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On Some Products Taken over Prime Numbers

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Abstract

We study expressions of the type $\prod_p p^{\lfloor \frac{x}{f(p)} \rfloor}$, where x is a nonnegative real number, f is an arithmetic function satisfying some conditions, and the product is over the primes p. We begin by proving that such expressions can be expressed by using the lcm function, without reference to prime numbers; we illustrate this result with several examples. The rest of the paper is devoted to studying two particular cases related to $f(m) = m$ and $f(m) = m - 1$. In both cases, we find arithmetic properties and analytic estimates for the underlying expressions.

1 Introduction and notation

Throughout this paper, we let $\mathbb{N}^*$ denote the set of positive integers and $\mathscr{P}$ the set of prime numbers. The letter p is reserved for primes. For a given prime number p, we let ϑ_p denote the usual p-adic valuation. For $x \in \mathbb{R}$, we let $\lfloor x \rfloor$ denote the integer part of x. For $N, b \in \mathbb{N}$, with $b \geq 2$, the expansion of N in base b is denoted by $N = \overline{a_k a_{k-1} \cdots a_1 a_0}_{(b)}$, meaning that $N = a_0 + ba_1 + b^2 a_2 + \cdots + b^k a_k$ (with $k \in \mathbb{N}$, $a_0, a_1, \dots, a_k \in \{0, 1, \dots, b-1\}$ and $a_k \neq 0$). In such a context, we let $S_b(N)$ denote the sum of base-b digits of N; that is, $S_b(N) := a_0 + a_1 + \cdots + a_k$. Further, we let τ, π, and θ, respectively, denote the divisor-counting function, the prime-counting function, and the Chebyshev theta function, defined as follows:

$$\tau(n) := \sum_{d \mid n} 1 \quad , \quad \pi(x) := \sum_{p \leq x} 1 \quad , \text{ and } \quad \theta(x) := \sum_{p \leq x} \log p \qquad (\forall n \in \mathbb{N}^*, \forall x \in \mathbb{R}^+).$$

It is known that for $n \geq 3$, we have $\tau(n) = n^{O(1/\log\log n)}$ (see e.g., [5, Proposition 7.12, page 101]). So, a fortiori,

$$\tau(n) = O(n^{1/3}). \tag{1}$$

On the other hand, the prime number theorem states that $\pi(x) \sim_{+\infty} \frac{x}{\log x}$. Other equivalent statements are $\theta(x) \sim_{+\infty} x$ and $\log \operatorname{lcm}(1, 2, \dots, n) \sim_{+\infty} n$ (see e.g., [5, Chapter 4]). The weaker estimates $\pi(x) = O(x/\log x)$, $\theta(x) = O(x)$, and $\log \operatorname{lcm}(1, 2, \dots, n) = O(n)$ are called Chebyshev's estimates.

In number theory, it is common that a prime factorization of some special numbers N produces, as exponents of each prime p, expressions of the form $\lfloor \frac{u_N}{f(p)} \rfloor$ or a sum of such expressions. The most famous example is perhaps the Legendre formula, stating that for natural numbers n, we have

$$n! = \prod_p p^{\lfloor \frac{n}{p} \rfloor + \lfloor \frac{n}{p^2} \rfloor + \lfloor \frac{n}{p^3} \rfloor + \cdots}, \tag{2}$$

which may be also reformulate in terms of base expansions as follows:

$$n! = \prod_p p^{\frac{n - S_p(n)}{p-1}}. \tag{3}$$

See e.g., [6, pages 76-77]. Another famous example is the formula of the least common multiple of the first consecutive positive integers:

$$\operatorname{lcm}(1, 2, \dots, n) = \prod_p p^{\lfloor \frac{\log n}{\log p} \rfloor} \qquad (\forall n \in \mathbb{N}). \tag{4}$$

Among other examples which are less known, we can cite the following

$$\operatorname{lcm}\{i_1 i_2 \cdots i_k \ ; \ k \in \mathbb{N}, i_1, i_2, \dots, i_k \in \mathbb{N}^*, i_1 + i_2 + \cdots + i_k \leq n\} = \prod_p p^{\lfloor \frac{n}{p} \rfloor}, \tag{5}$$

which is pointed out in the book of Cahen and Chabert [2, page 246] and also by Farhi [3] in the context of the integer-valued polynomials. Basing on the remark that in Formulas (2), (4), and (5), the right-hand side (which is a product taken over the primes) is interpreted without reference to prime numbers, we may naturally ask if an expression of a general type $\prod_p p^{\lfloor \frac{x}{f_1(p)} \rfloor + \lfloor \frac{x}{f_2(p)} \rfloor + \cdots}$, where $x \in \mathbb{R}^+$ and $(f_i)_i$ is a sequence of positive functions satisfying some regularity conditions, possesses the same property; that is, it has an interpretation without reference to the primes. In this paper, we study only the case of the products

$$\pi_f(x) := \prod_p p^{\lfloor \frac{x}{f(p)} \rfloor},$$

for which we affirmatively answer the previous question under some hypotheses on f. After giving several applications of our result, we focus our study on the two particular cases $f(p) = p$ and $f(p) = p - 1$. Because in both cases, there is no loss of generality to take x an integer, we are led define

$$\rho_n := \prod_p p^{\lfloor \frac{n}{p} \rfloor} \quad \text{and} \quad \sigma_n := \prod_p p^{\lfloor \frac{n}{p-1} \rfloor}$$

for $n \in \mathbb{N}$. These are, respectively, the sequences A048803 and A091137 of [7].

