Journal of Integer Sequences, Vol. 26 (2023),
Article 23.1.3

# Some New Identities for Arctangents and Chebyshev Polynomials

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**Abstract**

In this paper, we use some properties of Chebyshev polynomials and trigonometric functions to study four classes of sums of products of two arctangents involving these polynomials. We also give some infinite series identities concerning Fibonacci and Lucas numbers.

## 1 Introduction

For all integers $n \geq 1$ and all real $x$, the Chebyshev polynomials of the first and second kind, $T_n(x)$ and $U_n(x)$, are defined by the second-order linear recurrences

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \text{ and } U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x),$$

where the first two terms are $T_0(x) = 1$, $T_1(x) = x$, $U_0(x) = 1$ and $U_1(x) = 2x$.

Explicit formulae for $T_n(x)$ and $U_n(x)$ can be expressed as follows:

$$T_n(x) = \left(\alpha^n + \beta^n\right)/2 \text{ and } U_n(x) = \left(\alpha^{n+1} - \beta^{n+1}\right)/2\sqrt{x^2-1},$$

where $\alpha = \alpha(x) = x + \sqrt{x^2-1}$, $\beta = \beta(x) = x - \sqrt{x^2-1}$.

It is common knowledge that the Chebyshev polynomials $T_n(x)$ and $U_n(x)$ play an important role in approximation theory, and so many scholars have studied their properties and obtained many interesting conclusions. In particular, Kim and his team [4–9] have done a lot of important research work. You can also find many papers on Chebyshev polynomials in the literature [10–17]. For example, Ma and Lv [10] computed the reciprocal sums of Chebyshev polynomials. For $k = 1$, 2 and 3, they considered the summations

$$\sum_{a=1}^{q-1} T_a^{-2k}\big(\cos \pi h/q\big) \quad \text{and} \quad \sum_{a=1}^{q-1} U_{a-1}^{-2k}\big(\cos \pi h/q\big),$$

where $q$ is an odd number and $h$ is an integer co-prime to $q$.

Zhang [12] studied convolution sums involving $T_n(x)$, and proved the following identities:

$$\sum_{a_1+a_2+\cdots+a_{k+1}=n+k+1} \prod_{i=1}^{k+1} T_{a_i}(x) = (1/2^k \cdot k!) \cdot \sum_{h=0}^{k+1} (-x)^h \cdot \binom{k+1}{h} \cdot U_{n+2k+1-h}^{(k)}(x),$$

where $U_n^{(k)}(x)$ denotes the $k$-th derivative of $U_n(x)$ with respect to $x$, and the summation is over all $\big(k+1\big)$-dimension non-negative integer coordinates $\big(a_1, a_2, \ldots, a_{k+1}\big)$ such that $a_1 + a_2 + \cdots + a_{k+1} = n + k + 1$.

Zhang and Chen [13] proved the following result:

$$\sum_{a_1+a_2+\cdots+a_{h+1}=n} U_{a_1}(x) U_{a_2}(x) \cdots U_{a_{h+1}}(x)$$
$$= (1/2^h \cdot h!) \cdot \sum_{j=1}^{h} C\big(h, j\big)/x^{2h-j} \sum_{i=0}^{n} \big(n - i + j\big)!/\big(n - i\big)! \cdot \binom{2h + i - j - 1}{i} \cdot U_{n-i+j}(x)/x^i,$$

where $C(h, i)$ is the second order non-linear recurrence sequence defined by $C(h, 0) = 0$, $C(h, h) = 1$, $C(h + 1, 1) = 1 \cdot 3 \cdot 5 \cdots (2h - 1) = (2h - 1)!$ and

$$C(h + 1, i + 1) = (2h - 1 - i) \cdot C(h, i + 1) + C(h, i)$$

for all $1 \le i \le h - 1$.

Adegoke [14] studied sums of arctangents involving Fibonacci numbers and Lucas numbers and obtained the following result:

$$\sum_{r=p}^{\infty} \arctan \lambda F_{2j} L_{4jr+2k}/\big(F_{4jr+2k}^2 - F_{2j}^2 + \lambda^2\big) = \arctan \lambda/F_{4jp+2k-2j};$$
$$\sum_{r=p}^{\infty} \arctan \lambda F_{2j} F_{4jr+2k-1}/\big(L_{8jr+4k-2} - L_{4j} + \lambda^2\big) = \arctan \lambda/L_{4jp+2k-2j-1},$$

where $\lambda \in \mathbb{R}$, $j$, $k$, $p \in \mathbb{Z}$ and $j \neq 0$.

