Periodic Weighted Sums of Binomial Coefficients

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Abstract

Using elementary methods, we establish old and new relations between binomial coefficients, Fibonacci numbers, Catalan numbers, Pell numbers, and more.

1 Introduction

We present a general theorem on linear recurrences and we use it to produce results both old and new about weighted sums of binomial coefficients. One highlight is this first equation, which connects the Fibonacci numbers, the binomial coefficients, and the Legendre symbol:

$$F_{2n} = \sum_{k=0}^{n} \left(\frac{k}{5}\right)\binom{2n}{n+k} = \frac{1}{5^n}\sum_{k=0}^{2n}(-1)^{k+1}\left(\frac{k}{5}\right)\binom{4n}{2n+k}. \tag{1}$$

The first equality can also be derived from a more complicated expression of Andrews [1] from 1969, but the second equality is new.

Also of interest are the non-negative integer solutions (X_n, Y_n) to the Pell equation $X^2 - 3Y^2 = 1$. We show that

$$2X_n + 2^n = 3 \sum_{j=-n}^{n} (-1)^j \binom{2n}{n+6j} \tag{2}$$

and also that

$$Y_n = \binom{2n}{n+1} - \binom{2n}{n+5} - \binom{2n}{n+7} + \binom{2n}{n+11} + \binom{2n}{n+13} - \cdots \tag{3}$$

and while the formula for X_n follows from Merca [10], the formula for Y_n is new.

To provide some background, we note that we began this research project by looking at the Binet formula for F_{2n}, which is

$$F_{2n} = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^{2n} - \left(\frac{1-\sqrt{5}}{2} \right)^{2n} \right). \tag{4}$$

Since $(1+\sqrt{5})/2 = 2\cos \pi/5$ and $(1-\sqrt{5})/2 = 2\cos 3\pi/5$, then the above equation becomes

$$F_{2n} = \frac{1}{\sqrt{5}} ((2\cos \pi/5)^{2n} - (2\cos 3\pi/5)^{2n}), \tag{5}$$

and we then realized that we could apply a cosine formula to expand the above equation into a more interesting form, which eventually gave us the first part of equation (1). After applying this same technique to other sequences such as $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ mentioned earlier, we soon recognized that the Binet formula for these sequences was not necessary; in each case we only needed the linear recurrence formula itself.

In what follows, we will establish a general theorem which covers a large class of linear recurrence sequences. To be precise, so long as the coefficients of the linear recurrence satisfy an easily-verified property, then our theorem will give us an identity involving a weighted sum of binomial coefficients as seen in equations (1), (2), and (3) in which the weights, typically 1 and -1, appear in a pattern (hence the title of our paper). Sometimes this periodic pattern of weights matches up with a Legendre or Kronecker symbol as seen in equation (1), but other times as seen in equation (3) the best way to express the pattern is to write it out directly. Furthermore, the values of these periodic weights depend *only* on the initial values of the linear recurrence sequence. We then use this theorem to derive new identities for the Fibonacci numbers, the Pell numbers, the Catalan numbers, and more. Many of our identities now appear in the On-Line Encyclopedia of Integer Sequences (OEIS) [13].

2   Main result

To motivate some definitions, let us return back to our numbers F_{2n}, which satisfy the equation

$$F_{2n} = 3F_{2(n-1)} - 1F_{2(n-2)} \qquad \text{for } n \geq 2. \tag{6}$$

We say that $(F_{2n})_{n\geq 0}$ is a *linear recurrence sequence*, and since each term in equation (6) is defined by its two previous terms, we say that the sequence has *order* 2. The set of coefficients on the right of equation (6), in this case $\{3, -1\}$, is called the *signature* of this linear recurrence. From equation (6) we now define the *characteristic polynomial* for the sequence $(F_{2n})_{n\geq 0}$ to be

$$x^2 - (3x - 1).$$

Of course, all these definitions generalize nicely to linear recurrences of arbitrary order.

Now, one of the roots of our characteristic polynomial is $((1 + \sqrt{5})/2)^2 = (2\cos\pi/5)^2$. Another way of expressing this is to say that $x^2 - (3x - 1)$ is the *minimal polynomial* for $(2\cos\pi/5)^2$.

With this in mind, we are ready to present our main theorem.

Theorem 1. *Suppose we have a linear recurrence sequence $(A_n)_{n\geq 0}$ such that its characteristic polynomial is the minimal polynomial for $(2\cos\pi/q)^2$ for some integer $q \geq 3$. Then, we have*

$$A_n = A_0 \binom{2n}{n} + \sum_{k=1}^{n} \omega_k \binom{2n}{n+k}, \tag{7}$$

where we define ω_k in terms of $A_0, A_1, A_2, \ldots$ as follows: $\omega_0 = 2A_0$, and

$$\omega_k = \sum_{j=0}^{k} (-1)^j \binom{2k-j}{j} \frac{2k}{2k-j} A_{k-j} \qquad \text{for } k \geq 1. \tag{8}$$

Furthermore, we have that $\omega_{k'} = \omega_k$ for $k' \equiv \pm k \pmod{q}$, and if q is even then $\omega_{k'} = -\omega_k$ for $k' \equiv q/2 - k \pmod{q}$.

For the proof, see Section 4 at the end of this article.

We will find it useful to write out the first few values of ω_k. We use equation (8), along with our definition of ω_0 as $2A_0$, to get the values in Table 1.

$$\omega_0 = 2A_0$$
$$\omega_1 = A_1 - 2A_0$$
$$\omega_2 = A_2 - 4A_1 + 2A_0$$
$$\omega_3 = A_3 - 6A_2 + 9A_1 - 2A_0$$

Table 1: Initial values of ω_k from Theorem 1.

