated identities follow. □

Theorem 7 can be generalized in the following way.

Theorem 8. *If n is a positive integer and p is an integer, then we have*

$$\sum_{k=0}^{n}(-1)^{n-k}\frac{n}{n+k}\binom{n+k}{n-k}(2\pm\sqrt{5}F_{pm})^k=\frac{\pm\sqrt{5}}{2}F_{pmn},\qquad p,\ m,\ n \text{ odd},$$
$$\sum_{k=0}^{n}(-1)^{n-k}\frac{n}{n+k}\binom{n+k}{n-k}(2\pm L_{pm})^k=\frac{(\pm1)^n}{2}L_{pmn},\qquad p \text{ odd},\ m \text{ even},$$
$$\sum_{k=0}^{n}(-1)^{n-k}\frac{n}{n+k}\binom{n+k}{n-k}(2\pm L_{pm})^k=\frac{(\pm1)^n}{2}L_{pmn},\qquad p \text{ even}.$$

Proof. Combine (8) with (11) and (12). □

Some particular cases of Theorems 7 and 8 stated in the next Example.

Example 9. We have

$$\sum_{k=0}^{n}\frac{n}{n+k}\binom{n+k}{n-k}\alpha^{-3k}=\sum_{k=0}^{n}(-1)^{k+1}\frac{n}{n+k}\binom{n+k}{n-k}\alpha^{3k}=\frac{\sqrt{5}F_n}{2},\qquad n\text{ odd},$$
$$\sum_{k=0}^{n}\frac{n}{n+k}\binom{n+k}{n-k}4^k\alpha^{-k}=\sum_{k=0}^{n}(-1)^{k+1}\frac{n}{n+k}\binom{n+k}{n-k}4^k\alpha^{k}=\frac{\sqrt{5}F_{3n}}{2},\qquad n\text{ odd},$$
$$\sum_{k=0}^{n}\frac{n}{n+k}\binom{n+k}{n-k}=\frac{L_{2n}}{2},\qquad \sum_{k=0}^{n}(-1)^{n-k}\frac{n}{n+k}\binom{n+k}{n-k}5^k=\frac{L_{2n}}{2},$$
$$\sum_{k=0}^{n}\frac{n}{n+k}\binom{n+k}{n-k}5^k=\frac{L_{4n}}{2},\qquad \sum_{k=0}^{n}(-1)^{n-k}\frac{n}{n+k}\binom{n+k}{n-k}9^k=\frac{L_{4n}}{2}.$$

Lemma 10. *If p is a non-zero integer, then*

$$T_n\Big(\frac{\sqrt{5}F_p}{L_p}\Big)=\cosh\Big(n\operatorname{arctanh}\Big(\frac{2}{\sqrt{5}F_p}\Big)\Big),\qquad p\text{ odd},\tag{16}$$
$$T_n\Big(\frac{L_p}{\sqrt{5}F_p}\Big)=\cosh\Big(n\operatorname{arctanh}\Big(\frac{2}{L_p}\Big)\Big),\qquad p\text{ even}.\tag{17}$$

Proof. Setting $x=\sqrt{5}F_p/L_p$ in (4) and making use of $5F_p^2-4(-1)^{p+1}=L_p^2$ with p odd produces

$$T_n\Big(\frac{\sqrt{5}F_p}{L_p}\Big)=\frac{(\sqrt{5}F_p-2)^n+(\sqrt{5}F_p+2)^n}{2L_p^n}\tag{18}$$

from which (16) follows upon using the identity

$$(x-y)^n+(x+y)^n=2\big(\sqrt{x^2-y^2}\big)^n\cosh\Big(n\operatorname{arctanh}\Big(\frac{y}{x}\Big)\Big).$$

□

For low values of p, it is easier to use (18) directly for evaluation. Thus, at $p=1$ we recover (14) while $p=3$ gives (13). On account of (15), Formula (17) also implies

$$T_n\Big(\frac{L_{2p}}{\sqrt{5}F_{2p}}\Big)=\frac{(5F_p^2)^n+(L_p^2)^n}{2(\sqrt{5}F_{2p})^n}\tag{19}$$

for every non-zero integer p.

We also note that

$$\sum_{k=0}^{2n}\Big(\frac{4}{5}\Big)^k\binom{2n+k}{2n-k}\frac{4^k-(-1)^kL_{2p}^{2k}}{(2n+k)F_{2p}^{2k}}=0,\qquad p\neq 0,$$
$$\sum_{k=0}^{2n}\Big(\frac{4}{5}\Big)^k\binom{2n+k}{2n-k}\frac{4^k+(-1)^kL_{2p}^{2k}}{(2n+k)F_{2p}^{2k}}=\frac{625^nF_p^{8n}+L_p^{8n}}{2n\,25^nF_{2p}^{4n}},\qquad p\neq 0.$$

Theorem 11. *If p is a non-zero integer and n is a positive integer, then we have*

$$\sum_{k=0}^{n}\left(\frac{16}{5}\right)^k \frac{n}{n+k}\binom{n+k}{n-k}F_{2p}^{2(n-k)} = \frac{25^n F_p^{4n}+L_p^{4n}}{2\cdot 5^n},$$

$$\sum_{k=0}^{n}(-1)^{n-k}\left(\frac{4}{5}\right)^k \frac{n}{n+k}\binom{n+k}{n-k}\left(\frac{F_{2p}}{L_{2p}}\right)^{2(n-k)} = \frac{25^n F_p^{4n}+L_p^{4n}}{2\cdot 5^n L_{2p}^{2n}}.$$

In particular, we obtain

$$\sum_{k=0}^{n}\left(\frac{16}{5}\right)^k \frac{n}{n+k}\binom{n+k}{n-k} = \frac{25^n+1}{2\cdot 5^n},$$

$$\sum_{k=0}^{n}(-1)^{n-k}\left(\frac{36}{5}\right)^k \frac{n}{n+k}\binom{n+k}{n-k} = \frac{25^n+1}{2\cdot 5^n}.$$

Proof. Combine (19) with (9) and (10), respectively, while setting $x = L_{2p}/(\sqrt{5}F_{2p})$. □

Remark 12. We note the following relations:

$$\sum_{k=0}^{n}\left(\frac{5}{16}\right)^{n-k} \frac{n}{n+k}\binom{n+k}{n-k}F_{2p}^{2(n-k)} = \sum_{k=0}^{n}\binom{2n}{2k}\frac{L_{2p}^{2(n-k)}}{4^{2n-k}},$$

$$\sum_{k=0}^{n}\left(-\frac{5}{4}\right)^{n-k} \frac{n}{n+k}\binom{n+k}{n-k}\left(\frac{F_{2p}}{L_{2p}}\right)^{2(n-k)} = \sum_{k=0}^{n}\binom{2n}{2k}\frac{4^{k-n}}{L_{2p}^{2k}}.$$

Theorem 13. *If p is an odd integer and n is a positive integer, then*

$$\sum_{k=0}^{n}(-4)^k \frac{n}{n+k}\binom{n+k}{n-k}\frac{L_{pk+t}}{L_p^k} = \cosh\left(n\,\mathrm{arctanh}\left(\frac{2}{\sqrt{5}F_p}\right)\right)\cdot\begin{cases} L_t, & \text{if } n \text{ is even;}\\ -\sqrt{5}F_t, & \text{otherwise;}\end{cases}$$

$$\sum_{k=0}^{n}(-4)^k \frac{n}{n+k}\binom{n+k}{n-k}\frac{F_{pk+t}}{L_p^k} = \cosh\left(n\,\mathrm{arctanh}\left(\frac{2}{\sqrt{5}F_p}\right)\right)\cdot\begin{cases} F_t, & \text{if } n \text{ is even;}\\ -\frac{L_t}{\sqrt{5}}, & \text{otherwise.}\end{cases}$$

