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Partition-Theoretic Interpretation for Certain Truncated Series

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Abstract

We obtain generating functions for two new partition functions and provide partition-theoretic interpretations of the truncated pentagonal number theorem and two identities of Gauss.

1 Introduction

In 2012, Andrews and Merca [2] proved the truncated pentagonal number theorem

$$\frac{1}{(q;q)_\infty}\sum_{\tau=0}^{\nu-1}(-1)^\tau q^{\tau(3\tau+1)/2}\left(1-q^{2\tau+1}\right) = 1+(-1)^{\nu-1}\sum_{n=1}^{\infty}\frac{q^{\binom{\nu}{2}+(\nu+1)n}}{(q;q)_n}\begin{bmatrix} n-1 \\ \nu-1 \end{bmatrix},$$

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where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \qquad (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$$

and

$$\begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} M \\ N \end{bmatrix}_q = \begin{cases} 0, & \text{if } N<0 \text{ or } N>M; \\ \dfrac{(q;q)_M}{(q;q)_N(q;q)_{M-N}}, & \text{otherwise.} \end{cases}$$

As a consequence of the above result, they have obtained a family of inequalities for the partition function $p(n)$ [1]. Namely, for each $\nu \geq 1$, we have

$$(-1)^{\nu-1}\sum_{\tau=0}^{\nu-1}(-1)^\tau\big(p\left(n-\tau(3\tau+1)/2\right)-p\left(n-\tau(3\tau+5)/2-1\right)\big) \geq 0 \tag{1}$$

with strict inequality if $n \geq \nu(3\nu+1)/2$. They have also given the partition interpretation for the truncated sum in (1) as

$$(-1)^{\nu-1}\sum_{\tau=0}^{\nu-1}(-1)^\tau\big(p\left(n-\tau(3\tau+1)/2\right)-p\left(n-\tau(3\tau+5)/2-1\right)\big) = M_\nu(n),$$

where $M_\nu(n)$ counts the number of partitions of n in which ν is the smallest integer that is not a part and there are more parts greater than ν than there are less than ν.

Inspired by the above work, Guo and Zeng [10] considered two identities of Gauss, namely

$$1+2\sum_{\tau=1}^{\infty}(-1)^\tau q^{\tau^2} = \frac{(q;q)_\infty}{(-q;q)_\infty} = \phi(-q) \tag{2}$$

and

$$\sum_{\tau=0}^{\infty}(-1)^\tau q^{\tau(2\tau+1)}\left(1-q^{2\tau+1}\right) = \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} = \psi(-q) \tag{3}$$

to derive the truncated identities

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\left(1+2\sum_{\tau=1}^{\nu}(-1)^\tau q^{\tau^2}\right) = 1+(-1)^\nu\sum_{n=\nu+1}^{\infty}\frac{(-q;q)_\nu(-1;q)_{n-\nu}q^{(\nu+1)n}}{(q;q)_n}\begin{bmatrix} n-1 \\ \nu \end{bmatrix} \tag{4}$$

and

$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{\tau=0}^{\nu-1}(-1)^\tau q^{\tau(2\tau+1)}\left(1-q^{2\tau+1}\right)$$
$$= 1+(-1)^{\nu-1}\sum_{n=\nu}^{\infty}\frac{(-q;q^2)_\nu(-q;q^2)_{n-\nu}q^{2(\nu+1)n-\nu}}{(q^2;q^2)_n}\begin{bmatrix} n-1 \\ \nu-1 \end{bmatrix}_{q^2}, \tag{5}$$

for all $\nu \geq 1$.

An *overpartition* of the nonnegative integer n is a partition of n in which the first occurrence of a part may be overlined (see Corteel and Lovejoy [7]). We denote the number of overpartitions of n by $\overline{p}(n)$. Let $\mathrm{pod}(n)$ be the number of partitions of n wherein odd parts are distinct (see Hirschhorn and Sellers [12]). The reciprocals of (2) and (3) gives the generating functions for $\overline{p}(n)$ and $\mathrm{pod}(n)$, respectively. Let $p_{2,4}(n)$ be the number of partitions of n with parts $\not\equiv 2 \pmod 4$ and note that the generating function for $p_{2,4}(n)$ is same as that of $\mathrm{pod}(n)$.

Let us define $\mathrm{pp}(n)$ to be the number of bipartitions of n, and $p_o(n)$ be the number of partitions of n into odd parts. The generating functions for $\mathrm{pp}(n)$ and $p_o(n)$ are given by

$$\sum_{n=0}^{\infty} \mathrm{pp}(n) q^n = \frac{1}{(q;q)_\infty^2}$$

and

$$\sum_{n=0}^{\infty} p_o(n) q^n = \frac{1}{(q;q^2)_\infty} = \frac{(q^2;q^2)}{(q;q)_\infty},$$

respectively.

From (4) and (5), Guo and Zeng [10] obtained the following inequalities

$$(-1)^\nu \left(\overline{p}(n) + 2 \sum_{\tau=1}^{\nu} (-1)^\tau \overline{p}(n-\tau^2) \right) \geq 0 \tag{6}$$

with strict inequality if $n \geq (\nu+1)^2$ and

$$(-1)^{\nu-1} \sum_{\tau=0}^{\nu-1} (-1)^\tau \big(\mathrm{pod}\,(n-\tau(2\tau+1)) - \mathrm{pod}\,(n-(\tau+1)(2\tau+1)) \big) \geq 0 \tag{7}$$

with strict inequality if $n \geq \nu(2\nu+1)$.