We begin with the arithmetic study of ρ_n and σ_n by establishing several arithmetic properties concerning them; in particular, we obtain a nontrivial divisor and a nontrivial multiple for σ_n. Moreover, we determine the p-adic valuations of the integers $\frac{\sigma_n}{n!}$ when the prime p is large enough compared to $\sqrt{n}$; we discover that the prime numbers of the form $\lfloor \frac{n}{k} + 1 \rfloor$ ($k \in \mathbb{N}^*$, $k < \sqrt{n+1} + 1$) play a vital role in the arithmetic nature of the σ_n's. In another direction, we find asymptotic estimates for $\log \rho_n$ and $\log \sigma_n$.

2 An expression of π_f using the least common multiple

Our result of expressing π_f in terms of the lcm's without reference to prime numbers is the following:

Theorem 1. *Let $f : \mathbb{N}^* \to \mathbb{R}_+$ be an arithmetic function such that $f(\mathbb{N}^* \setminus \{1\}) \subset \mathbb{R}_+^*$ (i.e., f does not vanish except at 1 eventually). Consider the set $\mathbb{N}^* \setminus \{1\}$ equipped with the partial order relation "$\mid$" of divisibility ($a \mid b \Leftrightarrow a$ divides b) and the set $\mathbb{R}_+^*$ equipped with the usual total order relation "$\leq$", and suppose that the map*

$$\begin{aligned} \widetilde{f} :\ \mathbb{N}^* \setminus \{1\} &\longrightarrow \mathbb{R}_+^* \\ n &\longmapsto \frac{f(n)}{\log n} \end{aligned}$$

is nondecreasing with respect to these two orders. Then for $x \in \mathbb{R}^+$ we have

$$\prod_p p^{\lfloor \frac{x}{f(p)} \rfloor} = \operatorname{lcm}\{i_1 i_2 \cdots i_k \ ; \ k \in \mathbb{N}, i_1, i_2, \ldots, i_k \in \mathbb{N}^*, f(i_1) + f(i_2) + \cdots + f(i_k) \leq x\}.$$

In order to present a clean proof of Theorem 1, we use the following lemma:

Lemma 2. *Let $f : \mathbb{N}^* \to \mathbb{R}_+$ as in Theorem 1. Then, for prime numbers p and positive integers a, we have*

$$\vartheta_p(a) \leq \frac{f(a)}{f(p)}.$$

Proof. Let p be a prime number and a be a positive integer. Since the inequality of the lemma is trivial when $\vartheta_p(a) = 0$, we may suppose that $\vartheta_p(a) \geq 1$; that is $p \mid a$. So according to our assumptions on f, we have that $\frac{f(p)}{\log p} \leq \frac{f(a)}{\log a}$, which translates into $\frac{\log a}{\log p} \leq \frac{f(a)}{f(p)}$. Hence, $\vartheta_p(a) \leq \frac{\log a}{\log p} \leq \frac{f(a)}{f(p)}$, as required. □

Proof of Theorem 1. Let $x \in \mathbb{R}_+$ be fixed. For a given prime number p, the p-adic valuation of the left-hand side of the identity of Theorem 1 is equal to $\lfloor \frac{x}{f(p)} \rfloor$, while the p-adic valuation of the right-hand side of the same identity is equal to $\ell_p := \max\{\vartheta_p(i_1 i_2 \dots i_k); k \in \mathbb{N}, i_1, \dots, i_k \in \mathbb{N}^*, f(i_1) + \cdots + f(i_k) \leq x\}$. So, we have to show that $\ell_p = \lfloor \frac{x}{f(p)} \rfloor$ for primes p. To do so, we prove the two inequalities $\ell_p \geq \lfloor \frac{x}{f(p)} \rfloor$ and $\ell_p \leq \lfloor \frac{x}{f(p)} \rfloor$.

First, for a given prime number p, let us show that $\ell_p \geq \lfloor \frac{x}{f(p)} \rfloor$. By considering the particular natural number $k = \lfloor \frac{x}{f(p)} \rfloor$ and the particular positive integers $i_1 = i_2 = \cdots = i_k = p$, we get $f(i_1) + f(i_2) + \cdots + f(i_k) = kf(p) = \lfloor \frac{x}{f(p)} \rfloor f(p) \leq x$. Thus, according to the definition of ℓ_p we have $\ell_p \geq \vartheta_p(i_1 i_2 \cdots i_k) = \vartheta_p(p^k) = k = \lfloor \frac{x}{f(p)} \rfloor$, as required.

Now, for a given prime number p, let us show that $\ell_p \leq \lfloor \frac{x}{f(p)} \rfloor$. For $k \in \mathbb{N}$ and $i_1, i_2, \dots, i_k \in \mathbb{N}^*$, with $f(i_1) + f(i_2) + \cdots + f(i_k) \leq x$, we have

$$\begin{aligned} \vartheta_p(i_1 i_2 \cdots i_k) &= \vartheta_p(i_1) + \vartheta_p(i_2) + \cdots + \vartheta_p(i_k) \\ &\leq \frac{f(i_1)}{f(p)} + \frac{f(i_2)}{f(p)} + \cdots + \frac{f(i_k)}{f(p)} \qquad \text{(according to Lemma 2)} \\ &= \frac{f(i_1) + f(i_2) + \cdots + f(i_k)}{f(p)} \\ &\leq \frac{x}{f(p)}; \end{aligned}$$

but since $\vartheta_p(i_1 i_2 \cdots i_k) \in \mathbb{N}$, it follows that: $\vartheta_p(i_1 i_2 \cdots i_k) \leq \lfloor \frac{x}{f(p)} \rfloor$. The definition of ℓ_p concludes that $\ell_p \leq \lfloor \frac{x}{f(p)} \rfloor$, as required. This completes the proof. □

Remark 3. Let us put ourselves in the situation of Theorem 1.

1. If the map $\widetilde{f}$ is nondecreasing in the usual sense; i.e., with respect to the usual orders of the two sets $\mathbb{N}^* \setminus \{1\}$ and $\mathbb{R}_+^*$, then it remains nondecreasing in the sense imposed by Theorem 1. This immediately follows from the implication: $a \mid b \Rightarrow a \leq b$, $\forall a, b \in \mathbb{N}^*$.