The coefficients of A_n in Table 1 appear in reverse order as sequence A127677 in the OEIS.

Now, Theorem 1 applies to a large class of linear recurrences. One such example that we will see in Section 3.4 is the third-order sequence

$$A_n = 5A_{n-1} - 6A_{n-2} + 1A_{n-3}$$

with signature $\{5, -6, 1\}$ and characteristic polynomial $x^3 - (5x^2 - 6x + 1)$, because that polynomial is the minimal polynomial for $(2\cos \pi/q)^2$ with $q = 7$. For convenience, we list in Table 2 all such cases for linear recurrences with order no more than three, indexed by the corresponding values of q.

q	Signature	q	Signature
5	$\{3, -1\}$	10	$\{5, -5\}$
7	$\{5, -6, 1\}$	12	$\{4, -1\}$
8	$\{4, -2\}$	14	$\{7, -14, 7\}$
9	$\{6, -9, 1\}$	18	$\{6, -9, 3\}$

Table 2: All signatures of length 2 or 3 for minimal polynomials for $(2\cos \pi/q)^2$.

Thus, for any entry in the OEIS with one of these signatures, we can write the terms in that sequence as a weighted sum of binomial coefficients as given in equation (7) from Theorem 1. We can do the same for signatures of length 4 or more, but those are not as interesting. Let us now look at some examples.

3 Applications

3.1 The Fibonacci numbers

As mentioned above, the sequence $(F_{2n})_{n\geq 0}$ has signature $\{3, -1\}$ which is the first entry in our Table 2. This gives us the following formula for F_{2n} as a weighted sum of binomial coefficients.

Theorem 2. *For F_n the Fibonacci numbers, we have*

$$F_{2n} = \sum_{k=0}^{n} \binom{2n}{n+k}\left(\frac{k}{5}\right), \tag{9}$$

where $\left(\dfrac{k}{5}\right)$ represents the Legendre symbol.

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Remark 3. As pointed out by our helpful referee, the appearance of the Legendre symbol in Theorem 2 should not come as a complete surprise. If we were to apply the fairly well-known cosine equation (32) directly to our Binet formula (5) for F_{2n}, we would get

$$F_{2n} = \frac{1}{\sqrt{5}} \sum_{k=1}^{n} \binom{2n}{n+k} \left(2 \cos 2k\pi/5 - 2 \cos 6k\pi/5\right). \tag{10}$$

For k a multiple of 5, we have the expression $(2 \cos 2k\pi/5 - 2 \cos 6k\pi/5)$ equals 0, but otherwise it has the same values as the quadratic Gauss sum

$$g(k;5) = \sum_{n=0}^{4} e^{2\pi i k n^2/5}.$$

From a theorem by Berndt, Evans, and Williams [3, Theorem 1.5.2], we learn that this satisfies

$$g(k;5) = \left(\frac{k}{5}\right)\sqrt{5} \qquad \text{for } k \text{ not a multiple of 5,}$$

and so using this in equation (10) would give us a direct proof of our Theorem 2.

Proof of Theorem 2. If we set $A_n = F_{2n}$ then A_n satisfies the recurrence $A_n = 3A_{n-1} - A_{n-2}$. Thanks to Table 2, we can apply Theorem 1 with $q = 5$ and with initial values $A_0 = 0$ (because $F_{2n} = 0$ at $n = 0$) and $A_1 = 1$ (because $F_{2n} = 1$ at $n = 1$). From Table 1 we have the following values for ω_k:

$$\omega_0 = 2A_0 = 0,$$
$$\omega_1 = A_1 - 2A_0 = 1$$
$$\omega_2 = A_2 - 4A_1 + 2A_0 = -1$$

Theorem 1 also tells us that $\omega_k = \omega_{5-k} = \omega_{5+k}$. Using this, along with the three values already given above, we conclude that $\omega_k = 1$ for $k \equiv 1, 4 \pmod{5}$, and $\omega_k = -1$ for $k \equiv 2, 3 \pmod{5}$, and $\omega_k = 0$ for $k \equiv 0 \pmod{5}$. This means that ω_k has the same values as the Legendre symbol $\left(\frac{k}{5}\right)$, and since we also have $A_0 = 0$ then equation (7) of Theorem 1 gives us our desired equation (9) for $A_n = F_{2n}$. $\qquad\square$

As we mentioned above, our equation (9) for the Fibonacci numbers is not entirely new. Andrews [1] used complex numbers to show that

$$F_n = \sum_{\alpha=-\infty}^{\infty} (-1)^\alpha \binom{n}{\lfloor (n-1-5\alpha)/2 \rfloor},$$

where $\lfloor\ \rfloor$ represents the greatest integer function. From this, we can obtain our equation (9) with a bit of work. However, our method is both more general and more direct.

3.2 Binomial transforms of the Pell and Pell-Lucas numbers

The sequence $(F_{2n})_{n\geq 0}$ from the previous section has the nice property that it is the *binomial transform* of the "regular" Fibonacci sequence. In other words,

$$F_{2n} = \sum_{i=0}^{n} \binom{n}{i} F_i.$$

The following theorem will allow us to consider two other sequences of numbers that are also binomial transforms.