Mahon and Horadam [15] studied sums of arctangents involving Pell and Pell-Lucas polynomials and obtained the following result:

$$\sum_{r=1}^{n} \arctan 2x/P_{2r-1}(x) = \pi/2 - \arctan 1/P_{2n}(x);$$
$$\sum_{r=1}^{n} \arctan(-1)^{r-1}/P_{2r}(x) = \arctan P_n(x)/P_{n+1}(x),$$

where $x \in \mathbb{R}$, $n \in \mathbb{Z}$ and $n \geq 1$.

We observe that Chebyshev polynomials have similar properties to Pell and Pell-Lucas polynomials, so it is natural to wonder whether Chebyshev polynomials have similar identities. Inspired by [14] and [15], in this paper, we use the properties of Chebyshev polynomials and trigonometric functions to study the sums of products of two arctangents involving Chebyshev polynomials, and give some infinite series identities for Fibonacci and Lucas numbers. We prove the following results:

**Theorem 1.** *Let $n$, $k$ be positive integers. Then for all real $x > 1$, we have the identities*

$$\begin{aligned}
&\sum_{k=1}^{n} \arctan 2xT_{2k}(x)/\big(T_{2k}^2(x) + x^2 - 2\big) \arctan 2\big(x^2 - 1\big)U_{2k-1}(x)/\big(T_{2k}^2(x) + x^2\big) \\
&\qquad = \arctan^2 1/x - \arctan^2 1/T_{2n+1}(x); \\
&\sum_{k=1}^{n} \arctan 2xT_{2k-1}(x)/\big(T_{2k-1}^2(x) + x^2 - 2\big) \arctan 2\big(x^2 - 1\big)U_{2k-2}(x)/\big(T_{2k-1}^2(x) + x^2\big) \\
&\qquad = \pi^2/16 - \arctan^2 1/T_{2n}(x).
\end{aligned}$$

**Theorem 2.** *Let $n$, $k$ be positive integers. Then for all real $x > 1$, we have the identities*

$$\begin{aligned}
&\sum_{k=1}^{n} \arctan 2xU_{2k}(x)/\big(U_{2k}^2(x) - 2\big) \arctan 2T_{2k+1}(x)/U_{2k}^2(x) \\
&\qquad = \arctan^2 1/2x - \arctan^2 1/U_{2n+1}(x); \\
&\sum_{k=1}^{n} \arctan 2xU_{2k-1}(x)/\big(U_{2k-1}^2(x) - 2\big) \arctan 2T_{2k}(x)/U_{2k-1}^2(x) \\
&\qquad = \pi^2/16 - \arctan^2 1/U_{2n}(x).
\end{aligned}$$

**Theorem 3.** *Let $n$, $k$ be positive integers. Then for all real $x > 1$, we have the identities*

$$\sum_{k=1}^{n} \arctan\big(1-x^2\big)/2xT_k^2(x) \arctan\big(2T_k^2(x)+x^2-1\big)/\big(x^2-1\big)U_{2k-1}(x)$$
$$= \arctan^2 T_n(x)/T_{n+1}(x) - \arctan^2 1/x;$$
$$\sum_{k=1}^{n} \arctan 2x^2\big(1-x^2\big)/\big(2x^2-1\big)T_k^2(x) \arctan\big(T_k^2(x)+2x^2\big(x^2-1\big)\big)/x\big(x^2-1\big)U_{2k-1}(x)$$
$$= \arctan^2 T_{n-1}(x)/T_{n+1}(x) + \arctan^2 T_n(x)/T_{n+2}(x) - \pi^2/16 - \arctan^2 1/\big(2x^2-1\big).$$