2. More generally than the previous item, if the restriction of the map $\widetilde{f}$ on $\mathbb{N}^* \setminus \{1, 2\}$ is nondecreasing in the usual sense and $\widetilde{f}(2) \leq \widetilde{f}(4)$, then $\widetilde{f}$ is nondecreasing in the sense imposed by Theorem 1.

Now, from Theorem 1, we derive the following corollary in which the condition imposed on f is made simpler.

Corollary 4. *Let $f : \mathbb{N}^* \to \mathbb{R}_+$ be an arithmetic function satisfying $f(\mathbb{N}^* \setminus \{1\}) \subset \mathbb{R}_+^*$. Suppose that the map*

$$\begin{array}{ccc} \mathbb{N}^* \setminus \{1\} & \longrightarrow & \mathbb{R}_+^* \\ n & \longmapsto & \dfrac{f(n)}{n} \end{array}$$

is nondecreasing in the usual sense (i.e., with respect to the usual order of $\mathbb{R}$). Then for $x \in \mathbb{R}_+$ we have

$$\prod_p p^{\lfloor \frac{x}{f(p)} \rfloor} = \mathrm{lcm}\{i_1 i_2 \cdots i_k \ ; \ k \in \mathbb{N}, i_1, i_2, \dots, i_k \in \mathbb{N}^*, f(i_1) + f(i_2) + \dots + f(i_k) \leq x\}.$$

Proof. We use Theorem 1 together with Item 2 of Remark 3. We remark that $\widetilde{f}$ (defined as in Theorem 1) is the product of the two functions: $n \mapsto \frac{f(n)}{n}$ (assumed to be nondecreasing in the usual sense on $\mathbb{N}^* \setminus \{1\}$) and $n \mapsto \frac{n}{\log n}$, which is nondecreasing on $\mathbb{N}^* \setminus \{1, 2\} = \{3, 4, 5, \dots\}$. So, $\widetilde{f}$ is nondecreasing on $\mathbb{N}^* \setminus \{1, 2\}$ in the usual sense. In addition, we have

$$\widetilde{f}(2) = \frac{f(2)}{\log 2} = \frac{f(2)}{2} \cdot \frac{2}{\log 2} = \frac{f(2)}{2} \cdot \frac{4}{\log 4} \leq \frac{f(4)}{4} \cdot \frac{4}{\log 4}$$

(since $n \mapsto \frac{f(n)}{n}$ is assumed to be nondecreasing in the usual sense on $\mathbb{N}^* \setminus \{1\}$). That is, we have $\widetilde{f}(2) \leq \frac{f(4)}{\log 4} = \widetilde{f}(4)$. The conclusion follows from Item 2 of Remark 3 and Theorem 1. $\square$

2.1 Some applications

1. By applying Theorem 1 for $f(m) = \log m$, we obtain that for $x \in \mathbb{R}_+$, we have

$$\begin{aligned} \prod_p p^{\lfloor \frac{x}{\log p} \rfloor} &= \mathrm{lcm}\{i_1 i_2 \cdots i_k \ ; \ k \in \mathbb{N}, i_1, i_2, \dots, i_k \in \mathbb{N}^*, \log i_1 + \log i_2 + \dots + \log i_k \leq x\} \\ &= \mathrm{lcm}\{i_1 i_2 \cdots i_k \ ; \ k \in \mathbb{N}, i_1, i_2, \dots, i_k \in \mathbb{N}^*, i_1 i_2 \cdots i_k \leq e^x\} \\ &= \mathrm{lcm}\left(1, 2, \dots, \lfloor e^x \rfloor\right). \end{aligned}$$

   In particular, by taking $x = \log n$ ($n \in \mathbb{N}^*$), we obtain the following well-known formula:

$$\prod_p p^{\lfloor \frac{\log n}{\log p} \rfloor} = \mathrm{lcm}\left(1, 2, \dots, n\right).$$

2. By applying Corollary 4 for the function $f(m) = m$, we obtain in particular that for all $n \in \mathbb{N}$, we have

$$\prod_p p^{\lfloor \frac{n}{p} \rfloor} = \mathrm{lcm}\{i_1 i_2 \cdots i_k \ ; \ k \in \mathbb{N}, i_1, i_2, \dots, i_k \in \mathbb{N}^*, i_1 + i_2 + \dots + i_k \leq n\}, \quad (6)$$

   which is already pointed out by Cahen and Chabert [2] and by Farhi [3].

3. (Generalization of (6)). Let $\alpha \geq 1$. By applying Corollary 4 for the function $f(m) = m^\alpha$, we obtain in particular that for all $n \in \mathbb{N}$, we have

$$\prod_p p^{\lfloor \frac{n}{p^\alpha} \rfloor} = \mathrm{lcm}\{i_1 i_2 \cdots i_k \ ; \ k \in \mathbb{N}, i_1, i_2, \dots, i_k \in \mathbb{N}^*, i_1^\alpha + i_2^\alpha + \cdots + i_k^\alpha \leq n\}.$$

4. For all $n, k \in \mathbb{N}$, with $n \geq k$, let us define, as in [3],

$$q_{n,k} := \mathrm{lcm}\{i_1 i_2 \cdots i_k \ ; \ i_1, i_2, \dots, i_k \in \mathbb{N}^*, i_1 + i_2 + \cdots + i_k \leq n\}.$$