Theorem 4. *For $A_n = 4A_{n-1} - 2A_{n-2}$ with initial values A_0 and A_1, we have*

$$A_n = A_0 \binom{2n}{n} + \sum_{k=1}^{n} \omega_k \binom{2n}{n+k}, \tag{11}$$

with ω_k repeating modulo 8 as given below in Table 3, with $\omega_0 = 2A_0$ and $\omega_1 = A_1 - 2A_0$.

$k \pmod 8$	0	1	2	3	4	5	6	7	8
ω_k	ω_0	ω_1	0	$-\omega_1$	$-\omega_0$	$-\omega_1$	0	ω_1	ω_0

Table 3: Values of ω_k for signature $\{4, -2\}$.

Proof. Since the signature $\{4, -2\}$ appears in Table 2, we can apply Theorem 1 with $q = 8$. The values for ω_0 and ω_1 come to us from Table 1, and the remaining values of ω_k follow from the conclusion of Theorem 1 which tells us that $\omega_k = \omega_{8\pm k}$ and $\omega_k = -\omega_{4-k}$. $\qquad \square$

With this theorem in hand, we can now produce two results, one new and one old, about the binomial transforms of the Pell and the Pell-Lucas numbers.

Corollary 5. *For P_n the Pell numbers $0, 1, 2, 5, 12, 29, \ldots$ from [A000129](#), we have*

$$\sum_{i=0}^{n} \binom{n}{i} P_i = \sum_{k=0}^{n} \left(\frac{k}{8}\right) \binom{2n}{n+k} \tag{12}$$

where $\left(\dfrac{k}{8}\right)$ represents the Kronecker symbol.

Proof. If we set A_n equal to the binomial transform of P_n as seen on the left of equation (12), then the sequence $(A_n)_{n\geq 0}$ begins with $0, 1, 4, 14, 48, 164, \ldots$ and is given by [A007070](#) where we also learn that it has signature $\{4, -2\}$. Hence, we can apply Theorem 4 with $\omega_0 = 2A_0 = 0$ and $\omega_1 = A_1 - 2A_0 = 1$. Furthermore, Table 3 from Theorem 4 becomes

$k \pmod 8$	0	1	2	3	4	5	6	7	8
ω_k	0	1	0	-1	0	-1	0	1	0

and from this we see that $\omega_k = \left(\dfrac{k}{8}\right)$, as desired. $\qquad\square$

Corollary 6. *For Q_n the Pell-Lucas numbers $2, 2, 6, 14, 34, 81, \ldots$ from* [A002203](#)*, we have*

$$\sum_{i=0}^{n}\binom{n}{i}Q_i = 2\binom{2n}{n} + 4\sum_{j\geq 1}(-1)^j\binom{2n}{n+4j}. \tag{13}$$

Proof. The binomial transform of Q_n is $2, 4, 12, 40, 136, \ldots$, which is the sequence [A056236](#) with signature $\{4, -2\}$. Hence, if we let A_n equal the binomial transform of Q_n as seen on the left of equation (13), then we can again apply Theorem 4, this time with $\omega_0 = 2A_0 = 4$ and $\omega_1 = A_1 - 2A_0 = 0$. Thus, Table 3 from Theorem 4, starting at $k = 1$, becomes

$k \pmod 8$	1	2	3	4	5	6	7	8
ω_k	0	0	0	-4	0	0	0	4

and from this we obtain our desired formula. $\qquad\square$

We note that Corollary 6 is not a new result. From Merca [10, Corollary 8] we have the formula

$$(2+\sqrt{2})^n + (2-\sqrt{2})^n = 2\binom{2n}{n} + 4\sum_{j\geq 1}(-1)^j\binom{2n}{n+4j},$$

and as seen in [A056236](#) the binomial transform of Q_n is indeed equal to $(2+\sqrt{2})^n + (2-\sqrt{2})^n$.

As a special case of Corollary 6, we have the following relationship between a weighted sum of binomial coefficients (on the left) and a periodic weighted sum of binomial coefficients (on the right).

Corollary 7. *For Q_n the Pell-Lucas numbers $2, 2, 6, 14, 34, 81, \ldots$ as given by* [A002203](#)*, we have*

$$\sum_{i=0}^{4n}\binom{4n}{i}Q_i = 2(-1)^n\sum_{j=0}^{2n}(-1)^j\binom{8n}{4j}. \tag{14}$$

Proof. We can split up the right-hand side of equation (14) as follows:

$$2(-1)^n\sum_{j=0}^{n-1}(-1)^j\binom{8n}{4j} + 2\binom{8n}{4n} + 2(-1)^n\sum_{j=n+1}^{2n}(-1)^j\binom{8n}{4j}.$$

In the first sum we replace j with $n - j$, and in the second sum we replace j with $n + j$, giving us

$$2(-1)^n \sum_{j=1}^{n} (-1)^{n-j} \binom{8n}{4n - 4j} + 2\binom{8n}{4n} + 2(-1)^n \sum_{j=1}^{n}(-1)^{n+j} \binom{8n}{4n + 4j}.$$

We now use the symmetry of the binomial coefficients to combine the two sums to obtain

$$2\binom{8n}{4n} + 4 \sum_{j \geq 1}(-1)^j \binom{8n}{4n + 4j},$$

and by equation (13) in Corollary 6 this is equal to $\sum_{i=0}^{4n} \binom{4n}{i}Q_i$, as desired. $\qquad\square$

3.3 Solutions to Pell's equation

The non-negative integer solutions (X_n, Y_n) to the Pell equation $X^2 - 3Y^2 = 1$ are given by A001075 for X_n, and A001353 for Y_n, where the first sequence begins $1, 2, 7, 26, 97, \ldots$, and the second is $0, 1, 4, 15, 56, \ldots$. These are well-known linear recurrence sequences, and both of them have signature $\{4, -1\}$ which corresponds to $q = 12$ in Table 2. With this in mind, we present the following theorem and then we show how it gives us the equations (2) and (3) for X_n and Y_n as mentioned in the introduction to this paper.