**Theorem 4.** *Let $n$, $k$ be positive integers. Then for all real $x > 1$, we have the identities*

$$\sum_{k=1}^{n} \arctan 1/2xU_k^2(x) \arctan\big(2U_k^2(x)-1\big)/U_{2k+1}(x)$$
$$= \arctan^2 U_n(x)/U_{n+1}(x) - \arctan^2 1/2x;$$
$$\sum_{k=1}^{n} \arctan 2x^2/\big(2x^2-1\big)U_k^2(x) \arctan\big(U_k^2(x)-2x^2\big)/xU_{2k+1}(x)$$
$$= \arctan^2 U_n(x)/U_{n+2}(x) + \arctan^2 U_{n-1}(x)/U_{n+1}(x) - \arctan^2 1/\big(4x^2-1\big).$$

In the above theorems, we only consider the case $x > 1$. If $x < -1$, we have $-x > 1$, $\alpha(x) = -(-x) + \sqrt{(-x)^2-1} = -\beta(-x)$, $\beta(x) = -(-x) - \sqrt{(-x)^2-1} = -\alpha(-x)$. From the definition of $T_n(x)$ and $U_n(x)$, we obtain $T_n(x) = (-1)^n \cdot T_n(-x)$, $U_n(x) = (-1)^n \cdot U_n(-x)$. By observing the above theorems, we find that the results are the same in both cases, so we do not discuss them here. Taking $n \to \infty$, from our theorems we can deduce the following:

**Corollary 5.** *Let $k$ be an integer. Then for all real $x > 1$, we have*

$$\sum_{k=1}^{\infty} \arctan 2xT_{2k}(x)/\big(T_{2k}^2(x)+x^2-2\big) \arctan 2\big(x^2-1\big)U_{2k-1}(x)/\big(T_{2k}^2(x)+x^2\big)$$
$$= \arctan^2 1/x;$$
$$\sum_{k=1}^{\infty} \arctan 2xT_{2k-1}(x)/\big(T_{2k-1}^2(x)+x^2-2\big) \arctan 2\big(x^2-1\big)U_{2k-2}(x)/\big(T_{2k-1}^2(x)+x^2\big)$$
$$= \pi^2/16.$$

**Corollary 6.** *Let $k$ be an integer. Then for all real $x > 1$, we have*

$$\sum_{k=1}^{\infty} \arctan 2xU_{2k}(x)/\big(U_{2k}^2(x)-2\big) \arctan 2T_{2k+1}(x)/U_{2k}^2(x) = \arctan^2 1/2x;$$
$$\sum_{k=1}^{\infty} \arctan 2xU_{2k-1}(x)/\big(U_{2k-1}^2(x)-2\big) \arctan 2T_{2k}(x)/U_{2k-1}^2(x) = \pi^2/16.$$

**Corollary 7.** *Let $k$ be an integer. Then for all real $x > 1$, we have*

$$\sum_{k=1}^{\infty} \arctan\bigl(1-x^2\bigr)/2xT_k^2(x) \arctan\bigl(2T_k^2(x)+x^2-1\bigr)/\bigl(x^2-1\bigr)U_{2k-1}(x)$$
$$= \arctan^2\beta - \arctan^2 1/x;$$
$$\sum_{k=1}^{\infty} \arctan 2x^2\bigl(1-x^2\bigr)/\bigl(2x^2-1\bigr)T_k^2(x) \arctan\bigl(T_k^2(x)+2x^2\bigl(x^2-1\bigr)\bigr)/x\bigl(x^2-1\bigr)U_{2k-1}(x)$$
$$= 2\arctan^2\beta^2 - \pi^2/16 - \arctan^2 1/\bigl(2x^2-1\bigr),$$

*where $\beta = \beta(x) = x - \sqrt{x^2-1}$.*

**Corollary 8.** *Let $k$ be an integer. Then for all real $x > 1$, we have*

$$\sum_{k=1}^{\infty} \arctan 1/2xU_k^2(x) \arctan\bigl(2U_k^2(x)-1\bigr)/U_{2k+1}(x) = \arctan^2\beta - \arctan^2 1/2x;$$
$$\sum_{k=1}^{\infty} \arctan 2x^2/\bigl(2x^2-1\bigr)U_k^2(x) \arctan\bigl(U_k^2(x)-2x^2\bigr)/xU_{2k+1}(x)$$
$$= 2\arctan^2\beta^2 - \arctan^2 1/\bigl(4x^2-1\bigr),$$

