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Nontrivial Lower Bounds for the p-adic Valuation of Certain Rational Numbers

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Abstract

We show that the p-adic valuation (where p is a given prime number) of certain rational numbers is unusually large. This generalizes very recent results of the author and of Dubickas, which are both related to the special case $p = 2$. The crucial point for obtaining our main result is the fact that the p-adic valuation of the rational numbers in question is unbounded from above. We confirm this fact by three different methods; the first two are elementary, while the third one relies on p-adic analysis.

1 Introduction and notation

Throughout this paper, we let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ denote the set of non-negative integers. For $x \in \mathbb{R}$, we let $\lfloor x \rfloor$ denote the integer part of x. For a given prime number p and a given non-zero rational number r, we let $\nu_p(r)$ denote the usual p-adic valuation of r; if in addition r is positive then we let $\log_p(r)$ denote its logarithm to the base p (i.e., $\log_p(r) := \frac{\log r}{\log p}$). Next, the least common multiple of given positive integers $u_1, u_2, \dots, u_n$ ($n \in \mathbb{N}$) is denoted by $\mathrm{lcm}(u_1, u_2, \dots, u_n)$. In several places of this paper, we will use the estimate $\nu_p(n) \leq \log_p(n)$ (for primes p and all $n \in \mathbb{N}$). We also often use the formula $\nu_p\left(\mathrm{lcm}(1, 2, \dots, n)\right) = \left\lfloor \log_p(n) \right\rfloor$ (for primes p and all $n \in \mathbb{N}$). At the end of the paper, we need to use the p-adic logarithm function which we denote by L_p (to differentiate from the notation $\log_p$, which is reserved to denote the logarithm to the base p). With the usual notation $\mathbb{Q}_p$ for the field of p-adic numbers, $\mathbb{C}_p$ for the field of the p-adic complex numbers, and $|\cdot|_p$ for the usual p-adic absolute value on $\mathbb{C}_p$, recall that L_p can be defined by

$$-L_p(1-x) := \sum_{k=1}^{+\infty} \frac{x^k}{k} \qquad (\forall x \in \mathbb{C}_p, |x|_p < 1).$$

(See [5]). The fundamental property of L_p is that it satisfies the functional equation

$$L_p(uv) = L_p(u) + L_p(v)$$

(for all $u, v \in \mathbb{C}_p$, with $|u-1|_p < 1$ and $|v-1|_p < 1$).

The author [2, 3] has obtained nontrivial lower bounds for the 2-adic valuation of the rational numbers of the form $\sum_{k=1}^{n} \frac{2^k}{k}$ $(n \in \mathbb{N})$. The stronger one is

$$\nu_2\left(\sum_{k=1}^{n} \frac{2^k}{k}\right) \geq n - \lfloor \log_2(n) \rfloor \qquad (\forall n \in \mathbb{N}). \tag{1}$$

The author [3] has also posed the problem of generalizing Eq. (1) to other prime numbers p other than $p = 2$. Dubickas [1] has found arguments for improving and optimizing Eq. (1) by relying solely on the fact that the sequence $\left(\nu_2\left(\sum_{k=1}^{n} \frac{2^k}{k}\right)\right)_{n\geq 1}$ is unbounded from above. However, he did not establish a method to prove this fact without relying on Eq. (1). The main result in [1] states that

$$\nu_2\left(\sum_{k=1}^{n} \frac{2^k}{k}\right) \geq (n+1) - \log_2(n+1), \tag{2}$$

for all $n \in \mathbb{N}$, with equality if and only if $n = 2^\alpha - 1$ $(\alpha \in \mathbb{N})$.