Proof. Set $x = \sqrt{5}F_p/L_p$ in (7), taking the upper signs, and then use (16). □

Corollary 14. *If p is an odd integer and n is a positive integer, then*

$$\sum_{k=0}^{n}(-4)^k \frac{n}{n+k}\binom{n+k}{n-k}\frac{L_{pk}}{L_p^k} = 0, \qquad n \text{ odd},$$

$$\sum_{k=0}^{n}(-4)^k \frac{n}{n+k}\binom{n+k}{n-k}\frac{F_{pk}}{L_p^k} = 0, \qquad n \text{ even}.$$

Corollary 15. *If n is a positive integer, then we have the following combinatorial results:*

$$\sum_{k=0}^{n}(-4)^k\frac{n}{n+k}\binom{n+k}{n-k}L_{k+t}=\begin{cases}\frac{1}{2}L_tL_{3n}, & \text{if } n \text{ is even;}\\ -\frac{5}{2}F_tF_{3n}, & \text{otherwise;}\end{cases}$$

$$\sum_{k=0}^{n}(-4)^k\frac{n}{n+k}\binom{n+k}{n-k}F_{k+t}=\begin{cases}\frac{1}{2}F_tL_{3n}, & \text{if } n \text{ is even;}\\ -\frac{1}{2}L_tF_{3n}, & \textit{otherwise.}\end{cases}$$

The corresponding identities to Theorem 13 for p even are stated next.

Theorem 16. *If p is a non-zero even integer, n is a positive integer and t is an integer, then we have the following sum relations:*

$$\sum_{k=0}^{\lfloor n/2\rfloor}\left(\frac{16}{5F_p^2}\right)^k\left(\binom{n+2k}{n-2k}\frac{L_{2pk+t}}{n+2k}-\frac{4}{F_p}\binom{n+2k+1}{n-2k-1}\frac{F_{p(2k+1)+t}}{n+2k+1}\right)$$
$$=\frac{1}{n}\cosh\left(n\operatorname{arctanh}\left(\frac{2}{L_p}\right)\right)\cdot\begin{cases}L_t, & \text{if } n \text{ is even;}\\ -\sqrt{5}F_t, & \text{otherwise;}\end{cases}$$

$$\sum_{k=0}^{\lfloor n/2\rfloor}\left(\frac{16}{5F_p^2}\right)^k\left(\binom{n+2k}{n-2k}\frac{F_{2pk+t}}{n+2k}-\frac{4}{5F_p}\binom{n+2k+1}{n-2k-1}\frac{L_{p(2k+1)+t}}{n+2k+1}\right)$$
$$=\frac{1}{\sqrt{5}\,n}\cosh\left(n\operatorname{arctanh}\left(\frac{2}{L_p}\right)\right)\cdot\begin{cases}\sqrt{5}F_t, & \text{if } n \text{ is even;}\\ -L_t, & \text{otherwise.}\end{cases}$$

The non-alternating versions of the identities in Theorem 13 are derived from (16) and (7) by considering the bottom sign combinations. This is left to the interested reader.

Example 17. Noting that

$$\sum_{k=0}^{n}(-2)^k\frac{n}{n+k}\binom{n+k}{n-k}\big((1+x)^k\mp(1-x)^k\big)=\big((-1)^n\mp 1\big)T_n(x), \tag{20}$$

with $x=\sqrt{5}/2$ and (13) we get

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{n}{n+k}\binom{n+k}{n-k}F_{3k}=\frac{1-(-1)^n}{2}F_n, \tag{21}$$

as well as

$$\sum_{k=0}^{n}(-1)^k\frac{n}{n+k}\binom{n+k}{n-k}L_{3k}=\frac{1+(-1)^n}{2}L_n. \tag{22}$$

As we will see, identities (21) and (22) are special cases of Theorem 20 below.

Example 18. We derive the inverse relation of (21) and (22). Starting with (20) we set $x = -\sqrt{5}$, and make use of $T_n(-\sqrt{5}) = (-1)^n T_n(\sqrt{5})$ together with (14) to derive the following identities:

$$\sum_{k=1}^{n} (-4)^k \frac{n}{n+k} \binom{n+k}{n-k} F_k = \frac{(-1)^n - 1}{2} F_{3n},$$
$$\sum_{k=0}^{n} (-4)^k \frac{n}{n+k} \binom{n+k}{n-k} L_k = \frac{1 + (-1)^n}{2} L_{3n}.$$

Example 19. It is obvious that from (20) more appealing relations can be derived. We give just four examples:

$$\begin{aligned}\sum_{k=0}^{n} (-4)^k \frac{n}{n+k} \binom{n+k}{n-k} \big(L_{2k} \pm (-1)^k L_k\big) &= \big((-1)^n \pm 1\big)\big(T_n(\alpha^3) + T_n(\beta^3)\big) \\ &= \big((-1)^n \pm 1\big) \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 4^k L_{3(n-k)}, \\ \sum_{k=0}^{n} (-4)^k \frac{n}{n+k} \binom{n+k}{n-k} \big(F_{2k} \pm (-1)^k F_k\big) &= \frac{(-1)^n \pm 1}{\sqrt{5}} \big(T_n(\alpha^3) - T_n(\beta^3)\big) \\ &= \frac{(-1)^n \pm 1}{\sqrt{5}} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 4^k F_{3(n-k)}, \\ \sum_{k=0}^{n} \Big(-\frac{4}{3}\Big)^k \frac{n}{n+k} \binom{n+k}{n-k} L_{2k} &= \big((-1)^n + 1\big) T_n\Big(\frac{\sqrt{5}}{3}\Big), \\ \sum_{k=0}^{n} \Big(-\frac{4}{3}\Big)^k \frac{n}{n+k} \binom{n+k}{n-k} F_{2k} &= \frac{(-1)^n - 1}{\sqrt{5}} T_n\Big(\frac{\sqrt{5}}{3}\Big).\end{aligned}$$

Theorem 20. *If n is a positive integer and t is an integer, then we have*

$$\sum_{k=0}^{n} (-1)^{n-k} \frac{n}{n+k} \binom{n+k}{n-k} L_{3k+t} = \begin{cases} \frac{5}{2} F_t F_n, & \text{if } n \text{ is odd;} \\ \frac{1}{2} L_t L_n, & \text{otherwise;} \end{cases}$$
$$\sum_{k=0}^{n} (-1)^{n-k} \frac{n}{n+k} \binom{n+k}{n-k} F_{3k+t} = \begin{cases} \frac{1}{2} L_t F_n, & \text{if } n \text{ is odd;} \\ \frac{1}{2} F_t L_n, & \text{otherwise.} \end{cases}$$

Proof. Set $x = \sqrt{\alpha^3}/2$ and $x = \sqrt{\beta^3}/2$, in turn, in (10) and use

$$T_{2n}\Big(\frac{\sqrt{\alpha^3}}{2}\Big) = \begin{cases} \frac{\sqrt{5}}{2} F_n, & \text{if } n \text{ is odd;} \\ \frac{1}{2} L_n, & \text{otherwise;} \end{cases} \qquad T_{2n}\Big(\frac{\sqrt{\beta^3}}{2}\Big) = \begin{cases} -\frac{\sqrt{5}}{2} F_n, & \text{if } n \text{ is odd;} \\ \frac{1}{2} L_n, & \text{otherwise.} \end{cases}$$

to obtain

$$\sum_{k=0}^{n}(-1)^{n-k}\frac{n}{n+k}\binom{n+k}{n-k}(\alpha^{3k+t}+\lambda\beta^{3k+t})=\begin{cases}\frac{\sqrt{5}}{2}F_n(\alpha^t-\lambda\beta^t), & \text{if } n \text{ is odd;}\\ \frac{1}{2}L_n(\alpha^t-\lambda\beta^t), & \text{otherwise;}\end{cases}$$

from which the stated results follow upon setting $\lambda=1$ and $\lambda=-1$. □

Note that Example 17 is the special case ($t=0$) of Theorem 20.