These numbers were studied by Farhi [3] in a context related to integer-valued polynomials. By applying Corollary 4 for the function $f(m) = m - 1$, we obtain that for all $n \in \mathbb{N}$, we have

$$\begin{aligned}\prod_p p^{\lfloor \frac{n}{p-1} \rfloor} &= \mathrm{lcm}\{i_1 i_2 \cdots i_k \ ; \ k \in \mathbb{N}, i_1, i_2, \dots, i_k \in \mathbb{N}^*, \\ &\qquad (i_1 - 1) + (i_2 - 1) + \cdots + (i_k - 1) \leq n\} \\ &= \mathrm{lcm}\{i_1 i_2 \cdots i_k \ ; \ k \in \mathbb{N}, i_1, i_2, \dots, i_k \in \mathbb{N}^*, i_1 + i_2 + \cdots + i_k \leq n + k\} \\ &= \mathrm{lcm}\{q_{n+k,k} \ ; \ k \in \mathbb{N}\}, \end{aligned} \tag{7}$$

which remarkably represents the least common multiple of the n^{th} diagonal of the arithmetic triangle of the $q_{i,j}$'s, beginning as follows (see [3])

1							
1	1						
1	2	1					
1	6	2	1				
1	12	12	2	1			
1	60	12	12	2	1		
1	60	360	24	12	2	1	
1	420	360	360	24	12	2	1

Table 1: The triangle of the $q_{n,k}$'s for $0 \leq k \leq n \leq 7$.

For a given $n \in \mathbb{N}$, let $D_n = (d_{n,k})_{k \in \mathbb{N}}$ denote the sequence of the n^{th} diagonal of the above triangle, that is

$$d_{n,k} := q_{n+k,k} = \mathrm{lcm}\{i_1 i_2 \cdots i_k \ ; \ i_1, i_2, \dots, i_k \in \mathbb{N}^*, i_1 + i_2 + \cdots + i_k \leq n + k\}. \tag{8}$$

In order to simplify Formula (7), we show that the sequences D_n ($n \in \mathbb{N}$) are all nondecreasing in the divisibility sense and eventually constant. More precisely, we have the following proposition:

Proposition 5. *For all $n, k \in \mathbb{N}$, we have*

$$d_{n,k} \quad \textit{divides} \quad d_{n,k+1}.$$

If in addition $k \geq n$, then we have

$$d_{n,k} = d_{n,n}.$$

Proof. Let $n, k \in \mathbb{N}$ be fixed. If $i_1 i_2 \cdots i_k$ is a member of the list of lcm defining $d_{n,k}$, then $1 \cdot i_1 i_2 \cdots i_k$ satisfies $1 + (i_1 + i_2 + \cdots + i_k) \leq 1 + (n + k) = n + (k + 1)$, so the same element is a member of the list of the lcm defining $d_{n,k+1}$. Hence, $d_{n,k}$ divides $d_{n,k+1}$, as required.

Now, let us prove the second part of the proposition. So, suppose that $k \geq n$ and let us prove that $d_{n,k} = d_{n,n}$. It follows from an immediate induction leaning on the result of the first part of the proposition (proved above) that $d_{n,n} \mid d_{n,k}$. So, it remains to prove that $d_{n,k} \mid d_{n,n}$. Let $i_1, i_2, \dots, i_k \in \mathbb{N}^*$ such that $i_1 + i_2 + \cdots + i_k \leq n + k$. Let $\ell \in \mathbb{N}$ denote the number of indices i_r $(1 \leq r \leq k)$ which are equal to 1; so we have exactly $(k - \ell)$ indices i_r which are ≥ 2. Thus, we have

$$i_1 + i_2 + \cdots + i_k \geq \ell + 2(k - \ell) = 2k - \ell.$$

But since $i_1 + i_2 + \cdots + i_k \leq n + k$, we derive that $2k - \ell \leq n + k$, which gives $\ell \geq k - n$. This proves that we have at least $(k - n)$ indices i_r which are equal to 1. By assuming, without loss of generality, that those indices are $i_{n+1}, i_{n+2}, \dots, i_k$ (i.e., $i_{n+1} = i_{n+2} = \cdots = i_k = 1$), we get

$$i_1 i_2 \cdots i_n = i_1 i_2 \cdots i_k$$

and

$$i_1 + i_2 + \cdots + i_n = (i_1 + i_2 + \cdots + i_k) - (k - n) \leq (n + k) - (k - n) = 2n.$$

This shows that each product $i_1 i_2 \cdots i_k$ occurring in the definition of $d_{n,k}$ reduces (by permuting the i_r's and eliminate those of them which are equal to 1) to a product $j_1 j_2 \cdots j_n$ which occurs in the definition of $d_{n,n}$. Consequently $d_{n,k} \mid d_{n,n}$, as required. This completes the proof of the proposition. □

Using Proposition 5, for $n \in \mathbb{N}$ we have

$$\begin{aligned} \mathrm{lcm}\{q_{n+k,k} \ ; \ k \in \mathbb{N}\} &= \mathrm{lcm}\{d_{n,k} \ ; \ k \in \mathbb{N}\} \\ &= d_{n,n} \\ &= \mathrm{lcm}\{i_1 i_2 \cdots i_n \ ; \ i_1, i_2, \dots, i_n \in \mathbb{N}^*, i_1 + i_2 + \cdots + i_n \leq 2n\}. \end{aligned}$$

This proves the following interesting corollary, simplifying Formula (7):

Corollary 6. *For $n \in \mathbb{N}$, we have*

$$\prod_p p^{\lfloor \frac{n}{p-1} \rfloor} = \mathrm{lcm}\{i_1 i_2 \cdots i_n \ ; \ i_1, i_2, \dots, i_n \in \mathbb{N}^*, i_1 + i_2 + \cdots + i_n \leq 2n\}.$$ □

3 Arithmetic results on the numbers ρ_n and σ_n

A certain number of arithmetic properties concerning the numbers ρ_n and σ_n are either immediate or quite easy to prove. We have gathered them in the following proposition:

Proposition 7. *For natural numbers n, we have*

(i) $\rho_n \mid \rho_{n+1}$, $\sigma_n \mid \sigma_{n+1}$, *and* $\rho_n \mid \sigma_n$;

(ii) $\rho_n \mid n!$;

(iii) $n! \mid \sigma_n$ *and* $\sigma_n \mid (2n)!$;

(iv) $\sigma_{2n+1} = 2\sigma_{2n}$.