Andrews and Merca [3] revised (4) and (5) as follows:

$$\frac{(-q;q)_\infty}{(q;q)_\infty} \left(1 + 2 \sum_{\tau=1}^{\nu} (-1)^\tau q^{\tau^2} \right) = 1 + 2(-1)^\nu \frac{(-q;q)_\nu}{(q;q)_\nu} \sum_{\tau=0}^{\infty} \frac{q^{(\nu+1)(\nu+\tau+1)}(-q^{\nu+\tau+2};q)_\infty}{(1-q^{\nu+\tau+1})(q^{\nu+\tau+2};q)_\infty}$$

and

$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{\tau=0}^{2\nu-1} (-q)^{\tau(\tau+1)/2} = 1 - (-1)^\nu \frac{(-q;q^2)_\nu}{(q^2;q^2)_{\nu-1}} \sum_{\tau=0}^{\infty} \frac{q^{\nu(2\tau+2\nu+1)}(-q^{2\tau+2\nu+3};q^2)_\infty}{(q^{2\nu+2\tau+2};q^2)_\infty}.$$

From these two identities they deduced interpretations of the sums in the inequalities (6) and (7):

$$(-1)^\nu \left(\overline{p}(n) + 2 \sum_{\tau=1}^{\nu} (-1)^\tau \overline{p}(n-\tau^2) \right) = \overline{M}_\nu(n),$$

where $\overline{M}_\nu(n)$ counts the number of overpartitions of n in which the first part greater than ν appears at least $\nu+1$ times and

$$(-1)^{\nu-1}\sum_{\tau=0}^{\nu-1}(-1)^{\tau}\big(\mathrm{pod}\,(n-\tau(2\tau+1))-\mathrm{pod}\,(n-(\tau+1)(2\tau+1))\big)=\mathrm{MP}_\nu(n),$$

where $\mathrm{MP}_\nu(n)$ is the number of partitions of n in which the first part greater than $2\nu-1$ is odd and appears exactly ν times. All other odd parts appear at most once.

We have also provided another interpretations for the sums on the left-hand side of inequalities (6) and (7) in Theorem (9) and Theorem (8), respectively.

To know more about truncated identities one can read Burnette and Kolitsch [5], Chan et al. [6], He et al. [11], Kolitsch [13], Mao [14], Merca et al. [16], Wang and Yee [17, 18], Yee [19].

Furthermore, Andrews and Merca [3] conjectured that

Conjecture 1. For ν even or n odd,

$$(-1)^{\nu-1}\sum_{i=0}^{\nu-1}(-1)^{i}\big(p\,(n-i(2i+1))-p\,(n-(i+1)(2i+1))\big)\leq M_\nu(n). \tag{8}$$

We provide a partition-theoretic interpretation for the truncated sum in inequality (8) and show that the sum is bounded above by $p(n-\nu(2\nu+1))$.

In this article, we present several partition-theoretic interpretations of new truncated sums related to the pentagonal number theorem and two identities of Gauss, (2) and (3). Additionally, we derive inequalities for various partition functions. Before we state our main results, let us define modular partitions and Durfee rectangles.

MacMahon [1, p. 13] introduced the concept of modular partitions, which are a modification of Young diagrams used to represent partitions of integers. Given positive integers m and n, there exist unique integers $b\geq 0$ and $1\leq r\leq m$ such that $n=mb+r$. The m-modular partitions represent this decomposition of n by using a row of b boxes, each containing the number m, and a single box in the first column containing the number r.

The *m-modular Young diagram* of a partition π is a graphical representation of π whose jth row is m-modular partition of the jth part π_j. For example, Figure 1 below shows the 3-modular Young diagram for the partition $16+14+7+5+2$ with shape $6+5+3+2+1$.

For a Young diagram of partition π, define the *ν-Durfee rectangle*, ν being a nonnegative integer, to be the largest rectangle which fits in the graph whose width minus its height is ν. In Fig. 1, the 3-Durfee rectangle of the partition is the shaded rectangle of size 2×5.

For a fixed $\nu\geq 0$ and $n\geq 0$, define $M(a,m,\nu;n)$ to be the number of partitions of n into the parts $\equiv a \pmod m$ such that $0\leq a<m$ and all parts $\leq m\nu+a$ occur as a part at least once and the parts below the $(\nu+2)$-Durfee rectangle in the m-modular graph are strictly less than the width of the rectangle. For example, the partition $20+17+11+8+5+2$ is counted by $M(2,3,2;63)$. Whereas, the partition $23+17+17+8+5+5+2$ is not counted by $M(2,3,2;77)$.