Theorem 8. *For $A_n = 4A_{n-1} - A_{n-2}$ with initial values A_0 and A_1, we have*

$$A_n = A_0 \binom{2n}{n} + \sum_{k=1}^{n} \omega_k \binom{2n}{n + k}, \tag{15}$$

with ω_k repeating modulo 12 as given below in Table 4, with $\omega_1 = A_1 - 2A_0$ and $\omega_2 = A_0$.

$k \pmod{12}$	0	1	2	3	4	5	6	7	8	9	10	11	12
ω_k	$2\omega_2$	ω_1	ω_2	0	$-\omega_2$	$-\omega_1$	$-2\omega_2$	$-\omega_1$	$-\omega_2$	0	ω_2	ω_1	$2\omega_2$

Table 4: Values of ω_k for signature $\{4, -1\}$.

Proof. Thanks to Table 2, we can apply Theorem 1 with $q = 12$. The values $\omega_0 = 2A_0$ and $\omega_1 = A_1 - 2A_0$ are given to us in Table 1. As for ω_2, we see in Table 1 that $\omega_2 = A_2 - 4A_1 + 2A_0$, but since $A_2 = 4A_1 - A_0$ then this becomes just $\omega_2 = A_0$. Hence, we can write $\omega_0 = 2\omega_2$. The remaining values of ω_k follow from the conclusions of Theorem 1 that $\omega_k = \omega_{12 \pm k} = -\omega_{6-k}$, allowing us to fill in the rest of Table 4. $\qquad\square$

Thanks to Theorem 8, we have the following corollary.

Corollary 9. *For X_n and Y_n the non-negative solutions to $X^2 - 3Y^2 = 1$, then X_n satisfies equation (2) and Y_n satisfies equation (3).*

As we mentioned in the introduction, equation (2) for X_n comes from Merca [10]; to be precise, it follows from his Theorem 3 with $n = 6$. However, we believe that our equation (3) for Y_n is new.

Proof of Corollary 9. We begin with X_n, which has $X_0 = 1$ and $X_1 = 2$. From Theorem 8, we have $\omega_1 = 0$ and $\omega_2 = 1$ and so Table 4 gives us the following values:

$k \pmod{12}$	0	1	2	3	4	5	6	7	8	9	10	11	12
ω_k	2	0	1	0	-1	0	-2	0	-1	0	1	0	2

To reveal a hidden pattern, we write ω_k as the sum of two periodic sequences, as shown here. We have dropped some of the 0's for legibility.

$k \pmod{12}$	0	1	2	3	4	5	6	7	8	9	10	11	12
ω_k	2	0	1	0	-1	0	-2	0	-1	0	1	0	2
first sequence	-1		1		-1		1		-1		1		-1
second sequence	3						-3						3

Thus, equation (15) tells us that

$$X_n = \binom{2n}{n} - \sum_{i=1}^{n}(-1)^i\binom{2n}{n+2j} + 3\sum_{j=1}^{n}(-1)^j\binom{2n}{n+6j}. \tag{16}$$

From Lewis [9, p. 200], after adjusting for his notation we have the following formula:

$$2^{2n}\sin^{2n}z = \binom{2n}{n} + \sum_{h=1}^{n}(-1)^h\binom{2n}{n+h}2\cos 2hz,$$

and if we let $z = \pi/4$, divide by 2, and then renumber the sum, we obtain

$$2^{n-1} = \frac{1}{2}\binom{2n}{n} + \sum_{i=1}^{n}(-1)^i\binom{2n}{n+2i}.$$

(We could also obtain this from Merca [10, Corollary 8]). When we substitute this into equation (16) and multiply by two, we get

$$2X_n = 3\binom{2n}{n} - 2^n + 3\sum_{j=1}^{n}(-1)^j \cdot 2\binom{2n}{n+6j}.$$

Since $2\binom{2n}{n+6j} = \binom{2n}{n-6j} + \binom{2n}{n+6j}$, we can re-write that sum on the right to include the terms for $j < 0$, and as for the $j = 0$ term we simply bring $3\binom{2n}{n}$ into the sum. After simplifying, this gives us equation (2) at the beginning of this paper.

As for Y_n, we have $Y_0 = 0$ and $Y_1 = 1$ and so Theorem 8 tells us that in this case we have $\omega_1 = 1$ and $\omega_2 = 0$. Thanks to Table 4, we quickly assemble the following chart.

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$k \pmod{12}$	0	1	2	3	4	5	6	7	8	9	10	11	12
ω_k	0	1	0	0	0	-1	0	-1	0	0	0	1	0

From this we immediately get the expression for Y_n in equation (3) from the introduction. $\square$

3.4 Catalan numbers and a linear recurrence of order three

For a particular class of linear recurrences, we have the following theorem.