Theorem 21. *If n is a positive integer and p is a non-zero integer, then we have*

$$\sum_{k=0}^{n}(-1)^{n-k}\frac{n}{n+k}\binom{n+k}{n-k}\Big(\frac{2L_{2p}}{\sqrt{5}F_{2p}}\Big)^{2k}=\frac{1}{2}\bigg(\Big(\frac{\sqrt{5}F_p}{L_p}\Big)^{2n}+\Big(\frac{L_p}{\sqrt{5}F_p}\Big)^{2n}\bigg).$$

Proof. Set $x=L_{2p}/(\sqrt{5}F_{2p})$ in (10) and then use (19). □

Theorem 22. *If n is a non-negative integer and p and q are non-zero integers, then we have*

$$\sum_{k=0}^{n}(-1)^{(p-q)(n-k)}\frac{n}{n+k}\binom{n+k}{n-k}\left(\frac{F_{p+q}F_{p-q}}{F_pF_q}\right)^{2k}=\frac{F_p^{4n}+F_q^{4n}}{2(F_qF_p)^{2n}}.$$

Proof. Set $x=\dfrac{(-i)^{p-q+1}F_{p+q}F_{p-q}}{2F_qF_p}$, where $i=\sqrt{-1}$, in (10) and use

$$T_n\left(\frac{i^{p-q+1}F_{p+q}F_{p-q}}{2F_qF_p}\right)=i^{n(p-q+1)}\frac{F_p^{2n}+(-1)^{n(p-q+1)}F_q^{2n}}{2(F_qF_p)^n}.$$

□

Theorem 23. *If n is a non-negative integer and p is a non-zero integer, then we have*

$$\sum_{k=0}^{n}(-1)^{p(n-k)}\frac{n}{n+k}\binom{n+k}{n-k}\left(\frac{F_{3p}}{F_{2p}}\right)^{2k}=\frac{L_p^{4n}+1}{2L_p^{2n}},$$

$$\sum_{k=0}^{n}(-1)^{(p+1)(n-k)}\frac{n}{n+k}\binom{n+k}{n-k}5^{n-k}\left(\frac{L_{3p}}{F_{2p}}\right)^{2k}=\frac{5^{2n}F_p^{4n}+1}{2F_p^{2n}}.$$

Proof. Set $x=\dfrac{(-i)^{p+1}F_{3p}}{2F_{2p}}$ and $x=\dfrac{(-i)^pL_{3p}}{2\sqrt{5}F_{2p}}$, in turn, in (10) and use

$$T_n\left(\frac{(-i)^{p+1}F_{3p}}{2F_{2p}}\right)=(-i)^{n(p+1)}\frac{L_p^{2n}+(-1)^{n(p+1)}}{2L_p^n},$$

$$T_n\left(\frac{(-i)^pL_{3p}}{2\sqrt{5}F_{2p}}\right)=(-i)^{np}\frac{5^nF_p^{2n}+(-1)^{np}}{2\sqrt{5^n}F_p^n}.$$

□

Lemma 24. *If p and q are integers, then the following identities involving Chebyshev polynomials $T_n(x)$ hold:*

$$T_{2n}\left(\sqrt{\frac{(-1)^{q+1}F_q^2}{4F_pF_{p+q}}\alpha^{2p+q}}\right) = \frac{(-1)^n}{2}\left((-1)^{nq}\frac{F_{p+q}^n}{F_p^n}\alpha^{qn} + \frac{F_p^n}{F_{p+q}^n}\beta^{qn}\right), \quad p \neq 0,\, p \neq -q, \tag{23}$$

$$T_{2n}\left(\sqrt{\frac{(-1)^{q+1}F_q^2}{4F_pF_{p+q}}\beta^{2p+q}}\right) = \frac{(-1)^n}{2}\left((-1)^{nq}\frac{F_{p+q}^n}{F_p^n}\beta^{qn} + \frac{F_p^n}{F_{p+q}^n}\alpha^{qn}\right), \quad p \neq 0,\, p \neq -q,$$

$$T_{2n}\left(\sqrt{\frac{5(-1)^{q+1}F_q^2}{4L_pL_{p+q}}\alpha^{2p+q}}\right) = \frac{(-1)^n}{2}\left((-1)^{nq}\frac{L_{p+q}^n}{L_p^n}\alpha^{qn} + \frac{L_p^n}{L_{p+q}^n}\beta^{qn}\right),$$

$$T_{2n}\left(\sqrt{\frac{5(-1)^{q+1}F_q^2}{4L_pL_{p+q}}\beta^{2p+q}}\right) = \frac{(-1)^n}{2}\left((-1)^{nq}\frac{L_{p+q}^n}{L_p^n}\beta^{qn} + \frac{L_p^n}{L_{p+q}^n}\alpha^{qn}\right).$$

Theorem 25. *If n is a positive integer and p, q and t are integers with $p \neq 0$ and $q \neq 0$, then we have the following combinatorial results:*

$$\sum_{k=0}^{n} \frac{(-1)^{(n-k)q}\, n}{n+k}\binom{n+k}{n-k} F_{p+q}^{n-k}F_p^{n-k}F_q^{2k}L_{k(2p+q)+t} = \frac{1}{2}\left(F_{p+q}^{2n}L_{t+qn} + F_p^{2n}L_{t-qn}\right), \tag{24}$$

$$\sum_{k=0}^{n} \frac{(-1)^{(n-k)q}\, n}{n+k}\binom{n+k}{n-k} F_{p+q}^{n-k}F_p^{n-k}F_q^{2k}F_{k(2p+q)+t} = \frac{1}{2}\left(F_{p+q}^{2n}F_{t+qn} + F_p^{2n}F_{t-qn}\right), \tag{25}$$

$$\sum_{k=0}^{n} \frac{(-1)^{(n-k)q}\, n}{n+k}\binom{n+k}{n-k} 5^kL_{p+q}^{n-k}L_p^{n-k}F_q^{2k}L_{k(2p+q)+t} = \frac{1}{2}\left(L_{p+q}^{2n}L_{t+qn} + L_p^{2n}L_{t-qn}\right), \tag{26}$$

$$\sum_{k=0}^{n} \frac{(-1)^{(n-k)q}\, n}{n+k}\binom{n+k}{n-k} 5^kL_{p+q}^{n-k}L_p^{n-k}F_q^{2k}F_{k(2p+q)+t} = \frac{1}{2}\left(L_{p+q}^{2n}F_{t+qn} + L_p^{2n}F_{t-qn}\right). \tag{27}$$

