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Intrinsic Properties of a Non-Symmetric Number Triangle

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Abstract

Several authors are currently working on generalized Appell polynomials and their applications in the framework of hypercomplex function theory in $\mathbb{R}^{n+1}$. A few years ago, two of the authors of this paper introduced a prototype of these generalized Appell polynomials, which heavily draws on a one-parameter family of non-symmetric number triangles $\mathcal{T}(n)$, $n \geq 2$. In this paper, we prove several new and interesting properties of finite and infinite sums constructed from entries of $\mathcal{T}(n)$, similar to the ordinary Pascal triangle, which is not a part of that family. In particular, we obtain a recurrence relation for a family of finite sums, analogous to the ordinary Fibonacci sequence, and derive its corresponding generating function.

1 Introduction

Almost ten years ago, Falcão and Malonek [7] proved a set of basic properties of a one-parameter family of non-symmetric Pascal triangles. Such family, obtained as result of studies on generalized Appell polynomials in the framework of hypercomplex function theory in $\mathbb{R}^{n+1}$ [6], is given by the infinite array of rational numbers

$$T_s^k(n) = \binom{k}{s} \frac{\left(\frac{n+1}{2}\right)_{k-s} \left(\frac{n-1}{2}\right)_s}{(n)_k}, \ n, k = 1, 2, \ldots, \ s = 0, 1, \ldots, k, \tag{1}$$

where $(a)_r := a(a+1)\cdots(a+r-1)$, for any integer $r \geq 1$ is the Pochhammer symbol with $(a)_0 := 1, a \geq 0$. If $n = 1$, then $T_0^k(1) = 1$ and $T_s^k(1) = 0$, $s > 0$. We let $\mathcal{T}(n)$, $n \geq 2$, denote the family of triangles whose elements are given by (1).

The paper is organized as follows: first we recall several results related to interesting properties of $\mathcal{T}(n)$, specifically those relating to sums of row entries and series along its main diagonal. Next, we highlight the significant role played by sums over entries in the anti-diagonals of $\mathcal{T}(n)$, which are similar to those found in the Fibonacci sequence within the ordinary Pascal triangle. Naturally, the interconnections between all entries of $\mathcal{T}(n)$ and those along the main diagonal rely on several well-known and lesser-known combinatorial identities, which we use extensively throughout the paper. The final result of the paper is some surprising revelation about a property of such an analog of the Fibonacci sequence for $\mathcal{T}(n)$.

2 Preliminary results

Here we recall the most relevant properties of the family $\mathcal{T}(n)$. Relations between adjacent elements of $\mathcal{T}(n)$ were obtained by Falcão and Malonek [7], while Cação et al. [4] derived results concerning the main diagonal elements of the triangle, i.e., the sequence of numbers

$$\mathcal{T}_k(n) := T_k^k(n) = \frac{\left(\frac{n-1}{2}\right)_k}{(n)_k}, \ k = 0, 1, 2, \ldots, \ n = 2, 3, \ldots. \tag{2}$$

The first result [4, Proposition 8] shows the relation between any element $T_s^k(n)$ of the triangle $\mathcal{T}(n)$ and the main diagonal elements $\mathcal{T}_m(n)$, $m = s, s+1, \ldots, k$

Theorem 1. *For $k = 0, 1, 2, \ldots$ and $r = 0, \ldots, k$, we have*

$$T^k_{k-r}(n) = (-1)^r \binom{k}{r} \sum_{s=0}^{r} \binom{r}{s} (-1)^s \mathcal{T}_{k-s}(n). \tag{3}$$

The next result [4, Proposition 11] concerns a recurrence satisfied by the sequence $(D_k(n))_{k\geq 0}$ consisting of alternating partial sums of the main diagonal elements, i.e.,

$$D_k(n) := \sum_{s=0}^{k} (-1)^s \mathcal{T}_s(n), \tag{4}$$

whose first few elements are

$$1, \frac{n+1}{2n}, \frac{3n+1}{4n}, \frac{(n+1)(5n+7)}{8n(n+2)}, \frac{n(11n+28)+9}{16n(n+2)}, \frac{(n+1)(7n(3n+16)+107)}{32n(n+2)(n+4)}, \frac{n(n(43n+281)+485)+151}{64n(n+2)(n+4)}, \ldots.$$