*where $\beta = \beta(x) = x - \sqrt{x^2-1}$.*

*Remark* 9. Here we give several values of $T_n(x)$ and $U_n(x)$ as follows:

$$T_n\bigl(3/2\bigr) = L_{2n}/2; T_n\bigl(7/2\bigr) = L_{4n}/2; \qquad U_n\bigl(3/2\bigr) = F_{2n+2}; U_n\bigl(7/2\bigr) = F_{4n+4}/3.$$

$$T_n\bigl(\sqrt{5}/2\bigr) = \begin{cases} L_n/2, & \text{if } n \text{ is even;} \\ \sqrt{5}F_n/2, & \text{if } n \text{ is odd.} \end{cases} \qquad U_n\bigl(\sqrt{5}/2\bigr) = \begin{cases} L_{n+1}, & \text{if } n \text{ is even;} \\ \sqrt{5}F_{n+1}, & \text{if } n \text{ is odd.} \end{cases}$$

$$T_n\bigl(\sqrt{5}\bigr) = \begin{cases} L_{3n}/2, & \text{if } n \text{ is even;} \\ \sqrt{5}F_{3n}/2, & \text{if } n \text{ is odd.} \end{cases} \qquad U_n\bigl(\sqrt{5}\bigr) = \begin{cases} L_{3n+3}/4, & \text{if } n \text{ is even;} \\ \sqrt{5}F_{3n+3}/4, & \text{if } n \text{ is odd.} \end{cases}$$

where $F_n$ and $L_n$ denote the famous Fibonacci and Lucas numbers.

From the first formula of Corollary 5 and Corollary 6, we can also deduce the following four Corollaries about Fibonacci and Lucas numbers.

**Corollary 10.** *Let $k$ be an integer. Then for $x = \sqrt{5}/2$, we have the identities*

$$\sum_{k=1}^{\infty} \arctan 2\sqrt{5}L_{2k}/\bigl(L_{2k}^2-3\bigr) \arctan 2\sqrt{5}F_{2k}/\bigl(L_{2k}^2+5\bigr) = \arctan^2 2/\sqrt{5};$$
$$\sum_{k=1}^{\infty} \arctan \sqrt{5}L_{2k+1}/\bigl(L_{2k+1}^2-2\bigr) \arctan \sqrt{5}F_{2k+1}/L_{2k+1}^2 = \arctan^2 1/\sqrt{5}.$$

**Corollary 11.** *Let $k$ be an integer. Then for $x = 3/2$, we have the identities*

$$\sum_{k=1}^{\infty} \arctan 6L_{4k}/\big(L_{4k}^2 + 1\big) \arctan 10F_{4k}/\big(L_{4k}^2 + 9\big) = \arctan^2 2/3;$$
$$\sum_{k=1}^{\infty} \arctan 3F_{4k+2}/\big(F_{4k+2}^2 - 2\big) \arctan L_{4k+2}/F_{4k+2}^2 = \arctan^2 1/3.$$

**Corollary 12.** *Let $k$ be an integer. Then for $x = \sqrt{5}$, we have the identities*

$$\sum_{k=1}^{\infty} \arctan 4\sqrt{5}L_{6k}/\big(L_{6k}^2 + 12\big) \arctan 8\sqrt{5}F_{6k}/\big(L_{6k}^2 + 20\big) = \arctan^2 \sqrt{5}/5;$$
$$\sum_{k=1}^{\infty} \arctan 8\sqrt{5}L_{6k+3}/\big(L_{6k+3}^2 - 32\big) \arctan 16\sqrt{5}F_{6k+3}/L_{6k+3}^2 = \arctan^2 \sqrt{5}/10.$$

**Corollary 13.** *Let $k$ be an integer. Then for $x = 7/2$, we have the identities*

$$\sum_{k=1}^{\infty} \arctan 14L_{8k}/\big(L_{8k}^2 + 41\big) \arctan 30F_{8k}/\big(L_{8k}^2 + 49\big) = \arctan^2 2/7;$$
$$\sum_{k=1}^{\infty} \arctan 21F_{8k+4}/\big(F_{8k+4}^2 - 18\big) \arctan 9L_{8k+4}/F_{8k+4}^2 = \arctan^2 1/7.$$

