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A Study of Second-Order Linear Recurrence Sequences via Continuants

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Abstract

This paper presents a reinterpretation of a second-order linear recurrence sequence as a sequence of continuants derived from the convergents to a continued fraction. As a result, we are able to derive the generating function and Binet formula for continuants. Using this result, we provide a continuant-based formulation for well-known identities associated with Lucas sequences.

1 Introduction

Consider the d-periodic Fibonacci numbers, defined by the second-order linear recurrence

$$f_\nu = a_r f_{\nu-1} + b_r f_{\nu-2}, \quad \text{where } \nu \equiv r \pmod{d},$$

with initial values $f_0 = 0$ and $f_1 = 1$. Such a sequence was studied by Carson [3], who, using matrix manipulation, managed to rewrite the sequence in the form:

$$f_\nu = a f_{\nu-d} + b f_{\nu-2d},$$

for some constants a and b. This extends an earlier partial result by Lehmer [11]. An alternative method via continuants was provided by Panario et al. [15]. In the same paper, they also derived the generating function and Binet formula for d-periodic Fibonacci numbers, supplementing earlier work by Edson et al. [6]. Our paper will follow this approach closely. Other related studies include the papers by Tan [19] and Yayenie [21], which focus on the special case of $d = 2$.

The concept of continuants was initially introduced by Euler during the development of a theory for continued fractions. In particular, continuants appear in the numerators and denominators of the convergents to a continued fraction. Subsequently, Muir [14] studied continuants as determinants of tridiagonal matrices. A major contribution by Muir is the notation that we employ in this paper. Another perspective on continuants is through a combinatorial approach, where they are viewed as a tiling problem, as discussed by Benjamin et al. [2]. However, the main inspiration for the content of this paper stems from Perron's book [16] on continued fractions.

The goal of this paper is to highlight the similarities between continuants and second-order linear recurrence sequences, with a focus on d-periodic Fibonacci numbers. Additionally, we address a problem posed by Panario et al. [15], seeking an application for generalized continuants. We hope that this will give continuants the attention they deserve.

While not a prerequisite to understand this paper, we assume here that readers are familiar with Lucas sequences, especially with the Fibonacci numbers, as most of our results are analogous to classical theorems on Fibonacci numbers. This familiarity may help the readers to grasp the motivation behind the results. Some selected articles that may be useful include [4, 5, 7, 9, 17, 20].

2 Generalized continuants

Suppose that (a_ν) and (b_ν) are sequences of positive integers. Consider the generalized continued fraction

$$\xi_\lambda = b_\lambda + \frac{a_{\lambda+1}}{b_{\lambda+1}} + \frac{a_{\lambda+2}}{b_{\lambda+2}} + \cdots = b_\lambda + \cfrac{a_{\lambda+1}}{b_{\lambda+1} + \cfrac{a_{\lambda+2}}{b_{\lambda+2} + \cdots}}.$$

The convergents to the continued fraction are the numbers obtained by truncating the continued fraction. For example, the ν-th convergent of ξ_λ is exactly

$$\xi_{\nu,\lambda} = b_\lambda + \frac{a_{\lambda+1}}{b_{\lambda+1}} + \cdots + \frac{a_{\lambda+\nu}}{b_{\lambda+\nu}}.$$

Define the ν-th generalized continuants

$$A_{\nu,\lambda} = K\begin{pmatrix} a_{\lambda+1}, \dots, a_{\lambda+\nu} \\ b_\lambda, b_{\lambda+1}, \dots, b_{\lambda+\nu} \end{pmatrix},$$
$$B_{\nu,\lambda} = K\begin{pmatrix} a_{\lambda+2}, \dots, a_{\lambda+\nu} \\ b_{\lambda+1}, b_{\lambda+2}, \dots, b_{\lambda+\nu} \end{pmatrix},$$

as the numerator and denominator, respectively, of the ν-th convergent. In other words,

$$\frac{A_{\nu,\lambda}}{B_{\nu,\lambda}} = b_\lambda + \frac{a_{\lambda+1}}{b_{\lambda+1}} + \cdots + \frac{a_{\lambda+\nu}}{b_{\lambda+\nu}}.$$

By the choice of sequences (a_ν) and (b_ν), both $A_{\nu,\lambda}$ and $B_{\nu,\lambda}$ are integers. We can slightly extend ν to negative numbers by setting the initial values

$$\begin{aligned} &A_{-1,\lambda} = 1, \quad A_{0,\lambda} = b_\lambda, \\ &B_{-1,\lambda} = 0, \quad B_{0,\lambda} = 1, \quad B_{1,\lambda} = b_{\lambda+1}. \end{aligned}$$

The ordinary continuants correspond to the case when $a_\nu = 1$ is constant. To simplify our notation, we set $A_{\nu,0} = A_\nu$ and $B_{\nu,0} = B_\nu$. Observe that $A_{\nu,\lambda}$ and $B_{\nu,\lambda}$ are related via the identity $B_{\nu,\lambda} = A_{\nu-1,\lambda+1}$. Furthermore, there are more general identities connecting $A_{\nu,\lambda}$ and $B_{\nu,\lambda}$, notably an analogue of Cassini's identity for Fibonacci numbers.

Theorem 1 (Cassini's Identity)**.** *For non-negative integers λ, ν and μ, we have*

$$\begin{aligned} A_{\nu+\lambda+\mu-1}B_{\nu-1,\lambda} &= A_{\nu+\lambda-1}B_{\nu+\mu-1,\lambda} + (-1)^{\nu-1}a_{\lambda+1}\cdots a_{\lambda+\nu}A_{\lambda-1}B_{\mu-1,\nu+\lambda}, \\ B_{\nu+\lambda+\mu-1}B_{\nu-1,\lambda} &= B_{\nu+\lambda-1}B_{\nu+\mu-1,\lambda} + (-1)^{\nu-1}a_{\lambda+1}\cdots a_{\lambda+\nu}B_{\lambda-1}B_{\mu-1,\nu+\lambda}. \end{aligned}$$

Corollary 2 (Catalan's Identity)**.** *For non-negative integers λ and ν, we have*

$$\begin{aligned} A_{\nu+\lambda} &= A_\lambda B_{\nu,\lambda} + a_{\lambda+1}A_{\lambda-1}B_{\nu-1,\lambda+1}, \\ B_{\nu+\lambda} &= B_\lambda B_{\nu,\lambda} + a_{\lambda+1}B_{\lambda-1}B_{\nu-1,\lambda+1}. \end{aligned}$$

Corollary 3 (d'Ocagne's Identity)**.** *For non-negative integers λ and ν, we have*

$$\begin{aligned} A_\lambda B_{\nu-1,\lambda-\nu} &= A_{\lambda-1}B_{\nu,\lambda-\nu} + (-1)^{\nu-1}a_\lambda\cdots a_{\lambda-\nu+1}A_{\lambda-\nu-1}, \\ B_\lambda B_{\nu-1,\lambda-\nu} &= B_{\lambda-1}B_{\nu,\lambda-\nu} + (-1)^{\nu-1}a_\lambda\cdots a_{\lambda-\nu+1}B_{\lambda-\nu-1}. \end{aligned}$$

Corollary 4. *For integer $\nu \geq -1$, we have*

$$\begin{aligned} A_{\nu+2} &= b_{\nu+2}A_{\nu+1} + a_{\nu+2}A_\nu, \\ B_{\nu+2} &= b_{\nu+2}B_{\nu+1} + a_{\nu+2}B_\nu. \end{aligned}$$

Corollary 5. *For non-negative integers λ and ν, we have*

$$\begin{aligned} A_{\nu,\lambda} &= b_\lambda A_{\nu-1,\lambda+1} + a_{\lambda+1}A_{\nu-2,\lambda+2}, \\ B_{\nu,\lambda} &= b_{\lambda+1}B_{\nu-1,\lambda+1} + a_{\lambda+2}B_{\nu-2,\lambda+2}. \end{aligned}$$

From Corollary 4, we can express A_ν and B_ν as a determinant of tridiagonal matrices.

Corollary 6. *For non-negative integer ν, we have*

$$A_\nu = \begin{vmatrix} b_0 & -1 & 0 & \dots & 0 & 0 \\ a_1 & b_1 & -1 & \dots & 0 & 0 \\ 0 & a_2 & b_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_{\nu-1} & -1 \\ 0 & 0 & 0 & \dots & a_\nu & b_\nu \end{vmatrix}, \quad B_\nu = \begin{vmatrix} b_1 & -1 & 0 & \dots & 0 & 0 \\ a_2 & b_2 & -1 & \dots & 0 & 0 \\ 0 & a_3 & b_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_{\nu-1} & -1 \\ 0 & 0 & 0 & \dots & a_\nu & b_\nu \end{vmatrix}.$$

Alternatively, we can write A_ν and B_ν as a product of matrices.

