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Pattern-Avoidance and Fuss–Catalan Numbers

Per Alexandersson
Department of Mathematics
Stockholm University
S-106 91 Stockholm
Sweden
per.w.alexandersson@gmail.com

Frether Getachew Kebede
Department of Mathematics
Addis Ababa University
1176 Addis Ababa
Ethiopia
frigetach@gmail.com

Samuel Asefa Fufa
Department of Mathematics
Addis Ababa University
1176 Addis Ababa
Ethiopia
samuel.asefa@aau.edu.et

Dun Qiu
Center for Combinatorics, LPMC
Nankai University
Tianjin 300071
P. R. China
qiudun@nankai.edu.cn

Abstract

We study a subset of permutations where entries are restricted to having the same remainder as the index, modulo some integer $k \geq 2$. We show that by also imposing the classical 132- or 213-avoidance restriction on the permutations, we recover the Fuss–Catalan numbers and some special cases of the Raney numbers.

Surprisingly, an analogous statement also holds when we impose the mod k restriction on a Catalan family of subexcedant functions.

Finally, we completely enumerate all combinations of mod-k-alternating permutations that avoid two patterns of length 3. This is analogous to the systematic study by Simion and Schmidt, of permutations avoiding two patterns of length 3.

1 Introduction

It is classically known that the number of permutations of length n avoiding a permutation-pattern of length 3 is counted by the Catalan number, C_n. There is a plethora of other combinatorial objects that are counted by C_n; see, e.g., [18]. One classical Catalan family are the *Dyck paths*—paths using steps in $\{(1,0),(0,1)\}$, starting at $(0,0)$, ending at (n,n), and never going below the line $y=x$. This family can be generalized to counting paths that never go below $y=kx$ and end at (kn,n) for some fixed positive integer k. The number of such paths is the *Fuss–Catalan number*:

$$C_n^{k+1} := \frac{1}{(k+1)n+1}\binom{(k+1)n+1}{n} = \frac{1}{kn+1}\binom{(k+1)n}{n}.$$

In this paper, we find several *Fuss*-generalizations of pattern-avoiding permutations by imposing additional restrictions on the entries in the permutations.

A parity-alternating permutation starting with an odd integer (PAP) sends even integers to even integers and odd integers to odd. This set of permutations has been studied previously in [19] when considering the ascent statistic and signed Eulerian numbers, and later in [6] where parity-alternating derangements are enumerated.

We generalize the notion of parity-alternating permutations to *mod-k-alternating permutations*. Here, we require that π_i is congruent to $i \pmod k$, where $k \geq 1$ is a fixed integer. This notion is not to be confused with the generalization in [13], where the size of blocks of entries with the same parity are restricted (but not restricted to be 1).

We enumerate the mod-k-alternating permutations under pattern-avoidance, for each of the patterns in $\{132, 213, 231, 312\}$. Moreover, we also enumerate the mod-k-alternating permutations that avoid two patterns of length 3. The number of mod-k-alternating permutations of length n avoiding σ is denoted $\mathfrak{mp}_\sigma(n,k)$.

We also consider subexcedant functions. These are simply words of positive integers, $f_1, f_2, \ldots, f_n$, such that $f_i \leq i$. There is a (specific) bijection between permutations of length n and subexcedant functions of length n (introduced in [12]) so one can study subexcedant functions instead of permutations. The mod-k-alternating restriction translates under this bijection to the exact same restriction on the subexcedant function. In other words, $f_i \equiv i \pmod k$; see Proposition 20.

1.1 Main results

In Section 4 we prove the following theorem. In particular, this shows that $\mathfrak{mp}_\sigma(n,k)$ is a Raney number and that $\mathfrak{mp}_\sigma(km,k)$ is a Fuss–Catalan number.

Theorem 1 (Corollary 15 below)**.** *Let $\sigma \in \{132, 213\}$ and $n, k \geq 1$. Let m and j be defined via $n = km + j$ with $0 \leq j < k$. Then*

$$\mathfrak{mp}_\sigma(n,k) = \frac{j+1}{km+j+1}\binom{(k+1)m+j}{km+j}.$$

In particular

$$\mathfrak{mp}_\sigma(km,k) = \frac{1}{(k+1)m+1}\binom{(k+1)m+1}{m} = \frac{1}{km+1}\binom{(k+1)m}{m}.$$

In Section 5, we prove our second main result. We let $\mathfrak{mp}_C(n,k)$ denote the set of mod-k-alternating permutations whose subexcedant function satisfy the following two properties:

- $f_i \equiv i \pmod{k}$,
- $1 \le f_i \le f_{i-1} + 1$ for all $i \in \{2,3,\ldots,n\}$.

The first property is analogous to the mod-k-alternating property on permutations. Subexcedant functions that satisfy the second property are enumerated by the Catalan numbers. The second condition is therefore a natural analog to permutations avoiding a fixed-length-3 pattern. We have the following result:

Theorem 2 (Corollary 28 below)**.** *Let m and j be defined via $n = km + j$ with $0 \le j < k$. Then*

$$\mathfrak{mp}_C(n,k) = \frac{j+1}{km+j+1}\binom{(k+1)m+j}{km+j}$$

and in particular

$$\mathfrak{mp}_C(km,k) = \frac{1}{(k+1)m+1}\binom{(k+1)m+1}{m} = \frac{1}{km+1}\binom{(k+1)m}{m}.$$

We prove these two enumerative results by showing that the combinatorial objects satisfy certain recursions. These recursions can then be used to produce a canonical bijection between $\mathfrak{mp}_{132}(n,k)$ and $\mathfrak{mp}_C(n,k)$, for example. However, this bijection is not a restriction of any classical bijection that sends subexcedant functions to permutations.

Finally, in Section 6 we systematically enumerate all families of mod-k-alternating permutations avoiding two (different) patterns of length 3.

2 Preliminaries

Given a word $w = [w_1,\ldots,w_\ell]$ and an integer n, we let $n + w$ denote the word

$$[w_1 + n, w_2 + n, \ldots, w_\ell + n].$$

This convention is extended to the expressions $w - n$ and $n - w$ in a natural manner. If $\alpha = [\alpha_1,\ldots,\alpha_r]$ and $\beta = [\beta_1,\ldots,\beta_s]$ are words, then $[\alpha,\beta]$ denotes the concatenation $[\alpha_1,\ldots,\alpha_r,\beta_1,\ldots,\beta_s]$. We use several variants of this convention. We use the term *subword* to denote a sequence of consecutive entries in a word.

The set $\{1, 2, \ldots, n\}$ is denoted by $[n]$ (the context should make it clear that this is not to be interpreted as the word of one letter). The set of permutations of $[n]$ is denoted by S_n. We use brackets for permutations in one-line notation and regular parenthesis to denote cycles. We make use of four well-known involutions on S_n, *inverse*, *reverse*, *flip*, and *revflip*; for $\pi = [\pi_1, \ldots, \pi_n] \in S_n$, we let

$$\mathrm{rev}(\pi) \coloneqq [\pi_n, \pi_{n-1}, \ldots, \pi_1], \ \mathrm{flip}(\pi) \coloneqq n + 1 - \pi, \quad \text{and} \quad \mathrm{revflip}(\pi) \coloneqq \mathrm{flip}(\mathrm{rev}(\pi)).$$

For permutations $\pi, \omega \in S_n$, we think of multiplication as a composition of functions so that $(\pi \circ \omega)(k) = \pi(\omega(k))$ whenever $1 \leq k \leq n$. Given π, we associate it with its permutation matrix;

$$M_\pi \coloneqq \left(\delta_{i,\pi(j)}\right)_{1 \leq i,j \leq n}.$$

The permutation matrix of the composition $\pi \circ \omega$ is $M_\pi M_\sigma$. We follow the convention in [7] and illustrate permutation matrices by letting row indices start at the bottom, as in Figure 1. From now on, this is the picture we have in mind when referring to permutation matrices.

$$M_\pi = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

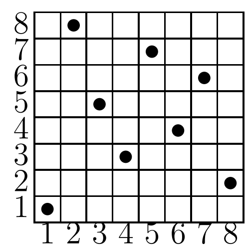


Figure 1: The permutation matrix M_π associated with $\pi = [1, 8, 5, 3, 7, 4, 6, 2]$, where the bottom row is the first row. The rightmost figure is simply the graph of the function $i \mapsto \pi(i)$ in Cartesian coordinates.

2.1 Permutation patterns

A permutation $\pi \in S_n$ is said to contain the pattern $\sigma \in S_k$ if there is a subsequence of π that is order-isomorphic to σ. Otherwise, π is said to avoid σ and $S_\sigma(n)$ denotes the set of σ-avoiding permutations over $[n]$. We simply write the pattern as a word. For example, $\pi = [1, 8, 5, 3, 7, 4, 6, 2]$ contains the pattern 231 as the subsequence $3, 6, 2$ in π. However, $[1, 7, 6, 4, 2, 3, 5, 8]$ avoids the pattern 231.