## 2 Lemmas

To complete the proofs of our theorems, we need several simple lemmas. The proof of these lemmas requires the properties of the Chebyshev polynomials of the first and second kind, $T_n(x)$ and $U_n(x)$. All these can be found in [2,3], and we do not repeat them. First, we have the following:

**Lemma 14.** *Let $k$, $\lambda$ be positive integers with $k \geq \lambda$. Then for all real $x$, we have*

$$\begin{aligned} T_{k+\lambda}(x) - T_{k-\lambda}(x) &= 2\big(x^2 - 1\big)U_{k-1}(x)U_{\lambda-1}(x), \\ T_{k+\lambda}(x) + T_{k-\lambda}(x) &= 2T_k(x)T_\lambda(x), \\ T_{k+\lambda}(x)T_{k-\lambda}(x) &= T_k^2(x) + T_\lambda^2(x) - 1. \end{aligned}$$

*Proof.* See [3, p. 393]. □

**Lemma 15.** *Let $k$, $\lambda$ be positive integers with $k \geq \lambda$. Then for all real $x$, we have*

$$\begin{aligned} U_{k+\lambda}(x) - U_{k-\lambda}(x) &= 2T_{k+1}(x)U_{\lambda-1}(x), \\ U_{k+\lambda}(x) + U_{k-\lambda}(x) &= 2U_k(x)T_\lambda(x), \\ U_{k+\lambda}(x)U_{k-\lambda}(x) &= U_k^2(x) - U_{\lambda-1}^2(x). \end{aligned}$$

*Proof.* See [3, p. 388]. □

Apart from that, we also need to use the following properties of the arctangent (see [1, p. 80]):

$$\arctan x + \arctan y = \arctan\bigl(x+y\bigr)/\bigl(1-xy\bigr) \quad \bigl(xy > 1\bigr),$$
$$\arctan x - \arctan y = \arctan\bigl(x-y\bigr)/\bigl(1+xy\bigr) \quad \bigl(xy < -1\bigr).$$

# 3 Proofs of the theorems

In this section, we use the two basic lemmas and the properties of arctangents to prove our main results. First we prove Theorem 1. From Lemma 14 and the properties of arctangents, we have

$$\arctan 1/T_{k-\lambda}(x) + \arctan 1/T_{k+\lambda}(x) = \arctan 2T_k(x)T_\lambda(x)/\bigl(T_k^2(x) + T_\lambda^2(x) - 2\bigr), \tag{1}$$

$$\arctan 1/T_{k-\lambda}(x) - \arctan 1/T_{k+\lambda}(x) = \arctan 2\bigl(x^2-1\bigr)U_{k-1}(x)U_{\lambda-1}(x)/\bigl(T_k^2(x) + T_\lambda^2(x)\bigr). \tag{2}$$

Taking $\lambda = 1$ in (1), (2) and replacing $k$ by $2k$, we have

$$\arctan 1/T_{2k-1}(x) + \arctan 1/T_{2k+1}(x) = \arctan 2xT_{2k}(x)/\bigl(T_{2k}^2(x) + x^2 - 2\bigr),$$
$$\arctan 1/T_{2k-1}(x) - \arctan 1/T_{2k+1}(x) = \arctan 2\bigl(x^2-1\bigr)U_{2k-1}(x)/\bigl(T_{2k}^2(x) + x^2\bigr).$$

Multiplying the two equations and summing for $k$ from 1 to $n$, we may deduce the identity

$$\sum_{k=1}^{n} \arctan 2xT_{2k}(x)/\bigl(T_{2k}^2(x) + x^2 - 2\bigr) \arctan 2\bigl(x^2-1\bigr)U_{2k-1}(x)/\bigl(T_{2k}^2(x) + x^2\bigr)$$
$$= \arctan^2 1/x - \arctan^2 1/T_{2n+1}(x). \tag{3}$$

This proves the first formula in Theorem 1. Similarly, replacing $k$ by $2k-1$, from the method of proving (3), we can deduce the second formula. This proves Theorem 1.