Corollary 7. *For non-negative integer ν, we have*

$$\begin{pmatrix} A_\nu & A_{\nu-1} \\ B_\nu & B_{\nu-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \prod_{j=1}^{\nu} \begin{pmatrix} b_j & 1 \\ a_j & 0 \end{pmatrix}.$$

Proof. The proofs of all the results above can be found in Perron's book [16], so we omit them here. It is worth noting that it is easier to prove the corollaries before proving Theorem 1. □

Remark 8. The results presented here exhibit similarities to those of Fibonacci numbers. This connection is not surprising since choosing $(a_\nu) = (b_\nu) = (1)$ leads to $(B_\nu) = (F_{\nu+1})$, which is just the usual Fibonacci sequence.

3 Periodic continuants

In this section and the remainder of this paper, we consider the case when (a_ν) and (b_ν) are both periodic with the same period d for $\nu \geq 1$, i.e., $a_{\nu+d} = a_\nu$ and $b_{\nu+d} = b_\nu$ for all $\nu \geq 1$. This type of continued fraction arises frequently in various contexts. For instance, when N is a positive integer, the continued fraction expansion of $\sqrt{N}$ is periodic. Under this assumption,

$$K\begin{pmatrix} a_{d+2}, \ldots, a_{d+\nu} \\ b_{d+1}, b_{d+2}, \ldots, b_{d+\nu} \end{pmatrix} = K\begin{pmatrix} a_2, \ldots, a_\nu \\ b_1, b_2, \ldots, b_\nu \end{pmatrix},$$

so $B_{\nu,d} = B_\nu$ (in fact, $A_{\nu,\lambda}$ and $B_{\nu,\lambda}$ are periodic with period d in the second argument). Using this fact and Corollary 3, we can derive an identity that will be useful later on.

Corollary 9. *For non-negative integers λ and ν with $d \mid (\lambda - \nu)$, we have*

$$A_{\lambda-1}B_\nu - A_\lambda B_{\nu-1} = (-1)^\nu a_\lambda \cdots a_{\lambda-\nu+1} A_{\lambda-\nu-1},$$
$$B_{\lambda-1}B_\nu - B_\lambda B_{\nu-1} = (-1)^\nu a_\lambda \cdots a_{\lambda-\nu+1} B_{\lambda-\nu-1}.$$

3.1 An equivalent recurrence relation

When (a_ν) and (b_ν) are periodic with period d, we can write (B_ν) as a second-order linear recurrence relation,

$$B_\nu = b_r B_{\nu-1} + a_r B_{\nu-2}, \quad \text{where } \nu \equiv r \pmod{d},$$

with the appropriate initial values. Motivated by Carson's result [3], we seek a recurrence relation of the form

$$B_{\nu+2d} = C_d B_{\nu+d} + D_d B_\nu,$$

for some integers C_d and D_d. The next theorem provides the desired values.

Theorem 10. *If (a_ν) and (b_ν) are periodic with period d, then*

$$B_{\nu+2d} = \frac{B_{2d-1}}{B_{d-1}} B_{\nu+d} + (-1)^{d-1} a_1 \cdots a_d B_\nu.$$

Proof. Using Corollary 2 with $\lambda = d$ and $2d$, we get

$$B_{\nu+d} = B_d B_\nu + a_1 B_{d-1} B_{\nu-1,1},$$
$$B_{\nu+2d} = B_{2d} B_\nu + a_1 B_{2d-1} B_{\nu-1,1}.$$

By canceling out $a_1 B_{\nu-1,1}$, we obtain

$$\begin{aligned} B_{\nu+2d} &= B_{2d} B_\nu + B_{2d-1} \left(\frac{B_{\nu+d} - B_d B_\nu}{B_{d-1}} \right). \\ &= \frac{B_{2d-1}}{B_{d-1}} B_{\nu+d} + \left(\frac{B_{d-1} B_{2d} - B_{2d-1} B_d}{B_{d-1}} \right) B_\nu. \end{aligned}$$

Again using Corollary 2 with $\lambda = d$ and $\nu = d-1$, we get

$$B_{2d-1} = B_d B_{d-1} + a_1 B_{d-1} B_{d-2,1},$$

so $C_d = B_{2d-1}/B_{d-1} = B_d + a_1 B_{d-2,1}$ is a non-negative integer. On the other hand, using Corollary 9 with $\lambda = 2d$ and $\nu = d$, we get

$$B_{2d} B_{d-1} - B_{2d-1} B_d = (-1)^{d-1} a_1 \cdots a_d B_{d-1}.$$

Thus, $D_d = (-1)^{d-1} a_1 \cdots a_d$ is an integer. □

One can check that the constants are indeed the same as those found by Carson [3]. From Theorem 10, it is immediate to deduce the next corollary.

Corollary 11. *Let $\Delta = C_d^2 + 4D_d$. For $n \geq 1$, we have*

$$B_{(n+1)d-1} = \frac{C_d B_{nd-1} + \sqrt{\Delta B_{nd-1}^2 + 4(-D_d)^n B_{d-1}^2}}{2}.$$

Proof. If we choose $\lambda = \mu = d$ and $\nu = nd$ with $n \geq 1$ in Theorem 1, then

$$B_{(n+1)d-1}^2 - B_{nd-1} B_{(n+2)d-1} = B_{(n+1)d-1}^2 - C_d B_{nd-1} B_{(n+1)d-1} - D_d B_{nd-1}^2 = (-D_d)^n B_{d-1}^2.$$

Solving this quadratic equation for $B_{(n+1)d-1}$ gives the desired equation. □

Remark 12. For the corresponding identities for Lucas sequences, refer to the book by Andrica and Bagdasar [1].

3.2 Generating function and Binet's formula

A benefit of having a single recurrence relation for (B_ν) is that we can compute its generating function and Binet formula easily. We follow the proof by Panario et al. [15] for the next two theorems. Firstly, let

$$G(x) = \sum_{n=0}^{\infty} B_n x^n,$$

be the required generating function. Then

$$\begin{aligned}(1 - C_d x^d - D_d x^{2d})G(x) &= \sum_{n=0}^{\infty} B_n x^n - C_d \sum_{n=d}^{\infty} B_{n-d} x^n - D_d \sum_{n=2d}^{\infty} D_d B_{n-2d} x^n \\ &= \sum_{n=0}^{2d-1} B_n x^n - C_d \sum_{n=0}^{d-1} B_n x^{n+d}.\end{aligned}$$

Therefore, we arrive at our generating function.

Theorem 13. *The generating function of B_n is*

$$G(x) = \frac{\sum\limits_{n=0}^{2d-1} B_n x^n - C_d \sum\limits_{n=0}^{d-1} B_n x^{n+d}}{1 - C_d x^d - D_d x^{2d}}.$$

Using this generating function, we can then calculate the Binet formula for B_ν. Since the continued fraction has period d, it is perhaps not surprising that the Binet formula depends on the residue modulo d. Let $\Delta = C_d^2 + 4D_d$ be the discriminant of the equation $D_d z^2 + C_d z - 1 = 0$. We assume throughout this paper that $\Delta \neq 0$. The next theorem gives the Binet formula for (B_ν).

Theorem 14 (Binet's formula)**.** *Suppose that $\Delta \neq 0$. Let α and β be the positive and negative roots of $D_d z^2 + C_d z - 1 = 0$, respectively. For integers n and r such that $n \geq 0$ and $r \geq -1$, we have*

$$B_{nd+r} = (-D_d)^{n-1}\left(\frac{\alpha^n - \beta^n}{\alpha - \beta} B_{d+r} - \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} B_r\right).$$

Proof. Consider the generating function of the subsequence (B_{nd+r}) given by

$$G_r(x) = \frac{B_r x^r + B_{d+r} x^{d+r} - C_d B_r x^{d+r}}{1 - C_d x^d - D_d x^{2d}}.$$

Let α and β be the positive and negative roots of $D_d z^2 + C_d z - 1 = 0$, respectively, or equivalently,

$$\alpha = \frac{-C_d + \sqrt{\Delta}}{2D_d}, \quad \beta = \frac{-C_d - \sqrt{\Delta}}{2D_d}.$$

Note that, $\alpha+\beta=-C_d/D_d$ and $\alpha\beta=-1/D_d$. By partial fraction decomposition, we have

$$\frac{1}{1-C_dx^d-D_dx^{2d}}=\frac{1}{\alpha-\beta}\left(\frac{\alpha}{1+D_d\alpha x^d}-\frac{\beta}{1+D_d\beta x^d}\right).$$

Substituting this into $G_r(x)$, we obtain

$$\begin{aligned}G_r(x)&=\frac{B_rx^r+B_{d+r}x^{d+r}-C_dB_rx^{d+r}}{\alpha-\beta}\sum_{n=0}^{\infty}(-1)^nD_d^n(\alpha^{n+1}-\beta^{n+1})x^{nd}\\&=\sum_{n=0}^{\infty}(-1)^n\frac{D_d^n(\alpha^{n+1}-\beta^{n+1})B_r}{\alpha-\beta}x^{nd+r}+\sum_{n=0}^{\infty}(-1)^n\frac{D_d^n(\alpha^{n+1}-\beta^{n+1})(B_{d+r}-C_dB_r)}{\alpha-\beta}x^{(n+1)d+r}.\end{aligned}$$

By replacing n with $n-1$ in the second sum, it becomes

$$\sum_{n=0}^{\infty}(-1)^{n-1}\frac{D_d^{n-1}(\alpha^n-\beta^n)(B_{d+r}-C_dB_r)}{\alpha-\beta}x^{nd+r}.$$