Theorem 2. *The sequence $(D_k(n))_{k\geq 0}$, satisfies the recurrence relation*

$$(n+1)D_{k+1}(n) - 2(k+n+1)D_{k+2}(n) + (2k+n+1)D_k(n) = 0, \tag{5}$$

with initial conditions

$$D_0(n) = 1, \ D_1(n) = \frac{n+1}{2n}.$$

We end this section by observing that the triangular array $T^k_s(n)$ can be easily written as a scaled integer triangle, for some values of n. In fact, it is easy to show, using the well-known properties of the Pochhammer symbol, that

$$T^k_s(2n+1) = \frac{\binom{n+k-s}{n}\binom{n+s-1}{n-1}}{\binom{k+2n}{2n}}.$$

Therefore, for odd values of n, the entries of the triangle $\binom{k+n-1}{n-1} T^k_s(n)$ are integers, as illustrated in Table 1.

Triangles	First rows				OEIS link
$\binom{k+2}{2} T^k_s(3)$	1				triangle A004736
	2	1			
	3	2	1		
	4	3	2	1	
$\binom{k+4}{4} T^k_s(5)$	1				triangle A104633
	3	2			
	6	6	3		
	10	12	9	4	
$\binom{k+6}{6} T^k_s(7)$	1				triangle A103252
	4	3			
	10	12	6		
	20	30	24	10	

Table 1: Some particular scaled triangles obtained from $\mathcal{T}(n)$, for odd n.

When n is even, determining the scale factor becomes more challenging (Cação et al. [3] provide further details). Table 2 illustrates the particular cases $n = 2$ and $n = 4$, as well as the limiting case $n = \infty$.

Triangles	First rows				OEIS link
$2^{m_k} T_s^k(2)$	1				$m_k =$ A283208
	3	1			
	5	2	1		
	35	15	9	5	
$2^{M_k} T_s^k(4)$	1				$M_{k-1} = m_{k-1} +$ A050605
	5	3			
	7	6	3		
	21	21	15	7	
$2^k T_s^k(\infty)$	1				Pascal triangle A007318
	1	1			
	1	2	1		
	1	3	3	1	

Table 2: Other integer triangles obtained by scaling $\mathcal{T}(n)$.

3 Filling in the gap: Fibonacci-like sequences

The study conducted by Falcão and Malonek [7] did not take into consideration the relationships among the entries located along the rising diagonals of $\mathcal{T}(n)$ (also called northeast diagonals [10] or anti-diagonals [2]). The aim of this paper is to fill in this gap with a surprising result presented in Theorem 3. From the qualitative point of view, it shows once more the particular nature of $\mathcal{T}(n)$ as a combinatorial object arising from generalized Appell polynomials in hypercomplex analysis.

Consider the triangle $\mathcal{L}(n)$ obtained after re-indexing of the triangle $\mathcal{T}(n)$ as illustrated in Figure 1. The elements $L_s^k(n)$ of $\mathcal{L}(n)$ are given by

$$L_s^k(n) := T_s^{k-s}(n), \; k = 0, 1, 2, \ldots, \; s = 0, 1, \ldots, \lfloor \tfrac{k}{2} \rfloor, \; n = 2, 3, \ldots. \tag{6}$$

Table 3 shows the first few lines of the array $\mathcal{L}(n)$, where each line has pairwise repeating lengths. In the limiting case $n = \infty$, one can recognize well-known scaled triangles, as illustrated in Table 4.

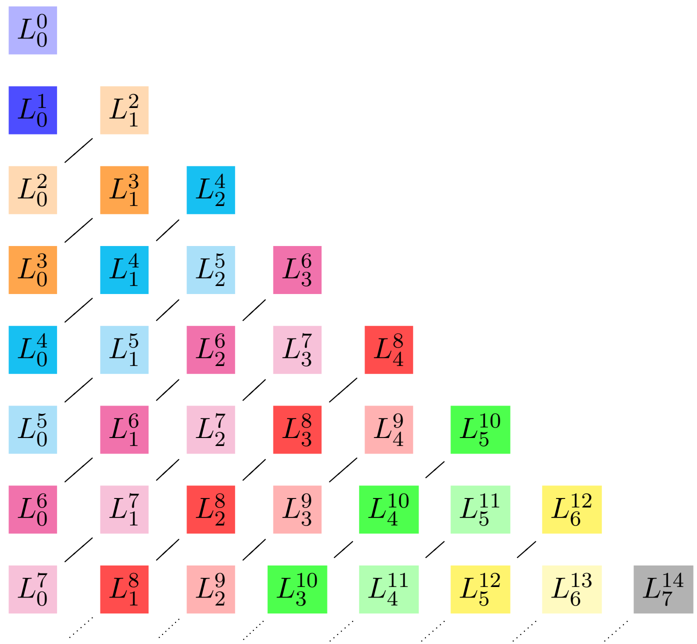

Figure 1: A re-indexing of the triangle $\mathcal{T}(n)$.