A pattern of length 3 is the first non-trivial case to consider and we use a one-line notation without the parentheses to represent such a pattern. In Figure 3, we show the structure of 132-avoiding permutations. Any permutation of length n avoiding the pattern 132 must be of the form

$$\pi = [\alpha_1, \ldots, \alpha_j, \ n, \ \beta_1, \ldots, \beta_{n-1-j}],$$

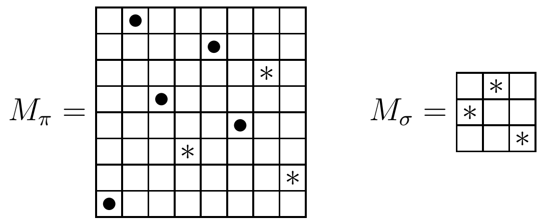


Figure 2: The permutation matrices associated with $\pi = [1, 8, 5, 3, 7, 4, 6, 2]$ and $\sigma = 231$. Here we see that π contains the pattern σ (there are other instances of this pattern in π).

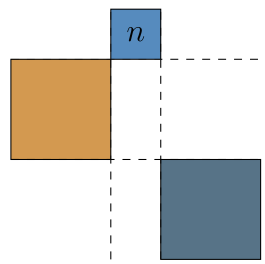


Figure 3: The structure of a permutation matrix that avoids the pattern 132. The larger regions on the left and right are also 132-avoiding which explains the "Catalan structure".

where $\min(\alpha_1, \ldots, \alpha_j) > \max(\beta_1, \ldots, \beta_{n-1-j})$.

Permutations avoiding one pattern of length 3 have been studied extensively and it is well-known (first proved by Knuth [8, 9]) that for all $n \geq 0$, we have

$$|S_{123}(n)| = |S_{132}(n)| = |S_{213}(n)| = |S_{231}(n)| = |S_{312}(n)| = |S_{321}(n)| = \frac{1}{n+1}\binom{2n}{n}.$$

For patterns σ of length 3, the inverse map, the rev map, and the revflip map are bijections between some of the sets $S_\sigma(n)$; see Figure 4.

3 Fuss–Catalan numbers and Raney numbers

The main result in this section is Proposition 3, where we give a closed-form formula for numbers $a_k(n)$ (defined in (1) below) that generalize the Fuss–Catalan numbers. The recurrence defining the $a_k(n)$ shows up in several places in the later sections.

The *Fuss–Catalan numbers*, C_n^p, generalize the classical Catalan numbers. Note that for $p = 2$ we recover the classical Catalan numbers; see Table 1. There is an extensive literature on Catalan numbers and we refer readers to Stanley's large collection of Catalan objects and information [18].

If we set

$$F_p(z) := \sum_{n \geq 0} \frac{1}{pn+1}\binom{pn+1}{n} z^n,$$

then Lambert [10, 11] proved that

$$F_p(z) = 1 + z(F_p(z))^p.$$

A modern proof can be found in [4, Eq. 7.68]. It follows that for all $\alpha \in \mathbb{N}^p_{\geq 0}$, we have

$$C^p_n = \sum_{\alpha} C^p_{\alpha_1} \cdot C^p_{\alpha_2} \cdot \dots \cdot C^p_{\alpha_p} \qquad C^p_0 = 1,$$

where the sum of the entries in α equals $n-1$. Note that this relation uniquely defines the sequence $(C^p_n)_{n\geq 0}$.

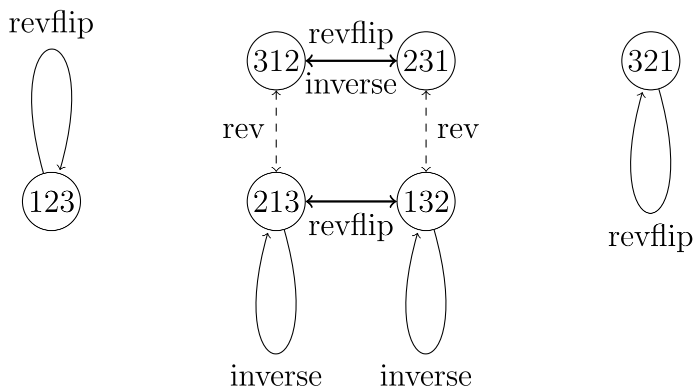


Figure 4: Bijections between sets of permutations that avoid a pattern of length 3; see [16] for details. There are more well-known bijections but the ones shown above also respect the parity-alternating property.

The Fuss–Catalan numbers can be modeled as the number of non-crossing partitions of certain polygons using $(p+1)$-gons; see [3, p. 78]. Another interpretation using lattice paths was mentioned in the introduction.

	1	2	3	4	5	6	7	8
C^1_n	1	1	1	1	1	1	1	1
C^2_n	1	2	5	14	42	132	429	1430
C^3_n	1	3	12	55	273	1428	7752	43263
C^4_n	1	4	22	140	969	7084	53820	420732
C^5_n	1	5	35	285	2530	23751	231880	2330445

Table 1: The values of C^p_n, with $n = 1, 2, \dots, 8$ and $p = 1, 2, \dots, 5$.

The *Raney numbers* (introduced in [15]) may be defined as

$$R_m(p,r) := \frac{r}{mp+r}\binom{mp+r}{m}$$

and are sometimes called the *two-parameter Fuss–Catalan numbers*. Note that $C^p_n = R_n(p,1)$, so the Raney numbers contain the Fuss–Catalan numbers as a sub-family.

Proposition 3. *Fix $k \in \mathbb{N}$ and let the sequence $(a_k(n))_{n=0}^{\infty}$ be defined by the recursion*

$$a_k(n+1) = \sum_{\substack{0 \le i \le n \\ k | i}} a_k(i) a_k(n-i), \qquad a_k(0) \coloneqq 1. \tag{1}$$

The closed-form expression for $a_k(n)$ is then given by

$$a_k(km+j) = \frac{j+1}{km+j+1} \binom{(k+1)m+j}{km+j} \tag{2}$$

where we write $n = km + j$ with $0 \le j < k$. In particular,

$$a_k(km+j) = R_m(k+1, j+1) \text{ and } a_k(km) = C_m^{k+1} \tag{3}$$

so the $a_k(n)$ form a sub-family of the Raney numbers, which includes the Fuss–Catalan numbers.

Proof. Let us first set

$$A_k(t) \coloneqq \sum_{m \ge 0} a_k(km) t^m \text{ and } B_{k,j}(t) \coloneqq \sum_{m \ge 0} a_k(km+j) t^m \text{ for } 0 \le j \le k.$$

Our first goal is to obtain functional equations for the $B_{k,j}(t)$. We shall first prove that

$$B_{k,j}(t) = (A_k(t))^{j+1} \qquad \text{and} \qquad A_k(t) = 1 + t \cdot (A_k(t))^{k+1}. \tag{4}$$

Let j now be fixed with $1 \le j \le k$. The recursion in (1) can then be rewritten as

$$\sum_{m \ge 0} a_k(km+j) t^m = \sum_{m \ge 0} t^m \sum_i a_k(ki) \cdot a_k(km + j - 1 - ki)$$

by first substituting $n \mapsto km + j - 1$ and then multiplying by t^m on both sides followed by summing over m. We then have

$$B_{k,j}(t) = \sum_{m \ge 0} \sum_i a_k(ki) t^i \cdot a_k(km - ki + j - 1) t^{m-i} = A_k(t) \cdot B_{k,j-1}(t).$$

A simple inductive argument over j concludes the proof and the first set of relations in (4) is established. On one hand, we now have $B_{k,k}(t) = (A_k(t))^{k+1}$. On the other hand, the definition of the $B_{k,j}$ tells us that $1 + tB_{k,k}(t) = A_k(t)$. So

$$A_k(t) = 1 + t(A_k(t))^{k+1}$$

which is the second relation in (4).

Recall that Lagrange inversion (see [20, Thm. 5.1.1]) states that if $x(t)$ satisfies the functional equation $x(t) = t \cdot \phi(x(t))$ with $\phi(0) \neq 0$, then

$$f(x(t))\Big|_{t^n} = \frac{1}{n} f'(x)\phi(x)^n \Big|_{x^{n-1}}. \tag{5}$$

The recursion for $A_k(t)$ we had before is precisely of the form $x(t) = t \cdot \phi(x(t))$ when $x(t) = A_k(t) - 1$, $f(x) = 1 + x$ and $\phi(x) = (1 + x)^{k+1}$. Hence, Lagrange inversion gives

$$A_k(t)\Big|_{t^m} = \frac{1}{m}(1 + x)^{(k+1)m}\Big|_{x^{m-1}},$$

which gives the $j = 0$ case of (2).