The goal of this paper is twofold. On the one hand, we expand and improve the arguments in [1] to establish a general result providing to us nontrivial lower bounds for the p-adic valuation of a sum of rational numbers under some conditions (see Theorem 2). On the other hand, we solve the problem posed in [3] by generalizing Eqs. (1) and (2) to other prime numbers. More precisely, we show (in different ways) that for all prime numbers p and all non-multiple integers a of p, the sequence

$$\left(\nu_p\left(\sum_{k=1}^{n}\left(\frac{1}{a^k} + \frac{1}{(p-a)^k}\right)\frac{p^k}{k}\right)\right)_{n\geq 1} \tag{3}$$

is unbounded from above. Then, by using our general theorem 2, we derive an optimal lower bound for the sequence in Eq. (3). It must be noted that the crucial point of the unboundedness from above of the sequence in Eq. (3) is established by three methods. The first two are elementary and effective while the third one relies on the p-adic analysis and it is ineffective; precisely, it uses the function L_p described above. Personally, we consider that the deep reason why the sequence in Eq. (3) is unbounded from above is rather given by the third method.

2 The results and the proofs

Our main result is the following:

Theorem 1. *Let p be a prime number, and let a be an integer that is not a multiple of p. Then*

$$\nu_p\left(\sum_{k=1}^{n}\left(\frac{1}{a^k}+\frac{1}{(p-a)^k}\right)\frac{p^k}{k}\right) \geq (n+1)-\log_p\left(\frac{n+1}{2}\right) \tag{4}$$

for all positive integers n. In addition, this inequality becomes an equality if and only if n has the form $n = 2p^{\alpha}-1$ ($\alpha \in \mathbb{N}_0$).

Note that Theorem 1 generalizes the recent results of the author [2, 3] and Dubickas [1], which are both related to the particular case $p = 2$. In particular, if we take $p = 2$ and $a = 1$ in Theorem 1, we exactly obtain (after some obvious simplifications) the main result of [1], stating that

$$\nu_2\left(\sum_{k=1}^{n}\frac{2^k}{k}\right) \geq (n+1)-\log_2(n+1) \qquad (\forall n \in \mathbb{N}),$$

with equality if and only if n has the form $(2^{\alpha}-1)$ ($\alpha \in \mathbb{N}$).

The proof of Theorem 1 is based in part on the following result, which can be useful in other situations for establishing a lower bound on the p-adic valuation of certain sums of rational numbers. It must be also noted that the result below is obtained by generalizing the arguments in [1].

Theorem 2. *Let p be a fixed prime number and $(r_n)_{n\geq 1}$ be a sequence of rational numbers such that the sequence $\left(\nu_p\left(\sum_{k=1}^{n} r_k\right)\right)_{n\geq 1}$ is unbounded from above. Let also $(\ell_k)_{k\geq 2}$ be an increasing real sequence satisfying the property*

$$\ell_k \leq \nu_p(r_k) \qquad (\forall k \geq 2). \tag{5}$$

Then we have

$$\nu_p\left(\sum_{k=1}^{n} r_k\right) \geq \min_{k\geq n+1} \nu_p(r_k) \geq \ell_{n+1} \tag{6}$$

for all positive integers n. In addition, the inequality $\nu_p\left(\sum_{k=1}^{n} r_k\right) \geq \ell_{n+1}$ becomes an equality if and only if we have

$$\nu_p(r_{n+1}) = \ell_{n+1}. \tag{7}$$

Our main result (i.e., Theorem 1) is proven in two steps. In the first one, we suppose (in the situation of Theorem 1) that the sequence $\left(\nu_p\left(\sum_{k=1}^{n}\left(\frac{1}{a^k}+\frac{1}{(p-a)^k}\right)\frac{p^k}{k}\right)\right)_{n\geq 1}$ is unbounded from above, and we apply for it Theorem 2 to establish the lower bound (4) and characterize the n's for which it is attained. In the second one, we focus on proving the unboundedness from above of the sequence in question. This is accomplished using three different methods. The first method relies on two identities, one combinatorial and the other arithmetic. The second method utilizes a specific functional equation and Taylor polynomials. Lastly, the third method employs p-adic analysis, specifically the p-adic logarithm function.

Let us begin by proving Theorem 2.