Theorem 10. *For A_n a sequence satisfying $A_n = 5A_{n-1} - 6A_{n-2} + A_{n-3}$ and with initial values $A_0, A_1,$ and A_2, we have*

$$A_n = A_0 \binom{2n}{n} + \sum_{k=1}^{n} \omega_k \binom{2n}{n+k}, \tag{17}$$

with

$$\omega_i = \begin{cases} 2A_0, & \text{for } k \equiv 0 \pmod{7}; \\ A_1 - 2A_0, & \text{for } k \equiv 1, 6 \pmod{7}; \\ A_2 - 4A_1 + 2A_0, & \text{for } k \equiv 2, 5 \pmod{7}; \\ 3A_1 - A_0 - A_2, & \text{for } k \equiv 3, 4 \pmod{7}. \end{cases} \tag{18}$$

Proof. Since the signature $\{5, -6, 1\}$ appears in Table 2, we can apply Theorem 1, this time with $q = 7$. The values for ω_0, ω_1, and ω_2 are given to us in Table 1, and from that same equation we also have

$$\omega_3 = A_3 - 6A_2 + 9A_1 - 2A_0,$$

but since $A_3 = 5A_2 - 6A_1 + A_0$ then this becomes

$$\omega_3 = -A_2 + 3A_1 - A_0,$$

as desired. Finally, Theorem 1 also tells us that $\omega_k = \omega_{7\pm k}$, thus concluding our proof. $\square$

With this in hand, we can present two new results related to Catalan paths.

Corollary 11. *For B_n the numbers $1, 1, 2, 5, 14, 42, \ldots$ from [A080937](#), we have*

$$B_n = \frac{1}{n+1}\binom{2n}{n} + \sum_{j \geq 1} 4\binom{2n}{n+7j} - \sum_{j \geq 1} \binom{2n+2}{n+7j+1} \tag{19}$$

where the first term on the right is also known as the nth Catalan number ([A000108](#)).

Proof. From [A080937](#) we learn that the numbers B_n have the recurrence $B_n = 5B_{n-1} - 6B_{n-2} + B_{n-3}$, and this signature $\{5, -6, 1\}$ appears in Table 2 with $q = 7$. Thus, we can once again apply Theorem 1 (this time with $q = 7$) and from Table 1 we learn that ω_0, ω_1, ω_2, and ω_3 are $2, -1, 0, 0$ respectively. The remaining values of ω_k follow from the fact that $\omega_k = \omega_{7\pm k}$. Here are those values for ω_k, starting at $k = 0$.

$k \pmod 7$	0	1	2	3	4	5	6	7
ω_k	2	-1	0	0	0	0	-1	2

Thus, equation (7) from Theorem 1 tells us that

$$B_n = \binom{2n}{n} - \binom{2n}{n+1} + \sum_{j \geq 1} \left(2\binom{2n}{n+7j} - \binom{2n}{n+7j-1} - \binom{2n}{n+7j+1} \right).$$

A simple calculation verifies that $\binom{2n}{n} - \binom{2n}{n+1}$ is equal to the Catalan number $\frac{1}{n+1}\binom{2n}{n}$, and as for the expression inside the sum, we can easily verify that

$$2\binom{2n}{n+7j} - \binom{2n}{n+7j-1} - \binom{2n}{n+7j+1} = 4\binom{2n}{n+7j} - \binom{2n+2}{n+7j+1}.$$

This gives us equation (19) for B_n, as desired. $\qquad\square$

From Corollary 11 we can obtain an unexpected equation for a type of Catalan paths. We recall [7, p. 152] that the nth Catalan number (call it C_n) counts the total number of non-negative paths of length $2n$ that start and end at 0, with each step ± 1. Our numbers B_n from Corollary 11, as described in [A080937](#), count the total number of such paths that have maximum height 5 or less. Hence, if we re-write equation (19), recalling again that $C_n = \frac{1}{n+1}\binom{2n}{n}$, we obtain the following.

Corollary 12. *The number of Catalan paths of length $2n$ with maximum height at least 6 is*

$$\sum_{j \geq 1} \left(\binom{2n+2}{n+7j+1} - 4\binom{2n}{n+7j} \right). \tag{20}$$

This formula gives us the (new) sequence [A359311](#), which begins $0, 0, 0, 0, 0, 0, 1, 12, 89, \dots$. We can only imagine that there is an easy combinatorial proof of our Corollary 12, although we are not clever enough to find it.

3.5   More binomial transforms

As we saw in Section 3.2, the binomial transforms of certain sequences (in that case, the Pell and Pell-Lucas sequences) can be expressed as weighted sums of binomial coefficients. We can do the same for the binomial transform of F_{2n} and L_{2n}. These transforms are given in [A093131](#) and [A020876](#) respectively, and both have signature $\{5, -5\}$. This leads us to the following theorem and its three corollaries.

Theorem 13. *For $A_n = 5A_{n-1} - 5A_{n-2}$ with initial values A_0 and A_1, we have*

$$A_n = A_0 \binom{2n}{n} + \sum_{k=1}^{n} \omega_k \binom{2n}{n+k}, \qquad (21)$$

with ω_k repeating modulo 10 as given in Table 5, with $\omega_0 = 2A_0$, $\omega_1 = A_1 - 2A_0$, and $\omega_2 = A_1 - 3A_0$.

$k \pmod{10}$	0	1	2	3	4	5	6	7	8	9	10
ω_k	ω_0	ω_1	ω_2	$-\omega_2$	$-\omega_1$	$-\omega_0$	$-\omega_1$	$-\omega_2$	ω_2	ω_1	ω_0

Table 5: Values of ω_k for signature $\{5, -5\}$.