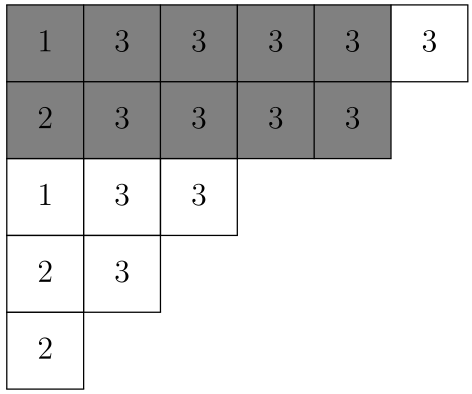

Figure 1: The 3-modular Young diagram of 16+14+7+5+2

Theorem 2. *For a fixed $\nu \geq 0$ and positive integers a, m such that $a < m$, we have*

$$\sum_{n=0}^{\infty} M(a,m,\nu;n)q^n = q^{a+\nu(m\nu+m+2a)/2} \sum_{\tau=0}^{\infty} \frac{q^{\tau(m\nu+m\tau+m+a)}}{(q^m;q^m)_\tau (q^a;q^m)_{\nu+\tau+1}}. \tag{9}$$

The case $m = 2, a = 1$ of Theorem (2) can be found in Merca et al. [16].

For a fixed $\nu > 0$ and $n \geq 0$, define $N_\nu(n)$ to be the number of partitions of n such that all parts $\leq \nu - 1$ occur at least once and the parts below the ν-Durfee rectangle in the Young diagram are strictly less than the width of the rectangle. For example, the partition $6+5+3+3+2+1$ is counted by $N_3(20)$. Whereas, the partition $6+5+5+2+1+1$ is not counted by $N_3(20)$. The generating function for $N_\nu(n)$ as stated in Theorem (3) can be found in Ballantine and Merca [4].

Theorem 3. *For a fixed $\nu \geq 1$, we have*

$$\sum_{n=0}^{\infty} N_\nu(n)q^n = q^{\nu(\nu-1)/2} \sum_{\tau=0}^{\infty} \frac{q^{\tau(\nu+\tau)}}{(q;q)_\tau (q;q)_{\nu+\tau-1}}. \tag{10}$$

Let $\nu \geq 0$ and $n \geq 0$ be fixed integers. Define $p(a,m,\nu;n)$ to be the number of partitions of n in which the parts that are congruent to a modulo m form a partition counted by $M(a,m,\nu;n-\beta)$, where β is the sum of the parts that are not congruent to a modulo m.

Theorem 4. *For $\nu \geq 1$ and $n > \nu(3\nu+5)/2$, we have*

$$\begin{aligned}(-1)^\nu &\left(\sum_{\tau=0}^{\nu} (-1)^\tau p\left(n - \tau(3\tau+1)/2\right) - \sum_{\tau=0}^{\nu-1} (-1)^\tau p\left(n - \tau(3\tau+5)/2 - 1\right) \right) \\ &= p(2,3,\nu;n) + p(1,3,\nu;n). \end{aligned} \tag{11}$$

For example, let $\nu = 2$ and $n = 17$.

Partitions counted by $p(2,3,2;17)$	Partitions counted by $p(1,3,2;17)$
$8+5+2+2$	$7+5+4+1$
$8+5+2+1+1$	$7+4+4+1+1$
	$7+4+3+2+1$
	$7+4+3+1+1+1$
	$7+4+2+2+1+1$
	$7+4+2+1+1+1+1$
	$7+4+1+1+1+1+1+1$

Hence, $p(17)-p(15)+p(10)-p(16)+p(12)=9=p(2,3,2;17)+p(1,3,2;17)$.

We can find alternative partition-theoretic interpretations of the sum on the left-hand side of equation (11) in [5] and [15].

Theorem 5. *For $\nu \geq 1$ and $n > \nu(2\nu+3)$, we have*

$$(-1)^\nu \left(\sum_{\tau=0}^{\nu}(-1)^\tau p\left(2n-\tau(2\tau+1)\right)-\sum_{\tau=0}^{\nu-1}(-1)^\tau p\left(2n-(\tau+1)(2\tau+1)\right)\right)$$
$$=(-1)^\nu p_o(n)+p(3,4,\nu;2n)+p(1,4,\nu;2n) \tag{12}$$

and

$$(-1)^\nu \left(\sum_{\tau=0}^{\nu}(-1)^\tau p\left(2n+1-\tau(2\tau+1)\right)-\sum_{\tau=0}^{\nu-1}(-1)^\tau p\left(2n+1-(\tau+1)(2\tau+1)\right)\right)$$
$$=p(3,4,\nu;2n+1)+p(1,4,\nu;2n+1). \tag{13}$$

Theorem 6. *For $\nu \geq 1$ and $n > \nu(2\nu+3)$, we have*

$$(-1)^\nu \left(\sum_{\tau=0}^{\nu}(-1)^\tau p_o\left(2n+1-\tau(2\tau+1)\right)-\sum_{\tau=0}^{\nu-1}(-1)^\tau p_o\left(2n+1-(\tau+1)(2\tau+1)\right)\right)$$
$$=p_o(3,4,\nu;2n+1)+p_o(1,4,\nu;2n+1), \tag{14}$$

where $p_o(a,4,\nu;n)$ ($a \in \{1,3\}$), counts partitions of n into odd parts in which the parts $\equiv a \pmod 4$ form a partition counted by $M(a,4,\nu;n-\alpha)$, α is the sum of parts $\not\equiv a \pmod 4$.

The following corollary is an immediate consequence of Theorem (5) and Theorem (6).