$L_s^k(n)$	$s=0$	$s=1$	$s=2$	$s=3$	$\dots$
$k=0$	1				
$k=1$	$\frac{n+1}{2n}$				
$k=2$	$\frac{n+3}{2^2 n}$	$\frac{n-1}{2n}$			
$k=3$	$\frac{(n+3)(n+5)}{2^3 n(n+2)}$	$\frac{2(n-1)}{2^2 n}$			
$k=4$	$\frac{(n+5)(n+7)}{2^4 n(n+2)}$	$\frac{3(n-1)(n+3)}{2^3 n(n+2)}$	$\frac{n-1}{2^2 n}$		
$k=5$	$\frac{(n+5)(n+7)(n+9)}{2^5 n(n+2)(n+4)}$	$\frac{4(n-1)(n+5)}{2^4 n(n+2)}$	$\frac{3(n^2-1)}{2^3 n(n+2)}$		
$k=6$	$\frac{(n+7)(n+9)(n+11)}{2^6 n(n+2)(n+4)}$	$\frac{5(n-1)(n+5)(n+7)}{2^5 n(n+2)(n+4)}$	$\frac{6(n^2-1)}{2^4 n(n+2)}$	$\frac{(n-1)(n+3)}{2^3 n(n+2)}$	
$k=7$	$\frac{(n+7)(n+9)(n+11)(n+13)}{2^7 n(n+2)(n+4)(n+6)}$	$\frac{6(n-1)(n+7)(n+9)}{2^6 n(n+2)(n+4)}$	$\frac{10(n-1)(n+1)(n+5)}{2^5 n(n+2)(n+4)}$	$\frac{4(n^2-1)}{2^4 n(n+2)}$	
$\vdots$					

Table 3: Values of $L_s^k(n)$ in the first few rising diagonals of the array $\mathcal{T}(n)$.

Triangles	First rows	OEIS link
$2^k L_s^k(\infty)$	1 1 1 2 1 4 1 6 4 1 8 12 1 10 24 8 1 12 40 32 1 14 60 80 16	A128099 Row sums are the Jacobsthal numbers A001045
$2^{k-s} L_s^k(\infty)$	1 1 1 1 1 2 1 3 1 1 4 3 1 5 6 1 1 6 10 4 1 7 15 10 1	A011973 triangle of coefficients of (one version of) Fibonacci polynomials

Table 4: Some particular scaled triangles obtained from $\mathcal{L}(\infty)$.

One of the most interesting patterns of the ordinary Pascal triangle is that the sum of the elements of its anti-diagonals gives rise to the Fibonacci sequence. Following this idea, we construct the sequence $(S_k(n))_{k\geq 0}$

$$S_k(n) := \sum_{s=0}^{\lfloor k/2 \rfloor} L_s^k(n), \tag{7}$$

consisting of the sum of the elements $L_s^k(n)$ in the k-th rising diagonal of the array $\mathcal{T}(n)$.

The next result shows that the sequence $(S_k(n))_{k\geq 0}$, built according to the construction of the Fibonacci sequence from the elements of the ordinary Pascal triangle, is identical to the sequence $(D_k(n))_{k\geq 0}$, built of the alternating partial sums of the main diagonal elements of the triangle $\mathcal{T}(n)$.

Theorem 3. *The sequence $(S_k(n))_{k\geq 0}$ given by (7) is identical to the sequence $(D_k(n))_{k\geq 0}$, given by (4).*

Proof. We first assume that k is even, i.e., $k = 2m$ ($m \in \mathbb{N}_0$). Then, from (6) and (7) we have

$$S_{2m}(n) = \sum_{s=0}^{m} T_s^{2m-s} = \sum_{\ell=0}^{m} T_{m-\ell}^{m+\ell}.$$