If we instead choose $f(x) = (1 + x)^{j+1}$ but keep $x(t) = A_k(t) - 1$ and $\phi(x) = (1 + x)^{k+1}$ in (5), then

$$B_{k,j}(t)\Big|_{t^m} = (A_k(t))^{j+1}\Big|_{t^m} = \frac{j+1}{m}(1 + x)^{(k+1)m+j}\Big|_{x^{m-1}} = \frac{j+1}{m}\binom{(k+1)m + j}{m - 1}.$$

This proves the remaining cases in (2). □

Lemma 4. *For a fixed r in $\{0, 1, \ldots, k - 1\}$ and $m \geq 1$, we have*

$$a_k(km) = \sum_{j=0}^{m-1} a_k(kj + r)a_k(k(m - j) - r - 1) \tag{6}$$

where $a_k(n)$ is the sequence in Proposition 3.

Proof. In the notation of Proposition 3 it is straightforward to prove that

$$A_k(t) = 1 + tB_{k,r}(t)B_{k,k-1-r}(t)$$

by using (4). Comparing coefficients now gives the desired formula. □

4 Mod-k-alternating permutations and pattern avoidance

4.1 Mod-k-alternating permutations

Let $\mathrm{P}(n)$ denote the set of parity alternating permutations (PAP) starting with an odd entry and let $\mathfrak{p}(n)$ be the cardinality of $\mathrm{P}(n)$. Moreover, we let $\mathrm{P}_\sigma(n)$ be the set of PAPs avoiding the permutation pattern σ. We also let $\mathrm{P}_{\sigma,\tau}(n)$ be the set of PAPs avoiding patterns σ and τ.

Similarly, $\mathrm{P}^*(n)$ denotes the set of parity-alternating permutations starting with an *even* entry; $\mathrm{P}^*_\sigma(n)$ and $\mathrm{P}^*_{\sigma,\tau}(n)$ are defined analogously.

The following lemma is straightforward to verify.

Lemma 5. *For $m \geq 0$, the maps*

$$\begin{aligned}\mathrm{rev} &: \mathrm{P}_{132}(2m) \to \mathrm{P}^*_{231}(2m)\\ \mathrm{rev} &: \mathrm{P}_{231}(2m) \to \mathrm{P}^*_{132}(2m)\\ \mathrm{rev} &: \mathrm{P}_{132}(2m+1) \to \mathrm{P}_{231}(2m+1)\\ \mathrm{rev} &: \mathrm{P}_{231}(2m+1) \to \mathrm{P}_{132}(2m+1)\end{aligned}$$

are bijections.

For instance, 34561278 is mapped to 87216543, and 5234167 is mapped to 7614325 under the reverse map. Note that 34561278 and 5234167 are 132-avoiding and 87216543 and 7614325 are 231-avoiding.

Definition 6. Given $k \geq 2$, a *mod-k-alternating permutation* of size n is a permutation $\pi \in S_n$ such that

$$\pi(i) \equiv i \pmod{k} \qquad \text{for all } i = 1, 2, \ldots, n.$$

Let $\mathrm{MP}(n,k)$ be the set of such mod-k-alternating permutations and we set

$$\mathrm{MP}_\sigma(n,k) \coloneqq \mathrm{MP}(n,k) \cap S_\sigma(n).$$

Note that it can be easily seen that $\mathrm{MP}(n,k)$ is closed under compositions and taking inverses, so that it is a subgroup of S_n.

For $k = 2$ we recover the classical parity-alternating permutations starting with an odd integer. Note that

$$\mathfrak{mp}(n,k) \coloneqq |\mathrm{MP}(n,k)| = \left(\left\lceil \frac{n}{k} \right\rceil !\right)^j \left(\left\lfloor \frac{n}{k} \right\rfloor !\right)^{k-j}$$

where j is the remainder when n is divided by k. We set $\mathfrak{mp}_\sigma(n,k) \coloneqq |\mathrm{MP}_\sigma(n,k)|$.

Example 7. For example, $\mathrm{MP}(7,3)$ has $(3!)^1 \cdot (2!)^2 = 24$ elements:

$$\begin{array}{lllll}
[1,2,3,4,5,6,7] & [1,2,6,4,5,3,7] & [1,5,3,4,2,6,7] & [1,5,6,4,2,3,7] & [1,2,3,7,5,6,4]\\
[1,2,6,7,5,3,4] & [1,5,3,7,2,6,4] & [1,5,6,7,2,3,4] & [4,2,3,1,5,6,7] & [4,2,6,1,5,3,7]\\
[4,5,3,1,2,6,7] & [4,5,6,1,2,3,7] & [4,2,3,7,5,6,1] & [4,2,6,7,5,3,1] & [4,5,3,7,2,6,1]\\
[4,5,6,7,2,3,1] & [7,2,3,1,5,6,4] & [7,2,6,1,5,3,4] & [7,5,3,1,2,6,4] & [7,5,6,1,2,3,4]\\
[7,2,3,4,5,6,1] & [7,2,6,4,5,3,1] & [7,5,3,4,2,6,1] & [7,5,6,4,2,3,1]. &
\end{array}$$

We shall also introduce some notation for mod-k-alternating permutations starting with the remainder r mod k.

Definition 8. Let $\mathrm{MP}(n,k,r)$ denote the set of permutations of length n that satisfy

$$\pi(i) \equiv (r + i - 1) \pmod{k} \qquad \text{for all } 1 \leq i \leq n. \tag{7}$$

Moreover, we set $\mathrm{MP}_\sigma(n,k,r) \coloneqq \mathrm{MP}(n,k,r) \cap S_\sigma(n)$, $\mathfrak{mp}(n,k,r) \coloneqq |\mathrm{MP}(n,k,r)|$, and $\mathfrak{mp}_\sigma(n,k,r) \coloneqq |\mathrm{MP}_\sigma(n,k,r)|$.

Lemma 9. *We have* $\mathfrak{mp}(n,k,r) = 0$ *unless* $r = 1$ *or* $n = mk$ *for some integer* m.

Proof. Suppose $2 \leq r \leq k$ and $\pi \in \mathrm{MP}(n,k,r)$. By (7), the number of positions in π where the entry has a remainder r always exceeds the number of entries with a remainder $r-1$, unless n is a multiple of k. This is due to the fact that the first position has a remainder r, so that the remainders must form the pattern

$$[r, r+1, r+2, \dots, 0, 1, \dots, r-1, r, \dots].$$

Now, the number of entries with index equal to $r \bmod k$ is never larger than the number of entries with index equal to $r-1 \bmod k$. This is because the entries are from $[n]$. Since these two counts must agree for π, the statement follows. $\square$

Lemma 10. *The cardinality of* $\mathrm{MP}(nk,k,r)$ *is* $(n!)^k$.

Proof. This is straightforward. $\square$

Lemma 11. *We have* $\mathfrak{mp}_\sigma(n,k) = \mathfrak{mp}_{\mathrm{revflip}(\sigma)}(n,k)$. *Moreover,*

$$\mathfrak{mp}_\sigma(n,k,r) = \mathfrak{mp}_{\mathrm{revflip}(\sigma)}(n,k,2-r). \tag{8}$$

In particular, $\mathfrak{mp}_{132}(n,k)$ *and* $\mathfrak{mp}_{213}(n,k)$ *have the same cardinality.*

Proof. Note that it is enough to prove (8), and that it suffices to show that

$$\mathrm{revflip} : \mathrm{MP}_\sigma(n,k,r) \to \mathrm{MP}_{\mathrm{revflip}(\sigma)}(n,k,2-r)$$

is a bijection.

For $\pi \in \mathrm{MP}_\sigma(n,k,r)$, we have that the last element in π has remainder $r-1+n \bmod k$. Hence, the first element of $\mathrm{revflip}(\pi)$ has remainder $n+1-(r-1+n) = 2-r \bmod k$. The remaining properties are now straightforward to verify; that is,

$$\mathrm{revflip}(\pi) \in \mathrm{MP}_{\mathrm{revflip}(\sigma)}(n,k,2-r).$$

$\square$

Lemma 12. *We have* $\mathfrak{mp}_\sigma(n,k) = \mathfrak{mp}_{\sigma^{-1}}(n,k)$.