Corollary 7. *With strict inequality if $n > \nu(2\nu+3)$, we have*

1. *For n odd or ν even,*

$$(-1)^{\nu-1}\sum_{\tau=0}^{\nu-1}(-1)^\tau\big(p\left(n-\tau(2\tau+1)\right)-p\left(n-(\tau+1)(2\tau+1)\right)\big) \leq p(n-\nu(2\nu+1)). \tag{15}$$

2. For odd n,

$$(-1)^{\nu-1}\sum_{\tau=0}^{\nu-1}(-1)^{\tau}\big(p_o\left(n-\tau(2\tau+1)\right)-p_o\left(n-(\tau+1)(2\tau+1)\right)\big)\leq p_o\left(n-\nu(2k+1)\right). \tag{16}$$

Let $\nu \geq 0$ and $n \geq 0$ be fixed integers. The notation $p_{2,4}(a,4,\nu;n)$ denotes the number of partitions of n into parts that are not congruent to 2 modulo 4, and in which the parts that are congruent to a modulo 4 form a partition counted by $M(a,4,\nu;n-\alpha)$, where α is the sum of the parts that are not congruent to a modulo 4.

Theorem 8. *For $\nu \geq 1$ and $n > \nu(2\nu+3)$, we have*

$$\begin{aligned}(-1)^{\nu}&\left(\sum_{\tau=0}^{\nu}(-1)^{\tau}p_{2,4}\left(n-\tau(2\tau+1)\right)-\sum_{\tau=0}^{\nu-1}(-1)^{\tau}p_{2,4}\left(n-(\tau+1)(2\tau+1)\right)\right)\\&=p_{2,4}(3,4,\nu;n)+p_{2,4}(1,4,\nu;n).\end{aligned} \tag{17}$$

Theorem 9. *For $\nu \geq 1$ and $n \geq (\nu+1)^2$, we have*

$$(-1)^{\nu}\left(\overline{p}(n)+2\sum_{\tau=1}^{\nu}(-1)^{\tau}\overline{p}(n-\tau^2)\right)=2\overline{p}(1,2,\nu;n), \tag{18}$$

where $\overline{p}(1,2,\nu;n)$ counts overpartitions of n in which the non-overlined odd parts form a partition counted by $M(1,2,\nu;n-\alpha)$, α is the sum of parts other than non-overlined odd parts.

Theorem 10. *For $\nu \geq 1$ and $n \geq (\nu+1)^2$, we have*

$$(-1)^{\nu}\left(\mathrm{pp}(2n)+2\sum_{\tau=1}^{\nu}(-1)^{\tau}\,\mathrm{pp}(2n-\tau^2)\right)=(-1)^{\nu}p(n)+2\,\mathrm{pp}_e(2n) \tag{19}$$

and

$$(-1)^{\nu}\left(\mathrm{pp}(2n+1)+2\sum_{\tau=1}^{\nu}(-1)^{\tau}\,\mathrm{pp}(2n+1-\tau^2)\right)=2\,\mathrm{pp}_e(2n+1), \tag{20}$$

where $\mathrm{pp}_e(n)$ counts bipartitions (π_1,π_2) such that π_1 is a unrestricted partition and π_2 is a partition in which odd parts are counted by $M(1,2,\nu;n-|\pi_1|-\alpha)$, $|\pi_1|$ is the sum of parts of π_1 and α is the sum of even parts of π_2.

The following corollary immediately follows from Theorem (10).

Corollary 11. *For n odd or ν even,*

$$(-1)^{\nu}\left(\mathrm{pp}(n)+2\sum_{\tau=1}^{\nu}(-1)^{\tau}\,\mathrm{pp}(n-\tau^2)\right)\geq 0, \tag{21}$$

with strict inequality if $n \geq (\nu+1)^2$.

Theorem 12. *For each $n \geq 1$, we have*

$$p(n) = p(2,3,0;n) + p(1,3,0;n) \tag{22}$$

and

$$p_{2,4}(n) = p_{2,4}(3,4,0;n) + p_{2,4}(1,4,0;n). \tag{23}$$

Theorem 13. *For $\nu \geq 1$ and $n \geq 0$, we have*

$$\sum_{\tau=0}^{\infty}(-1)^{\tau} p\left(n - \frac{(\nu+\tau)(\nu+\tau-1)}{2}\right) = N_{\nu}(n). \tag{24}$$

For example, let $\nu = 2$ and $n = 10$. Partitions counted by $N_2(10)$ are $1+1+1+1+1+1+1+1+1+1$, $9+1$, $8+1+1$, $7+2+1$, $7+1+1+1$, $6+2+1+1$, $6+1+1+1+1$, $5+2+2+1$, $5+2+1+1+1$, $5+1+1+1+1+1$, $4+2+2+1+1$, $4+2+1+1+1+1$, $4+1+1+1+1+1+1$, $3+2+2+2+1$, $3+2+2+1+1+1$, $3+2+1+1+1+1+1$, $3+1+1+1+1+1+1+1$, $5+4+1$, $4+4+1+1$. Hence, $p(9) - p(7) + p(4) - p(0) = 19 = N_2(10)$.

The following inequality is obvious from a previous Theorem.

Corollary 14. *For $\nu \geq 1$ and $n \geq 0$,*

$$\sum_{\tau=0}^{\infty}(-1)^{\tau} p\left(n - \frac{(\nu+\tau)(\nu+\tau-1)}{2}\right) \geq 0, \tag{25}$$

where $p(x) = 0$ if x is a negative integer.