This combined with the first sum gives the expression

$$G_r(x)=\sum_{n=0}^{\infty}(-1)^n\left(\frac{D_d^n(\alpha^{n+1}-\beta^{n+1})B_r-D_d^{n-1}(\alpha^n-\beta^n)(B_{d+r}-C_dB_r)}{\alpha-\beta}\right)x^{nd+r}.$$

The terms containing B_r simplifies to

$$D_d^n(\alpha^{n+1}-\beta^{n+1})+C_dD_d^{n-1}(\alpha^n-\beta^n)=D_d^{n-1}(\alpha^{n-1}-\beta^{n-1}).$$

Therefore, we arrive at

$$G_r(x)=\sum_{n=0}^{\infty}(-D_d)^{n-1}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}B_{d+r}-\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}B_r\right)x^{nd+r}.$$

The theorem holds by comparing the coefficient of x^{nd+r}. □

In particular, when $r=-1$, one observes that B_{nd-1}/B_{d-1} and B_{2nd-1}/B_{nd-1} behave like the Lucas sequence of the first kind and second kind, respectively. Hence, new identities can be discovered by utilizing known identities involving Fibonacci and Lucas numbers. For example, the identity $L_n^2-5F_n^2=4(-1)^n$ yields the following identity, which can be verified directly:

$$\frac{B_{2nd-1}^2}{B_{nd-1}^2}-\frac{\Delta B_{nd-1}^2}{B_{d-1}^2}-=4(-D_d)^n.$$

Remark 15. We also have another related identity

$$\frac{B_{2nd-1}^2}{B_{nd-1}^2}+\frac{\Delta B_{nd-1}^2}{B_{d-1}^2}=\frac{2B_{4nd-1}^2}{B_{2nd-1}^2}.$$

3.2.1 Example: bi-periodic Fibonacci numbers

When $d = 2$, the recurrence relation for (B_ν) can be written as

$$B_\nu = \begin{cases} b_1 B_{\nu-1} + a_1 B_{\nu-2}, & \text{if } \nu \text{ is odd;} \\ b_2 B_{\nu-1} + a_2 B_{\nu-2}, & \text{if } \nu \text{ is even,} \end{cases}$$

with initial values $B_{-1} = 0$ and $B_0 = 1$. The first few terms of B_ν are

$$B_1 = b_1, \quad B_2 = b_1 b_2 + a_2, \quad B_3 = b_1^2 b_2 + a_1 b_1 + a_2 b_1,$$
$$B_4 = b_1^2 b_2^2 + a_2^2 + 2a_2 b_1 b_2 + a_1 b_1 b_2.$$

We calculate

$$C_2 = \frac{B_3}{B_1} = b_1 b_2 + a_1 + a_2, \quad D_2 = -a_1 a_2.$$

Hence (B_ν) can be defined using a single recurrence relation

$$B_\nu = (b_1 b_2 + a_1 + a_2) B_{\nu-2} - a_1 a_2 B_{\nu-4},$$

with initial values as stated above. Its generating function is

$$G(x) = \frac{1 + b_1 x - a_2 x^2}{1 - (b_1 b_2 + a_1 + a_2) x^2 + a_1 a_2 x^4},$$

and its Binet formula is

$$B_{2n+r} = (a_1 a_2)^{n-1} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} B_{r+2} - \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} B_r \right).$$

where α and β are the positive and negative roots of $a_1 a_2 z^2 - (b_1 b_2 + a_1 + a_2) z + 1 = 0$, respectively.

Consider the convergents to the continued fraction of $\sqrt{8} = [2, \overline{1, 4}]$. The denominators of these convergents are given by the sequence $1, 1, 5, 6, 29, 35, 169, 204, 985, \ldots$ (see A041011 in the *On-Line Encyclopedia of Integer Sequences* (OEIS)). This sequence is generated by the following second-order linear recurrence relation:

$$B_\nu = 6B_{\nu-2} - B_{\nu-4}.$$

Its generating function is

$$G(x) = \frac{1 + x - x^2}{1 - 6x^2 + x^4}.$$

The roots of $1 - 6x + x^2 = 0$ are $3 \pm 2\sqrt{2}$. Thus, the Binet formula is given by

$$B_{2n+r} = \frac{(3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n}{4\sqrt{2}} B_{r+2} - \frac{(3 + 2\sqrt{2})^{n-1} - (3 - 2\sqrt{2})^{n-1}}{4\sqrt{2}} B_r.$$

We will return to this example a few more times throughout this paper.

3.3 Limits

We use Theorem 14 to compute the limit of consecutive term, which we need for later. Observe that, $|\alpha| < |\beta|$. Hence we deduce the next two theorems.

Theorem 16. *Let α and β be as defined in Theorem 14. For integer $r \geq 0$, we have*

$$\lim_{n\to\infty} \frac{B_{nd+r}}{B_{nd+r-1}} = \frac{\beta B_{d+r} - B_r}{\beta B_{d+r-1} - B_{r-1}}.$$

Proof. Take $n \longrightarrow \infty$ in the expression below

$$\frac{B_{nd+r}}{B_{nd+r-1}} = \frac{(\alpha(\alpha/\beta)^{n-1} - \beta)B_{d+r} - ((\alpha/\beta)^{n-1} - 1)B_r}{(\alpha(\alpha/\beta)^{n-1} - \beta)B_{d+r-1} - ((\alpha/\beta)^{n-1} - 1)B_{r-1}},$$

and use the fact that $(\alpha/\beta)^n \longrightarrow 0$. □

Theorem 17. *Let α and β be as defined in Theorem 14. For integer $r \geq -1$, we have*

$$\lim_{n\to\infty} \frac{B_{(n+1)d+r}}{B_{nd+r}} = -D_d\beta.$$

Proof. Take $n \longrightarrow \infty$ in the expression below

$$\frac{B_{(n+1)d+r}}{B_{nd+r}} = \frac{-D_d(\alpha^2(\alpha/\beta)^{n-1} - \beta^2)B_{d+r} - (\alpha(\alpha/\beta)^{n-1} - \beta)B_r}{(\alpha(\alpha/\beta)^{n-1} - \beta)B_{d+r} - ((\alpha/\beta)^{n-1} - 1)B_r},$$

and use the fact that $(\alpha/\beta)^n \longrightarrow 0$. □

3.4 Extending to negative indices

Replacing n with $-n$ in Binet's formula (Theorem 14) yields the next theorem.

Theorem 18. *For integers n and r such that $n \geq 0$ and $r \geq -1$, we have*

$$B_{-nd+r} = -(-D_d)^{n-1}\left(\frac{\alpha^n - \beta^n}{(-D_d)^n(\alpha - \beta)}B_{d+r} - \frac{\alpha^{n+1} - \beta^{n+1}}{(-D_d)^{n-1}(\alpha - \beta)}B_r\right).$$

In particular, when $r = -1$, $(-D_d)^n B_{-nd-1} = -B_{nd-1}$.

3.5 Some curious telescoping sums

Many interesting sums involving Fibonacci numbers arise from telescoping sums. We demonstrate here some of the analogous sums based on continuants.

3.5.1 Telescoping sum 1

When $d = 2$, we have encountered this sequence before. It has the form

$$B_\nu = \begin{cases} b_1 B_{\nu-1} + a_1 B_{\nu-2}, & \text{if } \nu \text{ is odd;} \\ b_2 B_{\nu-1} + a_2 B_{\nu-2}, & \text{if } \nu \text{ is even,} \end{cases}$$

with initial values $B_{-1} = 0$ and $B_0 = 1$. The next theorem is a continuant version of the Millin series [13] for Fibonacci numbers.

Theorem 19. *If $d = 2$, then*

$$\sum_{n=1}^{\infty} \frac{(a_1 a_2)^{2^{n-1}}}{B_{2^{n+1}-1}} = \frac{1}{b_1 \beta}.$$

Proof. By choosing $\lambda = 2^{n+1}$ and $\nu = 2^n$ with $n \geq 1$ in Corollary 9, we get

$$\frac{B_{2^n}}{B_{2^n-1}} - \frac{B_{2^{n+1}}}{B_{2^{n+1}-1}} = \frac{(a_1 a_2)^{2^{n-1}}}{B_{2^{n+1}-1}}.$$

The LHS is telescoping, hence summing from 1 to N, we have

$$\frac{B_2}{B_1} - \frac{B_{2^{N+1}}}{B_{2^{N+1}-1}} = \sum_{n=1}^{N} \frac{(a_1 a_2)^{2^{n-1}}}{B_{2^{n+1}-1}}.$$

When $N \longrightarrow \infty$, the second term on the LHS approaches a finite limit

$$\frac{\beta B_2 - 1}{\beta B_1},$$

by Theorem 16. Therefore,

$$\sum_{n=1}^{\infty} \frac{(a_1 a_2)^{2^{n-1}}}{B_{2^{n+1}-1}} = \frac{B_2}{B_1} - \frac{\beta B_2 - 1}{\beta B_1} = \frac{1}{b_1 \beta}.$$

$\square$

In the case of $\sqrt{8}$, this sum becomes

$$\sum_{n=1}^{\infty} \frac{1}{B_{2^{n+1}-1}} = \frac{1}{6} + \frac{1}{204} + \frac{1}{235416} + \cdots = \frac{1}{3 + 2\sqrt{2}} = 3 - 2\sqrt{2}.$$

3.5.2 Telescoping sum 2

For general d, (B_ν) has the form

$$B_\nu = b_r B_{\nu-1} + a_r B_{\nu-2}, \quad \text{where } \nu \equiv r \pmod{d},$$

for the appropriate initial values.