Proof of Theorem 2. Let n be a fixed positive integer. Let us show the first inequality of (6). Since, by hypothesis, the sequence $\left(\nu_p\left(\sum_{k=1}^{N} r_k\right)\right)_{N\geq 1}$ is unbounded from above then there exists $m \in \mathbb{N}$, with $m > n$, such that

$$\nu_p\left(\sum_{k=1}^{m} r_k\right) > \nu_p\left(\sum_{k=1}^{n} r_k\right).$$

Then, by using the elementary properties of the p-adic valuation, we have on the one hand

$$\nu_p\left(\sum_{k=n+1}^{m} r_k\right) = \nu_p\left(\sum_{k=1}^{m} r_k - \sum_{k=1}^{n} r_k\right) = \min\left(\nu_p\left(\sum_{k=1}^{m} r_k\right), \nu_p\left(\sum_{k=1}^{n} r_k\right)\right) = \nu_p\left(\sum_{k=1}^{n} r_k\right),$$

and on the other hand

$$\nu_p\left(\sum_{k=n+1}^{m} r_k\right) \geq \min_{n+1\leq k\leq m} \nu_p(r_k) \geq \min_{k\geq n+1} \nu_p(r_k).$$

By comparing these two results, we deduce that

$$\nu_p\left(\sum_{k=1}^{n} r_k\right) \geq \min_{k\geq n+1} \nu_p(r_k),$$

which is the first inequality of (6). The second inequality of (6) is immediately derived from its first inequality, combined with the properties of the sequence $(\ell_k)_{k\geq 2}$. Indeed, we have

$$\begin{aligned}
\nu_p\left(\sum_{k=1}^{n} r_k\right) &\geq \min_{k\geq n+1} \nu_p(r_k) && \text{(by the first inequality of (6))}\\
&\geq \min_{k\geq n+1} \ell_k && \text{(by using (5))}\\
&= \ell_{n+1} && \text{(since } (\ell_k)_k \text{ is increasing by hypothesis)},
\end{aligned}$$

confirming the second inequality of (6).

Now, let us prove the second part of Theorem 2. If $\nu_p\left(\sum_{k=1}^{n} r_k\right) = \ell_{n+1}$ then we have (according to Eq. (6), proved above) $\min_{k\geq n+1} \nu_p(r_k) = \ell_{n+1}$. Then, using Eq. (5) and the increase of $(\ell_k)_k$, we have

$$\begin{aligned}
\ell_{n+1} = \min_{k\geq n+1} \nu_p(r_k) = \min\left(\nu_p(r_{n+1}), \min_{k\geq n+2} \nu_p(r_k)\right) &\geq \min\left(\nu_p(r_{n+1}), \min_{k\geq n+2} \ell_k\right)\\
&= \min\left(\nu_p(r_{n+1}), \ell_{n+2}\right).
\end{aligned}$$

But since $\ell_{n+1} < \ell_{n+2}$, we must have $\nu_p(r_{n+1}) \leq \ell_{n+1}$, implying (according to Eq. (5)) that $\nu_p(r_{n+1}) = \ell_{n+1}$, as required. Conversely, suppose that $\nu_p(r_{n+1}) = \ell_{n+1}$ and let us show that $\nu_p\left(\sum_{k=1}^n r_k\right) = \ell_{n+1}$. If $\nu_p\left(\sum_{k=1}^n r_k\right) \neq \ell_{n+1}$, we must have (in view of Eq. (6))

$$\nu_p\left(\sum_{k=1}^n r_k\right) > \ell_{n+1} = \nu_p(r_{n+1});$$

consequently, we get

$$\nu_p\left(\sum_{k=1}^{n+1} r_k\right) = \min\left(\nu_p\left(\sum_{k=1}^n r_k\right), \nu_p(r_{n+1})\right) = \nu_p(r_{n+1}) = \ell_{n+1} < \ell_{n+2},$$

contradicting Eq. (6) (applied for the positive integer $(n+1)$ instead of n). Hence

$$\nu_p\left(\sum_{k=1}^n r_k\right) = \ell_{n+1},$$

as required. This confirms the second part of Theorem 2 and completes this proof. □

Next, we have the following fundamental result:

Theorem 3. *Let p be a prime number and a be an integer that is not a multiple of p. Then the sequence*

$$\left(\nu_p\left(\sum_{k=1}^n \left(\frac{1}{a^k} + \frac{1}{(p-a)^k}\right)\frac{p^k}{k}\right)\right)_{n\geq 1}$$

is unbounded from above.

