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The Thue–Morse Sequence in Base $3/2$

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Abstract

We discuss the base 3/2 representation of the natural numbers. We prove that the sum-of-digits function of the representation is a fixed point of a 2-block substitution on an infinite alphabet, and that this implies that sum-of-digits function modulo 2 of the representation is a fixed point $x_{3/2}$ of a 2-block substitution on $\{0, 1\}$. We prove that $x_{3/2}$ is invariant for taking the binary complement, and present a list of conjectured properties of $x_{3/2}$, which we think will be hard to prove. Finally, we make a comparison with a variant of the base 3/2 representation, and give a general result on p-q-block substitutions.

1 Introduction

A natural number N is written in base $3/2$ if N has the form

$$N = \sum_{i \geq 0} d_i \left(\frac{3}{2}\right)^i, \tag{1}$$

with digits $d_i = 0, 1$ or 2.

Base $3/2$ representations are also known as sesquinary representations of the natural numbers; see Propp [6]. We write these expansions as

$$\mathrm{SQ}(N) = d_R(N) \cdots d_1(N) d_0(N) = d_R \cdots d_1 d_0.$$

We have, for example, $\mathrm{SQ}(7) = 211$, since $2 \cdot (9/4) + (3/2) + 1 = 7$. See A024629 for the continuation of Table 1. Ignoring leading 0's, the base $3/2$ representation of a number N is unique (see Section 3).

N	0	1	2	3	4	5	6	7	8	9	10
SQ(N)	0	1	2	20	21	22	210	211	212	2100	2101

Table 1: Base 3/2 expansions for $N = 1, \ldots, 10$.

For $N \geq 0$ let

$$s_{3/2}(N) := \sum_{i=0}^{i=R} d_i(N)$$

be the sum-of-digits function of the base 3/2 expansions. We have (see A244040)

$$s_{3/2} = 0, 1, 2, 2, 3, 4, 3, 4, 5, 3, 4, 5, 5, 6, 7, 4, 5, 6, 5, 6, 7, 7, 8, 9, 5, 6, 7, 5, 6, 7, 7, 8, 9, 8, 9, 10, \ldots$$

In this note we study the base 3/2 analogue of the Thue–Morse sequence A010060 (where the base equals 2), i.e., the sequence (see A357448)

$$(x_{3/2}(N)) := (s_{3/2}(N) \bmod 2) \; = \; 0, 1, 0, 0, 1, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 1, 0, 1, 1, \ldots$$

The Thue Morse sequence is the fixed point starting with 0 of the substitution $0 \to 01,\ 1 \to 10$. This might be called a 1-2-block substitution.

Let $p \leq q$ be two natural numbers. A *p-q-block substitution* κ on an alphabet A is a map $\kappa:\ A^p \to A^q$. A p-q-block substitution κ acts on $(A^p)^*$ by defining

$$\kappa(w_1 w_2 \cdots w_{pm-1} w_{pm}) = \kappa(w_1 \cdots w_p) \cdots \kappa(w_{pm-p+1} \cdots w_{pm})$$

for $w_1 w_2 \cdots w_{pm-1} w_{pm} \in (A^p)^*$ and $m = 1, 2, \ldots$. Its action extends to infinite sequences $x = x_0 x_1 \cdots$ by defining $\kappa : x \mapsto y$ by $y_{qm} \cdots y_{qm+q-1} = \kappa(x_{pm} \cdots x_{pm+p-1})$ for $m = 0, 1, \ldots$.

Theorem 1. *The sequence $x_{3/2}$ is a fixed point of the 2-3-block substitution*

$$\kappa : \begin{cases} 00 & \to & 010 \\ 01 & \to & 010 \\ 10 & \to & 101 \\ 11 & \to & 101 \end{cases}$$

Theorem 1 will be proved in Section 2.2.

2 Sum of digits function and Thue–Morse in base 3/2

2.1 Sum of digits function in base 3/2

Let $s_{3/2} = (0, 1, 2, 2, 3, 4, 3, 4, 5, 3, 4, 5, 5, 6, 7, 4, 5, \ldots)$ be the sum-of-digits function of the base 3/2 expansions. To describe this sequence, we extend the notion of a p-q-block substitution to alphabets of infinite cardinality.