Proof. Since the signature $\{5, -5\}$ appears in Table 2, we apply Theorem 1 with $q = 10$. The values for ω_0 and ω_1 are given to us in Table 1. As for ω_2, we learn from Table 1 that $\omega_2 = A_2 - 4A_1 + 2A_0$, but since $A_2 = 5A_1 - 5A_0$ then this becomes $\omega_2 = A_1 - 3A_0$. The other values follow from the conclusion of Theorem 1 which tells us that $\omega_k = \omega_{10\pm k}$ and that $\omega_k = -\omega_{5-k}$. $\qquad\square$

We can now prove two nice identities for the binomial transforms of F_{2n} and of L_{2n}, respectively. Both of these next two corollaries feature an equality between a weighted sum of binomial coefficients (on the left) and a periodic weighted sum of binomial coefficients (on the right).

Corollary 14. *For F_n the Fibonacci numbers, we have*

$$\sum_{i=0}^{n} \binom{n}{i} F_{2i} = \sum_{k=0}^{n} (-1)^{k+1} \left(\frac{k}{5}\right) \binom{2n}{n+k} \qquad (22)$$

and also

$$5^n F_{2n} = \sum_{k=0}^{2n} (-1)^{k+1} \left(\frac{k}{5}\right) \binom{4n}{2n+k}, \qquad (23)$$

where $\left(\dfrac{k}{5}\right)$ represents the Legendre symbol.

We note that both of the formulas in Corollary 14 are new, and equation (23) gives us the second part of equation (1) from the opening paragraph of this paper.

Proof. The binomial transform of F_{2n} on the left of equation (22) is $0, 1, 5, 20, 75, \ldots$, as seen in A093131 where we also learn that it has signature $\{5, -5\}$. If we let A_n equal that binomial transform, then we can apply Theorem 13, and since $A_0 = 0$ and $A_1 = 1$ we will get $\omega_0 = 0$ and $\omega_1 = \omega_2 = 1$. Thus, Table 5 from Theorem 13 becomes

$k \pmod{10}$	0	1	2	3	4	5	6	7	8	9	10
ω_k	0	1	1	-1	-1	0	-1	-1	1	1	0

and from this we see that $\omega_k = (-1)^{k+1}\left(\frac{k}{5}\right)$, giving us equation (22).

As for equation (23), we replace n with $2n$ in equation (22) and we note from A093131 that $\sum_{i=0}^{2n}\binom{2n}{i}F_{2i}$ is equal to $5^n F_{2n}$. $\qquad\square$

Corollary 15. *For L_n the Lucas numbers, we have*

$$\sum_{i=0}^{n}\binom{n}{i}L_{2i} = 5\sum_{j\geq 0}(-1)^j\binom{2n}{n+5j} - 5\binom{2n-1}{n} \qquad \text{for } n \geq 1, \tag{24}$$

and also

$$5^n L_{2n} = 5\sum_{j\geq 0}(-1)^j\binom{4n}{2n+5j} - 5\binom{4n-1}{2n} \qquad \text{for } n \geq 1. \tag{25}$$

We note that these two identities are not new; equation (24) follows from Merca [10, Corollary 8], and equation (25) is just a special case of (24). However, our approach is rather different from the one in Merca's article.

Proof. If we let A_n equal the binomial transform of L_{2n} then the sequence $(A_n)_{n\geq 0}$ begins with $2, 5, 15, 50, 175, \ldots$ and is given by A020876 where we learn that it has signature $\{5, -5\}$. Hence, we can once again apply Theorem 13, and since $A_0 = 2$ and $A_1 = 5$ in this case, we get $\omega_0 = 4$, $\omega_1 = 1$, and $\omega_2 = -1$. Thus, Table 5 from Theorem 13 gives us the following table, where we have written ω_k as a sum of two periodic sequences in order to reveal a hidden pattern.

$k \pmod{10}$	1	2	3	4	5	6	7	8	9	10
ω_k	1	-1	1	-1	-4	-1	1	-1	1	4
first sequence	1	-1	1	-1	1	-1	1	-1	1	-1
second sequence					-5					5

As a result, Theorem 13 gives us

$$\sum_{i=0}^{n}\binom{n}{i}L_{2i} = 2\binom{2n}{n} - \sum_{k=1}^{n}(-1)^k\binom{2n}{n+k} + 5\sum_{j\geq 1}(-1)^j\binom{2n}{n+5j}. \tag{26}$$

If we adjust the right-hand side so that the two sums each start at 0 instead of 1, then this becomes

$$\sum_{i=0}^{n}\binom{n}{i}L_{2i} = -2\binom{2n}{n} - \sum_{k=0}^{n}(-1)^k\binom{2n}{n+k} + 5\sum_{j\geq 0}(-1)^j\binom{2n}{n+5j}, \tag{27}$$

and we do this because that middle sum is easily converted to a well-known identity. To be precise, if we replace $\binom{2n}{n+k}$ with $\binom{2n}{n-k}$, and then replace k with $n-k$, then the middle sum in equation (27) satisfies

$$\sum_{k=0}^{n}(-1)^k\binom{2n}{n+k} = \sum_{k=0}^{n}(-1)^k\binom{2n}{n-k} = \sum_{k=0}^{n}(-1)^{n-k}\binom{2n}{k},$$

and this can be simplified further to

$$(-1)^n\sum_{k=0}^{n}(-1)^k\binom{2n}{k}.$$

From Benjamin and Quinn [2, Identity 168] we learn that the above sum is equal to $\binom{2n-1}{n}$, and so we can now write equation (26) as

$$\sum_{i=0}^{n}\binom{n}{i}L_{2i} = -2\binom{2n}{n} - \binom{2n-1}{n} + 5\sum_{j\geq 0}(-1)^j\binom{2n}{n+5j}. \tag{28}$$

It is relatively easy to show that $-2\binom{2n}{n} - \binom{2n-1}{n}$ is equal to $-5\binom{2n-1}{n}$ so long as $n \geq 1$, and thus we obtain our desired equation (24).