We can use (3) with $k = m + \ell$ and $r = 2\ell$ to obtain

$$\begin{aligned}S_{2m}(n) &= \sum_{\ell=0}^{m} \binom{m+\ell}{2\ell} \sum_{s=0}^{2\ell} (-1)^s \binom{2\ell}{s} \mathcal{T}_{m+\ell-s}(n) \\ &= \sum_{\ell=0}^{m} \binom{m+\ell}{2\ell} \sum_{s=-\ell}^{\ell} (-1)^{\ell-s} \binom{2\ell}{\ell-s} \mathcal{T}_{m+s}(n).\end{aligned}$$

Reversing the order of summation, we get

$$S_{2m}(n) = \sum_{s=-m}^{m} (-1)^s \mathcal{T}_{m+s}(n) \Sigma_{s,m}$$

where

$$\Sigma_{s,m} := \sum_{\ell=|s|}^{m} (-1)^{\ell} \binom{m+\ell}{2\ell} \binom{2\ell}{\ell-s}.$$

Using the formula (cf. [8, Formula 5.21]) and replacing (r, m, k) by $(m+\ell, 2\ell, \ell-s)$ we can express $\Sigma_{s,m}$ as

$$\Sigma_{s,m} = \sum_{\ell=|s|}^{m} (-1)^{\ell} \binom{m+\ell}{\ell-s} \binom{m+s}{\ell+s} = \sum_{\ell=|s|}^{m} (-1)^{\ell} \binom{m+s}{s+\ell} \binom{m+\ell}{m+s}.$$

We now use the relation

$$\sum_{k} (-1)^k \binom{\ell}{m+k} \binom{s+k}{n} = (-1)^{\ell+m} \binom{s-m}{n-\ell}, \;\; \ell \in \mathbb{N}_0, \;\; m, n \in \mathbb{Z}$$

(cf. [8, Formula 5.24]) with (k, ℓ, m, n, s) replaced by $(\ell, m+s, s, m+s, m)$ to conclude that

$$\Sigma_{s,m} = (-1)^m,$$

which leads to

$$S_{2m}(n) = \sum_{s=-m}^{m} (-1)^{m+s} \mathcal{T}_{m+s}(n) = \sum_{s=0}^{2m} (-1)^s \mathcal{T}_s(n) = D_{2m}.$$

The case where k is odd can be handled using similar arguments. □

By combining the relation (5) with the coincidence of both sequences $(S_k(n))_{k\geq 0}$ and $(D_k(n))_{k\geq 0}$, we get immediately that the sequence $(S_k(n))_{k\geq 0}$, considered as the analog(7) of the Fibonacci sequence, can be characterized by a second-order recurrence with variable coefficients as follows:

Corollary 4. *For any integer $n \geq 2$ the elements $S_k(n)$ are recursively defined by*

$$S_{k+1}(n) = \frac{n+1}{2(k+n)} S_k(n) + \frac{2k+n-1}{2(k+n)} S_{k-1}(n), \text{ for } k \geq 1, \tag{8}$$

with initial conditions

$$S_0(n) = 1, \quad S_1(n) = \frac{n+1}{2n}. \tag{9}$$

The recurrence (8), which connects three consecutive terms of the sequence $(S_k(n))_{k\geq 0}$, enables the derivation of a differential equation, whose solution is the ordinary power series generating function of that sequence.

Theorem 5. *The generating function of the sequence $(S_k(n))_{k\geq 0}$, $n = 2, 3, \ldots$ can be written in terms of the Gauss hypergeometric function as*

$$F_n(x) = {}_2F_1(1, 1; 1; x)\ {}_2F_1(1, \tfrac{n-1}{2}; n; -x) = \frac{{}_2F_1(1, \frac{n-1}{2}; n; -x)}{1-x}.$$

Proof. Let F_n be the ordinary power series generating function of the sequence $(S_k(n))_{k\geq 0}$, i.e.,

$$F_n(x) = \sum_{k=0}^{\infty} S_k(n) x^k.$$

We rewrite the recurrence relation (8) in the following form:

$$(3 - n - 2k) S_{k-2}(n) - (1+n) S_{k-1}(n) + 2(n-1) S_k(n) + 2k S_k(n) = 0.$$

Then, we multiply by x^k and sum over $k \geq 2$ to obtain

$$(3-n)\sum_{k=2}^{\infty} S_{k-2}(n)x^k - 2\sum_{k=2}^{\infty} k S_{k-2}(n) x^k \\ - (1+n)\sum_{k=2}^{\infty} S_{k-1}(n)x^k + 2(n-1)\sum_{k=2}^{\infty} S_k(n)x^k + 2\sum_{k=2}^{\infty} k S_k(n) x^k = 0,$$

or, equivalently,

$$(3-n)x^2 F_n(x) - 2x^3 F_n'(x) - 4x^2 F_n(x) - (1+n)(xF_n(x) + S_0(n)) \\ + 2(n-1)(F_n(x) - S_1(n)x - S_0(n)) + 2(xF_n'(x) - xS_1(n)) = 0.$$