Proof. We have $\pi \in S_\sigma(n) \iff \pi^{-1} \in S_{\sigma^{-1}}(n)$ (this is classical; see [16, Lem. 1]) so $\pi \in \mathrm{MP}_\sigma(n,k)$ holds if and only if $\pi^{-1} \in \mathrm{MP}_{\sigma^{-1}}(n,k)$. $\square$

Proposition 13. *For* $k \geq 3$ *and* $n \geq 1$ *we have*

$$\mathfrak{mp}_{312}(n,k,r) = \begin{cases} 1, & \text{if } r = 1; \\ 2^{n/k-1}, & \text{if } r = 2 \text{ and } k \mid n; \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

Note that by Lemma 11, a corresponding statement holds for the pattern 231.

Proof. We shall prove the stronger statement that $\mathrm{MP}_{312}(n, k, 1)$ consists of only the identity permutation.

We proceed by induction over n and assume that the statement in (9) is true for permutations of length less than n; the base cases $n = 1$ and $n = 2$ are straightforward to verify. Note that since $k \geq 3$, $r = 0$ and $r = 2$ are different cases.

Suppose that $\pi \in \mathrm{MP}_{312}(n, k, r)$ is of the form

$$\pi = [\alpha, 1, \beta] \tag{10}$$

for some $n \geq 3$.

Case $r = 1$**.** If α is the empty sequence, then $[\beta - 1] \in \mathrm{MP}_{312}(n - 1, k, 1)$ which must be the identity permutation by induction. Hence, we can assume that the length, say s, of α is greater than 0 and we must therefore have $n \geq k + 1$.

Since π avoids 312, we have $\max(\alpha) < \min(\beta)$. In particular, α is constructed from

$$\{2, 3, \ldots, s + 1\}.$$

So $[\alpha, 1] \in \mathrm{MP}_{312}(s + 1, k, 1)$.

On the other hand, β consists of numbers in $\{s + 2, \ldots, n\}$ and

$$\beta - (s + 1) \in \mathrm{MP}_{312}(n - (s + 1), k, 1).$$

Thus, $\beta - (s + 1)$ is an identity permutation by the induction hypothesis. However, $[\alpha - 1]$ is an element in $\mathrm{MP}_{312}(s, k, 0)$, but this set is empty (by induction hypothesis). Hence, the case $r = 1$ is done.

Case $r \neq 1$**.** By Lemma 9, we can assume $k \mid n$ and $\alpha - 1 \in \mathrm{MP}_{312}(s, k, r - 1)$. But this set is empty unless $r = 2$ or $r = 3$ and $k \mid s$. The latter is impossible, since the form in (10) tells us $r + s \equiv 1 \pmod{k}$, which implies that $k \nmid s$. For the first case ($r = 2$) there is only one possibility for α, namely $[2, 3, \ldots, s + 1]$ by induction.

We also know that $[\beta - s - 1] \in \mathrm{MP}_{312}(n{-}s{-}1, k, 2)$. Moreover, we also need that $k \mid (n{-}s{-}1)$ and there are (by induction) $2^{(n-s-1)/k-1}$ options for β. Accounting for all possible values of $n{-}s{-}1$ in $\{0, k, 2k, \ldots, n{-}k\}$, we have $1{+}1{+}2{+}4{+}\cdots{+}2^{(n/k-1)-1} = 2^{n/k-1}$ ways to obtain π. $\square$

The following proposition is analogous to Proposition 13.

Proposition 14. *Let $k \geq 1$. With $a_k(n)$ defined as in Proposition 3, we have*

$$\mathfrak{mp}_{213}(n, k, r) = \begin{cases} a_k(n), & \text{if } r = 1 \text{ or } k \mid n; \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

By Lemma 11, a similar statement holds for the pattern 132.

Proof. If $k = 1$, then $r = 1$ and $\mathfrak{mp}_{213}(n, 1)$ is the Catalan number. From now on assume that $k \geq 2$.

For the case $r = 1$, let us consider $\pi \in \mathrm{MP}_{213}(n + 1, k)$. It must be of the form

$$\pi = [\alpha_1, \ldots, \alpha_j, 1, \beta_1, \ldots, \beta_{n-j}]$$

where we must have $\alpha_t > \beta_s$ for all relevant t, s; see Figure 5.

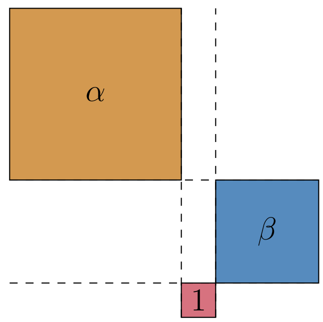


Figure 5: The structure of a 213-avoiding permutation.

Since 1 can only appear at some position $km + 1$ for some $m \in \mathbb{N}$, we must have that $k \mid j$. Hence, setting

$$\begin{aligned}
\alpha' &:= [\alpha_1 - (n - j + 1), \alpha_2 - (n - j + 1), \ldots, \alpha_j - (n - j + 1)] \\
\beta' &:= [\beta_1 - 1, \beta_2 - 1, \ldots, \beta_{n-j} - 1]
\end{aligned}$$

we have that $\alpha' \in \mathrm{MP}_{213}(j, k)$ if $k \mid n{+}1$ and $\alpha' \in \mathrm{MP}_{213}(j, k, r'{\neq}1)$ otherwise. In both cases, α' exists since $k \mid j$ and followed by inductive argument. In addition, we have

$$\beta' \in \mathrm{MP}_{213}(n - j, k).$$

Thus we have a bijective proof of the recursion in (1).

The "otherwise" case follows from Lemma 9. Hence, only the case $k \mid n$, $r \neq 1$ needs to be proved. Note that it suffices to show that $\mathfrak{mp}_{213}(km, k, r) = a_k(km)$.

It is straightforward to verify that (11) holds whenever $n \leq 2$. Now suppose π is in $\mathrm{MP}_{213}(n, k, r)$ for some $r \in \{2, 3, \ldots, k\}$ and $n \geq 3$ with $n = km$. Then π is of the form

$$[\alpha_1, 1, \alpha_2]$$

where $\min(\alpha_1) > \max(\alpha_2)$ as seen in Figure 5. Let α_1 have length s, so that α_2 has length $n - 1 - s$. Note that we must have $k \mid s + r - 1$ in order for the first entry in α_1 to have remainder r mod k. We then have

$$(\alpha_2 - 1) \in \mathrm{MP}_{213}(n - 1 - s, k, 1) \text{ and } (\alpha_1 - (n - s)) \in \mathrm{MP}_{213}(s, k, 1).$$

Conversely, given permutations $\tau_1 \in \mathrm{MP}_{213}(s,k,1)$ and $\tau_2 \in \mathrm{MP}_{213}(n-1-s,k,1)$ for some s satisfying $k \mid s+r-1$, we can construct a unique $\pi \in \mathrm{MP}_{213}(n,k,r)$ by reversing the process above. Hence, when $k \mid n$ we have

$$\mathfrak{mp}_{213}(n,k,r) = \sum_{\substack{s=1 \\ k \mid s+r-1}}^{n-1} \mathfrak{mp}_{213}(s,k,1) \cdot \mathfrak{mp}_{213}(n-s-1,k,1).$$

Now, setting $n = km$, $s = kj + r'$ (note that s can be expressed in this way for the fixed $r' = k - r + 1$), and using that $a_k(i) = \mathfrak{mp}_{213}(i,k,1)$, we get

$$\mathfrak{mp}_{213}(km,k,r) = \sum_{j=0}^{m-1} a_k(kj+r') \cdot a_k(km-(kj+r')-1).$$

By (6) in Lemma 4 we may now conclude that $\mathfrak{mp}_{213}(km,k,r) = a_k(km)$. $\square$

We can now give two new combinatorial interpretations of the Fuss–Catalan numbers.

Corollary 15. *Let $\sigma \in \{132, 213\}$ and $n, k \geq 1$. Let m and j be defined via $n = km + j$ with $0 \leq j < k$. Then*

$$\mathfrak{mp}_\sigma(n,k) = \frac{j+1}{km+j+1}\binom{(k+1)m+j}{km+j}$$

and in particular

$$\mathfrak{mp}_\sigma(km,k) = \frac{1}{(k+1)m+1}\binom{(k+1)m+1}{m} = \frac{1}{km+1}\binom{(k+1)m}{m}.$$

Proof. This follows from Proposition 3 and Equation (3). $\square$

4.2 PAPs avoiding one of 132, 213, 231, and 312

The case $k = 2$ is rather different from $k \geq 3$ as indicated by Proposition 13. In this subsection, we treat the $k = 2$ case for the patterns in $\{132, 213, 231, 312\}$.