Admitting Theorem 3 for the moment, our main result is obtained as an application of Theorem 2.

Proof of Theorem 1 by admitting Theorem 3. Let us put ourselves in the situation of Theorem 1. We apply Theorem 2 with $r_k := \left(\frac{1}{a^k} + \frac{1}{(p-a)^k}\right)\frac{p^k}{k}$ $(\forall k \in \mathbb{N})$ and $\ell_k := k - \log_p\left(\frac{k}{2}\right)$ $(\forall k \geq 2)$. The unboundedness from above of the sequence $\left(\nu_p\left(\sum_{k=1}^n r_k\right)\right)_{n\geq 1}$ is guaranteed by Theorem 3 (admitted for the moment). Next, the increase of the sequence $(\ell_k)_{k\geq 2}$ can be derived from the increase of the function $x \mapsto x - \log_p\left(\frac{x}{2}\right)$ on the interval $[2, +\infty)$. Finally, we have for

$$\begin{aligned}\nu_p(r_k) &= \nu_p\left(\left(\frac{1}{a^k} + \frac{1}{(p-a)^k}\right)\frac{p^k}{k}\right) \\ &= \nu_p\left(\frac{a^k + (p-a)^k}{(a(p-a))^k} \cdot \frac{p^k}{k}\right) \\ &= \nu_p\left(a^k + (p-a)^k\right) + k - \nu_p(k) \qquad (8)\end{aligned}$$

for all integers k (since a is coprime with p). If k is even, we use $\nu_p\left(a^k + (p-a)^k\right) \geq \nu_p(2)$ (for $p > 2$, this is obvious and for $p = 2$, observe that $a^k + (p-a)^k$ is even). So, we obtain

$$\nu_p(r_k) \geq k - \nu_p\left(\frac{k}{2}\right) \geq k - \log_p\left(\frac{k}{2}\right) = \ell_k$$

(because $k/2$ is a positive integer if k is even). However, if k is odd, we use $\nu_p\left(a^k + (p-a)^k\right) \geq 1$ (since $a^k + (p-a)^k \equiv a^k + (-a)^k \pmod p \equiv 0 \pmod p$). So, we again obtain

$$\nu_p(r_k) \geq 1 + k - \nu_p(k) \geq \log_p(2) + k - \log_p(k) = k - \log_p\left(\frac{k}{2}\right) = \ell_k.$$

Consequently, we have $\nu_p(r_k) \geq \ell_k$ for all integers $k \geq 2$. So, all the hypothesis of Theorem 2 are satisfied; thus we can apply it for our situation. Applying the first part of Theorem 2, we get

$$\nu_p\left(\sum_{k=1}^{n}\left(\frac{1}{a^k} + \frac{1}{(p-a)^k}\right)\frac{p^k}{k}\right) \geq \min_{k \geq n+1} \nu_p\left(\left(\frac{1}{a^k} + \frac{1}{(p-a)^k}\right)\frac{p^k}{k}\right) \geq (n+1) - \log_p\left(\frac{n+1}{2}\right)$$

for all positive integers n, thus confirming Inequality (4) of Theorem 1. Next, for a given positive integer n, the second part of Theorem 2 tells us that (4) becomes an equality if and only if we have

$$\nu_p(r_{n+1}) = (n+1) - \log_p\left(\frac{n+1}{2}\right),$$

which is equivalent (by using Eq. (8) and simplifying) to

$$\nu_p\left(a^{n+1} + (p-a)^{n+1}\right) - \nu_p(n+1) = -\log_p\left(\frac{n+1}{2}\right). \tag{9}$$

So, it remains to prove that Eq. (9) holds if and only if n has the form $n = 2p^\alpha - 1$ ($\alpha \in \mathbb{N}_0$). Let us prove this last fact.