Theorem 20. *If the period is d, then*

$$\sum_{n=1}^{\infty} \frac{(-D_d)^{n-1}}{B_{nd-1}B_{(n+1)d-1}} = \frac{\alpha}{B_{d-1}^2}.$$

Proof. By choosing $\lambda = \mu = d$ and $\nu = nd$ with $n \geq 1$, Theorem 1 becomes

$$\frac{B_{(n+1)d-1}}{B_{nd-1}} - \frac{B_{(n+2)d-1}}{B_{(n+1)d-1}} = \frac{(-1)^{nd}(a_1 \cdots a_d)^n B_{d-1}^2}{B_{nd-1}B_{(n+1)d-1}}.$$

The LHS is telescoping, so summing from 1 to N, we get

$$\frac{B_{2d-1}}{B_{d-1}} - \frac{B_{(N+2)d-1}}{B_{(N+1)d-1}} = \sum_{n=1}^{N} \frac{(-1)^{nd}(a_1 \cdots a_d)^n B_{d-1}^2}{B_{nd-1}B_{(n+1)d-1}}.$$

When $N \longrightarrow \infty$, the second term on the LHS approaches β by Theorem 17. Therefore,

$$\sum_{n=1}^{\infty} \frac{(-1)^{nd}(a_1 \cdots a_d)^n B_{d-1}^2}{B_{nd-1}B_{(n+1)d-1}} = C_d + D_d\beta = -D_d\alpha.$$

This proves the theorem after simplifications. □

It is a known fact that, for a positive integer N, the continued fraction of $\sqrt{N}$ is periodic (refer to Khrushchev's book [10] for a proof), say with period d. When (B_ν) represents the denominators of the convergents to $\sqrt{N}$, we have $C_d = B_{2d-1}/B_{d-1} = 2A_{d-1}$ and $D_d = (-1)^{d-1}$. This implies that $\alpha = (-1)^d A_{d-1} - (-1)^d \sqrt{A_{d-1}^2 - (-1)^d}$. Furthermore, there is an interesting interpretation of the number B_{nd-1} as a solution to the Pell's equation $x^2 - Ny^2 = 1$, as stated in the next two theorems.

Theorem 21. *The fundamental (minimal) solution to $x^2 - Ny^2 = 1$ is*

$$(x_1, y_1) = \begin{cases} (A_{d-1}, B_{d-1}), & \text{if } d \text{ is even;} \\ (A_{2d-1}, B_{2d-1}), & \text{if } d \text{ is odd.} \end{cases}$$

Theorem 22. *All the solutions to $x^2 - Ny^2 = 1$ are given by*

$$(x_n, y_n) = \begin{cases} (A_{nd-1}, B_{nd-1}), & \text{if } d \text{ is even;} \\ (A_{2nd-1}, B_{2nd-1}), & \text{if } d \text{ is odd,} \end{cases}$$

where n is a positive integer.

Proof. See Khrushchev's book [10] for proofs. □

Hence our telescoping sum can be reinterpreted as the next corollary.

Corollary 23. *If d is even and (x_n, y_n) is the n-th solution to $x^2 - Ny^2 = 1$, then*

$$\sum_{n=1}^{\infty} \frac{1}{y_n y_{n+1}} = \frac{x_1 - \sqrt{x_1^2 - 1}}{y_1^2}.$$

In the case when $N = 8$, the sum becomes

$$\sum_{n=1}^{\infty} \frac{1}{y_n y_{n+1}} = \frac{1}{6} + \frac{1}{210} + \frac{1}{7140} + \cdots = 3 - \sqrt{8}.$$

3.5.3 Telescoping sum 3

We assume for this part that (B_ν) is the sequence of denominators of the convergents to $\sqrt{N}$. In this case, $a_n = 1$ is constant and $D_d = (-1)^{d-1}$. The next corollary is similar to the previous one.

Corollary 24. *If d is even and (x_n, y_n) is the n-th solution to $x^2 - Ny^2 = 1$, then*

$$\sum_{n=1}^{\infty} \frac{1}{x_n x_{n+1}} = \frac{x_1 - \sqrt{x_1^2 - 1}}{x_1 \sqrt{x_1^2 - 1}}.$$

Proof. If we choose $\lambda = \mu = d$ and $\nu = (n-1)d$ with $n \geq 1$, then Theorem 1 becomes

$$\frac{B_{nd-1}}{A_{(n+1)d-1}} - \frac{B_{(n-1)d-1}}{A_{nd-1}} = \frac{(-1)^{(n-1)d}(a_1 \cdots a_d)^{n-1} A_{d-1} B_{d-1}}{A_{nd-1} A_{(n+1)d-1}}.$$

The LHS is telescoping, so summing from 1 to N, we get

$$\frac{B_{Nd-1}}{A_{(N+1)d-1}} - \frac{B_{-1}}{A_{d-1}} = \frac{B_{Nd-1}}{A_{(N+1)d-1}} = \sum_{n=1}^{N} \frac{(-1)^{(n-1)d}(a_1 \cdots a_d)^{n-1} A_{d-1} B_{d-1}}{A_{nd-1} A_{(n+1)d-1}}.$$

When $N \longrightarrow \infty$, using the same procedure as in Theorem 17, one checks that

$$\frac{B_{Nd-1}}{A_{(N+1)d-1}} \to -\frac{2B_{d-1}}{D_d^2 \beta(\alpha - \beta)}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{(-1)^{(n-1)d}(a_1 \cdots a_d)^{n-1} A_{d-1} B_{d-1}}{A_{nd-1} A_{(n+1)d-1}} = -\frac{2B_{d-1}}{D_d^2 \beta(\alpha - \beta)},$$

which can be simplified to

$$\sum_{n=1}^{\infty} \frac{(-D_d)^n}{A_{nd-1}A_{(n+1)d-1}} = -\frac{2\alpha}{A_{d-1}(\alpha-\beta)}.$$

We get our conclusion by noting that $D_d = -1$ and $\alpha - \beta = -2\sqrt{A_{d-1}^2 - 1}$. □

In the case when $N = 8$, we have the sum

$$\sum_{n=1}^{\infty} \frac{1}{x_n x_{n+1}} = \frac{1}{51} + \frac{1}{1683} + \frac{1}{57123} + \cdots = \frac{3\sqrt{2}-4}{12}.$$

3.5.4 Telescoping sum 4

Let (x_n) and (y_n) be any sequences . Consider the identity, which can be verified directly,

$$\frac{4(x_{n+1}y_n - x_n y_{n+1})(x_n x_{n+1} - y_n y_{n+1})}{(x_n - y_n)^2(x_{n+1} - y_{n+1})^2} = \frac{(x_n + y_n)^2}{(x_n - y_n)^2} - \frac{(x_{n+1} + y_{n+1})^2}{(x_{n+1} - y_{n+1})^2}.$$

We use this identity to prove the next theorem.

Theorem 25. *If the period is d, then*

$$\sum_{n=1}^{\infty} \frac{(-D_d)^{n-1} B_{(2n+1)d-1}}{B_{nd-1}^2 B_{(n+1)d-1}^2} = \frac{1}{B_{d-1}^3}.$$

Proof. Choose $x_n = \alpha^n$ and $y_n = \beta^n$ in the identity above to get

$$\frac{4(\alpha-\beta)(\alpha^{2n+1} - \beta^{2n+1})}{(-D_d)^n(\alpha^n - \beta^n)^2(\alpha^{n+1} - \beta^{n+1})^2} = \frac{(\alpha^n + \beta^n)^2}{(\alpha^n - \beta^n)^2} - \frac{(\alpha^{n+1} + \beta^{n+1})^2}{(\alpha^{n+1} - \beta^{n+1})^2}.$$

The RHS is telescoping, so summing from $n = 1$ to N, we obtain

$$\sum_{n=1}^{N} \frac{4(\alpha-\beta)(\alpha^{2n+1} - \beta^{2n+1})}{(-D_d)^n(\alpha^n - \beta^n)^2(\alpha^{n+1} - \beta^{n+1})^2} = \frac{(\alpha + \beta)^2}{(\alpha - \beta)^2} - \frac{(\alpha^{N+1} + \beta^{N+1})^2}{(\alpha^{N+1} - \beta^{N+1})^2}.$$

Taking $N \longrightarrow \infty$, the second term on RHS approaches 1. Therefore,

$$\sum_{n=1}^{\infty} \frac{4(\alpha-\beta)(\alpha^{2n+1} - \beta^{2n+1})}{(-D_d)^n(\alpha^n - \beta^n)^2(\alpha^{n+1} - \beta^{n+1})^2} = \frac{(\alpha + \beta)^2}{(\alpha - \beta)^2} - 1.$$

Rewriting this equation in terms of B_ν, we obtain

$$\sum_{n=1}^{N} \frac{4(-D_d)^{n-2} B_{(2n+1)d-1} B_{d-1}^3}{B_{nd-1}^2 B_{(n+1)d-1}^2} = \frac{C_d^2}{D_d^2} - \frac{\Delta}{D_d^2}.$$

We conclude our theorem upon simplification of this last equation. □

Suppose that (B_ν) are the denominators of the convergents to $\sqrt{N}$ for some positive integer N. The next corollary is immediate.