Proof. Let $n \in \mathbb{N}$ be fixed. The properties of Item (i) are trivial.

Item (ii) follows because for primes p, the term $\lfloor n/p \rfloor$ is the first term of $\lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \cdots$, which is the exact exponent of p in $n!$, which is also $\frac{n - S_p(n)}{p-1} \leq \frac{n}{p-1}$; so in particular $\leq \lfloor \frac{n}{p-1} \rfloor$, giving at the same time the first part of (iii).

Let us prove the second part of (iii). To do so, we use Corollary 6. For $i_1, i_2, \dots, i_n \in \mathbb{N}^*$ satisfying $i_1 + i_2 + \dots + i_n \leq 2n$, we have that $i_1 i_2 \cdots i_n \mid i_1! i_2! \cdots i_n! \mid (i_1 + i_2 + \dots + i_n)! \mid (2n)!$. Thus, $\mathrm{lcm}\{i_1 i_2 \cdots i_n \, ; \, i_1, i_2, \dots, i_n \in \mathbb{N}^*, i_1 + i_2 + \dots + i_n \leq 2n\} \mid (2n)!$; that is, according to Corollary 6, $\sigma_n \mid (2n)!$.

Finally, Item (iv) follows from the fact that $\vartheta_p(\sigma_{2n+1}) = \vartheta_p(2\sigma_{2n})$ for primes p (distinguish the two cases "p odd" and "$p = 2$"). This completes the proof of the proposition. □

In the following proposition, we shall improve Item (iii) of Proposition 7.

Proposition 8. *For natural numbers n, we have*

$$(n+1)! \ \mid \ \sigma_n \ \text{ and } \ \sigma_n \ \mid \ n! \, \mathrm{lcm}(1, 2, \dots, n, n+1).$$

Proof. Let $n \in \mathbb{N}$ be fixed. We have to show that for primes p, one has

$$\vartheta_p\left((n+1)!\right) \leq \vartheta_p\left(\sigma_n\right) \leq \vartheta_p\left(n! \, \mathrm{lcm}(1, 2, \dots, n, n+1)\right). \tag{9}$$

Let p be a fixed prime number and let us prove (9). By setting e the greatest nonnegative integer satisfying $p^e \leq n + 1$, we obtain

$$\vartheta_p(n!) = \sum_{i=1}^{e} \lfloor \frac{n}{p^i} \rfloor$$

and

$$\vartheta_p((n+1)!) = \sum_{i=1}^{e} \lfloor \frac{n+1}{p^i} \rfloor$$

(according to the Legendre formula),

$$\vartheta_p(\sigma_n) = \lfloor \frac{n}{p-1} \rfloor$$

(by definition of σ_n), and

$$\vartheta_p(\mathrm{lcm}(1, 2, \ldots, n+1)) = e.$$

So (9) reduces to

$$\sum_{i=1}^{e} \left\lfloor \frac{n+1}{p^i} \right\rfloor \leq \left\lfloor \frac{n}{p-1} \right\rfloor \leq \sum_{i=1}^{e} \left\lfloor \frac{n}{p^i} \right\rfloor + e. \tag{10}$$

On the one hand, we have

$$\sum_{i=1}^{e} \left\lfloor \frac{n+1}{p^i} \right\rfloor \leq \sum_{i=1}^{e} \frac{n+1}{p^i} = \frac{n+1}{p-1} \left(1 - \frac{1}{p^e}\right) \leq \frac{n}{p-1}$$

(since $p^e \leq n+1$). But since $\sum_{i=1}^{e} \lfloor \frac{n+1}{p^i} \rfloor$ is an integer, we derive the inequality

$$\sum_{i=1}^{e} \left\lfloor \frac{n+1}{p^i} \right\rfloor \leq \left\lfloor \frac{n}{p-1} \right\rfloor,$$

confirming the left inequality in (10). On the other hand, by using the refined inequality $\lfloor \frac{a}{b} \rfloor \geq \frac{a+1}{b} - 1$, which holds for all positive integers a, b, we have

$$\begin{aligned}
\left\lfloor \frac{n}{p-1} \right\rfloor - \sum_{i=1}^{e} \left\lfloor \frac{n}{p^i} \right\rfloor &\leq \frac{n}{p-1} - \sum_{i=1}^{e} \left(\frac{n+1}{p^i} - 1\right) \\
&= \frac{n}{p-1} - \frac{n+1}{p-1} \left(1 - \frac{1}{p^e}\right) + e \\
&= \frac{1}{p-1} \left(\frac{n+1}{p^e} - 1\right) + e.
\end{aligned}$$

But from the definition of e, we have $p^{e+1} > n+1$; that is, $\frac{n+1}{p^e} < p$. By inserting this into the last estimate, we get

$$\left\lfloor \frac{n}{p-1} \right\rfloor - \sum_{i=1}^{e} \left\lfloor \frac{n}{p^i} \right\rfloor < e + 1.$$

Next, since $\lfloor \frac{n}{p-1} \rfloor - \sum_{i=1}^{e} \lfloor \frac{n}{p^i} \rfloor \in \mathbb{Z}$, we conclude

$$\left\lfloor \frac{n}{p-1} \right\rfloor - \sum_{i=1}^{e} \left\lfloor \frac{n}{p^i} \right\rfloor \leq e,$$

confirming the right inequality of (10). This completes this proof. □

From Proposition 8, we derive an asymptotic estimate for the number $\log \sigma_n$ when n tends to infinity.