To prove our theorems, we rely on Gauss hypergeometric series [8, Eqn. 1.2.14]

$${}_2\phi_1\left(\begin{matrix}\alpha, \beta \\ \gamma\end{matrix}; q, z\right) = \sum_{n=0}^{\infty} \frac{(\alpha;q)_n(\beta;q)_n}{(q;q)_n(\gamma;q)_n} z^n$$

and Heine's transformation of ${}_2\phi_1$ series [8, Eqn. 1.4.5],

$${}_2\phi_1\left(\begin{matrix}\alpha, \beta \\ \gamma\end{matrix}; q, z\right) = \frac{(\gamma/\beta;q)_\infty(\beta z;q)_\infty}{(\gamma;q)_\infty(z;q)_\infty} \, {}_2\phi_1\left(\begin{matrix}\alpha\beta z/\gamma, \beta \\ \beta z\end{matrix}; q, \gamma/\beta\right).$$

Without explicitly mentioning them, we frequently use these two identities in our proofs.

2 Proofs of main theorems

In this section, we prove Theorems 2–13.

Proof of Theorem 2. For a fixed $\nu \geq 0$, the partitions counted by $M(a,m,\nu;n)$ with $(\nu+2)$-Durfee rectangle, in a m-modular Young diagram, of size $\tau \times (\nu+\tau+2)$ are composed of three pieces. The first piece is the Durfee rectangle with an enumerator of $q^{\tau(m\nu+m\tau+m+a)}$. Another piece is the one to the right of the Durfee rectangle, which consists of partitions with at most τ parts. The last piece is located under the Durfee rectangle, and contains partitions into parts $\leq \nu+\tau+1$. It is clear from the definition of $M(a,m,\nu;n)$ that the parts below and to the right of the Durfee rectangle are generated by $q^{a+\nu(m\nu+m+2a)/2}/(q^a;q^m)_{\nu+\tau+1}$ and $1/(q^m;q^m)_\tau$, respectively. Here, $q^{a+\nu(m\nu+m+2a)/2}$ accounts for all parts $\equiv a \pmod m$ that are $\leq m\nu+a$. □

Proof of Theorem 3. Proof of Theorem 3 is similar to that of 2, so we omit the details. □

Proof of Theorem 4. For $|q|<1$ and $\nu \geq 1$, Merca [15, Theorem 1.2] showed that

$$\begin{aligned}\frac{1}{(q;q)_\infty}&\sum_{\tau=0}^{\nu-1}(-1)^\tau q^{\tau(3\tau+1)/2}\\ &= 1+\sum_{\tau=1}^{\infty} q^\tau \begin{bmatrix} 2\tau-1 \\ \tau-1 \end{bmatrix} + (-1)^{\nu-1}\frac{q^{\nu(3\nu+1)/2}}{(q,q^3;q^3)_\infty}\sum_{\tau=0}^{\infty}\frac{q^{\tau(3\tau+3\nu+2)}}{(q^3;q^3)_\tau (q^2;q^3)_{\nu+\tau}}\end{aligned} \tag{26}$$

and

$$\begin{aligned}\frac{1}{(q;q)_\infty}&\sum_{\tau=0}^{\nu-1}(-1)^\tau q^{\tau(3\tau+5)/2+1}\\ &= \sum_{\tau=1}^{\infty} q^\tau \begin{bmatrix} 2\tau-1 \\ \tau-1 \end{bmatrix} + (-1)^{\nu-1}\frac{q^{\nu(3\nu+5)/2+1}}{(q^2,q^3;q^3)_\infty}\sum_{\tau=0}^{\infty}\frac{q^{\tau(3\tau+3\nu+4)}}{(q^3;q^3)_\tau (q;q^3)_{\nu+\tau+1}}.\end{aligned} \tag{27}$$

From (26) and (27), we have

$$\begin{aligned}&\frac{1}{(q;q)_\infty}\left(\sum_{\tau=0}^{\nu}(-1)^\tau q^{\tau(3\tau+1)/2} - \sum_{\tau=0}^{\nu-1}(-1)^\tau q^{\tau(3\tau+5)/2+1}\right)\\ &= 1+(-1)^\nu q^{2+\nu(3\nu+7)/2}\frac{(q^2;q^3)_\infty}{(q;q)_\infty}\sum_{\tau=0}^{\infty}\frac{q^{\tau(3\nu+3\tau+5)}}{(q^3;q^3)_\tau (q^2;q^3)_{\nu+\tau+1}}\\ &\quad + (-1)^\nu q^{1+\nu(3\nu+5)/2}\frac{(q;q^3)_\infty}{(q;q)_\infty}\sum_{\tau=0}^{\infty}\frac{q^{\tau(3\nu+3\tau+4)}}{(q^3;q^3)_\tau (q;q^3)_{\nu+\tau+1}}.\end{aligned} \tag{28}$$