Lemma 16. *Let $\pi \in \mathrm{P}_{132}(n)$ with $n \geq 2$. Then either n appears after $n-1$ in π or n is odd and appears at position 1.*

Proof. The case $n = 2$ is easy so suppose $n \geq 3$ and $\pi(1) \neq n$. Then the numbers to the left of n must be greater than the numbers from the right of n in π (since we must avoid the pattern 132; see Figure 3). In particular, $n-1$ must appear before n and we are done. $\square$

Proposition 17. *For $m \geq 0$, set*

$$a_m := \mathfrak{p}_{132}(2m), \quad b_m := \mathfrak{p}_{132}(2m+1),$$
$$a'_m := \mathfrak{p}_{231}(2m), \quad b'_m := \mathfrak{p}_{231}(2m+1).$$

We then have $a_m = a'_m$ and $b_m = b'_m$ for all m.

Proof. We already know that $b_m = b'_m$ since $\mathrm{rev} : \mathrm{P}_{132}(2m+1) \to \mathrm{P}_{231}(2m+1)$ is a bijection.

Claim: We have

$$a_m = \sum_{\substack{i+j+k=m-1\\ i,j,k\geq 0}} a_i a'_j a_k, \text{ and } a'_m = \sum_{\substack{i+j+k=m-1\\ i,j,k\geq 0}} a'_i a_j a'_k. \tag{12}$$

First set $n := 2m$ and suppose $\pi \in \mathrm{P}_{132}(2m)$. By Lemma 16, we know that it must be of the form

$$[\underbrace{\alpha_1, \alpha_2, \ldots, \alpha_{2i}}_{\alpha},\ n-1,\ \underbrace{\beta_1, \ldots, \beta_{2j}}_{\beta},\ n,\ \underbrace{\gamma_1, \ldots, \gamma_{2k}}_{\gamma}].$$

Moreover, since the permutation is 132-avoiding (see Figure 3), it must be of the form

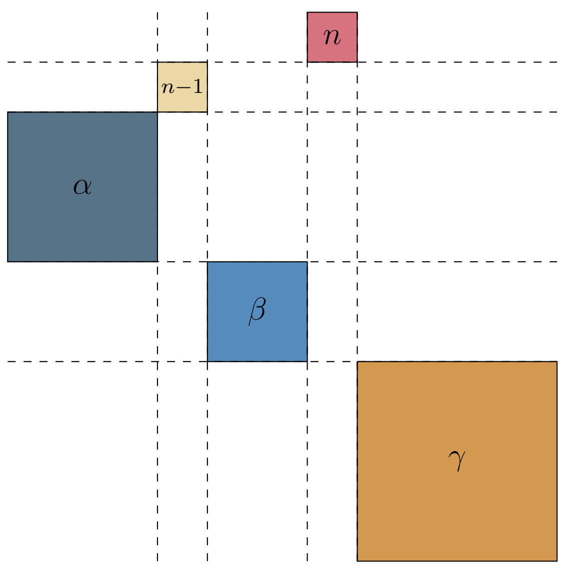


Figure 6: The structure of a 132-avoiding PAP with even length.

Now, the permutation matrices α and γ can be interpreted as smaller permutations in some $\mathrm{P}_{132}(2i)$ and $\mathrm{P}_{132}(2k)$, respectively. Moreover, the permutation β can be seen as an element in $\mathrm{P}^*_{132}(2j)$. By the reversal bijection, we have that β corresponds to some element in $\mathrm{P}_{231}(2j)$ whose cardinality is a'_j. Hence, we can conclude the first formula in (12).

Using a similar argument (just reverse everything) we can show the second recursive identity. An inductive argument over m now allows us to conclude that $a_m = a'_m$ for all $m \geq 0$. ∎

Corollary 18. *For any $\sigma \in \{132, 213, 231, 312\}$, we have*

$$\mathfrak{p}_\sigma(2m) = \frac{1}{2m+1}\binom{3m}{m} \quad \textit{and} \quad \mathfrak{p}_\sigma(2m+1) = \frac{1}{2m+1}\binom{3m+1}{m+1}.$$

Proof. The formulas follow from Corollary 15 for $\sigma = 132$. By Proposition 17, the statement holds for the pattern 231. By applying the revflip map we obtain the same counts for the other two patterns. $\square$

5 Mod-k-alternating subexcedant functions

Subexcedant functions and their close connection with permutations was initially studied in [12]. We refer to [1] and [2] as examples of applications of these objects.

Definition 19. A *subexcedant function* f on $[n]$ is a map $f : [n] \longrightarrow [n]$ such that

$$1 \le f(i) \le i \text{ for all } 1 \le i \le n.$$

We denote the set of all subexcedant functions on $[n]$ by $\mathcal{F}_n$.

One can easily see that $\mathcal{F}_n$ has cardinality $n!$. The bijection $\texttt{sefToPerm} : \mathcal{F}_n \longrightarrow S_n$ is defined as a product

$$\texttt{sefToPerm}(f) \coloneqq (n\ f(n)) \cdots (2\ f(2))(1\ f(1)) \tag{13}$$

of cycles of maximum length 2. This bijection is the one described in [12].

The inverse map is defined as follows:

$$\texttt{sefToPerm}^{-1}(\sigma)_j \coloneqq \begin{cases} \sigma(n), & \text{if } j = n; \\ \texttt{sefToPerm}^{-1}(\ (n\ \sigma(n)) \circ \sigma\ )_j, & \text{otherwise} \end{cases} \tag{14}$$

where $\sigma \in S_n$ and $j \in [n]$.

Proposition 20. *Suppose $\pi \in S_n$ and f_π is the corresponding element in $\mathcal{F}_n$. Then π is in* $\mathrm{MP}(n, k)$ *if and only if $f_\pi(i) \equiv i \pmod{k}$.*

Proof. First, suppose that $f_\pi(i) \equiv i \pmod{k}$ for some $\pi \in S_n$. In the product (13), we see that f_π has k disjoint cycle products, then the first product contains integers that are congruent to 1 (mod k), the second product contains integers that are congruent to 2 (mod k), and so on, with the last product containing integers that are congruent to k (mod k). Hence $\pi \in \mathrm{MP}(n, k)$.

On the other hand, suppose now that $\pi \in \mathrm{MP}(n, k)$ and consider $f_\pi = \texttt{sefToPerm}^{-1}(\pi)$. In the recursion in (14), we always interchange integers at positions i and j, where

$$j \equiv i \pmod{k}.$$

So (via an inductive argument) we must have that $f_\pi(i) \equiv i \pmod{k}$ for all i. $\square$

Example 21. The subexcedant function 12341634 corresponds to the permutation

$$[5, 2, 7, 8, 1, 6, 3, 4]$$

in $\mathrm{MP}(8, 4)$.

Definition 22. A word $w = (w_1, \ldots, w_n) \in \mathbb{N}^n$ is a *Catalan word* if

$$w_1 = 1 \text{ and } 1 \leq w_i \leq w_{i-1} + 1 \text{ whenever } 2 \leq i \leq n.$$

A permutation π in $\mathrm{P}(n)$ is called a *Catalan PAP* if f_π is a Catalan word. Let $\mathrm{P}_C(n)$ be the set of Catalan PAPs and let $\mathfrak{p}_C(n)$ be its cardinality. We define $\mathrm{MP}_C(n, k)$ in the same manner and set $\mathfrak{mp}_C(n, k) := |\mathrm{MP}_C(n, k)|$.

There are $\frac{1}{n+1}\binom{2n}{n}$ Catalan words of length n and we shall see below that Catalan words are essentially the same as the well-known *area sequences*.

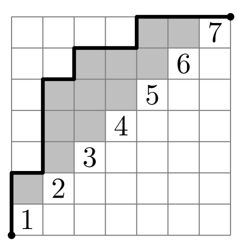
Definition 23. A sequence $(a_1, \ldots, a_n)$ is an *area sequence* if

$$a_1 = 0 \text{ and } 0 \leq a_i \leq a_{i-1} + 1 \text{ whenever } 2 \leq i \leq n.$$

Area sequences are in bijection with Catalan words. The array $(a_1, \ldots, a_n)$ is an area sequence if and only if $(a_1 + 1, a_2 + 1, \ldots, a_n + 1)$ is a Catalan word. The notion of area sequences plays an important role in *diagonal harmonics*; see [5].

There is a well-known bijection between Dyck paths and area sequences. We illustrate this bijection in the following example.

Example 24. The Dyck path $P = \texttt{nnennneneeneee}$ has the area sequence $\mathbf{a} = (0, 1, 1, 2, 3, 3, 2)$.

$P =$ 

The number of shaded boxes in row i (from the bottom) is given by a_i and this provides the bijection between area sequences and Dyck paths.

We denote the set of all Dyck paths of size n such that the number of boxes in each row above the path is a multiple of k by L_n^k; see Example 26.