Now we prove Theorem 2. From Lemma 15 and the properties of arctangents, we have

$$\arctan 1/U_{k-\lambda}(x) + \arctan 1/U_{k+\lambda}(x) = \arctan 2U_k(x)T_\lambda(x)/\bigl(U_k^2(x) - U_{\lambda-1}^2(x) - 1\bigr), \tag{4}$$

$$\arctan 1/U_{k-\lambda}(x) - \arctan 1/U_{k+\lambda}(x) = \arctan 2T_{k+1}(x)U_{\lambda-1}(x)/\bigl(U_k^2(x) - U_{\lambda-1}^2(x) + 1\bigr). \tag{5}$$

Taking $\lambda = 1$ in (4), (5) and replacing $k$ by $2k$, we have

$$\arctan 1/U_{2k-1}(x) + \arctan 1/U_{2k+1}(x) = \arctan 2xU_{2k}(x)/\big(U_{2k}^2(x) - 2\big),$$
$$\arctan 1/U_{2k-1}(x) - \arctan 1/U_{2k+1}(x) = \arctan 2T_{2k+1}(x)/U_{2k}^2(x).$$

Multiplying the two identities and summing for $k$ from 1 to $n$, we have

$$\sum_{k=1}^{n} \arctan 2xU_{2k}(x)/\big(U_{2k}^2(x) - 2\big) \arctan 2T_{2k+1}(x)/U_{2k}^2(x)$$
$$= \arctan^2 1/2x - \arctan^2 1/U_{2n+1}(x). \tag{6}$$

From Lemma 15 and (6), we can easily obtain the second formula. This proves Theorem 2.

Now we prove Theorem 3. From Lemma 14 and the properties of arctangents, we have

$$\arctan T_k(x)/T_{k+\lambda}(x) - \arctan T_{k-\lambda}(x)/T_k(x) = \arctan\big(1 - T_\lambda^2(x)\big)/2T_k^2(x)T_\lambda(x), \tag{7}$$

$$\arctan T_k(x)/T_{k+\lambda}(x) + \arctan T_{k-\lambda}(x)/T_k(x)$$
$$= \arctan\big(2T_k^2(x) + T_\lambda^2(x) - 1\big)/\big(x^2 - 1\big)U_{2k-1}(x)U_{\lambda-1}(x). \tag{8}$$

Taking $\lambda = 1$ in (7) and (8), we have

$$\arctan T_k(x)/T_{k+1}(x) - \arctan T_{k-1}(x)/T_k(x) = \arctan\big(1 - x^2\big)/2xT_k^2(x),$$
$$\arctan T_k(x)/T_{k+1}(x) + \arctan T_{k-1}(x)/T_k(x) = \arctan\big(2T_k^2(x) + x^2 - 1\big)/\big(x^2 - 1\big)U_{2k-1}(x).$$

Multiplying the above two formulae and summing for $k$ from 1 to $n$, we have the following identity

$$\sum_{k=1}^{n} \arctan\big(1 - x^2\big)/2xT_k^2(x) \arctan\big(2T_k^2(x) + x^2 - 1\big)/\big(x^2 - 1\big)U_{2k-1}(x)$$
$$= \arctan^2 T_n(x)/T_{n+1}(x) - \arctan^2 1/x. \tag{9}$$

This proves the first formula in Theorem 3. Taking $\lambda = 2$ in (7) and (8), from the method of proving (9), we can easily deduce the second formula. This proves Theorem 3.

Analogously, from Lemma 15 and the method of proving Theorem 3, we can easily obtain Theorem 4. This completes the proofs of our all results.

# 4 Applications

To further illustrate the application of our results, we take some specific values to obtain identities about Fibonacci and Lucas numbers.

• Taking $x = \sqrt{5}/2$, from the first formula of Corollary 5 and Corollary 6, we have the following identities

$$\sum_{k=1}^{\infty} \arctan \sqrt{5}T_{2k}\big(\sqrt{5}/2\big)/\big(T_{2k}^2\big(\sqrt{5}/2\big) - 3/4\big) \arctan U_{2k-1}\big(\sqrt{5}/2\big)/\big(2T_{2k}^2\big(\sqrt{5}/2\big) + 5/2\big)$$
$$= \arctan^2 2/\sqrt{5};$$
$$\sum_{k=1}^{\infty} \arctan \sqrt{5}U_{2k}\big(\sqrt{5}/2\big)/\big(U_{2k}^2\big(\sqrt{5}/2\big) - 2\big) \arctan 2T_{2k+1}\big(\sqrt{5}/2\big)/U_{2k}^2\big(\sqrt{5}/2\big)$$
$$= \arctan^2 1/\sqrt{5}.$$