Using Theorem 2 and Eq. (28), we arrive at (11). □

Proof of Theorem 5. From the Gauss second identity (3), we have

$$\begin{aligned}
&\frac{1}{\psi(-q)}\left(\sum_{\tau=0}^{\nu}(-1)^{\tau}q^{\tau(2\tau+1)}-\sum_{\tau=0}^{\nu-1}(-1)^{\tau}q^{(\tau+1)(2\tau+1)}\right)\\
&=1-\frac{1}{\psi(-q)}\sum_{\tau=\nu+1}^{\infty}(-1)^{\tau}q^{\tau(2\tau+1)}+\frac{1}{\psi(-q)}\sum_{\tau=\nu}^{\infty}(-1)^{\tau}q^{(\tau+1)(2\tau+1)}\\
&=1+(-1)^{\nu}\frac{q^{3+\nu(2\nu+5)}}{\psi(-q)}\sum_{\tau=0}^{\infty}(-1)^{\tau}q^{\tau(4\nu+2\tau+5)}+(-1)^{\nu}\frac{q^{1+\nu(2\nu+3)}}{\psi(-q)}\sum_{\tau=0}^{\infty}(-1)^{\tau}q^{\tau(4\nu+2\tau+3)}\\
&=1+(-1)^{\nu}\frac{q^{3+\nu(2\nu+5)}}{\psi(-q)}\lim_{\delta\to0}\ {}_2\phi_1\left(\begin{matrix}q^4,\ \frac{q^{4\nu+7}}{\delta}\\ 0\end{matrix};q^4,\delta\right)\\
&+(-1)^{\nu}\frac{q^{1+\nu(2\nu+3)}}{\psi(-q)}\lim_{\delta\to0}\ {}_2\phi_1\left(\begin{matrix}q^4,\ \frac{q^{4\nu+5}}{\delta}\\ 0\end{matrix};q^4,\delta\right)\\
&=1+(-1)^{\nu}\frac{q^{3+\nu(2\nu+5)}}{\psi(-q)}\lim_{\delta\to0}\frac{(q^{4\nu+7};q^4)_\infty}{(\delta;q^4)_\infty}\sum_{\tau=0}^{\infty}\frac{(-1)^{\tau}\delta^{\tau}q^{2\tau(\tau+1)}(q^{4\nu+7}/\delta;q^4)_\tau}{(q^4;q^4)_\tau(q^{4\nu+7};q)_\tau}\\
&\quad+(-1)^{\nu}\frac{q^{1+\nu(2\nu+3)}}{\psi(-q)}\lim_{\delta\to0}\frac{(q^{4\nu+5};q^4)_\infty}{(\delta;q^4)_\infty}\sum_{\tau=0}^{\infty}\frac{(-1)^{\tau}\delta^{\tau}q^{2\tau(\tau+1)}(q^{4\nu+5}/\delta;q^4)_\tau}{(q^4;q^4)_\tau(q^{4\nu+5};q)_\tau}\\
&=1+(-1)^{\nu}q^{3+\nu(2\nu+5)}\frac{(q^3;q^4)_\infty}{\psi(-q)}\sum_{\tau=0}^{\infty}\frac{q^{\tau(4\nu+4\tau+7)}}{(q^4;q^4)_\tau(q^3;q^4)_{\nu+\tau+1}}\\
&\quad+(-1)^{\nu}q^{1+\nu(2\nu+3)}\frac{(q;q^4)_\infty}{\psi(-q)}\sum_{\tau=0}^{\infty}\frac{q^{\tau(4\nu+4\tau+5)}}{(q^4;q^4)_\tau(q;q^4)_{\nu+\tau+1}}. \qquad (29)
\end{aligned}$$

Multiplying equation (29) by $(q^4;q^4)_\infty/(q^2;q^2)_\infty$ and using $\psi(-q)=\frac{(q;q)_\infty(q^4;q^4)_\infty}{(q^2;q^2)_\infty}$, we obtain

$$\begin{aligned}
&\frac{1}{(q;q)_\infty}\left(\sum_{\tau=0}^{\nu}(-1)^{\tau}q^{\tau(2\tau+1)}-\sum_{\tau=0}^{\nu-1}(-1)^{\tau}q^{(\tau+1)(2\tau+1)}\right)\\
&\qquad=\frac{(q^4;q^4)_\infty}{(q^2;q^2)_\infty}+(-1)^{\nu}q^{3+\nu(2\nu+5)}\frac{(q^3;q^4)_\infty}{(q;q)_\infty}\sum_{\tau=0}^{\infty}\frac{q^{\tau(4\nu+4\tau+7)}}{(q^4;q^4)_\tau(q^3;q^4)_{\nu+\tau+1}}\\
&\qquad\quad+(-1)^{\nu}q^{1+\nu(2\nu+3)}\frac{(q;q^4)_\infty}{(q;q)_\infty}\sum_{\tau=0}^{\infty}\frac{q^{\tau(4\nu+4\tau+5)}}{(q^4;q^4)_\tau(q;q^4)_{\nu+\tau+1}},
\end{aligned}$$

which implies that

$$
\begin{aligned}
&(-1)^\nu \sum_{n=0}^{\infty} p(n)q^n \left(\sum_{\tau=0}^{\nu} (-1)^\tau q^{\tau(2\tau+1)} - \sum_{\tau=0}^{\nu-1} (-1)^\tau q^{(\tau+1)(2\tau+1)} \right) \\
&\quad = (-1)^\nu \sum_{n=0}^{\infty} p_o(n) q^{2n} + \sum_{n=0}^{\infty} p(3,4,\nu;n)q^n + \sum_{n=0}^{\infty} p(1,4,\nu;n)q^n. \qquad (30)
\end{aligned}
$$

Equating coefficients of q^{2n} and q^{2n+1} on both sides of (30), we arrive at (12) and (13), respectively. □

Proof of Theorem 6. Proof of (14) is similar to that of (13) but instead of multiplying by $(q^4;q^4)_\infty/(q^2;q^2)_\infty$, we multiply by $(q^4;q^4)_\infty$. □