Proof. In (10) set $x = \sqrt{\frac{(-1)^{q+1}F_q^2}{4F_pF_{p+q}}\alpha^{2p+q}}$ and use (23) to obtain

$$\sum_{k=0}^{n}(-1)^{kq}\frac{n}{n+k}\binom{n+k}{n-k}F_{p+q}^{n-k}F_p^{n-k}F_q^{2k}\alpha^{k(2p+q)+t} = \frac{(-1)^{nq}}{2}F_{p+q}^{2n}\alpha^{qn+t} + \frac{(-1)^t}{2}F_p^{2n}\beta^{qn-t},$$

from which (24) and (25) follow. The proof of (26) and (27) is similar; in (10), set $x = \sqrt{\frac{5(-1)^{q+1}F_q^2}{4L_pL_{p+q}}\alpha^{2p+q}}$. □

Example 26. If n is a positive integer and t is an integer, then we get

$$\sum_{k=0}^{n}(-2)^{n-k}\frac{n}{n+k}\binom{n+k}{n-k}L_{5k+t}=\frac{1}{2}\big(L_{t-n}+4^nL_{n+t}\big),$$
$$\sum_{k=0}^{n}(-2)^{n-k}\frac{n}{n+k}\binom{n+k}{n-k}F_{5k+t}=\frac{1}{2}\big(F_{t-n}+4^nF_{n+t}\big),$$
$$\sum_{k=0}^{n}(-2)^{n-k}\frac{n}{n+k}\binom{n+k}{n-k}5^kL_{k+t}=\frac{1}{2}\big(4^nL_{t-n}+L_{n+t}\big),$$
$$\sum_{k=0}^{n}(-2)^{n-k}\frac{n}{n+k}\binom{n+k}{n-k}5^kF_{k+t}=\frac{1}{2}\big(4^nF_{t-n}+F_{n+t}\big).$$

3 Binomial Fibonacci and Lucas sums from identities involving $U_n(x)$

Using the fact that $\frac{d}{dx}T_n(x)=nU_{n-1}(x)$, from (7) we get

$$\sum_{k=1}^{n}(-2)^k\frac{k}{n+k}\binom{n+k}{n-k}(1\mp x)^{k-1}=\mp U_{n-1}(x), \tag{28}$$

and from (9) and (10)

$$\sum_{k=1}^{n}4^k\frac{k}{n+k}\binom{n+k}{n-k}(x^2-1)^{k-1}=\frac{U_{2n-1}(x)}{x}$$

and

$$\sum_{k=1}^{n}(-4)^k\frac{k}{n+k}\binom{n+k}{n-k}x^{2k}=(-1)^nxU_{2n-1}(x). \tag{29}$$

From here, setting $x=3/2$ in (28) and using $U_{n-1}(3/2)=F_{2n}$, we get

$$\sum_{k=1}^{n}\frac{k}{n+k}\binom{n+k}{n-k}=\frac{F_{2n}}{2}$$

and

$$\sum_{k=1}^{n}(-5)^{k-1}\frac{k}{n+k}\binom{n+k}{n-k}=\frac{(-1)^{n-1}}{2}F_{2n}.$$

Also, with $x=\sqrt{5}/2$ upon combining we produce

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{k}{n+k}\binom{n+k}{n-k}L_{3(k-1)}=\frac{1-(-1)^n}{2}L_n,$$

where we used the fact that

$$U_n\Big(\frac{\sqrt5}{2}\Big) = \begin{cases} L_{n+1}, & \text{if } n \text{ is even;} \\ \sqrt5 F_{n+1}, & \text{otherwise.} \end{cases}$$

In general, working with (8) we get the relations

$$\sum_{k=1}^{n} (-2)^{k-1} \frac{k}{n+k}\binom{n+k}{n-k}\big(1 \mp T_m(x)\big)^{k-1} = (\pm1)^{n-1}\frac{U_{nm-1}(x)}{2U_{m-1}(x)},$$

where we have used that

$$U_{m-1}\big(T_n(x)\big) = \frac{U_{mn-1}(x)}{U_{n-1}(x)}.$$

Lemma 27. *If n is a positive integer and p an integer, then we have the following identities involving Chebyshev polynomials of the second kind:*

$$\begin{aligned} U_n\Big(\frac{L_p}{2}\Big) &= \frac{F_{p(n+1)}}{F_p}, \quad p \text{ even}, \ p \neq 0, \\ U_n\Big(\frac{iL_p}{2}\Big) &= \frac{i^n F_{p(n+1)}}{F_p}, \quad p \text{ odd}, \\ U_n\Big(\frac{\sqrt5 F_p}{2}\Big) &= \begin{cases} L_{p(n+1)}/L_p, & \text{if } p \text{ is odd and } n \text{ is even;} \\ \sqrt5 F_{p(n+1)}/L_p, & \text{if } p \text{ and } n \text{ are odd;} \end{cases} \\ U_n\Big(\frac{i\sqrt5 F_p}{2}\Big) &= \begin{cases} i^n L_{p(n+1)}/L_p, & \text{if } p \text{ and } n \text{ are even;} \\ i^n\sqrt5 F_{p(n+1)}/L_p, & \text{if } p \text{ is even and } n \text{ is odd.} \end{cases} \end{aligned}$$

Theorem 28. *If n is a positive integer and p is an integer, then we have*

$$\begin{aligned} \sum_{k=1}^{n} (-1)^{(p-1)(n-k)} \frac{k}{n+k}\binom{n+k}{n-k} L_p^{2k-1} &= \frac{F_{2np}}{2F_p}, \qquad p \neq 0, \\ \sum_{k=1}^{n} (-1)^{p(n-k)} \frac{k}{n+k}\binom{n+k}{n-k} 5^{k-1} F_p^{2k-1} &= \frac{F_{2np}}{2L_p}. \end{aligned}$$

Proof. Evaluate (29) at $x = L_p/2$, $x = iL_p/2$, $x = \sqrt5 F_p/2$, and $x = i\sqrt5 F_p/2$, in turn, using Lemma 27. □

Lemma 29. *If n is a positive integer, then*

$$\begin{aligned} \sqrt{\frac{5}{2\alpha}}\, U_{2n-1}\Bigg(\sqrt{\frac{\alpha^5}{8}}\Bigg) &= 2^n\alpha^n - (-1)^n\frac{\beta^n}{2^n}, \\ \sqrt{\frac{5}{2\beta}}\, U_{2n-1}\Bigg(\sqrt{\frac{\beta^5}{8}}\Bigg) &= 2^n\beta^n - (-1)^n\frac{\alpha^n}{2^n}. \end{aligned} \tag{30}$$

Theorem 30. *If n is a positive integer and t is an integer, then we have*

$$\sum_{k=1}^{n}(-2)^{n-k}\frac{k}{n+k}\binom{n+k}{n-k}L_{5k+t}=\frac{1}{2}\left(4^nF_{t+n+3}-F_{t-n+3}\right),$$
$$\sum_{k=1}^{n}(-2)^{n-k}\frac{k}{n+k}\binom{n+k}{n-k}F_{5k+t}=\frac{1}{10}\left(4^nL_{t+n+3}-L_{n-t-3}\right).$$