Corollary 26. *If d is even and (x_n, y_n) are the n-th solution to $x^2 - Ny^2 = 1$, then*

$$\sum_{n=1}^{\infty} \frac{y_{2n+1}}{y_n^2 y_{n+1}^2} = \frac{1}{y_1^3}.$$

When $N = 8$, we have

$$\sum_{n=1}^{\infty} \frac{y_{2n+1}}{y_n^2 y_{n+1}^2} = \frac{35}{1^2 \cdot 6^2} + \frac{1189}{6^2 \cdot 35^2} + \frac{40391}{35^2 \cdot 204^2} + \cdots = 1.$$

3.5.5 Telescoping sum 5

Suppose that $D_d = 1$, so

$$B_{(n+1)d-1} - B_{(n-1)d-1} = C_d B_{nd-1}.$$

Motivated by the identity $F_n^2 - F_{n+1}F_{n-1} = (-1)^{n-1}$ for Fibonacci numbers, we deduce

$$\frac{B_{nd-1}^2}{B_{d-1}^2} - \frac{B_{(n+1)d-1}}{B_{d-1}} \cdot \frac{B_{(n-1)d-1}}{B_{d-1}} = (-1)^{n-1},$$

where $n \geq 1$. The application of this identity yields the next result.

Theorem 27. *If $D_d = 1$, then*

$$\sum_{n=1}^{\infty} \tan^{-1} \frac{B_{2d-1}}{B_{(2n+1)d-1}} = \tan^{-1} \frac{B_{d-1}}{B_{2d-1}}.$$

Proof. Using the addition formula for $\tan^{-1}$ (see §3.6.1),

$$\begin{aligned} \tan^{-1} \frac{B_{d-1}}{B_{2nd-1}} - \tan^{-1} \frac{B_{d-1}}{B_{2(n+1)d-1}} &= \tan^{-1} \frac{B_{d-1}(B_{2(n+1)d-1} - B_{2nd-1})}{B_{d-1}^2 + B_{2(n+1)d-1}B_{2nd-1}} \\ &= \tan^{-1} \frac{B_{2d-1}}{B_{(2n+1)d-1}}. \end{aligned}$$

The LHS is telescoping, so summing from 1 to N, we obtain

$$\tan^{-1} \frac{B_{d-1}}{B_{2d-1}} - \tan^{-1} \frac{B_{d-1}}{B_{2(N+1)d-1}} = \sum_{n=1}^{N} \tan^{-1} \frac{B_{2d-1}}{B_{(2n+1)d-1}}.$$

Taking $N \longrightarrow \infty$, the second term on the LHS vanishes, so

$$\sum_{n=1}^{\infty} \tan^{-1} \frac{B_{2d-1}}{B_{(2n+1)d-1}} = \tan^{-1} \frac{B_{d-1}}{B_{2d-1}}.$$

□

Remark 28. This theorem generalizes an earlier theorem by Hoggatt and Bruggles [9, Theorem 4].

Using a similar construction, we can get a corresponding sum for $\tanh^{-1}$.

Theorem 29. *If $D_d = 1$, then*

$$\sum_{n=2}^{\infty} \tanh^{-1} \frac{B_{2d-1}}{B_{2nd-1}} = \frac{1}{2} \ln \frac{B_{3d-1} + B_{d-1}}{B_{3d-1} - B_{d-1}}.$$

Proof. Using the addition formula for $\tanh^{-1}$ (see §3.6.2),

$$\begin{aligned} \tanh^{-1} \frac{B_{d-1}}{B_{(2n-1)d-1}} - \tanh^{-1} \frac{B_{d-1}}{B_{(2n+1)d-1}} &= \tanh^{-1} \frac{B_{d-1}(B_{(2n+1)d-1} - B_{(2n-1)d-1})}{B_{(2n+1)d-1}B_{(2n-1)d-1} - B_{d-1}^2} \\ &= \tanh^{-1} \frac{B_{2d-1}}{B_{2nd-1}}. \end{aligned}$$

The LHS is telescoping, so summing from 2 to N ($\tanh^{-1} 1$ is undefined), we obtain

$$\tanh^{-1} \frac{B_{d-1}}{B_{3d-1}} - \tanh^{-1} \frac{B_{d-1}}{B_{(2N+1)d-1}} = \sum_{n=2}^{N} \tanh^{-1} \frac{B_{2d-1}}{B_{2nd-1}}.$$

Taking $N \longrightarrow \infty$, the second term on the LHS vanishes, so

$$\sum_{n=2}^{\infty} \tanh^{-1} \frac{B_{2d-1}}{B_{2nd-1}} = \tanh^{-1} \frac{B_{d-1}}{B_{3d-1}} = \frac{1}{2} \ln \frac{B_{3d-1} + B_{d-1}}{B_{3d-1} - B_{d-1}},$$

where we used the fact that

$$\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x}.$$

□

Let us apply these theorems to the continued fraction of $\sqrt{2} = [1, \overline{2}]$. The denominators, (B_ν), of the convergents to $\sqrt{2}$ are $1, 2, 5, 12, 29, 70, 169, 408, 985, \ldots$. Each B_ν corresponds to the $(\nu+1)$-th term of the Pell number (see A000129 in the OEIS). The theorems above imply that

$$\sum_{n=1}^{\infty} \tan^{-1} \frac{2}{B_{2n}} = \tan^{-1} \frac{2}{5} + \tan^{-1} \frac{2}{29} + \tan^{-1} \frac{2}{169} + \cdots = \tan^{-1} \frac{1}{2},$$

and

$$\sum_{n=2}^{\infty} \tanh^{-1} \frac{2}{B_{2n-1}} = \tanh^{-1} \frac{1}{6} + \tanh^{-1} \frac{1}{35} + \tanh^{-1} \frac{1}{204} + \cdots = \tanh^{-1} \frac{1}{5} = \frac{1}{2} \ln \frac{3}{2}.$$

3.6 More series involving $\tan^{-1}$ and $\tanh^{-1}$

Besides the telescoping sums in the last section, another set of series can be generated using $\tan^{-1}$ and $\tanh^{-1}$. The results in this section are inspired by the papers of Castellanos [4, 5]. However, Castellanos' original method using Chebyshev polynomials is cumbersome. Therefore, we provide a direct method that is applicable to his results, while generating new ones.

3.6.1 Series involving $\tan^{-1}$

Let ζ be a number to be determined later. Using the addition formula for $\tan^{-1}$,

$$\tan^{-1}(x-y) = \frac{\tan^{-1} x - \tan^{-1} y}{1 + \tan^{-1} x \tan^{-1} y},$$

and the Gregory series,

$$\tan^{-1} z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{2k+1},$$

we get

$$\tan^{-1} \frac{(\alpha-\beta)\zeta}{\alpha\beta+\zeta^2} = \tan^{-1} \frac{\zeta}{\beta} - \tan^{-1} \frac{\zeta}{\alpha} = \sum_{k=0}^{\infty} \frac{(-1)^k(\alpha^{2k+1}-\beta^{2k+1})}{(2k+1)(\alpha\beta)^{2k+1}} \zeta^{2k+1}.$$

Since the series expansion for $\tan^{-1} z$ is only valid for $|z| < 1$, we require that $|\zeta| < \min(|\alpha|, |\beta|) = |\alpha|$. The next two theorems are direct consequences of the last equation.

Theorem 30. *Let ζ be a root of $D_d z^2 - \sqrt{3\Delta} z - 1 = 0$ that satisfies $|\zeta| < \alpha$. Then*

$$\frac{\pi B_{d-1}}{6\sqrt{\Delta}} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} B_{(2k+1)d-1}}{2k+1} \zeta^{2k+1}.$$

Proof. We choose ζ such that

$$\frac{(\alpha-\beta)\zeta}{\alpha\beta+\zeta^2} = \frac{1}{\sqrt{3}}.$$

We obtain our theorem upon simplification and using the fact $\tan^{-1}(1/\sqrt{3}) = \pi/6$. □

Theorem 31. *Let ζ be a root of $D_d z^2 - (\sqrt{2}-1)\sqrt{\Delta} z - 1 = 0$ that satisfies $|\zeta| < \alpha$. Then*

$$\frac{\pi B_{d-1}}{8\sqrt{\Delta}} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} B_{(2k+1)d-1}}{2k+1} \zeta^{2k+1}.$$

Proof. We choose ζ such that

$$\frac{(\alpha-\beta)\zeta}{\alpha\beta+\zeta^2} = \sqrt{2} - 1.$$

We obtain our theorem upon simplification and using the fact $\tan^{-1}(\sqrt{2}-1) = \pi/8$. □

3.6.2 Series involving $\tanh^{-1}$

We also have similar addition formula for $\tanh^{-1}$,

$$\tanh^{-1}(x-y)=\frac{\tanh^{-1}x-\tanh^{-1}y}{1-\tanh^{-1}x\tanh^{-1}y},$$

and Taylor series expansion,

$$\tanh^{-1}z=\sum_{k=0}^{\infty}\frac{z^{2k+1}}{2k+1}.$$

So one expects similar series using $\tanh^{-1}$. Again, take ζ as a number to be determined later, then

$$\tanh^{-1}\frac{(\alpha-\beta)\zeta}{\alpha\beta-\zeta^2}=\tanh^{-1}\frac{\zeta}{\beta}-\tanh^{-1}\frac{\zeta}{\alpha}=\sum_{k=0}^{\infty}\frac{\alpha^{2k+1}-\beta^{2k+1}}{(2k+1)(\alpha\beta)^{2k+1}}\zeta^{2k+1}.$$

The series expansion for $\tanh^{-1}z$ is also valid for $|z|<1$, so we have the same inequality condition $|\zeta|<|\alpha|$. The previous equation gives the next two theorems.