- Suppose that Eq. (9) holds. Then, we have

  $$\log_p\left(\frac{n+1}{2}\right) = \nu_p(n+1) - \nu_p\left(a^{n+1} + (p-a)^{n+1}\right) \in \mathbb{Z}.$$

  But since $\log_p\left(\frac{n+1}{2}\right) \geq 0$, we have even $\log_p\left(\frac{n+1}{2}\right) \in \mathbb{N}_0$. By setting $\alpha := \log_p\left(\frac{n+1}{2}\right) \in \mathbb{N}_0$, we get $n = 2p^\alpha - 1$, as required.

- Conversely, suppose that $n = 2p^\alpha - 1$ for some $\alpha \in \mathbb{N}_0$. Then we have $\nu_p(n+1) = \nu_p(2) + \alpha$ and $\log_p\left(\frac{n+1}{2}\right) = \alpha$. So Eq. (9) is equivalent to

  $$\nu_p\left(a^{n+1} + (p-a)^{n+1}\right) = \nu_p(2). \tag{10}$$

  To confirm Eq. (10), we distinguish two cases:

Case 1: (If $p = 2$). In this case, since a is coprime with p then a and $(p - a)$ are both odd, implying that $a^2 \equiv 1 \pmod 4$ and $(p-a)^2 \equiv 1 \pmod 4$. Then, because $n+1 = 2p^\alpha$ is even, we have also $a^{n+1} \equiv 1 \pmod 4$ and $(p-a)^{n+1} \equiv 1 \pmod 4$; thus $a^{n+1} + (p-a)^{n+1} \equiv 2 \pmod 4$, implying that $\nu_p\left(a^{n+1} + (p-a)^{n+1}\right) = 1 = \nu_p(2)$.

Case 2: (If p is odd). In this case, because $n + 1 = 2p^\alpha$ is even, we have $a^{n+1} + (p-a)^{n+1} \equiv a^{n+1} + (-a)^{n+1} \pmod p \equiv 2a^{n+1} \pmod p \not\equiv 0 \pmod p$ (since p is assumed odd and a is coprime with p). Thus $\nu_p\left(a^{n+1} + (p-a)^{n+1}\right) = 0 = \nu_p(2)$. Consequently, Formula (10) is confirmed in all cases. This completes the proof of the second part of Theorem 1 and completes this proof.

$\square$

The rest of the paper is now devoted to proving Theorem 3. We achieve this by three different methods.

2.1 The first method

We rely on two identities. The first one (due to Mansour [6]) is combinatorial and states that

$$\sum_{k=0}^{n} \frac{x^k y^{n-k}}{\binom{n}{k}} = \frac{n+1}{(x+y)\left(\frac{1}{x}+\frac{1}{y}\right)^{n+1}} \sum_{k=1}^{n+1} \frac{\left(x^k+y^k\right)\left(\frac{1}{x}+\frac{1}{y}\right)^k}{k} \tag{11}$$

(for $x, y \in \mathbb{R}^*$, with $x + y \neq 0$, and $n \in \mathbb{N}_0$). While the second one (due to the author [4]) is arithmetic and states that

$$\operatorname{lcm}\left(\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}\right) = \frac{\operatorname{lcm}(1, 2, \dots, n, n+1)}{n+1} \tag{12}$$

(for all $n \in \mathbb{N}_0$).

Using Eqs. (11) and (12), we are now ready to prove Theorem 3. Let p be a prime number and a be an integer non-multiple of p. By applying Eq. (11) for $x = a$ and $y = p - a$ and replacing n by $(n - 1)$ (where $n \in \mathbb{N}$), we get (after simplifying and rearranging)

$$\sum_{k=1}^{n} \left(\frac{1}{a^k} + \frac{1}{(p-a)^k}\right) \frac{p^k}{k} = \frac{p^{n+1}}{n\left(a(p-a)\right)^n} \sum_{k=0}^{n-1} \frac{a^k (p-a)^{n-1-k}}{\binom{n-1}{k}}. \tag{13}$$