Proof. Set $x=\sqrt{\beta^5/8}$ in (29) and use (30) to obtain

$$\sqrt{5}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{2^k}\frac{k}{n+k}\binom{n+k}{n-k}\beta^{5k+t}=(-2)^{n-1}\beta^{n+t+3}+\frac{(-1)^t}{2^{n+1}}\alpha^{n-t-3},$$

from which the results follow. □

Theorem 31. *If n is a non-negative integer and t is an integer, then we have*

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{k}{n+k}\binom{n+k}{n-k}L_{3k+t}=\begin{cases}\frac{1}{2}L_{t+3}L_n, & \text{if } n \text{ is odd;}\\ -\frac{5}{2}F_{t+3}F_n, & \text{otherwise;}\end{cases}$$
$$\sum_{k=1}^{n}(-1)^{k-1}\frac{k}{n+k}\binom{n+k}{n-k}F_{3k+t}=\begin{cases}\frac{1}{2}F_{t+3}L_n, & \text{if } n \text{ is odd;}\\ -\frac{1}{2}L_{t+3}F_n, & \text{otherwise.}\end{cases}$$

Proof. Set $x=\sqrt{\alpha^3}/2$ in (29) and use

$$U_{2n-1}\left(\frac{\sqrt{\alpha^3}}{2}\right)=\begin{cases}\sqrt{\alpha^3}L_n, & \text{if } n \text{ is odd;}\\ \sqrt{5\alpha^3}F_n, & \text{otherwise;}\end{cases}$$

to obtain

$$\sum_{k=1}^{n}(-1)^{k-1}\frac{k}{n+k}\binom{n+k}{n-k}\beta^{3k+t}=\frac{\beta^{t+3}}{2}\cdot\begin{cases}L_n, & \text{if } n \text{ is odd;}\\ \sqrt{5}F_n, & \text{otherwise;}\end{cases}$$

from which the results follow. □

Theorem 32. *If n is a non-negative integer and t is an integer, then we have*

$$\sum_{k=1}^{n}(-4)^{k-1}\frac{k}{n+k}\binom{n+k}{n-k}L_{k+t}=\begin{cases}-\frac{1}{2}L_{t+1}L_{3n}, & \text{if } n \text{ is odd;}\\ \frac{5}{2}F_{t+1}F_{3n}, & \text{otherwise;}\end{cases}$$
$$\sum_{k=1}^{n}(-4)^{k-1}\frac{k}{n+k}\binom{n+k}{n-k}F_{k+t}=\begin{cases}-\frac{1}{2}F_{t+1}L_{3n}, & \text{if } n \text{ is odd;}\\ \frac{1}{2}L_{t+1}F_{3n}, & \text{otherwise.}\end{cases}$$

Proof. Set $x = \sqrt{\alpha}$ in (29) and use

$$U_{2n-1}(\sqrt{\alpha}) = \begin{cases} \frac{\sqrt{\alpha}}{2} L_{3n}, & \text{if } n \text{ is odd;} \\ \frac{\sqrt{5\alpha}}{2} F_{3n}, & \text{otherwise,} \end{cases}$$

to obtain

$$\sum_{k=1}^{n} (-4)^{k-1} \frac{k}{n+k} \binom{n+k}{n-k} \beta^{k+t} = -\frac{\beta^{t+1}}{2} \cdot \begin{cases} L_{3n}, & \text{if } n \text{ is odd;} \\ \sqrt{5} F_{3n}, & \text{otherwise,} \end{cases}$$

from which the results follow. □

Theorem 33. *If n is a non-negative integer and p is a non-zero integer, then we have*

$$\sum_{k=1}^{n} (-1)^{n-k} \frac{k}{n+k} \binom{n+k}{n-k} \left(\frac{2L_{2p}}{\sqrt{5} F_{2p}} \right)^{2k} = (-1)^p \frac{L_{2p}(L_p^{4n} - 5^{2n} F_p^{4n})}{4(\sqrt{5} F_{2p})^{2n}}.$$

Proof. Set $x = L_{2p}/(\sqrt{5} F_{2p})$ in (29) and use

$$U_{n-1}\left(\frac{L_{2p}}{\sqrt{5} F_{2p}} \right) = \frac{(-1)^p (L_p^{2n} - 5^n F_p^{2n})}{4(\sqrt{5} F_{2p})^{n-1}}. \tag{31}$$

□

Theorem 34. *If n is a non-negative integer and p and q are non-zero integers, then we have*

$$\sum_{k=1}^{n} (-1)^{(p-q)(n-k)} \frac{k}{n+k} \binom{n+k}{n-k} \left(\frac{F_{p+q} F_{p-q}}{F_p F_q} \right)^{2k} = \frac{(F_p^{4n} - F_q^{4n}) F_{p+q} F_{p-q}}{2\big(F_p^2 + (-1)^{p-q} F_q^2\big)(F_q F_p)^{2n}}.$$

Proof. Set $x = \dfrac{(-i)^{p-q+1} F_{p+q} F_{p-q}}{2 F_q F_p}$ in (29) and use

$$U_{n-1}\left(\frac{i^{p-q+1} F_{p+q} F_{p-q}}{2 F_q F_p} \right) = \frac{F_p^{2n} + (-1)^{p-q} F_q^{2n}}{\big(F_p^2 + (-1)^{p-q} F_q^2\big)(F_q F_p)^{n-1}}. \tag{32}$$

□

Lemma 35. *We have*

$$
\begin{aligned}
&(-1)^{(q+1)(n+3/2)+\lfloor q/2\rfloor+1}\alpha^{p+q/2}\,U_{2n-1}\left(\sqrt{\frac{(-1)^{q+1}F_q^2}{4F_pF_{p+q}}\alpha^{2p+q}}\right)\\
&\qquad=\frac{F_{p+q}^{2n+1}\alpha^{qn+p}-F_p^{2n}F_{p+q}(-\beta)^{qn-p}+F_{p+q}^{2n}F_p\alpha^{qn+p+q}-F_p^{2n+1}(-\beta)^{qn-p-q}}{\big(F_pF_{p+q}\big)^{n-1/2}\big(F_{p+q}^2+F_pF_{p+q}L_q+(-1)^qF_p^2\big)}, \qquad (33)\\
&(-1)^{(q+1)(n+5/2)+\lfloor q/2\rfloor+p}\beta^{p+q/2}\,U_{2n-1}\left(\sqrt{\frac{(-1)^{q+1}F_q^2}{4F_pF_{p+q}}\beta^{2p+q}}\right),\\
&\qquad=\frac{F_{p+q}^{2n+1}\beta^{qn+p}-F_p^{2n}F_{p+q}(-\alpha)^{qn-p}+F_{p+q}^{2n}F_p\beta^{qn+p+q}-F_p^{2n+1}(-\alpha)^{qn-p-q}}{\big(F_pF_{p+q}\big)^{n-1/2}\big(F_{p+q}^2+F_pF_{p+q}L_q+(-1)^qF_p^2\big)},\\
&(-1)^{(q+1)(n+3/2)+\lfloor q/2\rfloor+1}\alpha^{p+q/2}\,U_{2n-1}\left(\sqrt{\frac{(-1)^{q+1}5F_q^2}{4L_pL_{p+q}}\alpha^{2p+q}}\right)\\
&\qquad=\frac{L_{p+q}^{2n+1}\alpha^{qn+p}-L_p^{2n}L_{p+q}(-\beta)^{qn-p}+L_{p+q}^{2n}L_p\alpha^{qn+p+q}-L_p^{2n+1}(-\beta)^{qn-p-q}}{\big(L_pL_{p+q}\big)^{n-1/2}\big(L_{p+q}^2+L_pL_{p+q}L_q+(-1)^qL_p^2\big)},\\
&(-1)^{(q+1)(n+3/2)+p}\beta^{p+q/2}\,U_{2n-1}\left(\sqrt{\frac{(-1)^{q+1}5F_q^2}{4L_pL_{p+q}}\beta^{2p+q}}\right)\\
&\qquad=\frac{L_p^{2n+1}\beta^{qn+p}-L_p^{2n}L_{p+q}(-\alpha)^{qn-p}+L_{p+q}^{2n}L_p\beta^{qn+p+q}-L_p^{2n+1}(-\alpha)^{qn-p-q}}{(L_pL_{p+q})^{n-1/2}\big(L_{p+q}^2+L_pL_{p+q}L_q+(-1)^qL_p^2\big)}.
\end{aligned}
$$