Theorem 32. *Let ζ be a root of $D_dz^2+2\sqrt{\Delta}z+1=0$ that satisfies $|\zeta|<\alpha$. Then*

$$\frac{B_{d-1}}{2\sqrt{\Delta}}\ln 3=-\sum_{k=0}^{\infty}\frac{B_{(2k+1)d-1}}{2k+1}\zeta^{2k+1}.$$

Proof. We choose ζ such that

$$\frac{(\alpha-\beta)\zeta}{\alpha\beta-\zeta^2}=\frac{1}{2}.$$

We obtain our theorem upon simplification and using the fact $2\tanh^{-1}(1/2)=\ln 3$. □

Theorem 33. *Let ζ be a root of $D_dz^2+3\sqrt{\Delta}z+1=0$ that satisfies $|\zeta|<\alpha$. Then*

$$\frac{B_{d-1}}{2\sqrt{\Delta}}\ln 2=-\sum_{k=0}^{\infty}\frac{B_{(2k+1)d-1}}{2k+1}\zeta^{2k+1}.$$

Proof. We choose ζ such that

$$\frac{(\alpha-\beta)\zeta}{\alpha\beta-\zeta^2}=\frac{1}{3}.$$

We obtain our theorem upon simplification and using the fact $2\tanh^{-1}(1/3)=\ln 2$. □

3.6.3 Example (continued)

For $\sqrt{8}$, the first two theorems give

$$\frac{\pi}{24\sqrt{2}} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} B_{4k+1}}{2k+1} \left(2\sqrt{6}-5\right)^{2k+1},$$
$$\frac{\pi}{32\sqrt{2}} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} B_{4k+1}}{2k+1} \left(\sqrt{23+16\sqrt{2}}-2\sqrt{2}-4\right)^{2k+1}.$$

On the other hand, the latter two theorems give

$$\frac{\ln 3}{8\sqrt{2}} = \sum_{k=0}^{\infty} \frac{B_{4k+1}}{2k+1} \left(\sqrt{33}-4\sqrt{2}\right)^{2k+1},$$
$$\frac{\ln 2}{8\sqrt{2}} = \sum_{k=0}^{\infty} \frac{B_{4k+1}}{2k+1} \left(\sqrt{73}-6\sqrt{2}\right)^{2k+1}.$$

3.7 Sums involving continuants

The Binet formula enables us to calculate an intriguing sum of involving B_ν, as stated in the next theorem.

Theorem 34. *Let $x \neq \alpha, \beta$. For integer $r \geq -1$, we have*

$$\sum_{n=1}^{N} x^n B_{nd+r} = \frac{x^{N+1}(B_{(N+1)d+r}/D_d + xB_{Nd+r}) - x(B_{d+r}/D_d + xB_r)}{(x-\alpha)(x-\beta)}.$$

Proof. It is straightforward to check that

$$\sum_{n=1}^{N} (xD_d)^n \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{x(xD_d)^{N+1}(\alpha^N - \beta^N) + (xD_d)^{N+1}(\alpha^{N+1} - \beta^{N+1}) - xD_d(\alpha - \beta)}{(\alpha-\beta)(x^2 D_d - xC_d - 1)},$$
$$\sum_{n=0}^{N-1} (xD_d)^n \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{x(xD_d)^{N+1}(\alpha^{N-1} - \beta^{N-1}) + (xD_d)^{N}(\alpha^{N} - \beta^{N}) - xD_d(\alpha - \beta)}{(\alpha-\beta)(x^2 D_d - xC_d - 1)}.$$

Hence, by applying Binet's formula (Theorem 14), we get

$$\begin{aligned}\sum_{n=1}^{N} x^n B_{nd+r} &= \sum_{n=1}^{N} \left((-D_d)^{-1}(-xD_d)^n \frac{\alpha^n - \beta^n}{\alpha - \beta} B_{d+r} - x(-xD_d)^{n-1} \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} B_r \right) \\ &= \frac{-x^{N+1}B_{(N+1)d+r} - x^{N+2}D_d B_{Nd+r} + xB_{d+r} + x^2 D_d B_r}{1 + xC_d - x^2 D_d} \\ &= \frac{x^{N+1}(B_{(N+1)d+r}/D_d + xB_{Nd+r}) - x(B_{d+r}/D_d + xB_r)}{(x-\alpha)(x-\beta)}.\end{aligned}$$

□

Remark 35. One could also apply telescoping sum directly to the identity

$$(x-\alpha)(x-\alpha)B_{kd+r} = x^{k+1}(B_{(k+1)d+r}/D_d + xB_{kd+r}) - x^k(B_{kd+r}/D_d + xB_{(k-1)d+r}),$$

although our derivation gives an explanation for this identity.

For the next series, we take $r=-1$.

Theorem 36. *We have*

$$\sum_{n=1}^{N}\binom{N}{n}(-1)^n(\alpha+\beta)^n B_{nd-1} = \frac{B_{2Nd-1}}{D_d^N}.$$

Proof. Note that $1-C_d z = Dz^2$ for $z=\alpha$ and β. Hence

$$\sum_{n=1}^{N}\binom{N}{n}(-C_d)^n\frac{\alpha^n-\beta^n}{\alpha-\beta} = \frac{(1-C_d\alpha)^N-(1-C_d\beta)^N}{\alpha-\beta} = D_d^N\frac{\alpha^{2N}-\beta^{2N}}{\alpha-\beta}.$$

The sum in question is then

$$\begin{aligned}\sum_{n=1}^{N}\binom{N}{n}(-1)^n(\alpha+\beta)^n B_{nd-1} &= \sum_{n=1}^{N}\binom{N}{n}\left(\frac{C_d}{D_d}\right)^n B_{nd-1}\\ &= -D_d^{N-1}\frac{\alpha^{2N}-\beta^{2N}}{\alpha-\beta}B_{d-1} = \frac{B_{2Nd-1}}{D_d^N}.\end{aligned}$$

□

3.7.1 Example (continued)

Returning to the example of $\sqrt{8}$, we get, for $x \neq 3\pm 2\sqrt{2}$, the sums

$$\sum_{n=1}^{N} x^n B_{2n+r} = \frac{x^{N+1}(-B_{2(N+1)+r}+xB_{2N+r}) - x(-B_{2+r}+xB_r)}{(x-3+2\sqrt{2})(x-3-2\sqrt{2})},$$

and

$$\sum_{n=1}^{N}\binom{N}{n}(-6)^n B_{2n-1} = (-1)^N B_{4N-1}.$$

3.8 Divisibility properties

In this section, we explore some of the divisibility behaviors of the continuants. For similar results associated with Lucas sequences, refer to Ribenboim's book [17].

3.8.1 Divisibility within the sequence

We saw in the proof of Theorem 10 that $B_{d-1} \mid B_{2d-1}$. This can be further extended to the following theorem.

Theorem 37. *If $m \mid n$, then $B_{md-1} \mid B_{nd-1}$.*

Proof. Using Corollary 2, when $\nu = nd - 1$, $\lambda = md$, then

$$B_{(m+n)d-1} = B_d B_{nd-1} + a_1 B_{md-1} B_{md-2,1}.$$

We get our conclusion by an inductive argument, which is omitted here. □

Stated differently, this theorem implies that the sequence $(B'_n) = (B_{nd-1})$ is a divisibility sequence. In fact, it is a strong divisibility sequence, i.e., $B'_{\gcd(m,n)} = \gcd(B'_m, B'_n)$. One can prove this using the same technique for Fibonacci sequence, so we leave the proof as an exercise.

3.8.2 Divisibility by primes

Let p be an odd prime. If $p \mid C_d$ and $p \mid D_d$, then $p \mid B_{nd+r}$ (this also holds for 2). For the case of $p \mid C_d$ and $p \nmid D_d$, the result is given in the next theorem.

Theorem 38. *If $p \mid C_d$, $p \nmid D_d$, and $n \geq 2$, then*

$$B_{nd+r} = \begin{cases} (-1/2)^{n-2} D_d \Delta^{(n-2)/2} B_r, & \text{if } n \text{ is even;} \\ (-1/2)^{n-1} \Delta^{(n-1)/2} B_{d+r}, & \text{if } n \text{ is odd,} \end{cases} \pmod{p}.$$

In particular, $B_{nd-1} \equiv 0 \pmod{p}$ when n is even.