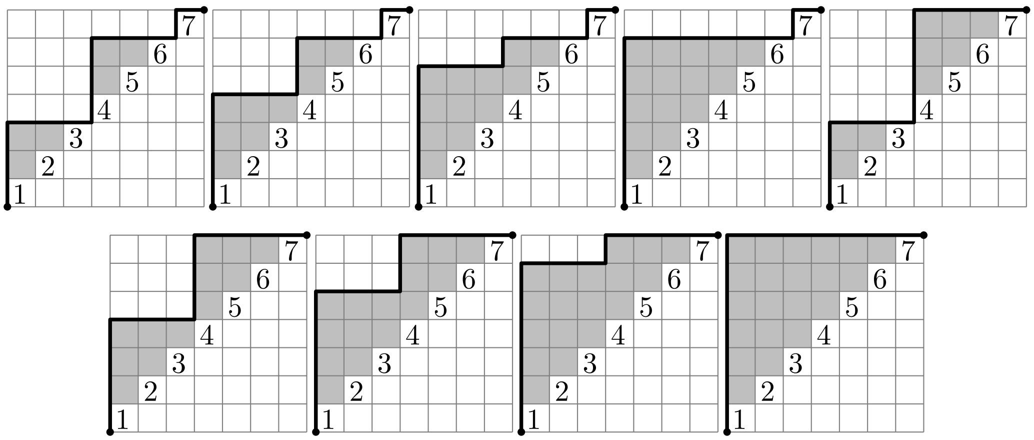
Corollary 25. *The set* $\mathrm{MP}_C(n, k)$ *has the same cardinality as* L_n^k.

Proof. Let $\pi \in \mathrm{MP}_C(n,k)$. By Proposition 20, the corresponding subexcedant function $f_\pi = w_1 w_2 \cdots w_n$ is a Catalan word with the property $i - w_i = \alpha_i k$ for all $i \in [n]$, where $\alpha_i \in \mathbb{N}$. Hence, $\mathbf{a} = (w_1 - 1, w_2 - 1, \ldots, w_n - 1)$ is an area sequence for some Dyck path. Now, the number of boxes in the i^{th} row above the Dyck path is given by

$$(i-1) - (w_i - 1) = \alpha_i k,$$

since the i^{th} row has $i-1$ boxes above the diagonal in the $n \times n$ grid. □

Example 26. Here are the 9 Dyck paths corresponding to the entries in $\mathrm{MP}_C(7,3)$.



Theorem 27. *Let $r_k(n) := \mathfrak{mp}_C(n,k)$. Then*

$$r_k(n+1) = \sum_{\substack{0 \le j \le n \\ k \mid j}} r_k(j) r_k(n-j), \qquad r_k(0) := 1.$$

Proof. Let π be in $\mathrm{MP}_C(n+1,k)$. Consider the last 1 in f_π. Then we have

$$f_\pi = \gamma_j, 1, \gamma_{n-j},$$

where γ_j is the subword having length j. Clearly, j is a multiple of k and $0 \le j \le n$. Finally $\gamma_j \in \mathrm{MP}_C(j,k)$ and $\gamma_{n-j} - 1 \in \mathrm{MP}_C(n-j,k)$, since γ_{n-j} starts with 2. □

In the same manner as the proof of Corollary 15, we have the following corollary.

Corollary 28. *Let m and j be defined via $n = km + j$ with $0 \le j < k$. Then*

$$\mathfrak{mp}_C(n,k) = \frac{j+1}{km+j+1} \binom{(k+1)m+j}{km+j}$$

and in particular,

$$\mathfrak{mp}_C(km,k) = \frac{1}{(k+1)m+1} \binom{(k+1)m+1}{m} = \frac{1}{km+1} \binom{(k+1)m}{m}.$$

The fact that our Dyck paths are counted by Raney numbers is not new; this is a classical interpretation of Raney numbers. We decided to include this interpretation mainly for context and to illustrate the surprising fact that a Catalan family with mod-k-restrictions again gives Raney numbers, just as for mod-k-alternating permutations and pattern-avoidance.

6 Mod-k-alternating permutations avoiding two patterns

The systematic study of permutations avoiding two patterns of length 3 was completed by Simion and Schmidt [16]. In this section, we do the same for parity-alternating permutations and mod-k-alternating permutations.

Due to Proposition 13, the situation for $k \geq 3$ is, in general, simpler than the $k = 2$ case. In Table 2, we present an overview of our results in the $k = 2$ case. We consider the sequence $(\mathfrak{p}_{\sigma,\tau}(n))_{n\geq 1}$ and separate it into the cases $n = 2m$ and $n = 2m + 1$. Closed-form formulas for the sequences $(\mathfrak{p}_{\sigma,\tau}(2m))_{m\geq 1}$ and $(\mathfrak{p}_{\sigma,\tau}(2m + 1))_{m\geq 0}$, where

$$\sigma, \tau \in \{123, 132, 213, 231, 312, 321\},$$

are shown in the table.

The bold letters refer to cases that are treated below. Note that Lemmas 11 and 12 can easily be generalized to sets of permutations avoiding two or more patterns. For example, $\mathfrak{p}_{\sigma,\tau}(n, k) = \mathfrak{p}_{\mathrm{revflip}(\sigma),\mathrm{revflip}(\tau)}(n, k)$. Cases marked with the same letter indicate situations that give the same enumeration after applying these lemmas. We emphasize that the formulas are only valid for permutations of length at least 1.

	123	**132**	**213**	**231**	**312**	**321**
123	*					
132	**A**; 2^{m-1}, 1	*				
213	**A**; 2^{m-1}, 1	**D**; 2^{m-1}, F_{2m+1}	*			
231	**B**; m, $\binom{m}{2}+1$	**E**; 2^{m-1}, 2^m	**E**; 2^{m-1}, 2^m	*		
312	**B**; m, $\binom{m}{2}+1$	**E**; 2^{m-1}, 2^m	**E**; 2^{m-1}, 2^m	**F**; F_n	*	
321	**C**; 0	**G**; $\binom{m}{2}+1$	**G**; $\binom{m}{2}+1$	**I**; 1	**I**; 1	*

Table 2: Pair of formulas in an entry represent the cases $n = 2m$ and $n = 2m+1$, respectively. A single formula covers to both cases. The Fibonacci numbers are denoted by F_n, with $F_1 = F_2 = 1$.

Remark 29. There are no permutations of length 5 or more that avoid 123 and 321. On the other hand, we only get the identity permutation that avoids 231 and 321 due to Proposition 13 or (in the case $k = 2$) a simple argument.

Only the cases A, D, and G are interesting for $k \geq 3$. Otherwise at least one pattern is covered by Proposition 13 and then the number of mod-k-alternating permutations over $[n]$ that avoid the two patterns is 1, for all n. The general formula for $\mathfrak{mp}_{\sigma,\tau}(n, k)$ in cases A, D, and G is given in the corresponding subsection.

6.1 Case A, (132, 123)

Lemma 30. *Any permutation in $S_{132,123}(n)$ starts with at least $n-1$.*

Proof. Suppose $\pi \in S_{132,123}(n)$ and $\pi(1) < n-1$. Then π contains either the 123 pattern $\pi(1)\ n{-}1\ n$ or the 132 pattern $\pi(1)\ n\ n{-}1$. Hence, $\pi(1) \geq n-1$. $\square$

Proposition 31. *For n odd, the set $\mathrm{P}_{132,123}(n)$ consists only of the permutation $[n, n-1, \ldots, 1]$. When $n = 2m$, we have $\mathfrak{p}_{132,123}(2m) = 2^{m-1}$.*

Proof. Suppose $\pi \in \mathrm{P}_{132,123}(n)$ for odd n. From Lemma 30, π starts with n and then the rest of the permutation must follow the same rule. Hence, π is the unique permutation $[n, n-1, \ldots, 1]$.

For the second claim, suppose that

$$\pi = [\alpha, 2m, \beta] \in \mathrm{P}_{132,123}(2m).$$

Then it is easy to verify that the permutations

$$[2m+1, 2m+2, \alpha, 2m, \beta] \text{ and } [2m+1, 2m, \alpha, 2m+2, \beta] \tag{15}$$

are elements in $\mathrm{P}_{132,123}(2m+2)$.

Now, for any $\pi' \in \mathrm{P}_{132,123}(2m+2)$, by Lemma 30 we have that $2m+1$ is in the very first position. In a similar manner, we must have that $\pi'(2) \in \{2m, 2m+2\}$. We have now shown that all elements in $\mathrm{P}_{132,123}(2m+2)$ are in one of the forms described in (15). By an inductive argument it follows that $\mathfrak{p}_{132,123}(2m) = 2^{m-1}$. $\square$

Proposition 32. *Let $k \geq 3$ and write $n = km + l$, where $0 \leq l < k$. We then have*

$$\mathfrak{mp}_{132,123}(km+l, k) = \begin{cases} 1, & \textit{if } (k,l) \in \{(3,1),(3,2),(4,2)\}; \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Suppose $\pi \in \mathrm{MP}_{132,123}(n,k)$ and consider the position of 1 in π. Since π avoids both 132 and 123, we must have that 1 is in the last or penultimate position. Consequently, $l \in \{1, 2\}$ and we shall treat these two cases separately.