Theorem 36. *If n is a positive integer, p, q, and t are integers, then we have the following sum evaluations*

$$
\begin{aligned}
&\sum_{k=1}^{n}(-1)^{q(n-k)}\frac{k}{n+k}\binom{n+k}{n-k}F_p^{n-k}F_{p+q}^{n-k}F_q^{2k-1}L_{(2p+q)k+t}\\
&\qquad=\frac{F_{p+q}^{2n}\big(F_{p+q}L_{p+t+qn}+F_pL_{q+p+t+qn}\big)-F_p^{2n}\big(F_{p+q}L_{p+t-qn}+F_pL_{q+p+t-qn}\big)}{2\big(F_{p+q}^2+F_pF_{p+q}L_q+(-1)^qF_p^2\big)}, \qquad (34)\\
&\sum_{k=1}^{n}(-1)^{q(n-k)}\frac{k}{n+k}\binom{n+k}{n-k}F_p^{n-k}F_{p+q}^{n-k}F_q^{2k-1}F_{(2p+q)k+t}\\
&\qquad=\frac{F_{p+q}^{2n}\big(F_{p+q}F_{p+t+qn}+F_pF_{q+p+t+qn}\big)-F_p^{2n}\big(F_{p+q}F_{p+t-qn}+F_pF_{p+q+t-qn}\big)}{2\big(F_{p+q}^2+F_pF_{p+q}L_q+(-1)^qF_p^2\big)}, \qquad (35)\\
&\sum_{k=1}^{n}(-1)^{q(n-k)}\frac{k}{n+k}\binom{n+k}{n-k}L_p^{n-k}L_{p+q}^{n-k}F_q^{2k-1}5^kF_{(2p+q)k+t}\\
&\qquad=\frac{L_{p+q}^{2n}\big(L_{p+q}L_{p+t+qn}+L_pL_{q+p+t+qn}\big)-L_p^{2n}\big(L_{p+q}L_{p+t-qn}+L_pL_{p+q+t-qn}\big)}{2\big(L_{p+q}^2+L_pL_{p+q}L_q+(-1)^qL_p^2\big)}, \qquad (36)
\end{aligned}
$$

and

$$\sum_{k=1}^{n}(-1)^{q(n-k)}\frac{k}{n+k}\binom{n+k}{n-k}L_p^{n-k}L_{p+q}^{n-k}F_q^{2k-1}5^{k-1}L_{(2p+q)k+t}$$
$$= \frac{L_{p+q}^{2n}\big(L_{p+q}F_{qn+p+t}+L_pF_{q+p+t+qn}\big)-L_p^{2n}\big(L_{p+q}F_{p+t-qn}+L_pF_{p+q+t-qn}\big)}{2\big(L_{p+q}^2+L_pL_{p+q}L_q+(-1)^qL_p^2\big)}. \tag{37}$$

Proof. In (29) set $x=\sqrt{\dfrac{(-1)^{q+1}F_q^2}{4F_pF_{p+q}}\alpha^{2p+q}}$ and use (33) to obtain

$$2\sum_{k=1}^{n}(-1)^{qk}\frac{k}{n+k}\binom{n+k}{n-k}F_p^{n-k}F_{p+q}^{n-k}F_q^{2k}\,\alpha^{(2p+q)k+t}=\frac{F_q}{2\big(F_{p+q}^2+F_pF_{p+q}L_q+(-1)^qF_p^2\big)}$$
$$\times\Big((-1)^{nq}F_{p+q}^{2n+1}\,2\alpha^{qn+p+t}-(-1)^{p+t}F_p^{2n}F_{p+q}\,2\beta^{qn-p-t}$$
$$+(-1)^{nq}F_{p+q}^{2n}F_p\,2\alpha^{qn+q+p+t}-(-1)^{p+q+t}F_p^{2n+1}\,2\beta^{qn-p-q-t}\Big),$$

from which (34) and (35) follow. The proof of (36) and (37) is similar; in (29), set $x=\sqrt{\dfrac{(-1)^{q+1}5F_q^2}{4L_pL_{p+q}}\alpha^{2p+q}}$. □

4 Some binomial-coefficient weighted Fibonacci and Lucas sums

In this section we present some Fibonacci and Lucas sums having binomial coefficients as weights.

Theorem 37. *If n is a positive integer and p is an integer, then we have the following identities*

$$\sum_{k=0}^{n}(-1)^{(p-1)(n-k)}\binom{n+k}{n-k}L_p^{2k}=\frac{F_{(2n+1)p}}{F_p},$$
$$\sum_{k=0}^{n}(-1)^{p(n-k)}\binom{n+k}{n-k}5^kF_p^{2k}=\frac{L_{(2n+1)p}}{L_p}.$$

Proof. Addition of each identity stated in Theorem 7 and its counterpart in Theorem 28 while making use of $L_rF_s+L_sF_r=2F_{r+s}$ and $L_rL_s+5F_rF_s=2L_{r+s}$ gives the proof. □

The identities stated below in Theorem 38 follow immediately upon addition of each identity in Theorem 25 to the respective corresponding identity in Theorem 36.

Theorem 38. *If n is a non-negative integer and p, q and t are integers, then we have*

$$\sum_{k=0}^{n}(-1)^{q(n-k)}\binom{n+k}{n-k}F_p^{n-k}F_{p+q}^{n-k}F_q^{2k}L_{(2p+q)k+t}=\frac{1}{2}\left(F_{p+q}^{2n}L_{t+qn}+F_p^{2n}L_{t-qn}\right)$$
$$+\frac{F_q\big(F_{p+q}^{2n}(F_{p+q}L_{p+t+qn}+F_pL_{q+p+t+qn})-F_p^{2n}(F_{p+q}L_{p+t-qn}+F_pL_{q+p+t-qn})\big)}{2\big(F_{p+q}^2+F_pF_{p+q}L_q+(-1)^qF_p^2\big)},$$

$$\sum_{k=0}^{n}(-1)^{q(n-k)}\binom{n+k}{n-k}F_p^{n-k}F_{p+q}^{n-k}F_q^{2k}F_{(2p+q)k+t}=\frac{1}{2}\left(F_{p+q}^{2n}F_{t+qn}-F_p^{2n}F_{t-qn}\right)$$
$$+\frac{F_q\big(F_{p+q}^{2n}(F_{p+q}F_{p+t+qn}+F_pF_{q+p+t+qn})-F_p^{2n}(F_{p+q}F_{t+p-qn}+F_pF_{q+p+t-qn})\big)}{2\big(F_{p+q}^2+F_pF_{p+q}L_q+(-1)^qF_p^2\big)},$$

