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On an Identity by Bruckman and Good

Hongshen Chua
Puchong 47170
Malaysia
hschua0622@hotmail.com

Abstract

This paper introduces a variation on an identity by Bruckman and Good. Using this identity, we are able to derive various well-known sums involving reciprocals of Fibonacci and Lucas numbers, including the case when the indices form an arithmetic progression. Moreover, we provide generalizations of the Millin series.

1 Introduction

Define the *generalized Lucas sequence* $(w_n)_{n\geq 0} = w_n(a, b; p, q)$ recursively by

$$w_0 = a, \quad w_1 = b, \quad w_n = pw_{n-1} - qw_{n-2},$$

where a, b, p, and q are complex numbers. Additionally, we define

$$u_n(p, q) = w_n(0, 1; p, q), \quad v_n(p, q) = w_n(2, p; p, q),$$

as the Lucas sequence of the first kind and second kind, respectively. The usual Fibonacci and Lucas numbers are given by $F_n = u_n(1, -1)$ and $L_n = v_n(1, -1)$, respectively. Let α and β, with $|\alpha| > |\beta|$, denote the roots of the characteristic equation $x^2 - px + q = 0$. The discriminant of this equation is $D = p^2 - 4q \neq 0$. Notably, we have $\alpha + \beta = p$, $\alpha\beta = q$, and $\alpha - \beta = \sqrt{D}$. The Binet formulas for w_n, u_n, and v_n are given by

$$w_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \quad u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad v_n = \alpha^n + \beta^n,$$

where $A = b - a\beta$ and $B = b - a\alpha$.

Many papers have been devoted to finding closed forms for sums involving reciprocals of generalized Lucas sequences, with a particular focus on the Millin series. The series

$$\sum_{i=0}^{\infty} \frac{1}{F_{2^i}} = \frac{7-\sqrt{5}}{2},$$

was proposed by Miller [24, p. 309] as a problem in Fibonacci Quarterly. It is worth noting that this series appeared in earlier papers by Lucas [20, Eq. (126)] and Brady [4, Eq. (3)]. In fact, they both gave a more general formula. For instance, Lucas demonstrated that

$$\sum_{i=0}^{\infty} \frac{q^{2^i r}}{u_{2^{i+1}r}} = \frac{\beta^r}{u_r},$$

where $r \geq 1$ is an integer. Other solutions include [7, 12, 16, 27].

On top of that, numerous authors have also taken interest in the sum when the indices form an arithmetic progression, say

$$\sum_{i=1}^{N} \frac{q^{ri}}{w_{ri+s} w_{r(i+M)+s}},$$

and its corresponding infinite version. Some examples include [3, 17, 22]. Furthermore, Adegoke [1] extended the sum to cases where the denominator contains two or more terms.

Traditionally, one can derive these results by applying telescoping sums to recurrence relations of generalized Lucas sequences. An alternative was provided by Bruckman and Good [7], via an identity that can be traced back to de Morgan, i.e.,

$$\frac{x_{k+1}y_k - x_k y_{k+1}}{(x_k - y_k)(x_{k+1} - y_{k+1})} = \frac{y_k}{x_k - y_k} - \frac{y_{k+1}}{x_{k+1} - y_{k+1}}.$$

However, Bruckman and Good only gave sums involving Fibonacci and Lucas numbers. Further extensions to Lucas sequences were added by Farhi [9]. Finally, we have to mention that de Morgan's identity is a special case of a more general identity by Duverney and Shiokawa [8, Cor. 4.1]. It could be worthwhile to use this identity to discover even more extensive series involving Lucas sequences.

In this paper, we consider two variations on the identity by Bruckman and Good, enabling us to establish numerous intriguing sums and rediscover some previously known ones. We will make repeated use of the following telescoping summation identities. For positive integers N and M, we have

$$\sum_{i=1}^{N}(x_i - x_{i+M}) = \sum_{i=1}^{M}(x_i - x_{i+N}), \tag{1}$$

and

$$\sum_{i=1}^{2N}(\pm 1)^i(x_i - x_{i+2M}) = \sum_{i=1}^{2M}(\pm 1)^i(x_i - x_{i+2N}). \tag{2}$$

Note that the usual telescoping sum is obtained by setting $M = 1$ in Eq. (1).

2 New reciprocal series of Lucas sequences

Let A and B be arbitrary complex numbers. The identity provided by Bruckman and Good [7] can be generalized for any sequences (x_n) and (y_n) as follows:

$$\frac{A(x_{i+j}y_i - x_i y_{i+j})}{(Ax_i - By_i)(Ax_{i+j} - By_{i+j})} = \frac{y_i}{Ax_i - By_i} - \frac{y_{i+j}}{Ax_{i+j} - By_{i+j}}.$$