Theorem 2. *The sequence $s_{3/2}$ is the fixed point starting with 0 of the 2-3-block substitution given by*

$$a, b \mapsto a, a+1, a+2 \quad \text{for } a = 0, 1, 2, \ldots \text{ and } b = 0, 1, 2, \ldots.$$

Proof. We have $d(0) = 0, d(1) = 1$ and from the uniqueness of the base 3/2 expansions it follows immediately that $d(3N + r) = d(2N) + r$ for $N \geq 0$ and $r = 0, 1, 2$.

Thus $s_{3/2}(3N) = s_{3/2}(2N)$, $s_{3/2}(3N+1) = s_{3/2}(2N)+1$, and $s_{3/2}(3N+2) = s_{3/2}(2N)+2$. This gives the result. □

Remark 3. The base-4/3 version of this sequence is A244041; the base-2 version is A000120; the base-3 version is A053735; the base-10 version is A007953.

2.2 Thue–Morse in base 3/2

Proof of Theorem 1. This follows directly from Theorem 2 by taking a and b modulo 2. □

Although iterates of $\kappa : 00 \to 010, 01 \to 010, 10 \to 101, 11 \to 101$ are undefined, we can generate the fixed point $x_{3/2}$ by iteration of a map κ' defined by $\kappa'(w) = \kappa(w)$ if w has even length, and $\kappa'(v) = \kappa(w)$ if $v = w0$ or $v = w1$ has odd length.

The fact that the iterates of κ are undefined causes difficulty in establishing properties of $x_{3/2}$. This is similar to the lack of progress in the last 25 years to prove the conjectures on the Kolakoski sequence, which is also a fixed point of a 2-block substitution (cf. the papers [2, 3]). Here is a property that is open for the Kolakoski sequence A000002, but can be proved for $x_{3/2}$.

Proposition 4. *If a word w occurs in $x_{3/2}$, then its binary complement $\overline{w}$ defined by $\overline{0} = 1, \overline{1} = 0$, also occurs in $x_{3/2}$.*

Proof. First one checks this for all 16 words of length 6 that occur in $x_{3/2}$. Note that then also $\overline{w}$ occurs for all w with $|w| \leq 6$, where $|w|$ denotes the length of w. Let u be a word of length $m \geq 7$. By adding at most 3 letters at the beginning and/or end of u one can obtain a word v with $|v| = 3n$ that occurs in $x_{3/2}$ at a position 0 modulo 3. But then Theorem 1 gives that $v = \kappa(w)$ for at least one word w occurring in $x_{3/2}$. The length of w is $|w| = 2n$. Since $\overline{\kappa(w)} = \kappa(\overline{w})$ the result follows by induction on $m = |u|$. For example, for $|u| = m = 7$, one has $|v| = 9$, and so $|w| = 6$. □

Here are some conjectured properties of $x_{3/2}$.

Conjecture 5. $x_{3/2}$ is reversal invariant, i.e., if the word $w = w_1 \cdots w_m$ occurs in $x_{3/2}$ then $\overleftarrow{w} = w_m \cdots w_1$ occurs in $x_{3/2}$.

Conjecture 6. $x_{3/2}$ is uniformly recurrent, i.e., each word that occurs in $x_{3/2}$ occurs infinitely often, with bounded gaps between consecutive occurrences.

Conjecture 7. The frequencies $\mu[w]$ of the words w occurring in $x_{3/2}$ exist. Two conjectured values: $\mu[00] = 1/10$, $\mu[01] = 4/10$.

Conjecture 8. μ is invariant for binary complements, i.e., $\mu[w] = \mu[\overline{w}]$ for all words w.

Conjecture 9. μ is reversal invariant, i.e., $\mu[w] = \mu[\overleftarrow{w}]$ for all words w.

Conjecture 10. (Shallit) The critical exponent (=largest number of repeated blocks) of $x_{3/2}$ is 5.