Note that

$$T_{2k-1}\big(\sqrt{5}/2\big) = \sqrt{5}F_{2k-1}/2, \quad T_{2k+1}\big(\sqrt{5}/2\big) = \sqrt{5}F_{2k+1}/2, \quad T_{2k}\big(\sqrt{5}/2\big) = L_{2k}/2,$$
$$U_{2k-1}\big(\sqrt{5}/2\big) = \sqrt{5}F_{2k}, \quad U_{2k-2}\big(\sqrt{5}/2\big) = L_{2k-1}, \quad U_{2k}\big(\sqrt{5}/2\big) = L_{2k+1}.$$

Substituting the values into the above two identities, respectively, we have

$$\sum_{k=1}^{\infty} \arctan 2\sqrt{5}L_{2k}/\big(L_{2k}^2 - 3\big) \arctan 2\sqrt{5}F_{2k}/\big(L_{2k}^2 + 5\big) = \arctan^2 2/\sqrt{5};$$
$$\sum_{k=1}^{\infty} \arctan \sqrt{5}L_{2k+1}/\big(L_{2k+1}^2 - 2\big) \arctan \sqrt{5}F_{2k+1}/L_{2k+1}^2 = \arctan^2 1/\sqrt{5}.$$

Similarly, using the above method, we also have

$$\sum_{k=1}^{\infty} \arctan 10F_{2k-1}/\big(5F_{2k-1}^2 - 3\big) \arctan 2L_{2k-1}/\big(5F_{2k-1}^2 + 5\big) = \pi^2/16;$$
$$\sum_{k=1}^{\infty} \arctan 5F_{2k}/\big(5F_{2k}^2 - 2\big) \arctan L_{2k}/5F_{2k}^2 = \pi^2/16.$$

• Take $x = 3/2$ and note that

$$T_{2k+1}\big(3/2\big) = L_{4k+2}/2, \quad T_{2k-1}\big(3/2\big) = L_{4k-2}/2, \quad T_{2k}\big(3/2\big) = L_{4k}/2,$$
$$U_{2k+1}\big(3/2\big) = F_{4k+4}, \quad U_{2k-1}\big(3/2\big) = F_{4k}, \quad U_{2k}\big(3/2\big) = F_{4k+2}.$$

Substituting the values into the first formula of Corollary 5, Corollary 6, Corollary 7 and

Corollary 8, respectively, we have

$$\sum_{k=1}^{\infty} \arctan 6L_{4k}/\big(L_{4k}^2+1\big) \arctan 10F_{4k}/\big(L_{4k}^2+9\big) = \arctan^2 2/3;$$

$$\sum_{k=1}^{\infty} \arctan 3F_{4k+2}/\big(F_{4k+2}^2-2\big) \arctan L_{4k+2}/F_{4k+2}^2 = \arctan^2 1/3;$$

$$\sum_{k=1}^{\infty} \arctan -5/3L_{2k}^2 \arctan\big(2L_{2k}^2+5\big)/5F_{4k} = \arctan^2\big(2/\sqrt{5}\big)/4 - \arctan^2 2/3;$$

$$\sum_{k=1}^{\infty} \arctan 1/3F_{2k+2}^2 \arctan\big(2F_{2k+2}^2-1\big)/F_{4k+4} = \arctan^2\big(2/\sqrt{5}\big)/4 - \arctan^2 1/3.$$

Similarly, from the method above, we also have

$$\sum_{k=1}^{\infty} \arctan 6L_{4k-2}/\big(L_{4k-2}^2+1\big) \arctan 10F_{4k-2}/\big(L_{4k-2}^2+9\big) = \pi^2/16;$$

$$\sum_{k=1}^{\infty} \arctan 3F_{4k}/\big(F_{4k}^2-2\big) \arctan L_{4k}/F_{4k}^2 = \pi^2/16;$$

$$\sum_{k=1}^{\infty} \arctan -45/7L_{2k}^2 \arctan\big(2L_{2k}^2+45\big)/15F_{4k} = 2\arctan^2\big(7-3\sqrt{5}\big)/2 - \pi^2/16 - \arctan^2 2/7;$$

$$\sum_{k=1}^{\infty} \arctan 9/7F_{2k+2}^2 \arctan\big(2F_{2k+2}^2-9\big)/3F_{4k+4} = 2\arctan^2\big(7-3\sqrt{5}\big)/2 - \arctan^2 1/8.$$