Since the RHS is telescoping, upon applications of Eq. (1) and Eq. (2), we obtain

$$\sum_{i=1}^{N} \frac{A(x_{i+M}y_i - x_i y_{i+M})}{(Ax_i - By_i)(Ax_{i+M} - By_{i+M})} = \sum_{i=1}^{M} \frac{A(x_{i+N}y_i - x_i y_{i+N})}{(Ax_i - By_i)(Ax_{i+N} - By_{i+N})},$$

$$\sum_{i=1}^{2N} \frac{(\pm1)^i A(x_{i+2M}y_i - x_i y_{i+2M})}{(Ax_i - By_i)(Ax_{i+2M} - By_{i+2M})} = \sum_{i=1}^{2M} \frac{(\pm1)^i A(x_{i+2N}y_i - x_i y_{i+2N})}{(Ax_i - By_i)(Ax_{i+2N} - By_{i+2N})}.$$

In particular, for any integer-valued function f,

$$\sum_{i=1}^{N} \frac{A(x^{f(i+M)}y^{f(i)} - x^{f(i)}y^{f(i+M)})}{(Ax^{f(i)} - By^{f(i)})(Ax^{f(i+M)} - By^{f(i+M)})} = \sum_{i=1}^{M} \frac{A(x^{f(i+N)}y^{f(i)} - x^{f(i)}y^{f(i+N)})}{(Ax^{f(i)} - By^{f(i)})(Ax^{f(i+N)} - By^{f(i+N)})},$$

$$\sum_{i=1}^{2N} \frac{(\pm1)^i A(x^{f(i+2M)}y^{f(i)} - x^{f(i)}y^{f(i+2M)})}{(Ax^{f(i)} - By^{f(i)})(Ax^{f(i+2M)} - By^{f(i+2M)})} = \sum_{i=1}^{2M} \frac{(\pm1)^i A(x^{f(i+2N)}y^{f(i)} - x^{f(i)}y^{f(i+2N)})}{(Ax^{f(i)} - By^{f(i)})(Ax^{f(i+2N)} - By^{f(i+2N)})}.$$

By setting $x = \alpha$ and $y = \beta$, we obtain the following theorems:

Theorem 1. *For an integer-valued function f and positive integers M and N, we have*

$$\sum_{i=1}^{N} \frac{q^{f(i)}u_{f(i+M)-f(i)}}{w_{f(i)}w_{f(i+M)}} = \sum_{i=1}^{M} \frac{q^{f(i)}u_{f(i+N)-f(i)}}{w_{f(i)}w_{f(i+N)}}.$$

Theorem 2. *For an integer-valued function f and positive integers M and N, we have*

$$\sum_{i=1}^{2N} \frac{(\pm1)^i q^{f(i)}u_{f(i+2M)-f(i)}}{w_{f(i)}w_{f(i+2M)}} = \sum_{i=1}^{2M} \frac{(\pm1)^i q^{f(i)}u_{f(i+2N)-f(i)}}{w_{f(i)}w_{f(i+2N)}}.$$

In particular, if $M = 1$, then

$$\sum_{i=1}^{N} \frac{Aq^{f(i)}u_{f(i+1)-f(i)}}{w_{f(i)}w_{f(i+1)}} = \frac{Aq^{f(1)}u_{f(N+1)-f(1)}}{w_{f(1)}w_{f(N+1)}} = \frac{\beta^{f(1)}}{w_{f(1)}} - \frac{\beta^{f(N+1)}}{w_{f(N+1)}},$$

and

$$\begin{aligned}\sum_{i=1}^{2N} \frac{(\pm1)^i Aq^{s(i)}u_{f(i+2)-f(i)}}{w_{f(i)}w_{f(i+2)}} &= \pm\frac{Aq^{f(1)}u_{f(2N+1)-f(1)}}{w_{f(1)}w_{f(2N+1)}} + \frac{Aq^{f(2)}u_{f(2N+2)-f(2)}}{w_{f(2)}w_{f(2N+2)}} \\ &= \pm\frac{\beta^{f(1)}}{w_{f(1)}} \mp \frac{\beta^{f(2N+1)}}{w_{f(2N+1)}} + \frac{\beta^{f(2)}}{w_{f(2)}} - \frac{\beta^{f(2N+2)}}{w_{f(2N+2)}}.\end{aligned}$$

Assuming further that $f(i) \to \infty$ as $i \to \infty$. By letting $N \to \infty$, each term containing N on the RHS vanishes, so we get the following infinite versions of the previous theorems.

Theorem 3. *Suppose that $f(i) \to \infty$ as $i \to \infty$. Then*

$$\sum_{i=1}^{\infty} \frac{q^{f(i)} u_{f(i+1)-f(i)}}{w_{f(i)} w_{f(i+1)}} = \frac{\beta^{f(1)}}{A w_{f(1)}}.$$

Remark 4. This was also derived by Hu et al. [17, Thm. 2]. For increments greater than 1, refer to Farhi's paper [9, Thm. 2].

Theorem 5. *Suppose that $f(i) \to \infty$ as $i \to \infty$. Then*

$$\sum_{i=1}^{\infty} \frac{(\pm 1)^i q^{f(i)} u_{f(i+2)-f(i)}}{w_{f(i)} w_{f(i+2)}} = \frac{\beta^{f(2)}}{A w_{f(2)}} \pm \frac{\beta^{f(1)}}{A w_{f(1)}}.$$

It is immediate to deduce from Theorem 5 the next result.