On the other hand, for all $n \in \mathbb{N}$, we have (according to Eq. (12))

$$1 = \frac{n}{\operatorname{lcm}(1, 2, \dots, n)} \operatorname{lcm}\left(\binom{n-1}{0}, \binom{n-1}{1}, \dots, \binom{n-1}{n-1}\right). \tag{14}$$

Then, for a given $n \in \mathbb{N}$, by multiplying Eqs. (13) and (14), we obtain

$$\sum_{k=1}^{n} \left(\frac{1}{a^k} + \frac{1}{(p-a)^k}\right) \frac{p^k}{k} = \frac{p^{n+1}}{\left(a(p-a)\right)^n \operatorname{lcm}(1,2,\dots,n)} \times \operatorname{lcm}\left(\binom{n-1}{0}, \binom{n-1}{1}, \dots, \binom{n-1}{n-1}\right) \sum_{k=0}^{n-1} \frac{a^k (p-a)^{n-1-k}}{\binom{n-1}{k}}.$$

But since the rational number

$$\operatorname{lcm}\left(\binom{n-1}{0}, \binom{n-1}{1}, \dots, \binom{n-1}{n-1}\right) \sum_{k=0}^{n-1} \frac{a^k (p-a)^{n-1-k}}{\binom{n-1}{k}}$$

is obviously an integer, we derive from the last identity that

$$\begin{aligned}\nu_p\left(\sum_{k=1}^{n} \left(\frac{1}{a^k} + \frac{1}{(p-a)^k}\right) \frac{p^k}{k}\right) &\geq \nu_p\left(\frac{p^{n+1}}{\left(a(p-a)\right)^n \operatorname{lcm}(1,2,\dots,n)}\right) \\ &= n+1-\nu_p\left(\operatorname{lcm}(1,2,\dots,n)\right) \quad (\text{since } a \text{ is not a multiple of } p) \\ &= n+1-\left\lfloor \log_p(n) \right\rfloor,\end{aligned}$$

confirming the unboundedness from above of the sequence $\left(\nu_p\left(\sum_{k=1}^{n}\left(\frac{1}{a^k}+\frac{1}{(p-a)^k}\right)\frac{p^k}{k}\right)\right)_{n\geq 1}$ (since $n+1-\left\lfloor \log_p(n)\right\rfloor \to +\infty$ as $n \to +\infty$). $\square$

2.2 The second method

Let p be a prime number and a be an integer non-multiple of p. For a given $n \in \mathbb{N}$, consider the rational function R_n defined by

$$R_n(X) := \sum_{k=1}^{n} \left(\frac{1}{a^k} + \frac{1}{(X-a)^k}\right) \frac{X^k}{k} = \sum_{k=1}^{n} \frac{\left(\frac{X}{a}\right)^k}{k} + \sum_{k=1}^{n} \frac{\left(\frac{X}{X-a}\right)^k}{k}.$$

Also consider the real function f defined at the neighborhood of 0 by

$$f(X) := -\log\left(1-X\right),$$

which satisfies the functional equation

$$f\left(\frac{X}{a}\right) + f\left(\frac{X}{X-a}\right) = 0 \tag{15}$$

and whose the n^{th} degree Taylor polynomial at 0 is $\sum_{k=1}^{n} \frac{X^k}{k}$.

On the one hand, according to the well-known properties of Taylor polynomials, the n^{th} degree Taylor polynomial of the function $X \overset{g}{\mapsto} f\left(\frac{X}{a}\right) + f\left(\frac{X}{X-a}\right)$ at 0 is the same with the n^{th} degree Taylor polynomial of

$$\sum_{k=1}^{n} \frac{\left(\frac{X}{a}\right)^k}{k} + \sum_{k=1}^{n} \frac{\left(\frac{X}{X-a}\right)^k}{k} = R_n(X).$$