3 Base $3/2$ and base $1/2 \cdot 3/2$

Many authors refer to the paper [1] from Akiyama, Frougny, and Sakarovitch for the properties of base 3/2 expansions (see, e.g., Propp [6] and Rigo and Stipulanti [7]). However, the q/p expansions studied in paper [1] are different from the 3/2 expansions that are usually considered as in Equation (1). In the paper [1]:

$$N = \sum_{i \geq 0} d_i \, \frac{1}{p} \left(\frac{q}{p} \right)^i, \tag{2}$$

with digits $d_i = 0, 1$ or 2. We write AFS(N) for the expansion of N.

Remark 11. There is a small notational problem here: Akiyama, Frougny, and Sakarovitch write about p/q expansions with $p > q$, but in this note we consider q/p expansions with $p \leq q$. This fits better with the p-q-block substitutions, and with the order of p and q in the alphabet.

Here is the table given in the paper [1] for the case 3/2:

N	0	1	2	3	4	5	6	7	8	9	10
AFS(N)	ε	2	21	210	212	2101	2120	2122	21011	21200	21202

Table 2: Base $1/2 \cdot 3/2$ expansions for $N = 1, \ldots, 10$.

These expansions will not even be found in the OEIS (at the moment).

The situation is clarified in the paper [5] by Frougny and Klouda. They consider both representations, called, respectively, base p/q and base $1/q \cdot p/q$ representations. In the present note these are called respectively base q/p and base $1/p \cdot q/p$ representations.

A combination of the results in [1] and [5] yields a proof of the uniqueness of the base 3/2 expansions (QS(N)). There is also a direct proof of uniqueness in the paper by Edgar et al. [4]; see Theorem 1.1.

Note that AFS(N) = QS($2N$) for $N > 0$. So uniqueness of the base 3/2 representation implies immediately uniqueness of the $1/2 \cdot 3/2$ representation AFS(N). This observation obviously extends to base q/p.

Next we consider the question whether also the sequence $y_{3/2}$, the sum-of-digits function modulo 2 of the base $1/2 \cdot 3/2$ representation, is a fixed point of a 2-block substitution. This is indeed the case, and this 2-block substitution is given by Rigo and Stipulanti in [7].

Theorem 12. ([7]) *$y_{3/2}$ is the fixed point with prefix 00 of the 2-3-block substitution*

$$\kappa' : \begin{cases} 00 & \to & 001 \\ 01 & \to & 000 \\ 10 & \to & 111 \\ 11 & \to & 110 \end{cases}$$

In the paper [7] the proof of Theorem 12 is based on a generalization of Cobham's theorem to what are called $\mathcal{S}$-automatic sequences built on tree languages with a periodic labeled signature. Here we consider a more direct route, based on a simple closure property of p-q-block substitutions. Recall that a coding is a letter to letter map from one alphabet to another.

Theorem 13. *Let $x = (x(N))$ be a fixed point of a p-q-block substitution. Let r be a positive integer. Then the sequence $(x(rN))$ is the fixed point of a coding of a p-q-block substitution.*

Proof. If x is a fixed point of a p-q-block substitution, then x is also a fixed point of a pr-qr-block substitution. As new alphabet, take the words of length r occurring in x. On this alphabet, the pr-qr-block substitution induces a p-q-block substitution in an obvious way. Mapping each word of length r to its first letter is a coding that gives the result. □

Alternative proof for Theorem 12. Apply Theorem 13 with $r = 2$. The 4-6-block substitution is given by

$$0010 \to 010101,\ 0100 \to 010010,\ 0101 \to 010010,\ 0110 \to 010101,$$
$$1001 \to 101010,\ 1010 \to 101101,\ 1011 \to 101101,\ 1101 \to 101010.$$

Coding $00 \mapsto a$, $01 \mapsto b$, $10 \mapsto c$, $11 \mapsto d$, this induces the 2-3-block substitution

$$ac \to bbb, ba \to bac, bb \to bac, bc \to bbb, cb \to ccc, cc \to cdb, cd \to cdb, db \to ccc.$$

If we code further $a, b \mapsto 0$, and $c, d \mapsto 1$, then we obtain κ' from Theorem 12. □

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