By using the initial conditions (9) in the above equation, we get the first order linear differential equation

$$F_n'(x) + \frac{1}{x-1}\left(\frac{n+1}{2} - \frac{n-1}{x^2+x}\right) F_n(x) = \frac{n-1}{x-x^3}. \tag{10}$$

The use of the standard integrating factor method for solving (10) leads to

$$F_n(x) = \frac{(n-1)(1+x)^{\frac{n-1}{2}}x^{1-n}}{1-x}\int x^{n-2}(1+x)^{-\frac{n+1}{2}}\,dx. \tag{11}$$

Since $(1+x)^{-a} = {}_2F_1(1,a;1;-x)$ (cf. [1, Formula 15.1.8]), we have

$$\begin{aligned}\int x^{n-2}(1+x)^{-\frac{n+1}{2}}\,dx &= \int x^{n-2}\sum_{k=0}^{\infty}\left(\frac{n+1}{2}\right)_k\frac{(-x)^k}{k!}\\ &= \frac{x^{n-1}}{n-1}\sum_{k=0}^{\infty}\frac{(n-1)_k\left(\frac{n+1}{2}\right)_k}{(n)_k}\frac{(-x)^k}{k!}\\ &= \frac{x^{n-1}}{n-1}\,{}_2F_1\left(n-1,\frac{n+1}{2};n;-x\right). \end{aligned}\tag{12}$$

Combining (11) and (12) we get

$$F_n(x) = \frac{(1+x)^{\frac{n-1}{2}}\,{}_2F_1(n-1,\frac{n+1}{2};n;-x)}{1-x}.$$

The result follows by applying Euler's transformation

$${}_2F_1(a,b;c;-x) = (1+x)^{c-a-b}\,{}_2F_1(c-a,c-b;c;-x).$$

□

We observe that the ODE (10) can be written as

$$\frac{2}{n-1}F_n'(x) + \frac{1}{x-1}\Big(\frac{n+1}{n-1} - \frac{2}{x^2+x}\Big)F_n(x) = \frac{2}{x-x^3},$$

which gives in the limit case $n=\infty$,

$$F_\infty(x) = -\frac{2}{x^2+x-2}.$$

Notice that $F_\infty(x)$ is also the generating function of the sequence $(2^{-k}J_{k+1})_{k\geq 0}$, where J_k denotes Jacobsthal numbers. In fact,

$$F_\infty(x) = \frac{2}{3}\Big(\frac{1}{1-x} + \frac{1}{2+x}\Big) = \sum_{k=0}^{\infty}\frac{1}{3}\left(2+\Big(-\frac{1}{2}\Big)^k\right)x^k,$$

and taking into account the Binet form of the Jacobsthal numbers $J_k = \frac{1}{3}\left(2^k - (-1)^k\right)$ (cf. [10]), we obtain

$$F_\infty(x) = \sum_{k=0}^{\infty}2^{-k}J_{k+1}x^k,$$

i.e., $S_k(\infty) = 2^{-k} J_{k+1}$. Furthermore, we immediately conclude that

$$\lim_{k\to\infty} S_k(\infty) = \frac{2}{3}.$$

We can obtain explicit expressions for $F_n(x)$ for certain values of n, by using properties of the Gauss hypergeometric function.

- $F_2(x) = \dfrac{2\big(-1+\sqrt{1+x}\big)}{(1-x)x}$
- $F_3(x) = \dfrac{-2x + 2(1+x)\log(1+x)}{(1-x)x^2}$
- $F_4(x) = \dfrac{2\left(8 - 8\sqrt{1+x} + x\left(12 + 3x - 8\sqrt{1+x}\right)\right)}{(1-x)x^3}$
- $F_5(x) = \dfrac{-2x\,(6 + x(9+2x)) + 12(1+x)^2\log(1+x)}{(-1+x)x^4}$

Finally, we refer to some limiting properties of $S_k(n)$.