Corollary 9. *We have*

$$\log \sigma_n \ \sim_{+\infty} \ n \log n.$$

Proof. According to Proposition 8, for $n \in \mathbb{N}^*$ we have

$$\log (n+1)! \leq \log \sigma_n \leq \log (n!) + \log \operatorname{lcm}(1, 2, \dots, n, n+1).$$

Then the asymptotic estimate of the corollary follows from the facts

$$\log (n+1)! \sim_{+\infty} \log(n!) \sim_{+\infty} n \log n$$

according to Stirling's formula, and

$$\log \operatorname{lcm}(1, 2, \dots, n, n+1) \sim_{+\infty} n$$

according to the prime number theorem. □

Note that the asymptotic estimate of the above corollary will be specified in Section 4.

We now turn to establish a result evaluating the p-adic valuations of the positive integers $\frac{\sigma_n}{n!}$ $(n \in \mathbb{N}^*)$ for sufficiently large prime numbers. We discover as a remarkable phenomenon that primes of a special type play a vital role. We have the following theorem:

Theorem 10. *Let n be a positive integer and p be a prime number such that*

$$\sqrt{n+1} < p \leq n+1.$$

Then we have

$$\vartheta_p\left(\frac{\sigma_n}{n!}\right) = \left\lfloor \frac{S_p(n)}{p-1} \right\rfloor \in \{0, 1\}.$$

Furthermore, the equality $\vartheta_p(\frac{\sigma_n}{n!}) = 1$ holds if and only if $S_p(n) \geq p-1$, which holds if and only if p has the form

$$p = \left\lfloor \frac{n}{k} + 1 \right\rfloor,$$

with $k \in \mathbb{N}^*$ and $k < \sqrt{n+1} + 1$.

Proof. By the definition of σ_n and the Legendre formula (3), we have that

$$\begin{aligned}
\vartheta_p\left(\frac{\sigma_n}{n!}\right) &= \vartheta_p\left(\sigma_n\right) - \vartheta_p\left(n!\right) \\
&= \left\lfloor \frac{n}{p-1} \right\rfloor - \frac{n - S_p(n)}{p-1} \\
&= \left\lfloor \frac{n}{p-1} - \frac{n - S_p(n)}{p-1} \right\rfloor \qquad \left(\text{since } \frac{n - S_p(n)}{p-1} = \vartheta_p(n!) \in \mathbb{Z}\right) \\
&= \left\lfloor \frac{S_p(n)}{p-1} \right\rfloor. \qquad (11)
\end{aligned}$$

Next, let us prove that $\lfloor \frac{S_p(n)}{p-1} \rfloor \in \{0,1\}$. The hypothesis on p insures that $n < p^2 - 1$, which implies that the representation of the positive integer n in base p has the form $n = \overline{a_1 a_0}_{(p)}$, with $a_0, a_1 \in \{0, 1, \ldots, p-1\}$ and $(a_0, a_1) \neq (p-1, p-1)$. Consequently, we have $S_p(n) = a_0 + a_1 < 2(p-1)$, implying that $\frac{S_p(n)}{p-1} < 2$; hence $\left\lfloor \frac{S_p(n)}{p-1} \right\rfloor \in \{0,1\}$, as required. This achieves the proof of the first part of the theorem, which immediately gives the equivalence between $\vartheta_p(\frac{\sigma_n}{n!}) = 1$ and $S_p(n) \geq p-1$. Now, let us prove the last part of the theorem.

Suppose that $S_p(n) \geq p-1$. As seen above, the representation of n in base p has the form $n = \overline{a_1 a_0}_{(p)} = a_0 + pa_1$, where $a_0, a_1 \in \{0, 1, \ldots, p-1\}$ and $(a_0, a_1) \neq (p-1, p-1)$. We will show that $k = a_1 + 1$ is suitable for the required form of p. By supposition, we have $a_0 + a_1 \geq p - 1$, implying that

$$p - 1 \leq \frac{a_0 + a_1 p}{a_1 + 1} < p,$$

which is equivalent to

$$\left\lfloor \frac{n}{a_1 + 1} \right\rfloor = p - 1.$$

Thus,

$$p = \left\lfloor \frac{n}{a_1 + 1} + 1 \right\rfloor.$$

Furthermore, we have $a_1 = \lfloor \frac{n}{p} \rfloor \leq \frac{n}{p} < \sqrt{n+1}$ (since $p > \sqrt{n+1} > \frac{n}{\sqrt{n+1}}$). Thus, $k = a_1 + 1$ satisfies the required properties; i.e., $p = \lfloor \frac{n}{k} + 1 \rfloor$ and $k < \sqrt{n+1} + 1$.

Conversely, suppose that there exists $k \in \mathbb{N}^*$, with $k < \sqrt{n+1} + 1$, such that $p = \lfloor \frac{n}{k} + 1 \rfloor$, and let us show that $S_p(n) \geq p - 1$. Setting $a_0 := n - (k-1)p$ and $a_1 := k - 1$, we first show that the representation of n in base p is $n = \overline{a_1 a_0}_{(p)}$. Since it is immediate that $n = a_0 + pa_1$, it just remains to prove that $a_0, a_1 \in \{0, 1, \ldots, p-1\}$. Since $k < \sqrt{n+1} + 1 < p + 1$ then $k - 1 < p$; that is $a_1 \in \{0, 1, \ldots, p-1\}$. Next, since $p = \lfloor \frac{n}{k} + 1 \rfloor$ then

$$p \leq \frac{n}{k} + 1 < p + 1,$$

implying that

$$p - k \leq n - (k-1)p < p.$$

Hence

$$p - k \leq a_0 < p.$$

But $p - k = (p-1) - a_1 \geq 0$; thus $a_0 \in \{0, 1, \ldots, p-1\}$. We have confirmed that the representation of n in base p is $n = \overline{a_1 a_0}_{(p)}$. Consequently, we have

$$S_p(n) = a_0 + a_1 = n - (k-1)(p-1).$$

Then, since $n \geq k(p-1)$ (because $\frac{n}{k} + 1 \geq \lfloor \frac{n}{k} + 1 \rfloor = p$), it follows that $S_p(n) \geq p - 1$, as required. This completes the proof of the theorem. □