- Take $x=\sqrt{5}$ and note that

$$T_{2k-1}\big(\sqrt{5}\big) = \sqrt{5}F_{6k-3}/2, \quad T_{2k+1}\big(\sqrt{5}\big) = \sqrt{5}F_{6k+3}/2, \quad T_{2k}\big(\sqrt{5}\big) = L_{6k}/2,$$
$$U_{2k-1}\big(\sqrt{5}\big) = \sqrt{5}F_{6k}/4, \quad U_{2k-2}\big(\sqrt{5}\big) = L_{6k-3}/4, \quad U_{2k}\big(\sqrt{5}\big) = L_{6k+3}/4.$$

Substituting the values into the first formula of Corollary 5 and Corollary 6, respectively, we have the following identities

$$\sum_{k=1}^{\infty} \arctan 4\sqrt{5}L_{6k}/\big(L_{6k}^2+12\big) \arctan 8\sqrt{5}F_{6k}/\big(L_{6k}^2+20\big) = \arctan^2 \sqrt{5}/5;$$

$$\sum_{k=1}^{\infty} \arctan 8\sqrt{5}L_{6k+3}/\big(L_{6k+3}^2-32\big) \arctan 16\sqrt{5}F_{6k+3}/L_{6k+3}^2 = \arctan^2 \sqrt{5}/10.$$

Analogously, from the above calculation method, we also obtain

$$\sum_{k=1}^{\infty} \arctan 20F_{6k-3}/\big(5F_{6k-3}^2+36\big) \arctan 8L_{6k-3}/\big(5F_{6k-3}^2+20\big) = \pi^2/16;$$

$$\sum_{k=1}^{\infty} \arctan 40F_{6k}/\big(5F_{6k}^2-32\big) \arctan 16L_{6k}/5F_{6k}^2 = \pi^2/16.$$

- Take $x = 7/2$ and note that

$$T_{2k+1}\big(7/2\big) = L_{8k+4}/2, \quad T_{2k-1}\big(7/2\big) = L_{8k-4}/2, \quad T_{2k}\big(7/2\big) = L_{8k}/2,$$
$$U_{2k-2}\big(7/2\big) = F_{8k-4}/3, \quad U_{2k-1}\big(7/2\big) = F_{8k}/3, \quad U_{2k}\big(7/2\big) = F_{8k+4}/3.$$

Substituting the values into the first formula of Corollary 5 and Corollary 6, respectively, we have the identities

$$\sum_{k=1}^{\infty} \arctan 14L_{8k}/\big(L_{8k}^2+41\big) \arctan 30F_{8k}/\big(L_{8k}^2+49\big) = \arctan^2 2/7;$$
$$\sum_{k=1}^{\infty} \arctan 21F_{8k+4}/\big(F_{8k+4}^2-18\big) \arctan 9L_{8k+4}/F_{8k+4}^2 = \arctan^2 1/7.$$

Analogously, from the above calculation method, we also have

$$\sum_{k=1}^{\infty} \arctan 14L_{8k-4}/\big(L_{8k-4}^2+41\big) \arctan 30F_{8k-4}/\big(L_{8k-4}^2+49\big) = \pi^2/16;$$

$$\sum_{k=1}^{\infty} \arctan 21F_{8k}/\big(F_{8k}^2-18\big) \arctan 9L_{8k}/F_{8k}^2 = \pi^2/16.$$

# 5 Acknowledgment

Funding: This work is supported by the N. S. F. (12126357) of P. R. China.

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2020 *Mathematics Subject Classification*: Primary 11B37; Secondary 33B10, 11B39.
*Keywords:* Chebyshev polynomial, trigonometric function, arctangent, Fibonacci number, Lucas number.

(Concerned with sequences A000032 and A000045.)

Received August 31 2022; revised versions received September 16 2022; December 20 2022. Published in *Journal of Integer Sequences*, January 1 2023.

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