Theorem 6. *Consider the sequence* $(S_k(n))_{k\geq 0}$, $n = 2, 3, \ldots$. *Then*

$$\lim_{k\to\infty} S_k(n) = {}_2F_1\left(1, \tfrac{n-1}{2}; n; -1\right).$$

Proof. The result is an immediate consequence of the Theorem 3, along with equations (4) and (2). In fact,

$$S_k(n) = D_k(n) = \sum_{s=0}^{k} (-1)^s \mathcal{T}_s(n) = \sum_{s=0}^{k} (-1)^s \frac{\left(\frac{n-1}{2}\right)_s}{(n)_s} = \sum_{s=0}^{k} \frac{(-1)^s}{s!} \frac{(1)_s \left(\frac{n-1}{2}\right)_s}{(n)_s},$$

which proves the desired result.

It is worth noting that the hypergeometric series ${}_2F_1(1, \frac{n-1}{2}; n; -1)$ converges absolutely [1], since $n - 1 - \frac{n-1}{2} > 0$. □

It is well known that the ratio of two consecutive ordinary Fibonacci numbers converges to the golden ratio. Here the corresponding property can be obtained as an immediate consequence of Theorem 6. Specifically, we have

$$\lim_{k\to\infty} \frac{S_{k+1}(n)}{S_k(n)} = 1.$$

4 Final remarks

It was initially unclear what kind of result could be expected from constructing an analog of the Fibonacci sequence using hypercomplex tools and following the usual rules of the ordinary Pascal triangle. Specifically, we were uncertain about the result of summing the rising diagonal elements of $\mathcal{T}(n)$. This gap was left unaddressed in our previous work [7]. However, as explained in Section 3, Theorem 3 provides the answer to this question and leads to the main recurrence relation given in (8)-(9).

Concluding our final remarks, we would like to highlight the potential for fruitful and interesting connections between real, complex, and hypercomplex analysis when considering hypercomplex polynomials [5, 9, 11] from the specific discrete viewpoint of combinatorial relations.

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References

[1] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, US Government Printing Office, 10th printing, 1972.

[2] L. Abrams, D. E. Fishking, and S. Valdes-Leon, Reflecting the Pascal matrix about its main antidiagonal, *Linear Multilinear Algebra* **47** (2000), 129–136.

[3] I. Cação, M. I. Falcão, and H. R. Malonek, Hypercomplex polynomials, Vietoris' rational numbers and a related integer numbers sequence, *Complex Anal. Oper. Theory* **11** (2017), 1059–1076.

[4] I. Cação, M. I. Falcão, H. R. Malonek, and G. Tomaz, Non-symmetric number triangles arising from hypercomplex function theory in $\mathbb{R}^{n+1}$, in O. Gervasi, B. Murgante, et al., eds., *Computational Science and Its Applications—ICCSA 2022 Workshops*, Lect. Notes in Comp. Sci., Vol. 13377, Springer, 2022, pp. 420–434.

[5] R. Delanghe, F. Sommen, and V. Souček, Clifford algebra and spinor-valued functions: a function theory for the Dirac operator, *Mathematics and its Applications*, Vol. 53, Kluwer Academic, 1992.

[6] M. I. Falcão, J. Cruz, and H. R. Malonek, Remarks on the generation of monogenic functions, in K. Gürlebeck and C. Könke, eds., *17th Inter. Conf. on the Appl. of Computer Science and Mathematics on Architecture and Civil Engineering*, Germany, 2006. Available at https://e-pub.uni-weimar.de/opus4/files/2939/f81_pdfa.pdf.

[7] M. I. Falcão and H. R. Malonek, A note on a one-parameter family of non-symmetric number triangles, *Opuscula Math.* **32** (2012), 661–673.

[8] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, Addison-Wesley Longman, 1994.

[9] K. Gürlebeck, K. Habetha, and W. Sprößig, *Application of Holomorphic Functions in Two or Higher Dimensions*, Birkhäuser Verlag, 2016.

[10] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley, 2019.

[11] H. R. Malonek, I. Cação, M. I. Falcão, and G. Tomaz, Harmonic analysis and hypercomplex function theory in co-dimension one, in A. Karapetyants, V. Kravchenko, and E. Liflyand, eds., *Modern Methods in Operator Theory and Harmonic Analysis—OTHA 2018*, Springer Proc. Math. Stat., Vol. 291, Springer, 2019, pp. 93–115.

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