But on the other hand, in view of Eq. (15), this n^{th} degree Taylor polynomial of g at 0 is zero. Comparing these two results, we deduce that the multiplicity of 0 in R_n is at least $(n+1)$. Consequently, $R_n(X)$ can be written as

$$R_n(X) = X^{n+1} \cdot \frac{U_n(X)}{a^n (X-a)^n \mathrm{lcm}(1, 2, \ldots, n)},$$

where $U_n \in \mathbb{Z}[X]$. In particular, we have

$$R_n(p) = p^{n+1} \cdot \frac{U_n(p)}{a^n (p-a)^n \mathrm{lcm}(1, 2, \ldots, n)}.$$

Next, because $U_n(p) \in \mathbb{Z}$ (since $U_n \in \mathbb{Z}[X]$) and a is not a multiple of p, then by taking the p-adic valuations in the two sides of the last identity, we derive that

$$\nu_p\left(R_n(p)\right) \geq n + 1 - \nu_p\left(\mathrm{lcm}(1, 2, \ldots, n)\right) = n + 1 - \left\lfloor \log_p(n) \right\rfloor,$$

implying that the sequence $\left(\nu_p\left(R_n(p)\right)\right)_{n \geq 1}$ is unbounded from above, as required by Theorem 3. $\square$

Remark 4. Curiously, the two previous methods give the same upper bound

$$\nu_p\left(\sum_{k=1}^{n}\left(\frac{1}{a^k} + \frac{1}{(p-a)^k}\right)\frac{p^k}{k}\right) \geq n + 1 - \left\lfloor \log_p(n) \right\rfloor.$$

Furthermore, this last estimate is remarkably very close to the optimal one of Theorem 1.

In the third method below, we will show the unboundedness of the sequence in Theorem 3 without providing any estimate!

2.3 The third method

Let p be a prime number and a be an integer non-multiple of p. For all $n \in \mathbb{N}$, set

$$r_n := \left(\frac{1}{a^n} + \frac{1}{(p-a)^n}\right)\frac{p^n}{n} \quad \text{and} \quad s_n := \sum_{k=1}^{n} r_k.$$

The property we have to show is that the sequence $(\nu_p(s_n))_{n\geq 1}$ is unbounded from above; in other words, we have that $\limsup_{n\to+\infty} \nu_p(s_n) = +\infty$. So, if we show the stronger property $\lim_{n\to+\infty} \nu_p(s_n) = +\infty$, then we are done. To do so, observe that

$$\begin{aligned}
\lim_{n\to+\infty} \nu_p(s_n) = +\infty &\Longleftrightarrow \lim_{n\to+\infty} |s_n|_p = 0 \\
&\Longleftrightarrow \lim_{n\to+\infty} s_n = 0 \qquad\qquad \text{(in the } p\text{-adic sense)} \\
&\Longleftrightarrow \sum_{k=1}^{+\infty} r_k = 0 \qquad\qquad \text{(in the } p\text{-adic sense).}
\end{aligned}$$

Consequently, it suffices to show that

$$\sum_{k=1}^{+\infty} \left(\frac{1}{a^k} + \frac{1}{(p-a)^k}\right) \frac{p^k}{k} = 0 \tag{16}$$

(in the p-adic sense). Let us show Eq. (16). By using the p-adic logarithm function (recalled in §1), we have

$$\begin{aligned}
\sum_{k=1}^{+\infty} \left(\frac{1}{a^k} + \frac{1}{(p-a)^k}\right) \frac{p^k}{k} &= \sum_{k=1}^{+\infty} \frac{\left(\frac{p}{a}\right)^k}{k} + \sum_{k=1}^{+\infty} \frac{\left(\frac{p}{p-a}\right)^k}{k} \\
&= -L_p\left(1 - \frac{p}{a}\right) - L_p\left(1 - \frac{p}{p-a}\right) \\
&= -\left[L_p\left(\frac{a-p}{a}\right) + L_p\left(\frac{-a}{p-a}\right)\right] \\
&= -L_p\left(\frac{a-p}{a} \cdot \frac{-a}{p-a}\right) \\
&= -L_p(1) = 0,
\end{aligned}$$

as required. The unboundedness from above of the sequence $(\nu_p(s_n))_{n\geq 1}$ follows. $\square$

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