$$\sum_{k=0}^{n}(-1)^{q(n-k)}\binom{n+k}{n-k}L_p^{n-k}L_{p+q}^{n-k}F_q^{2k}5^kF_{(2p+q)k+t}=\frac{1}{2}\left(L_{p+q}^{2n}F_{t+qn}+L_p^{2n}F_{t-qn}\right)$$
$$+\frac{F_q\big(L_{p+q}^{2n}(L_{p+q}L_{p+t+qn}+L_pL_{q+p+t+qn})-L_p^{2n}(L_{p+q}L_{p+t-qn}+L_pL_{q+p+t-qn})\big)}{2\big(L_{p+q}^2+L_pL_{p+q}L_q+(-1)^qL_p^2\big)},$$

$$\sum_{k=0}^{n}(-1)^{q(n-k)}\binom{n+k}{n-k}L_p^{n-k}L_{p+q}^{n-k}F_q^{2k}5^kL_{(2p+q)k+t}=\frac{1}{2}\left(L_{p+q}^{2n}L_{t+qn}+L_p^{2n}L_{t-qn}\right)$$
$$+\frac{5F_q\left(L_{p+q}^{2n+1}(L_{p+q}F_{p+t+qn}+L_pF_{q+p+t+qn})-L_p^{2n}(L_{p+q}F_{p+t-qn}+L_pF_{q+p+t-qn})\right)}{2(L_{p+q}^2+L_pL_{p+q}L_q+(-1)^qL_p^2)}.$$

Example 39. If n is a non-negative integer and t is an integer, then we have

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n+k}{n-k}L_{3k+t}=\begin{cases}L_{t+n}+L_{t+1}L_n, & \text{if } n \text{ is odd;}\\ L_{t+n}+5F_{t+1}F_n, & \text{otherwise;}\end{cases}$$
$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n+k}{n-k}F_{3k+t}=\begin{cases}F_{t+n}+F_{t+1}L_n, & \text{if } n \text{ is odd;}\\ F_{t+n}+L_{t+1}F_n, & \text{otherwise.}\end{cases}$$

In particular,

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n+k}{n-k}L_{3k}=\begin{cases}2L_n, & \text{if } n \text{ is odd;}\\ 2L_{n+1}, & \text{otherwise;}\end{cases}$$
$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n+k}{n-k}F_{3k}=\begin{cases}2F_{n+1}, & \text{if } n \text{ is odd;}\\ 2F_n, & \text{otherwise;}\end{cases}$$
$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n+k}{n-k}L_{3k-1}=\begin{cases}L_{n+2}, & \text{if } n \text{ is odd;}\\ L_{n-1}, & \text{otherwise;}\end{cases}$$
$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n+k}{n-k}F_{3k-1}=\begin{cases}F_{n-1}, & \text{if } n \text{ is odd;}\\ F_{n+2}, & \text{otherwise.}\end{cases}$$

Example 40. If n is a non-negative integer and t is an integer, then we have

$$\sum_{k=0}^{n}(-2)^{n-k}\binom{n+k}{n-k}L_{5k+t}=2^{2n+1}F_{t+n+1}-F_{t-n},$$
$$\sum_{k=0}^{n}(-2)^{n-k}\binom{n+k}{n-k}F_{5k+t}=\frac{1}{5}\left(2^{2n+1}L_{t+n+1}-L_{t-n}\right).$$

The identities in this section and many similar results can be obtained directly from Lemma 41 which is a consequence of (6), (10), and (29).

Lemma 41. *If n is a non-negative integer and x is a complex variable, then*

$$\sum_{k=0}^{n}(-4)^{k}\binom{n+k}{n-k}x^{2k}=(-1)^{n}U_{2n}(x). \tag{38}$$

Theorem 42. *If n is a non-negative integer and p is a non-zero integer, then we have*

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n+k}{n-k}\left(\frac{2L_{2p}}{\sqrt{5}F_{p}}\right)^{2k}=(-1)^{p}\,\frac{L_{p}^{4n+2}-5^{2n+1}F_{p}^{4n+2}}{4\cdot 5^{n}F_{2p}^{2n}}.$$

Proof. Set $x=L_{2p}/(\sqrt{5}F_{2p})$ in (38) and use (31). □

Theorem 43. *If n is a non-negative integer and p and q are non-zero integers, then we have*

$$\sum_{k=0}^{n}(-1)^{(p-q)(n-k)}\binom{n+k}{n-k}\left(\frac{F_{p+q}F_{p-q}}{F_{p}F_{q}}\right)^{2k}=\frac{F_{p}^{4n+2}+(-1)^{p-q}F_{q}^{4n+2}}{(F_{q}F_{p})^{2n}(F_{p}^{2}+(-1)^{p-q}F_{q}^{2})}.$$

Proof. Set $x=\dfrac{(-i)^{p-q+1}F_{p+q}F_{p-q}}{2F_{q}F_{p}}$ in (38) and use (32). □

5 Related combinatorial sums

This section does not deal with Fibonacci numbers but is inspired by a recent paper by Chu and Guo [6]. In this paper the authors study combinatorial sums of the form $\sum_{k=0}^{n}(-1)^{k}\frac{\binom{n+\lambda}{2k+\delta}}{\binom{n}{k}}$, where λ and n are nonnegative integers, $\delta\in\{0;1\}$. We show how these sums are related to the sums studied in this paper via Theorem 1.

Theorem 44. *For $0<n+m\le 2n+1$, we have the relation*

$$\sum_{k=0}^{n+m}\frac{(-2)^{k}}{(n+m+k)(n-m+k+2)}\frac{\binom{n+m+k}{n+m-k}}{\binom{n-m+k+1}{k}}=\frac{1}{2(n+1)(n+m)}\sum_{k=0}^{n}(-1)^{k}\frac{\binom{n+m}{2k}}{\binom{n}{k}}.$$

In particular,

$$\sum_{k=0}^{n}\frac{(-2)^k}{(n+k)(n+k+1)(n+k+2)}\frac{\binom{n+k}{n-k}}{\binom{n+k}{k}}=\frac{1}{2n(n+1)^2}\sum_{k=0}^{n}(-1)^k\frac{\binom{n}{2k}}{\binom{n}{k}},$$

$$\sum_{k=0}^{2n}\frac{(-2)^k}{(2n+k)(k+1)(k+2)}\binom{2n+k}{2n-k}=\frac{1}{4n(n+1)}\sum_{k=0}^{n}(-1)^k\frac{\binom{2n}{2k}}{\binom{n}{k}}.$$

Proof. From Theorem 1 upon replacing n by $n+m$ we get

$$\sum_{k=0}^{n+m}(-2)^k\frac{n+m}{n+m+k}\binom{n+m+k}{n+m-k}(1-x)^k=T_{n+m}(x).$$

Multiplying through by x^{n-m+1} and integrating from 0 to 1 results in

$$\sum_{k=0}^{n+m}(-2)^k\frac{n+m}{n+m+k}\binom{n+m+k}{n+m-k}\int_0^1 x^{n-m+1}(1-x)^k dx=\int_0^1 x^{n-m+1}T_{n+m}(x)dx.$$