As for equation (25), we replace n with $2n$ in equation (24) and we note from [A020875](https://oeis.org/A020875) that $\sum_{i=0}^{2n}\binom{2n}{i}L_{2i}$ is equal to $5^n L_{2n}$. From this we can obtain equation (25), as desired. $\square$

As a result of Corollary 15, we can produce the following nice result about sums of every tenth binomial coefficient. As far as we can tell, this is a new formula.

Corollary 16. *For L_n the Lucas numbers and for $n \geq 1$, we have*

$$\sum_{j=-n}^{n}\binom{4n}{2n+10j} = (2^{4n-1} + L_{4n} + 5^n L_{2n})/5. \tag{29}$$

Proof. We begin with a result from Shibukawa [12, equation (1.15)] with $r = 2$, giving us

$$L_{4n} = -2^{4n-1} + \frac{5}{2}\sum_{j=-n}^{n}\binom{4n}{2n+5j}.$$

We now re-write our equation (25) from Corollary 15 by using the identity $-5\binom{4n-1}{2n} = -\frac{5}{2}\binom{4n}{2n}$; this gives us

$$5^n L_{2n} = -\frac{5}{2}\binom{4n}{2n} + 5\sum_{j=0}^{n}(-1)^j\binom{4n}{2n+5j} \qquad \text{for } n \geq 1,$$

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and since each $\binom{4n}{2n+5j}$ is equal to $\binom{4n}{2n-5j}$, then we can re-write this again as

$$5^n L_{2n} = \frac{5}{2} \sum_{j=-n}^{n} (-1)^j \binom{4n}{2n+5j} \qquad \text{for } n \geq 1.$$

When we add this to Shibukawa's equation we get, after re-indexing our sum,

$$L_{4n} + 5^n L_{2n} = -2^{4n-1} + 5 \sum_{j=-n}^{n} \binom{4n}{2n+10j},$$

and this gives us our desired equation (29). $\qquad\qquad\square$

4 Proof of Theorem 1

We now give the rather technical proof of our main result.

Proof of Theorem 1. For $(A_n)_{n\geq 0}$ a linear recurrence as given, we let $f(x)$ represent its characteristic polynomial and we let M be its order (equivalently, M is the degree of $f(x)$). We are given that this $f(x)$ is the minimal polynomial for $(2\cos\pi/q)^2$ for some integer $q \geq 3$, and since $(2\cos\pi/q)^2 = 2\cos 2\pi/q + 2$, then $f(2x+2)$ is the minimal polynomial for $x = \cos 2\pi/q$. It is well known [5, 8] that $f(2x+2)$ has degree $M = \phi(q)/2$.

Furthermore, borrowing notation from Gürtaş [5], if we set

$$S(q) = \{p_i \mid \gcd(p_i, q) = 1 \text{ and } 1 \leq p_i < q/2\}$$

then the complete and distinct set of roots [14] for $f(2x+2)$ are $\{\cos 2\pi p_i/q \mid p_i \in S(q)\}$ and so likewise for $f(x)$ the roots are $\{(2\cos\pi p_i/q)^2 \mid p_i \in S(q)\}$. Now that we have the M distinct roots of the characteristic polynomial $f(x)$, we know from Dubeau et. al. [4, Theorem 1] or Kelly and Peterson [6, Theorem 3.7] that the Binet formula for A_n is

$$A_n = \alpha_1 (2\cos\pi p_1/q)^{2n} + \cdots + \alpha_M (2\cos\pi p_M/q)^{2n} = \sum_{i=1}^{M} \alpha_i (2\cos\pi p_i/q)^{2n}, \qquad (30)$$

with $p_1, \ldots p_M$ the distinct elements in the set $S(q)$ and with $\alpha_1, \ldots, \alpha_M$ determined by the initial values $A_0, A_1, A_2, \ldots A_{M-1}$.

We now call upon the well-known trigonometric power formula [15] for cosine,

$$\cos^{2n}\theta = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} \binom{2n}{k} \cos 2(n-k)\theta. \qquad (31)$$

We simplify the above sum by replacing k with $n-k$ and noting that $\binom{2n}{n-k} = \binom{2n}{n+k}$. After doing so, and then multiplying through by 2^{2n}, we obtain

$$(2\cos\theta)^{2n} = \binom{2n}{n} + \sum_{k=1}^{n} \binom{2n}{n+k} 2\cos 2k\theta. \qquad (32)$$

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Next, we apply equation (32) to each cosine in equation (30) to obtain

$$A_n = \sum_{i=1}^{M} \alpha_i \left(\binom{2n}{n} + \sum_{k=1}^{n} \binom{2n}{n+k} 2\cos 2k\pi p_i/q \right).$$

We pull out the first term, and switch the order of summation on the rest, to get

$$A_n = \binom{2n}{n} \sum_{i=1}^{M} \alpha_i + \sum_{k=1}^{n} \omega_k \binom{2n}{n+k}, \tag{33}$$

where ω_k is defined as

$$\omega_k = \sum_{i=1}^{M} \alpha_i \, 2\cos 2k\pi p_i/q. \tag{34}$$

For the first term in equation (33), we note that if we take $n = 0$ in equation (30), we obtain

$$A_0 = \sum_{i=1}^{M} \alpha_i,$$

and so equation (33) becomes

$$A_n = A_0 \binom{2n}{n} + \sum_{k=1}^{n} \omega_k \binom{2n}{n+k}, \tag{35}$$

as desired.