Proof of Theorem 8. Equation (17) follows from Eq. (29) and Theorem 2. □

Proof of Theorem 9. From the Gauss identity (2), we have

$$
\begin{aligned}
&\frac{1}{\phi(-q)} \left(1 + 2\sum_{\tau=1}^{\nu} (-1)^\tau q^{\tau^2} \right) \\
&= 1 - \frac{2}{\phi(-q)} \sum_{\tau=\nu+1}^{\infty} (-1)^\tau q^{\tau^2} \\
&= 1 + \frac{2(-1)^\nu q^{(\nu+1)^2}}{\phi(-q)} \sum_{\tau=0}^{\infty} (-1)^\tau q^{\tau^2+2\tau(\nu+1)} \\
&= 1 + \frac{2(-1)^\nu q^{(\nu+1)^2}}{\phi(-q)} \lim_{\delta\to 0} {}_2\phi_1 \left(\begin{matrix} q^2, \frac{q^{2\nu+3}}{\delta} \\ 0 \end{matrix} ; q^2, \delta \right) \\
&= 1 + \frac{2(-1)^\nu q^{(\nu+1)^2}}{\phi(-q)} \lim_{\delta\to 0} \frac{(q^{2\nu+3};q^2)_\infty}{(\delta;q^2)_\infty} \sum_{\tau=0}^{\infty} \frac{(-1)^\tau \delta^\tau q^{\tau(\tau+1)} (q^{2\nu+3}/\delta;q^2)_\tau}{(q^2;q^2)_\tau (q^{2\nu+3};q^2)_\tau} \\
&= 1 + \frac{2(-1)^\nu q^{(\nu+1)^2}}{\phi(-q)} (q^{2\nu+3};q^2)_\infty \sum_{\tau=0}^{\infty} \frac{q^{2\tau^2+(2\nu+3)\tau}}{(q^2;q^2)_\tau (q^{2\nu+3};q^2)_\tau} \\
&= 1 + \frac{2(-1)^\nu q^{(\nu+1)^2}}{(q;q)_\infty (q;q^2)_\infty} (q^{2\nu+3};q^2)_\infty \sum_{\tau=0}^{\infty} \frac{q^{\tau(2\nu+2\tau+3)}}{(q^2;q^2)_\tau (q^{2\nu+3};q^2)_\tau} \\
&= 1 + \frac{2(-1)^\nu q^{(\nu+1)^2}}{(q;q)_\infty} \sum_{\tau=0}^{\infty} \frac{q^{\tau(2\nu+2\tau+3)}}{(q^2;q^2)_\tau (q;q^2)_{\nu+\tau+1}} \\
&= 1 + \frac{2(-1)^\nu (-q;q)_\infty}{(q^2;q^2)_\infty} q^{(\nu+1)^2} \sum_{\tau=0}^{\infty} \frac{q^{\tau(2\nu+2\tau+3)}}{(q^2;q^2)_\tau (q;q^2)_{\nu+\tau+1}}. \qquad (31)
\end{aligned}
$$

Equation (18) follows from Eq. (31) and Theorem 2. □

Proof of Theorem 10. Multiplying equation (31) by $\frac{1}{(q^2;q^2)_\infty}$ and using $\phi(-q)=\frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}$, we obtain

$$\frac{1}{(q;q)_\infty^2}\left(1+2\sum_{\tau=1}^{\nu}(-1)^\tau q^{\tau^2}\right)=\frac{1}{(q^2;q^2)_\infty}+2(-1)^\nu\frac{q^{(\nu+1)^2}}{(q;q)_\infty(q^2;q^2)_\infty}\sum_{\tau=0}^{\infty}\frac{q^{\tau(2\nu+2\tau+3)}}{(q^2;q^2)_\tau(q;q^2)_{\nu+\tau+1}}$$

and the statements follow easily from Theorem 2 and the above equation. □

Proof of Theorem 12. From the pentagonal number theorem, we have

$$\begin{aligned}(q;q)_\infty&=\sum_{\tau=0}^{\infty}(-1)^\tau q^{\tau(3\tau+1)/2}+\sum_{\tau=-1}^{-\infty}(-1)^\tau q^{\tau(3\tau+1)/2}\\&=\sum_{\tau=0}^{\infty}(-1)^\tau q^{\tau(3\tau+1)/2}-\sum_{\tau=0}^{\infty}(-1)^\tau q^{\tau(3\tau+5)/2+1}.\end{aligned}\tag{32}$$