The left-hand side evaluated using the beta function $B(a,b)$ given by

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx,$$

and equals

$$\sum_{k=0}^{n+m}(-2)^k\frac{n+m}{n+m+k}\binom{n+m+k}{n+m-k}B(n-m+2,k+1)$$

with

$$B(n-m+2,k+1)=\frac{1}{n-m+k+2}\binom{n-m+k+1}{k}^{-1}.$$

The right-hand side is evaluated similarly using the representation from (2). We have

$$\begin{aligned}\int_0^1 x^{n-m+1}T_{n+m}(x)dx &= \sum_{k=0}^{\lfloor (n+m)/2\rfloor}(-1)^k\binom{n+m}{2k}\int_0^1 x^{2n-2k+1}(1-x^2)^k dx\\ &= \frac{1}{2}\sum_{k=0}^{\lfloor (n+m)/2\rfloor}(-1)^k\binom{n+m}{2k}\int_0^1 z^{n-k}(1-z)^k dz \qquad (z=x^2)\\ &= \frac{1}{2}\sum_{k=0}^{\lfloor (n+m)/2\rfloor}(-1)^k\binom{n+m}{2k}B(n-k+1,k+1)\\ &= \frac{1}{2(n+1)}\sum_{k=0}^{\lfloor (n+m)/2\rfloor}(-1)^k\frac{\binom{n+m}{2k}}{\binom{n}{k}}.\end{aligned}$$

The final expression follows as $\binom{n}{k}=0$ if $k>n$. □

Keeping in mind that

$$\frac{n}{n+k}\binom{n+k}{n-k} = \frac{\binom{n+k-1}{k}\binom{n}{k}}{\binom{2k}{k}},$$

Theorem 44 is actually an identity containing four binomial coefficients

$$\sum_{k=0}^{n+m} \frac{(-2)^k}{n-m+k+2} \frac{\binom{n+m+k-1}{k}}{\binom{n-m+k+1}{k}} \frac{\binom{n+m}{k}}{\binom{2k}{k}} = \frac{1}{2(n+1)} \sum_{k=0}^{n} (-1)^k \frac{\binom{n+m}{2k}}{\binom{n}{k}}.$$

The special case $m=0$ gives the combinatorial relation valid for all $n \geq 1$

$$\sum_{k=0}^{n} \frac{(-2)^k}{(n+k)(n+k+1)(n+k+2)} \frac{\binom{n}{k}}{\binom{2k}{k}} = \frac{1}{2n(n+1)} - \frac{1}{4(n+1)^2} \sum_{k=0}^{n} \frac{1}{\binom{n}{k}},$$

as Chu and Guo have shown [6, Prop. 5] that

$$\sum_{k=0}^{n} (-1)^k \frac{\binom{n}{2k}}{\binom{n}{k}} = n+1 - \frac{n}{2} \sum_{k=0}^{n} \frac{1}{\binom{n}{k}}.$$

We continue with some more examples based on Chu and Guo's results.

Example 45. When $m = 1, 2, 3, 4$, then we can use the following evaluation from [6]

$$\sum_{k=0}^{n} (-1)^k \frac{\binom{n+1}{2k}}{\binom{n}{k}} = 2 - \sum_{k=0}^{n} \frac{1}{\binom{n}{k}},$$

$$\sum_{k=0}^{n} (-1)^{k+1} \frac{\binom{n+2}{2k}}{\binom{n}{k}} = \frac{2}{n},$$

$$\sum_{k=0}^{n} (-1)^{k+1} \frac{\binom{n+3}{2k}}{\binom{n}{k}} = \frac{2(n-3)}{(n-1)n},$$

$$\sum_{k=0}^{n} (-1)^{k+1} \frac{\binom{n+4}{2k}}{\binom{n}{k}} = \frac{2(n^2-7n+16)}{(n-2)(n-1)n},$$

to get the following summation formulas:

$$\sum_{k=0}^{n+1} \frac{(-2)^k}{(n+1+k)^2} \frac{\binom{n+1+k}{n+1-k}}{\binom{n+k}{k}} = \frac{1}{2(n+1)^2}\left(2 - \sum_{k=0}^{n} \frac{1}{\binom{n}{k}}\right), \qquad n \geq 0,$$

$$\sum_{k=0}^{n+2} \frac{(-2)^k}{(n+k)(n+2+k)} \frac{\binom{n+2+k}{n+2-k}}{\binom{n-1+k}{k}} = \frac{-1}{n(n+1)(n+2)}, \qquad n \geq 1,$$

$$\sum_{k=0}^{n+3} \frac{(-2)^k}{(n-1+k)(n+3+k)} \frac{\binom{n+3+k}{n+3-k}}{\binom{n-2+k}{k}} = \frac{3-n}{(n-1)n(n+1)(n+3)}, \qquad n \geq 2,$$

$$\sum_{k=0}^{n+4} \frac{(-2)^k}{(n-2+k)(n+4+k)} \frac{\binom{n+4+k}{n+4-k}}{\binom{n-3+k}{k}} = \frac{-n^2+7n-16}{(n-2)(n-1)n(n+1)(n+4)}, \qquad n \geq 3.$$

Working with the second part of Theorem 1, applying the same arguments but using the elementary integral

$$\int_0^1 x^{n-m+1}(1+x)^k dx = \sum_{j=0}^{k} \frac{\binom{k}{j}}{n-m+2+j}$$

yields the next theorem involving a double sum.

Theorem 46. *For $m-1 \le n$, we have the relation*

$$\sum_{j=0}^{n+m} \sum_{k=0}^{n+m-j} \frac{(-2)^{k+j}}{(n+m+k+j)(n-m+j+2)} \binom{n+m+k+j}{n+m-k-j} \binom{k+j}{j}$$
$$= \frac{(-1)^{n+m}}{2(n+1)(n+m)} \sum_{k=0}^{n} (-1)^k \frac{\binom{n+m}{2k}}{\binom{n}{k}}.$$

In particular,

$$\sum_{j=0}^{n} \sum_{k=0}^{n-j} \frac{(-2)^{k+j}}{(n+k+j)(n+j+2)} \binom{n+k+j}{n-k-j} \binom{k+j}{j} = \frac{(-1)^n}{2n(n+1)} \sum_{k=0}^{n} (-1)^k \frac{\binom{n}{2k}}{\binom{n}{k}},$$
$$\sum_{j=0}^{2n} \sum_{k=0}^{2n-j} \frac{(-2)^{k+j}}{(2n+k+j)(j+2)} \binom{2n+k+j}{2n-k-j} \binom{k+j}{j} = \frac{1}{4n(n+1)} \sum_{k=0}^{n} (-1)^k \frac{\binom{2n}{2k}}{\binom{n}{k}}.$$

Similar double sums involving ratios of binomial coefficients were studied recently by Stenlund and Wan [20].

The analogous identities involving $\binom{n+m}{2k+1}$ should follow from (28) in conjunction with (3).

6 Conclusion

In this paper, we have derived a huge number of new combinatorial identities involving the binomial coefficients

$$\frac{n}{n+k}\binom{n+k}{n-k} \quad \text{and} \quad \frac{k}{n+k}\binom{n+k}{n-k},$$

and Fibonacci and Lucas numbers. The basic idea behind the proofs is to relate the sums to certain relations involving the Chebyshev polynomials $T_n(x)$ and $U_n(x)$, respectively, evaluated at specific arguments. Theorems 25, 36, and 38 offer additional appealing identities not discussed here. In addition, we have shown a connection between the sums studied in this paper and the combinatorial sums studied recently by Chu and Guo [6].

Extensions of the results presented in the first part of the paper to Fibonacci and Lucas polynomials should be possible without many efforts.

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