4 Analytic estimates of the numbers $\log \rho_n$ and $\log \sigma_n$

Throughout this section, we let c denote the absolute positive constant given by

$$c := \sum_p \frac{\log p}{p(p-1)} = 0.755\dots.$$

Our goal is to find asymptotic estimates for $\log \rho_n$ and $\log \sigma_n$ as n tends to infinity. The obtained main results are the following:

Theorem 11. *We have*

$$\log \rho_n = n \log n - (c+1)n + O\left(\sqrt{n}\right).$$

Theorem 12. *We have*

$$\log \sigma_n = n \log n - n + 2\sqrt{n} + o(\sqrt{n}).$$

To establish Theorem 11, we need the auxiliary results below. Theorem 12 is derived from Theorem 10 and a result of Bordellès et al. [1].

Lemma 13. *For $x \geq 1$, we have*

$$\sum_{p>x} \frac{\log p}{p(p-1)} = O\left(\frac{1}{x}\right).$$

Proof. Since $\frac{\log p}{p(p-1)} \leq 2\frac{\log p}{p^2}$ (for primes p), then it suffices to show that $\sum_{p>x} \frac{\log p}{p^2} = O(\frac{1}{x})$. According to the Abel summation formula (see e.g., [5, Proposition 1.4]), for positive real numbers x, y, with $x < y$ we have

$$\begin{aligned}
\sum_{x<p\leq y} \frac{\log p}{p^2} &= \left(\sum_{x<p\leq y} \log p\right) \frac{1}{y^2} - \int_x^y \left(\sum_{x<p\leq t} \log p\right) \left(\frac{1}{t^2}\right)' dt \\
&= \frac{\theta(y) - \theta(x)}{y^2} + 2\int_x^y \frac{\theta(t) - \theta(x)}{t^3}\, dt.
\end{aligned}$$

Then, by setting y to infinity, it follows (since $\theta(y) = O(y)$) that

$$\sum_{p>x} \frac{\log p}{p^2} = 2\int_x^{+\infty} \frac{\theta(t) - \theta(x)}{t^3}\, dt = 2\int_x^{+\infty} \frac{\theta(t)}{t^3}\, dt - \frac{\theta(x)}{x^2}.$$

Using finally $\theta(t) = O(t)$, we get

$$\sum_{p>x} \frac{\log p}{p^2} = O\left(\int_x^{+\infty} \frac{dt}{t^2}\right) + O\left(\frac{1}{x}\right) = O\left(\frac{1}{x}\right),$$

as required. The proof is complete. □

Lemma 13 above is used in the proof of the following proposition:

Proposition 14. *For positive integers n we have*

$$\sum_p \left(\left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots \right) \log p = c \cdot n + O\left(\sqrt{n}\right).$$

Proof. Let n be a fixed positive integer. For primes p, let e_p denote the greatest nonnegative integer satisfying $p^{e_p} \leq n$; explicitly $e_p = \lfloor \frac{\log n}{\log p} \rfloor$. So we have $p^{e_p+1} > n$. On the one hand, we have

$$\sum_p \left(\left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots \right) \log p \leq \sum_p \left(\frac{n}{p^2} + \frac{n}{p^3} + \cdots \right) \log p = \sum_p \frac{n}{p(p-1)} \log p;$$

that is

$$\sum_p \left(\left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots \right) \log p \leq c \cdot n. \tag{12}$$

On the other hand, we have (according to the definition of the e_p's)

$$\begin{aligned}
\sum_p \left(\left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots \right) \log p &= \sum_{p \leq \sqrt{n}} \left(\left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots + \left\lfloor \frac{n}{p^{e_p}} \right\rfloor \right) \log p \\
&\geq \sum_{p \leq \sqrt{n}} \left(\left(\frac{n}{p^2} - 1 \right) + \left(\frac{n}{p^3} - 1 \right) + \cdots + \left(\frac{n}{p^{e_p}} - 1 \right) \right) \log p \\
&= n \sum_{p \leq \sqrt{n}} \left(\frac{1}{p^2} + \frac{1}{p^3} + \cdots + \frac{1}{p^{e_p}} \right) \log p - \sum_{p \leq \sqrt{n}} (e_p - 1) \log p \\
&= n \sum_{p \leq \sqrt{n}} \left(\frac{1}{p(p-1)} - \frac{1}{p^{e_p}(p-1)} \right) \log p - \sum_{p \leq \sqrt{n}} (e_p - 1) \log p \\
&= n \sum_{p \leq \sqrt{n}} \frac{\log p}{p(p-1)} - n \sum_{p \leq \sqrt{n}} \frac{\log p}{p^{e_p}(p-1)} - \sum_{p \leq \sqrt{n}} (e_p - 1) \log p \\
&= n \left(c - \sum_{p > \sqrt{n}} \frac{\log p}{p(p-1)} \right) - n \sum_{p \leq \sqrt{n}} \frac{\log p}{p^{e_p}(p-1)} - \sum_{p \leq \sqrt{n}} (e_p - 1) \log p;
\end{aligned}$$

that is,

$$\sum_p \left(\left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots \right) \log p \geq c\, n - n \sum_{p > \sqrt{n}} \frac{\log p}{p(p-1)} - n \sum_{p \leq \sqrt{n}} \frac{\log p}{p^{e_p}(p-1)} - \sum_{p \leq \sqrt{n}} (e_p - 1) \log p. \tag{13}$$

But, by using Lemma 13, we have

$$\sum_{p>\sqrt{n}} \frac{\log p}{p(p-1)} = O\left(\frac{1}{\sqrt{n}}\right). \tag{14}$$