Case $l = 1$. Since $k \geq 3$, we see that 2 must appear to the left of 1 with at least one entry in between. Moreover, since π must avoid both 132 and 123, there cannot be more than one entry between 2 and 1, which implies that $k = 3$. The same reasoning as above shows that 3 must be placed between 2 and 1, so π ends with 231. We may now treat the first $n-3$ entries in π as an element in $\mathrm{MP}_{132,123}(n-3, k)$. By induction, we can then conclude that π is the unique permutation

$$\pi = [3m{+}1,\ \ 3m{-}1, 3m, 3m{-}2,\ \ldots,\ 8, 9, 7,\ 5, 6, 4,\ 2, 3, 1],$$

where the largest entry is first and the remaining entries appear in blocks of 3.

Case $l = 2$. It is straightforward to show that we must have 2 at the last position of π in order to avoid the two patterns. Now, a similar argument as above shows that

$$\pi = [\dots, 3, 1, 2] \text{ or } \pi = [\dots, 3, j, 1, 2]$$

where $4 \leq j \leq n$. The first option gives $k = 3$ while the second option forces $k = 4$ and $j = 4$.

In case $k = 3$, we have that π ends with $3, 1, 2$ and an inductive argument then gives

$$\pi = [3m{+}1, 3m{+}2,\ \ 3m, 3m{-}2, 3m{-}1,\ \ \dots,\ \ 9, 7, 8,\ \ 6, 4, 5,\ \ 3, 1, 2].$$

Similarly, the case $k = 4$ gives

$$\pi = [4m{+}1, 4m{+}2,\ \ 4m{-}1, 4m, 4m{-}3, 4m{-}2,\ \ \dots,\ \ 7, 8, 5, 6,\ \ 3, 4, 1, 2].$$

□

6.2 Case B, (231, 123)

For n odd, we have $\mathfrak{p}_{321,132}(n) = \mathfrak{p}_{231,123}(n)$ because the reversal map (Lemma 5) gives a bijection between the sets. The latter set is enumerated further down in Case G.

The case of even n is covered by the following lemma.

Lemma 33. *Suppose* $\pi \in \mathrm{P}_{231,123}(2m)$. *Then*

$$\pi = [k, k-1, k-2, \dots, 1, 2m, 2m-1, \dots, k+1]$$

for some odd k. *Consequently,* $\mathfrak{p}_{231,123}(2m) = m$.

Proof. Because π is 231-avoiding, all entries to the right of $2m$ must be larger than all entries to its left. Moreover, it is then straightforward to show that both these intervals must be decreasing in order for π to avoid the pattern 123. There are exactly m choices of the first entry in π—any odd number in $[2m]$. The lemma follows from these observations. □

6.3 Case D, (213, 132)

Proposition 34. *For* $k \geq 1$ *and* $0 \leq l < k$, *we have*

$$\mathfrak{mp}_{213,132}(km + l, k) = \begin{cases} 2^{m-1}, & \text{if } l = 0; \\ F_{2m+1}, & \text{otherwise.} \end{cases}$$

Proof. For $k = 1$, this reduces to proving that $S_{213,132}(m) = 2^{m-1}$. This has been done already by Simion and Schmidt [16]. We treat the case $k \geq 2$ in the remaining part of the proof.

Case $l \geq 1$. From the recursive formula $F_{2m+1} = F_{2m} + F_{2m-1}$ of the Fibonacci numbers, we can easily deduce the following recursive formula involving only the odd indexed Fibonacci numbers:

$$F_{2m+1} = 2F_{2(m-1)+1} + F_{2(m-2)+1} + F_{2(m-3)+1} + \cdots + F_3 + F_1. \tag{16}$$

Our goal is to show the same recursive structure for elements in $\mathrm{MP}_{213,132}(km+l,k)$.

Suppose $\pi \in \mathrm{MP}_{213,132}(km+l,k)$. We claim that π can be obtained in one of the following three ways, corresponding to the factor 2 in $2F_{2(m-1)+1}$ and the remaining terms in (16), respectively.

(1a) From $\pi' \in \mathrm{MP}_{213,132}(k(m-1)+l,k)$ by inserting the consecutive integers

$$k(m-1)+l+1,\ k(m-1)+l+2, \ldots, km+l-1,\ km+l$$

(in this order) immediately after the largest entry in π'.

(1b) From $\pi' \in \mathrm{MP}_{213,132}(k(m-1)+l,k)$ by inserting the two sequences of consecutive integers

$$k(m-1)+l+i,\ k(m-1)+l+i+1, \ldots, km+l,$$

and then

$$k(m-1)+l+1, \ldots, k(m-1)+l+i-1,$$

(in this order) for some $1 < i \leq k$ such that $k(m-1)+l+i \equiv 1 \pmod{k}$, in the beginning of π'. In fact, we have $i = k-l+1$.

(2) From $\pi'' \in \mathrm{MP}_{213,132}(ks+l,k)$ for any $1 \leq s < m-1$ by inserting the two sequences of consecutive integers

$$k(m-1)+l+i,\ k(m-1)+l+i+1, \ldots, km+l,$$

and then

$$ks+l+1,\ ks+l+2, \ldots, k(m-1)+l+i-1$$

in the beginning of π'', where i satisfies the same condition as in the previous case.

It is straightforward to verify that the three cases are mutually exclusive, and are indeed elements of $\mathrm{MP}_{213,132}(km+l,k)$. We must show that every element in $\mathrm{MP}_{213,132}(km+l,k)$ belongs to one of these cases.

Consider the largest element, $km+l$, in π. All elements to its left must appear in increasing order. Moreover, there cannot be any gaps as that would produce a 132-pattern. There are two cases to consider. The first one is if $km+l$ occurs after $k(m-1)+l$ and there are exactly $k-1$ elements, all are greater than $k(m-1)+l$, between them in increasing order. This implies that π belongs to case (1a) above. Otherwise, the entries to the left of $km+l$ form the interval

$$\underbrace{k(m-1)+l+i,\ k(m-1)+l+i+1,\ldots,km+l-1}_{k-i}, km+l$$

for a unique $i \in [k]$. The largest entry in π (to the right of $km+l$) is $k(m-1)+l+i-1$, so the entries between $km+l$ and $k(m-1)+l+i-1$ must form an increasing sequence of consecutive integers. The first one being of the form $ks+l+1$ for some $s \in [m-1]$. This puts π in either case (1b) (for $s = m-1$) or case (2).

This shows that we have the recursion

$$\begin{aligned}\mathfrak{mp}_{213,132}(km+l,k) = 2\,\mathfrak{mp}_{213,132}(k(m-1)+l,k) + \mathfrak{mp}_{213,132}(k(m-2)+l,k)\\ + \cdots + \mathfrak{mp}_{213,132}(k+l,k) + \mathfrak{mp}_{213,132}(l,k).\end{aligned}$$

Thus, (after checking initial conditions) we have that $(\mathfrak{mp}_{213,132}(km+l,k))_{m=0}^{\infty}$ satisfies the same recursion as in (16). Therefore, $\mathfrak{mp}_{213,132}(km+l,k) = F_{2m+1}$.

Case $l = 0$. Suppose that $\pi \in \mathrm{MP}_{213,132}(k(m+1),k)$.