Proof. We expand the term $(\alpha^n - \beta^n)/(\alpha - \beta)$ in Binet's formula (Theorem 14),

$$\frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{1}{(2D_d)^{n-1}} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2k+1} (-C_d)^{n-2k-1} \Delta^k.$$

If $p \mid C_d$ and $p \nmid D_d$, then

$$(-D_d)^{n-1} \frac{\alpha^n - \beta^n}{\alpha - \beta} = \begin{cases} 0, & \text{if } n \text{ is even;} \\ \Delta^{(n-1)/2}/2^{n-1}, & \text{if } n \text{ is odd,} \end{cases} \pmod{p}.$$

Hence the theorem follows. □

If $p \nmid C_d$ and $p \mid D_d$, then $\Delta \equiv C_d^2 \pmod{p}$. Using this fact, we conclude the following theorem.

Theorem 39. *If $p \nmid C_d$ and $p \mid D_d$, then*

$$B_{(p-1)d+r} \equiv B_r, \quad B_{(p-1)d-1} \equiv 0 \pmod{p}$$

Proof. When $p \mid D_d$, the term involving B_r vanishes modulo p. Consider again the expansion of $(\alpha^n - \beta^n)/(\alpha - \beta)$,

$$2^{p-1} B_{pd+r} \equiv \sum_{k=0}^{(p-1)/2} \binom{p}{2k+1} (-C_d)^{p-1} B_{d+r} \pmod{p}.$$

Since $\binom{p}{k} \equiv 0 \pmod{p}$ for $1 \le k \le p-1$, this simplifies to

$$B_{pd+r} \equiv B_{d+r} \pmod{p},$$

and the theorem follows. □

Suppose now that $p \nmid C_d D_d$. By setting $n = p+1$ in the expansion of $(\alpha^n - \beta^n)/(\alpha - \beta)$, we get

$$2^p B_{(p+1)d+r} = \sum_{k=0}^{(p-1)/2} \binom{p+1}{2k+1} C_d^{p-2k} \Delta^k B_{d+r} + 2D_d \sum_{k=0}^{(p-1)/2} \binom{p}{2k+1} C_d^{p-2k-1} \Delta^k B_r.$$

Taking modulo p on both sides,

$$2B_{(p+1)d+r} \equiv C_d B_{d+r} + C_d \Delta^{(p-1)/2} B_{d+r} + 2D_d \Delta^{(p-1)/2} B_r \pmod{p}.$$

If $p \mid \Delta$, then

$$2B_{(p+1)d+r} \equiv C_d B_{d+r} \pmod{p}.$$

This is equivalent to our next theorem.

Theorem 40. *If $p \mid \Delta$ and $p \nmid C_d D_d$, then*

$$2B_{pd+r} \equiv C_d B_r, \quad B_{pd-1} \equiv 0 \pmod{p}.$$

Otherwise, $\Delta^{(p-1)/2} \equiv (\Delta \mid p) \pmod{p}$, where $(\cdot \mid p)$ denotes the Legendre symbol modulo p. Hence we obtain our next result.

Theorem 41. *For an odd prime $p \nmid C_d D_d$, if Δ is a quadratic residue modulo p, then*

$$B_{(p+1)d+r} \equiv C_d B_{d+r} + D_d B_r, \quad B_{(p+1)d-1} \equiv C_d B_{d-1} \pmod{p}.$$

If Δ is a quadratic non-residue modulo p, then

$$B_{(p+1)d+r} \equiv -D_d B_r, \quad B_{(p+1)d-1} \equiv 0 \pmod{p}.$$

On the other hand, if we take $n = p$ with $p \nmid C_d D_d$, then

$$2^{p-1} B_{pd+r} = \sum_{k=0}^{(p-1)/2} \binom{p}{2k+1} C_d^{p-2k-1} \Delta^k B_{d+r} + 2D_d \sum_{k=0}^{(p-3)/2} \binom{p-1}{2k+1} C_d^{p-2k-2} \Delta^k B_r.$$

Taking modulo p on both sides and using the fact that $\binom{p-1}{k} \equiv (-1)^k \pmod{p}$, we get

$$\begin{aligned} B_{pd+r} &\equiv \Delta^{(p-1)/2} B_{d+r} - \frac{2D_d}{C_d} \sum_{k=0}^{(p-3)/2} \left(1 + \frac{4D_d}{C_d^2}\right)^k B_r \pmod{p} \\ &\equiv \Delta^{(p-1)/2} B_{d+r} + \frac{C_d}{2}(1 - \Delta^{(p-1)/2}) B_r \pmod{p}. \end{aligned}$$

Thus, we can deduce the following theorem.

Theorem 42. *For an odd prime $p \nmid C_d D_d$, if Δ is a quadratic residue modulo p, then*

$$B_{(p-1)d+r} \equiv B_r, \quad B_{(p-1)d-1} \equiv 0 \pmod{p}.$$

If Δ is a quadratic non-residue modulo p, then

$$B_{pd+r} \equiv C_d B_r - B_{d+r}, \quad B_{pd-1} \equiv -B_{d-1} \pmod{p}.$$

Remark 43. In particular, if $d = 1$, $C_d = (a+1)$ and $D_d = -a$ for some integer a, then $\Delta = (a+1)^2 - 4a = (a-1)^2$. For any prime $p \nmid C_d D_d \Delta$, then $(\Delta \mid p) = 1$ and

$$p \mid B_{p-2} = a^{p-2} \frac{a^{-(p-1)} - 1}{a^{-1} - 1} = \frac{a^{p-1} - 1}{a - 1}.$$

Therefore, $p \mid a^{p-1} - 1$ (this is trivial for $p \mid C_d D_d \Delta$), which is Fermat's little theorem.

Theorems 40 to 42 suggest a certain periodicity of B_ν modulo p. This period is commonly referred to as the Pisano period modulo p. While we do not delve into the full theory here, we will demonstrate a minor consequence of these theorems. Set $\sigma_p(n)$ as the order of n modulo p, i.e., the smallest positive integer k such that $n^k \equiv 1 \pmod{p}$. The next corollary gives the Pisano period modulo p.

Corollary 44. *The Pisano period of (B_ν) modulo p, where $p \nmid C_d D_d$, divides*

$$\begin{cases} pd\sigma_p(C_d/2), & \text{if } p \mid \Delta; \\ (p-1)d, & \text{if } \Delta \text{ is a quadratic residue modulo } p; \\ (p+1)d\sigma_p(-D_d), & \text{if } \Delta \text{ is a quadratic non-residue modulo } p. \end{cases}$$

Proof. If $p \mid \Delta$, then Theorem 40 implies that

$$B_{pd\sigma_p(C_d/2)+r} \equiv (C_d/2)^{\sigma_p(C_d/2)} B_r \equiv B_r \pmod{p}.$$

Thus, the Pisano period divides $pd\sigma_p(C_d/2)$. The other cases are similar. □

Remark 45. For a more detailed account of Pisano period involving Fibonacci numbers, refer to the papers by Falcón [7] and Wall [20].

3.8.3 Example (continued)

Using the example of $\sqrt{8}$ as before, we calculate $\Delta = 32$. We choose $p = 3$ and 7, since 32 is a quadratic non-residue modulo 3 and a quadratic residue modulo 7. Theorems 41 and 42 give us the following results, respectively:

$$B_{r+8} \equiv B_r \pmod 3, \quad B_{r+16} \equiv 6B_{r+2} - B_r \pmod 7,$$

and

$$B_{r+6} \equiv 6B_r - B_{r+2} \pmod 3, \quad B_{r+12} \equiv B_r \pmod 7.$$

3.9 Law of apparition and repetition

3.9.1 Law of apparition

The results in §3.8.2 resemble the law of apparition of Lucas sequences. For any prime p, define $\omega(p)$ as the rank of apparition of p, which is the minimum positive integer k such that $B_{kd-1} \equiv 0 \pmod p$. The next theorem summarizes what we know about $\omega(p)$.

Theorem 46 (Law of apparition)**.** *For a prime p, we have*

(i) If $p \mid C_d$ and $p \mid D_d$, then $\omega(p) = 1$.