Rewriting, we obtain

$$\begin{aligned}\frac{1}{(q;q)_\infty}&=1-\frac{1}{(q;q)_\infty}\sum_{\tau=1}^{\infty}(-1)^\tau q^{\tau(3\tau+1)/2}+\frac{1}{(q;q)_\infty}\sum_{\tau=0}^{\infty}(-1)^\tau q^{\tau(3\tau+5)/2+1}\\&=1+\frac{q^2}{(q;q)_\infty}\sum_{\tau=0}^{\infty}(-1)^\tau q^{\tau(3\tau+7)/2}+\frac{q}{(q;q)_\infty}\sum_{\tau=0}^{\infty}(-1)^\tau q^{\tau(3\tau+5)/2}\\&=1+\frac{q^2}{(q;q)_\infty}\lim_{\delta\to0}\ {}_2\phi_1\left(\begin{matrix}q^3,\frac{q^5}{\delta}\\0\end{matrix};q^3,\delta\right)+\frac{q}{(q;q)_\infty}\lim_{\delta\to0}\ {}_2\phi_1\left(\begin{matrix}q^3,\frac{q^4}{\delta}\\0\end{matrix};q^3,\delta\right)\\&=1+\frac{q^2}{(q;q)_\infty}\lim_{\delta\to0}\frac{(q^5;q^3)_\infty}{(\delta;q^3)_\infty}\sum_{\tau=0}^{\infty}\frac{(-1)^\tau\delta^\tau q^{3\tau(\tau+1)/2}\,(q^5/\delta;q^3)_\tau}{(q^3;q^3)_\tau(q^5;q^3)_\tau}\\&\quad+\frac{q}{(q;q)_\infty}\lim_{\delta\to0}\frac{(q^4;q^3)_\infty}{(\delta;q^3)_\infty}\sum_{\tau=0}^{\infty}\frac{(-1)^\tau\delta^\tau q^{3\tau(\tau+1)/2}\,(q^4/\delta;q^3)_\tau}{(q^3;q^3)_\tau(q^4;q^3)_\tau}\\&=1+q^2\frac{(q^5;q^3)_\infty}{(q;q)_\infty}\sum_{\tau=0}^{\infty}\frac{q^{\tau(3\tau+5)}}{(q^3;q^3)_\tau\,(q^5;q^3)_\tau}+q\frac{(q^4;q^3)_\infty}{(q;q)_\infty}\sum_{\tau=0}^{\infty}\frac{q^{\tau(3\tau+4)}}{(q^3;q^3)_\tau\,(q^4;q^3)_\tau}\\&=1+q^2\frac{(q^2;q^3)_\infty}{(q;q)_\infty}\sum_{\tau=0}^{\infty}\frac{q^{\tau(3\tau+5)}}{(q^3;q^3)_\tau(q^2;q^3)_{\tau+1}}+q\frac{(q;q^3)_\infty}{(q;q)_\infty}\sum_{\tau=0}^{\infty}\frac{q^{\tau(3\tau+4)}}{(q^3;q^3)_\tau(q;q^3)_{\tau+1}}.\end{aligned}\tag{33}$$

Using Eq. (33) and Theorem 2, we arrive at (22). Similarly one can prove (23) by using Gauss second identity (3). □

Proof of Theorem 13. First, we will prove the following lemma:

Lemma 15. *For $\nu \geq 1$ and $m \geq 0$, we have*

$$\sum_{n=0}^{\infty} N_\nu(n) q^n = \sum_{n=0}^{\infty} p(n) q^n \sum_{\tau=0}^{m} (-1)^\tau q^{(\nu+\tau)(\nu+\tau-1)/2} + (-1)^{m+1} \sum_{n=0}^{\infty} N_{\nu+m+1}(n) q^n. \tag{34}$$

Proof. From Theorem 3, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} N_\nu(n) q^n &= q^{\nu(\nu-1)/2} \sum_{\tau=0}^{\infty} \frac{q^{\tau(\nu+\tau)}}{(q;q)_\tau (q;q)_{\nu+\tau-1}} \\
&= q^{\nu(\nu-1)/2} \sum_{\tau=0}^{\infty} \frac{(1-q^{\nu+\tau}) q^{\tau(\nu+\tau)}}{(q;q)_\tau (q;q)_{\nu+\tau}} \\
&= q^{\nu(\nu-1)/2} \sum_{\tau=0}^{\infty} \frac{q^{\tau(\nu+\tau)}}{(q;q)_\tau (q;q)_{\nu+\tau}} - q^{\nu(\nu+1)/2} \sum_{\tau=0}^{\infty} \frac{q^{\tau(\nu+\tau+1)}}{(q;q)_\tau (q;q)_{\nu+\tau}} \\
&= q^{\nu(\nu-1)/2} \sum_{n=0}^{\infty} p(n) q^n - \sum_{n=0}^{\infty} N_{\nu+1}(n) q^n, \quad \text{(from [9, p. 91] and (10))}
\end{aligned} \tag{35}$$

which is $m=0$ case of (34). Now assume that (34) holds for some $m \geq 0$, then, from (34) and (35), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} N_\nu(n) q^n &= \sum_{n=0}^{\infty} p(n) q^n \sum_{\tau=0}^{m} (-1)^\tau q^{(\nu+\tau)(\nu+\tau-1)/2} \\
&\quad + (-1)^{m+1} \left(q^{(\nu+m)(\nu+m+1)/2} \sum_{n=0}^{\infty} p(n) q^n - \sum_{n=0}^{\infty} N_{\nu+m+2}(n) q^n \right) \\
&= \sum_{n=0}^{\infty} p(n) q^n \sum_{\tau=0}^{m+1} (-1)^\tau q^{(\nu+\tau)(\nu+\tau-1)/2} + (-1)^{m+2} \sum_{n=0}^{\infty} N_{\nu+m+2}(n) q^n,
\end{aligned}$$

which is the $m+1$ case of (34). □

For $n \leq (\nu+m)(\nu+m+1)/2$, equating coefficients of q^n on both sides of (34),

$$\sum_{\tau=0}^{m} (-1)^\tau p\left(n - \frac{(\nu+\tau)(\nu+\tau-1)}{2}\right) = N_\nu(n). \tag{36}$$

Now the statement (24) is obvious from (36) for sufficiently large m. □

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