By a similar argument as above, π either starts with

$$km+1,\ km+2,\ldots,k(m+1)-1,\ k(m+1)$$

or km is followed by this sequence of integers. Thus, we can obtain π from $\pi' \in \mathrm{MP}_{213,132}(km,k)$ by inserting

$$km+1,\ km+2,\ldots,k(m+1)-1,\ k(m+1)$$

either immediately after km, or at the beginning of π'. Hence,

$$\mathfrak{mp}_{213,132}(k(m+1),k) = 2\,\mathfrak{mp}_{213,132}(km,k).$$

We also have the initial condition $\mathfrak{mp}_{213,132}(k,k) = 1$ (the identity permutation) so it follows that $\mathfrak{mp}_{213,132}(km,k) = 2^{m-1}$. □

Corollary 35. *We have*

$$\mathfrak{p}_{213,132}(2m) = 2^{m-1} \qquad \textit{and} \qquad \mathfrak{p}_{213,132}(2m+1) = F_{2m+1}.$$

6.4 Case E, (231, 132)

Proposition 36. *We have*

$$\mathfrak{p}_{231,132}(2m) = 2^{m-1} \quad \textit{and} \quad \mathfrak{p}_{231,132}(2m+1) = 2^{m}.$$

Proof. If $\pi \in S_{231,132}(n)$, it is clear that n must be either in the very beginning or at the very end of π. Thus, $\pi \in \mathrm{P}_{231,132}(2m+1)$ is obtained from $\pi' \in \mathrm{P}_{231,132}(2m-1)$ by inserting $2m, 2m+1$ at the end of π' or by inserting $2m+1, 2m$ at the beginning of π'. Moreover, every $\pi \in \mathrm{P}_{231,132}(2m)$ can be obtained from some $\pi' \in \mathrm{P}_{231,132}(2m-1)$ by inserting $2m$ at the end of π'. Hence,

$$\mathfrak{p}_{231,132}(2m+1) = 2\,\mathfrak{p}_{231,132}(2m-1) \quad \text{and} \quad \mathfrak{p}_{231,132}(2m) = \mathfrak{p}_{231,132}(2m-1),$$

where $\mathfrak{p}_{231,132}(1) = 1$ and $\mathfrak{p}_{231,132}(2) = 1$. The statement then follows by a simple inductive argument. □

6.5 Case F, (312, 231)

For $k \geq 3$, the set $\mathrm{MP}_{312,231}(n,k)$ contains only the identity permutation due to Proposition 13.

For $k = 2$, we have the following proposition.

Proposition 37. *We have $\mathfrak{p}_{312,231}(n) = F_n$, the n^{th} Fibonacci number.*

Proof. It is easy to verify that this holds for $n \leq 3$ so suppose that $\pi \in \mathrm{P}_{312,231}(n)$ for some $n \geq 4$.

If $\pi(n) \neq n$, then $n-1$ must appear on the right of n; otherwise $n-1,\ n,\ \pi(n)$ forms a 231-pattern so n is immediately followed by $n-1$ in order to avoid 312.

Similarly, $n-2$ cannot be anywhere left of n, since then $n-2,\ n,\ \pi(n)$ would form a 231-pattern.

Now, if there was some element between $n-1$ and $n-2$, then there would be a 312-pattern in π. Hence, $n, n-1, n-2$ must be a subword of π.

It follows that any $\pi \in \mathrm{P}_{312,231}(n)$ can be constructed either from some $\pi' \in \mathrm{P}_{312,231}(n-1)$, by appending n at the very end, or by inserting $n, n-1$ immediately before $n-2$ in some $\pi'' \in \mathrm{P}_{312,231}(n-2)$. Thus, we have the classical Fibonacci recursion and the statement follows. □

Observe that for odd n we may apply Lemma 5 and then see that $\mathfrak{p}_{312,231}(n) = \mathfrak{p}_{132,213}(n)$, where the latter is F_n according to Case D.

6.6 Case G, (321, 132)

Lemma 38. *Let $\pi \in \mathrm{MP}_{321,132}(km+l, k)$, where $0 \leq l < k$ and $k \geq 2$. Then for any $j \in [m]$, the entries*

$$k(j-1)+1,\ k(j-1)+2,\ \ldots,\ kj-1,\ kj$$

form a subword of π. Moreover, the last l numbers in $[km+l]$ must appear in increasing order at the very end of π.

Proof. Suppose $k(j-1)+i$ is on the left of $k(j-1)+i-1$ for some $1 < i \leq k$. Then $\pi(1)$, $k(j-1)+i$, $k(j-1)+i-1$ form either a 132- or a 321-pattern, depending on the value of $\pi(1)$ (we know $\pi(1) \neq k(j-1)+i$ since $\pi(1) \equiv 1 \pmod{k}$). Therefore,

$$k(j-1)+1,\ k(j-1)+2,\ \ldots,\ kj-1,\ kj$$

appear in this order.

Now suppose that $k(j-1)+i-1$ and $k(j-1)+i$ are not adjacent for some $1 < i \leq k$. Then there are at least k entries between them since π is a mod-k-alternating permutation. Each such entry must be less than $k(j-1)+i-1$, otherwise we have a 132-pattern. Now consider the position of 1 in π and let t be the number succeeding $k(j-1)+i-1$. If 1 appears to the left of $k(j-1)+i-1$, then $1, \ldots,\ k(j-1)+i-1,\ t$ form a 132-pattern. If 1 is on the right, it can not be immediately after $k(j-1)+i-1$ since $i > 1$. Thus, $t \neq 1$ and $k(j-1)+i-1,\ t, \ldots,\ 1$ form a 321-pattern.

Now, it remains to show the last statement in the lemma. For $l = 0$, there is nothing more to prove. If $l > 0$ we can argue in the same manner as above, that $km+1, km+2, \ldots, km+l$ must form a subword of π. Since all other entries are partitioned into blocks of length k, we must have that the l largest entries appear at the very end. Otherwise, the entry immediately after $km+l$ would have the wrong remainder mod k. □

Proposition 39. *For any $k \geq 1$, $m \geq 0$, and $0 \leq l < k$, we have*

$$\mathfrak{mp}_{321,132}(km+l,k) = \binom{m}{2} + 1.$$

Proof. Let $\pi \in \mathrm{MP}_{321,132}(km+l,k)$. By the second statement in Lemma 38, we can simply disregard the last l entries and conclude that $\mathfrak{mp}_{321,132}(km+l,k) = \mathfrak{mp}_{321,132}(km,k)$. By the first property in Lemma 38, we note that there is a simple bijection from $\mathrm{MP}_{321,132}(km,k)$ to $S_{321,132}(m)$. We simply keep only the first entry in each block of k letters and standardize the result. Thus, $\mathfrak{mp}_{321,132}(km+l,k) = |S_{321,132}(m)|$. Finally, it has already been shown that $|S_{321,132}(m)| = \binom{m}{2} + 1$; see [16, Prop. 11]. □

Corollary 40. *We have*

$$\mathfrak{p}_{321,132}(2m) = \mathfrak{p}_{321,132}(2m+1) = \binom{m}{2} + 1.$$

7 Remarks

The attentive reader has most likely noticed that we have said nothing about parity–alternating permutations avoiding the single pattern 123, or the single pattern 321.

The sequence with the number of 123-avoiding PAPs of size n starts as

$$1, 1, 1, 3, 3, 10, 11, 37, 44, 146, 185, 603, 808, 2576, \ldots.$$

Similarly, for 321-avoiding PAPs, we get

$$1, 1, 1, 2, 3, 6, 11, 22, 44, 89, 185, 382, 808, 1702, \ldots.$$

Recall from Lemma 5 that the reversal map is a bijection between the sets $\mathrm{P}_{321}(2m+1)$ and $\mathrm{P}_{123}(2m+1)$, so $\mathfrak{p}_{321}(2m+1) = \mathfrak{p}_{123}(2m+1)$ for all $m \geq 0$.

We have not managed to produce recursive formulas for these sequences. The numbers above were obtained by brute-force enumeration—we construct the odd and even parts separately and then check if the merged permutation is 123- (or 321)-avoiding.

However, we have managed to find a property of 321-avoiding PAPs which may be exploited to produce a recursive formula. This property is proved in Proposition 42 but we need a definition first in order to state it.

Definition 41. A *left-to-right maximum* of $\pi \in S_n$ is an integer $i \in [n]$ such that $i > j$ for all $j \in [n]$ to the left of i in the one-line notation of π. We call the integers that are not left-to-right maxima *hole values*.

Proposition 42. *Let $\pi \in \mathrm{P}_{321}(n)$ be non-identity. Then*

1. *there exists a pair of hole values in π that are consecutive and*
2. *the highest such pair of numbers either stands in adjacent positions or has an even length sequence of consecutive left-to-right-maxima between the numbers.*

Proof. Let $i \in [1, n-1]$ be the highest position with no excedance in $\pi \in \mathrm{P}_{321}(n)$ till i. Then $\pi_j = j$, for all $j = 1, 2, \ldots, i-1$ and $\pi_{i+1} > i+1$ since $\pi_i > i$ and π is a 321-avoiding PAP. Thus, i and $i+1$ are hole values, which show the existence.

Now consider the highest such pair of consecutive hole values in π, say h and $h+1$. The integer $h+1$ is always to the right of h. Otherwise, $\pi_h, h+1, h$ forms a 321 pattern. If h and $h+1$ are adjacent, then we are done. Otherwise, there is an even number (at least two) of positions between h and $h+1$ since π is a PAP. Moreover, the numbers between h and $h+1$ are left-to-right maxima since π is 321-avoiding. If the numbers in between are consecutive, then we are done. Otherwise, we have a pair of consecutive hole values which are greater than h and $h+1$. This is a contradiction since h and $h+1$ are the highest. □

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