We now turn our attention to establishing equation (8) for ω_k. For this, we will use the Chebyshev polynomials $T_k(x)$ which have the useful property that $T_k(\cos\theta) = \cos k\theta$. We can write them as

$$T_k(x) = \sum_{j=0}^{\lfloor k/2 \rfloor} t_{k,j} x^{k-2j} \tag{36}$$

such that our Chebyshev coefficients $t_{k,j}$, as seen in the CRC handbook [16, § 6.10.6], satisfy

$$t_{k,j} = (-1)^j \binom{k-j}{j} 2^{k-2j-1} \frac{k}{k-j} \tag{37}$$

so long as $k - j \neq 0$. From our summation in equation (36), we see that $0 \leq j \leq \lfloor k/2 \rfloor$, so the only case when $k - j = 0$ is when both j and k are zero, and for that we define $t_{0,0} = 1$.

Next, we define $S_k(x)$ as

$$S_k(x) = 2T_{2k}(x/2), \tag{38}$$

which from equation (36) gives us that

$$S_k(x) = 2 \sum_{j=0}^{k} t_{2k,j} \frac{x^{2k-2j}}{2^{2k-2j}} \tag{39}$$

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and furthermore we have that $S_k(2\cos\theta) = 2T_{2k}(\cos\theta)$ and thanks to the property of the Chebyshev polynomials, this equals $2\cos 2k\theta$. This means

$$S_k(2\cos\pi p_i/q) = 2\cos 2k\pi p_i/q,$$

and so our formula for ω_k in equation (34) becomes

$$\omega_k = \sum_{i=1}^{M}\alpha_i \cdot S_k(2\cos\pi p_i/q).$$

Thanks to equation (39) this becomes

$$\omega_k = \sum_{i=1}^{M}\alpha_i \cdot 2\sum_{j=0}^{k}\frac{t_{2k,j}}{2^{2k-2j}}(2\cos\pi p_i/q)^{2k-2j}.$$

We switch the order of summation to get

$$\omega_k = 2\sum_{j=0}^{k}\frac{t_{2k,j}}{2^{2k-2j}}\sum_{i=1}^{M}\alpha_i(2\cos\pi p_i/q)^{2k-2j},$$

and thanks to our formula for A_n in equation (30), this becomes

$$\omega_k = 2\sum_{j=0}^{k}\frac{t_{2k,j}}{2^{2k-2j}}A_{k-j}.$$

We recall from our discussion above that $t_{0,0} = 1$ so this tells us that $\omega_0 = 2A_0$. For $k \geq 1$, we can use equation (37) to replace the coefficients $t_{2k,j}$, and so the above equation becomes

$$\omega_k = \sum_{j=0}^{k}(-1)^j\binom{2k-j}{j}\frac{2k}{2k-j}A_{k-j} \qquad \text{for } k \geq 1,$$

as desired.

Next, since

$$\cos 2(q\pm k)\pi p_i/q = \cos(2\pi p_i \pm 2k\pi p_i/q) = \cos 2k\pi p_i/q,$$

then from our definition of ω_k in equation (34) we have that $\omega_{k'} = \omega_k$ for $k' \equiv \pm k \pmod{q}$, as desired.

Finally, if q is even, then for $k' \equiv q/2 - k \pmod{q}$ we have

$$\cos 2k'\pi p_i/q = \cos 2(q/2-k)\pi p_i/q = \cos(\pi p_i + 2k\pi p_i/q) = -\cos 2k\pi p_i/q,$$

where the last equality holds because since q is even, then $p_i \in S(q)$ must be odd. $\qquad\square$

5   Conclusion

We are rather surprised by how many different identities with weighted sums of binomial coefficients we could obtain from our Theorem 1, all thanks to the cosine formula in equation (32) for $(2\cos\theta)^{2n}$. There is a similar formula for $(2\cos\theta)^{2n+1}$, and we encourage the reader to explore the additional identities that this could provide.

We might want to think about reversing the problem. Given a periodic sequence $(\omega_k)_{k\geq 0}$, what can we say about the periodic weighted sum

$$A_n = \sum_{k=1}^{n} \omega_k \binom{2n}{n+k},$$

and under what conditions on ω_k will $(A_n)_{n\geq 1}$ be a linear recurrence sequence?

Also, while Table 2 only covers signatures of length 2 or 3, our Theorem 1 applies to all signatures corresponding to minimal polynomials for $(2\cos\pi/q)^2$ for all $q \geq 3$. For $q = 11$, the signature $\{9, -28, 35, -15, 1\}$ has eight entries in the OEIS and it might be worthwhile to explore them. Another interesting example appears at $q = 60$. The coefficients of the minimal polynomial for $(2\cos\pi/60)^2$ match up perfectly with the signature for the sequence A126569 (which is related to the Cartan matrix for the Lie group E_8) and so we could establish a new identity for those terms as well.

Finally, we mention in closing another result on periodic binomial sums and the Fibonacci numbers. Moser [11] gives this identity (adapted slightly to match our notation) for F_{2n}, which is a nice counterpoint to our equation (1) at the beginning of the paper:

$$F_{2n} = \binom{2n-1}{n} - 2\binom{2n-1}{n+2} + \binom{2n-1}{n+4} - 2\binom{2n-1}{n+6} + \cdots .$$

6   Acknowledgments

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