Next, by using the fact $p^{e_p} > \frac{n}{p}$ (for primes p), we have

$$\sum_{p\leq\sqrt{n}} \frac{\log p}{p^{e_p}(p-1)} < \frac{1}{n}\sum_{p\leq\sqrt{n}} \frac{p}{p-1}\log p \leq \frac{2}{n}\sum_{p\leq\sqrt{n}} \log p = \frac{2}{n}\theta\left(\sqrt{n}\right) = O\left(\frac{1}{\sqrt{n}}\right), \tag{15}$$

and by using the fact $e_p - 1 < e_p := \lfloor \frac{\log n}{\log p} \rfloor \leq \frac{\log n}{\log p}$, we have

$$\sum_{p\leq\sqrt{n}} (e_p - 1)\log p < \sum_{p\leq\sqrt{n}} \log n = (\log n)\pi(\sqrt{n}) = O\left(\sqrt{n}\right). \tag{16}$$

Then, by substituting (14), (15), and (16) into (13), we get

$$\sum_{p} \left(\left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots\right) \log p \geq c\, n + O\left(\sqrt{n}\right). \tag{17}$$

Finally, (12) and (17) conclude to

$$\sum_{p} \left(\left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots\right) \log p = c \cdot n + O\left(\sqrt{n}\right),$$

as required. □

We are now able to prove Theorem 11.

Proof of Theorem 11. For sufficiently large integers n, we have, according to Legendre's formula,

$$\begin{aligned}\log \rho_n &= \sum_{p} \left\lfloor \frac{n}{p} \right\rfloor \log p = \sum_{p} \left(\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \cdots\right) \log p \\ &\qquad - \sum_{p} \left(\left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots\right) \log p \\ &= \log(n!) - \sum_{p} \left(\left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots\right) \log p.\end{aligned}$$

The weaker form of Stirling's approximation formula $\log(n!) = n\log n - n + O(\log n)$ and Proposition 14 imply that

$$\log \rho_n = n \log n - (c+1)n + O(\sqrt{n}),$$

as required. □

We now turn to estimate $\log \sigma_n$. To do so, we rely on Theorem 10 and a result of Bordellès et al. [1] (already conjectured by Kellner [4]), part of which is recalled below:

Theorem 15 (Corollary 1.6 of [1])**.** *We have*

$$\sum_{\substack{p>\sqrt{n} \\ S_p(n)\geq p}} 1 = \frac{2\sqrt{n}}{\log n} + o\left(\frac{\sqrt{n}}{\log n}\right)$$

as $n \to +\infty$.

Proof of Theorem 12. For a given positive integer n, according to Theorem 10 we have

$$\frac{\sigma_n}{n!} = \prod_{\substack{\sqrt{n+1}<p\leq n+1 \\ S_p(n)\geq p-1}} p = \prod_{\substack{\sqrt{n+1}<p\leq n+1 \\ S_p(n)=p-1}} p \cdot \prod_{\substack{p>\sqrt{n+1} \\ S_p(n)\geq p}} p$$

(remark that $S_p(n) \geq p$ implies $p \leq n$). Thus,

$$\log \sigma_n = \log(n!) + \sum_{\substack{\sqrt{n+1}<p\leq n+1 \\ S_p(n)=p-1}} \log p + \sum_{\substack{p>\sqrt{n+1} \\ S_p(n)\geq p}} \log p. \tag{18}$$

Now, on the one hand, we remark that $n \equiv S_p(n) \pmod{(p-1)}$ (for primes p), so for a prime p satisfying $\sqrt{n+1} < p \leq n+1$, the condition $S_p(n) = p-1$ is equivalent to $(p-1) \mid n$. Consequently,

$$\sum_{\substack{\sqrt{n+1}<p\leq n+1 \\ S_p(n)=p-1}} \log p \leq \sum_{d\mid n} \log(d+1) \leq \tau(n)\log(n+1) = O\left(n^{1/3}\log n\right) \tag{19}$$

(by (1)). On the other hand, by using Theorem 15, we have

$$\sum_{\substack{p>\sqrt{n+1} \\ S_p(n)\geq p}} \log p = \sum_{\substack{p>\sqrt{n} \\ S_p(n)\geq p}} \log p + O\left(\log n\right) = \left(\sum_{\substack{p>\sqrt{n} \\ S_p(n)\geq p}} 1\right)\log n + O\left(\log n\right) = 2\sqrt{n} + o\left(\sqrt{n}\right). \tag{20}$$

Then, by inserting (19) and (20) together with the Stirling approximation formula $\log(n!) = n\log n - n + O(\log n)$ into (18), we conclude that

$$\log \sigma_n = n\log n - n + 2\sqrt{n} + o(\sqrt{n}),$$

as required. □

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References

[1] O. Bordellès, F. Luca, P. Moree, and I. E. Shparlinski, Denominators of Bernoulli polynomials, *Mathematika* **64** (2018), 519–541.

[2] P.-J. Cahen and J.-L. Chabert, *Integer-Valued Polynomials*, Mathematical Surveys and Monographs, **48**, American Mathematical Society, 1997.

[3] B. Farhi, On the derivatives of the integer-valued polynomials, *Funct. Approx. Comment. Math.* **61** (2019), 227–241.

[4] B. C. Kellner, On a product of certain primes, *J. Number Theory* **179** (2017), 149–164.

[5] J.-M. De Koninck and F. Luca, *Analytic Number Theory: Exploring the Anatomy of Integers*, Graduate Studies in Mathematics, Vol. **134**, American Mathematical Society, 2012.

[6] Victor H. Moll, *Numbers and Functions. From a Classical-Experimental Mathematician's Point of View*, Student Mathematical Library, Vol. **65**, American Mathematical Society, 2012.

[7] N. J. A. Sloane et al., *The On-Line Encyclopedia of Integer Sequences*, 2022. Available at https://oeis.org/.

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