(ii) When $p = 2$, the following results hold:

- *If $2 \nmid C_d$ and $2 \mid D_d$, then*

$$\omega(2) = \begin{cases} \text{does not exist}, & \text{if } B_{d-1} \equiv 1 \pmod 2; \\ 1, & \text{if } B_{d-1} \equiv 0 \pmod 2. \end{cases}$$

- *If $2 \mid C_d$ (or $2 \mid \Delta$) and $2 \nmid D_d$, then $\omega(2) = 1$ or 2.*
- *If $2 \nmid C_d$ and $2 \nmid D_d$, then $\omega(2) = 1$ or 3.*

(iii) When p is an odd prime, the following results hold:

- *If $p \mid C_d$ and $p \nmid D_d$, then $\omega(p) = 1$ or 2.*
- *If $p \nmid C_d$ and $p \mid D_d$, then $\omega(p) \mid p - 1$.*
- *If $p \nmid C_d$, $p \nmid D_d$, and $p \mid \Delta$, then $\omega(p) = 1$ or p.*
- *If $p \nmid C_d$, $p \nmid D_d$, and $p \nmid \Delta$, then $\omega(p) \mid p - (\Delta \mid p)$.*

Proof. We have already dealt with the case of odd primes, so we can suppose $p = 2$. Using the fact $B_{2d-1} = C_d B_{d-1}$, we see that if $p \mid C_d$ (or $p \mid \Delta$) and $p \nmid D_d$, then $B_{2d-1} = 0$. Otherwise, if $p \nmid C_d$ and $p \nmid D_d$, then

$$B_{3d-1} \equiv B_{2d-1} + B_{d-1} \equiv 2B_{d-1} \equiv 0 \pmod 2.$$

However, if $p \nmid C_d$, but $p \mid D_d$, then for $n \geq 1$,

$$B_{(n+1)d-1} \equiv B_{nd-1} \pmod 2.$$

An inductive argument shows that $B_{nd-1} \equiv B_{d-1} \pmod 2$ for all $n \geq 1$. Thus, it is possible that $\omega(2)$ does not exists if $B_{d-1} \equiv 1 \pmod 2$. □

3.10 Lucas pseudoprime

Theorem 46 provides an extension to the notion of Lucas pseudoprime. For any positive integer n, by an abuse of notation, set $\epsilon(n) = (\Delta \mid n)$ as the Jacobi symbol modulo n. We say that n is a Lucas pseudoprime with respect to (B_ν), if $n \nmid C_d D_d \Delta$ and

$$B_{(n-\epsilon(n))d-1} \equiv 0 \pmod n.$$

If the congruence is false, we can conclude that n is composite. However, if the congruence holds, it does not necessarily imply that n is prime and hence the term 'pseudoprime'. For example, we take (B_ν) as the sequence of denominators of the convergent to $\sqrt{8}$ again. Choose $n = 35$, then the Jacobi symbol is $(32 \mid 35) = -1$. On the other hand,

$$B_{71} = 641614773393652358999201580 \equiv 0 \pmod{35}.$$

Hence 35 is a pseudoprime with respect to (B_ν).

3.10.1 Law of repetition

For the next part, we require an identity that can be traced back to Lagrange. Given any odd integer m, we have

$$X^m - Y^m = \sum_{k=0}^{(m-1)/2} \frac{m}{k} \binom{m-k-1}{k-1} (XY)^k \Big(X - Y \Big)^{m-2k},$$

This gives us an expansion for the term B_{mnd-1}/B_{d-1} as

$$\frac{B_{mnd-1}}{B_{d-1}} = \sum_{k=0}^{(m-1)/2} \frac{m}{k} \binom{m-k-1}{k-1} \Delta^{(m-2k-1)/2} (-D_d)^{nk} \left((-D_d)^{n-1} \frac{\alpha^n - \beta^n}{\alpha - \beta} \right)^{m-2k}.$$

We write $p^e \mid\mid N$ if e is the highest power of p dividing N. The next theorem gives the law of repetition.

Theorem 47 (Law of repetition)**.** *If* $p^e \mid\mid B_{nd-1}/B_{d-1}$ *and* $p \nmid m$*, then* $p^{e+f} \mid B_{p^f mnd-1}/B_{d-1}$*. In addition, if* $p \nmid D_d$*, then the power is exact.*

Proof. By induction, it suffices to establish the theorem for $f = 0$ and 1. The previous identity gives

$$\frac{B_{mnd-1}}{B_{d-1}} \equiv m(-D_d)^{n(m-1)/2}\frac{B_{nd-1}}{B_{d-1}} \pmod{(B_{nd-1}/B_{d-1})^3},$$
$$\frac{B_{pmnd-1}}{B_{d-1}} \equiv pm(-D_d)^{n(pm-1)/2}\frac{B_{nd-1}}{B_{d-1}} \pmod{(B_{nd-1}/B_{d-1})^3}.$$

Hence if $p^e \mid\mid B_{nd-1}/B_{d-1}$, then $p^e \mid B_{mnd-1}/B_{d-1}$ and $p^{e+1} \mid B_{pmnd-1}/B_{d-1}$. The second part is clear. □

3.11 Non-integer sequences

While all the theorems presented above assume that (a_ν) and (b_ν) are integer sequences, there is no reason not to extend them to real sequences as well. The only casualties are that C_d, D_d, and consequently B_ν are not necessarily integers. However, most of our results still apply except those involving divisibility properties in §3.8. We demonstrate here an example. Consider the sequence

$$B_\nu = \begin{cases} \sqrt{3}B_{\nu-1} + B_{\nu-2}, & \text{if } \nu \text{ is odd;} \\ B_{\nu-1} - B_{\nu-2}, & \text{if } \nu \text{ is even.} \end{cases}$$

The first few values of B_ν are

$$B_1 = \sqrt{3}, \quad B_2 = \sqrt{3} - 1, \quad B_3 = 3, \quad B_4 = 4 - 2\sqrt{3}.$$

We compute $\alpha = -(\sqrt{3} - \sqrt{7})/2$ and $\beta = -(\sqrt{3} + \sqrt{7})/2$, so using Binet's formula (Theorem 14), we have

$$B_{4k+1} = \sqrt{\frac{3}{7}}(\alpha^{2k-1} - \beta^{2k-1} - \sqrt{3}(\alpha^{2k} - \beta^{2k})).$$

Plugging this value into Theorem 27, we obtain

$$\sum_{n=1}^{\infty} \tan^{-1}\frac{3}{B_{4k+1}} = \tan^{-1}\frac{\sqrt{3}}{4} + \tan^{-1}\frac{\sqrt{3}}{19} + \tan^{-1}\frac{\sqrt{3}}{91} + \cdots = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}.$$

4 Further discussion

The results presented in this paper primarily focus on the denominator, $B_{\nu,\lambda}$. However, it should be possible to give similar results involving the numerator, $A_{\nu,\lambda}$. We leave that to the readers.

Apart from that, the Fibonacci numbers have a companion sequence called the Lucas numbers. By examining the initial values, we observe that $(A_{\nu,\lambda})$ is not the desired companion sequence. Therefore, it is still an open problem to construct the corresponding Lucas sequence for $(B_{\nu,\lambda})$.

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References

[1] D. Andrica and O. Bagdasar, *Recurrent Sequences*, Springer Nature, 2020.

[2] A. T. Benjamin, F. E. Su, and J. J. Quinn, Counting on continued fractions, *Math. Mag.* **73** (2000), 98–104.

[3] T. R. Carson, Periodic recurrence relations and continued fractions, *Fibonacci Quart.* **45** (2007), 357–361.

[4] D. Castellanos, Rapidly converging expansions with Fibonacci coefficients, *Fibonacci Quart.* **24** (1986), 70–82.

[5] D. Castellanos, A generalization of Binet's formula and some of its consequences, *Fibonacci Quart.* **27** (1989), 424–438.

[6] M. Edson, S. Lewis, and O. Yayenie, The k-Periodic Fibonacci sequence and an extended Binet's formula, *Integers* **11** (2011), #A32.

[7] S. Falcón and Á. Plaza, k-Fibonacci sequence modulo m, *Chaos Solitons Fractals* **41** (2009), 497–504.

[8] R. Frontczak and K. Prasad, Balancing polynomials, Fibonacci numbers and some new series for π, *Mediterr. J. Math.*, **20** (2023), Article 207.

[9] V. E. Hoggatt Jr. and I. D. Bruggles, A primer on the Fibonacci sequence: Part V, *Fibonacci Quart.* **2** (1964), 59–65.

[10] S. Khrushchev, *Orthogonal Polynomials and Continued Fractions*, Cambridge University Press, 2008.

[11] D. H. Lehmer, Fibonacci and related sequences in periodic tridiagonal matrices, *Fibonacci Quart.* **13** (1975), 150–158.

[12] E. Lucas, Théorie des fonctions numériques simplement périodiques, *Amer. J. Math.* **1** (1878), 184–196; 197–240; 289–321.

[13] D. A. Millin, Advanced problems H-237, *Fibonacci Quart.* **12** (1974), 309–312.

[14] T. Muir, *A Treatise on the Theory of Determinants*, Dover Publications, 1960.

[15] D. Panario, M. Sahin, and Q. Wang, A family of Fibonacci-like conditional sequences, *Integers* **14** (2013), #A78.

[16] O. Perron, *Die Lehre von den Kettenbrüchen*, Leipzig und Berlin, B. G. Teubner, 1913.

[17] P. Ribenboim, *The Little Book of Bigger Primes*, Springer, 2004.

[18] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, `https://oeis.org`, 2023.

[19] E. Tan, Some properties of the bi-periodic Horadam sequences, *Notes Number Theory Discrete Math.* **23** (2011), 56–65.

[20] D. D. Wall, Fibonacci series modulo m, *Amer. Math. Monthly* **67** (1960), 525–532.

[21] O. Yayenie, A note on generalized Fibonacci sequence, *Appl. Math. Comput.* **217** (2011), 5603–5611.

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