Theorem 6. *Suppose that $f(i) \to \infty$ as $i \to \infty$. Then*

$$\sum_{\substack{i=1 \\ i \text{ even}}}^{\infty} \frac{q^{f(i)} u_{f(i+2)-f(i)}}{w_{f(i)} w_{f(i+2)}} = \frac{\beta^{f(2)}}{A w_{f(2)}},$$

$$\sum_{\substack{i=1 \\ i \text{ odd}}}^{\infty} \frac{q^{f(i)} u_{f(i+2)-f(i)}}{w_{f(i)} w_{f(i+2)}} = \frac{\beta^{f(1)}}{A w_{f(1)}}.$$

3 Applications, Part I

3.1 When $f(i) = ri + s$ with integers r, s and $r > 0$

Using Theorems 1 to 5, we obtain the next corollaries as special cases.

Corollary 7. *Let r, s, M, and N be integers with r, M, and $N > 0$. Then*

$$u_{rM} \sum_{i=1}^{N} \frac{q^{ri}}{w_{ri+s} w_{r(i+M)+s}} = u_{rN} \sum_{i=1}^{M} \frac{q^{ri}}{w_{ri+s} w_{r(i+N)+s}},$$

$$u_{2rM} \sum_{i=1}^{2N} \frac{(\pm 1)^i q^{ri}}{w_{ri+s} w_{r(i+2M)+s}} = u_{2rN} \sum_{i=1}^{2M} \frac{(\pm 1)^i q^{ri}}{w_{ri+s} w_{r(i+2N)+s}}.$$

Corollary 8. *Let r and s be integers with $r > 0$. Then*

$$\sum_{i=1}^{\infty} \frac{q^{ri+s}}{w_{ri+s} w_{r(i+1)+s}} = \frac{\beta^{r+s}}{A u_r w_{r+s}},$$

$$\sum_{i=1}^{\infty} \frac{(\pm 1)^i q^{ri+s}}{w_{ri+s} w_{r(i+2)+s}} = \frac{\beta^{2r+s}}{A u_{2r} w_{2r+s}} \pm \frac{\beta^{r+s}}{A u_{2r} w_{r+s}}.$$

Remark 9. These corollaries include the results of Adegoke et al. [3]. For example, if we carry out the transformations $M \to 2M$ and $s \to s - rM$, then the first sum in Corollary 7 becomes

$$u_{2rM} \sum_{i=1}^{N} \frac{q^{ri}}{w_{r(i-M)+s} w_{r(i+M)+s}} = u_{rN} \sum_{i=1}^{2M} \frac{q^{ri}}{w_{r(i-M)+s} w_{r(i+N-M)+s}}.$$

This is equivalent to Theorem 1 of the aforementioned paper. The rest can be deduced analogously.

Example 10. To apply our ideas to the case of Fibonacci and Lucas numbers, we set $p = 1$ and $q = -1$, then $\alpha = \varphi = (1 + \sqrt{5})/2$, $\beta = 1 - \varphi = (1 - \sqrt{5})/2$, and $D = 5$. Corollary 8 gives

$$\begin{aligned}
\sum_{i=1}^{\infty} \frac{(-1)^{ri+s}}{F_{ri+s} F_{r(i+1)+s}} &= \frac{(1-\varphi)^{r+s}}{F_r F_{r+s}}, \\
\sum_{i=1}^{\infty} \frac{(-1)^{ri+s}}{L_{ri+s} L_{r(i+1)+s}} &= \frac{(1-\varphi)^{r+s}}{\sqrt{5} L_r L_{r+s}}, \\
\sum_{i=1}^{\infty} \frac{(\pm 1)^i (-1)^{ri+s}}{F_{ri+s} F_{r(i+2)+s}} &= \frac{(1-\varphi)^{2r+s}}{F_{2r} F_{2r+s}} \pm \frac{(1-\varphi)^{r+s}}{F_{2r} F_{r+s}}, \\
\sum_{i=1}^{\infty} \frac{(\pm 1)^i (-1)^{ri+s}}{L_{ri+s} L_{r(i+2)+s}} &= \frac{(1-\varphi)^{2r+s}}{\sqrt{5} F_{2r} L_{2r+s}} \pm \frac{(1-\varphi)^{r+s}}{\sqrt{5} F_{2r} L_{r+s}}.
\end{aligned}$$

Remark 11. The first two sums, when both r and s are even, can be found in Popov's paper [26, p. 263]. The general case for r being even and $s \neq 0$ was proved by Melham [22, Thms. 2, 3].

Example 12. Set $p = e + 1/e$, $q = 1$, $A = 1$, and $B = \pm 1$. Then $\alpha = e$ and $\beta = 1/e$. Corollary 8 gives

$$\begin{aligned}
\sum_{i=1}^{\infty} \frac{\sinh r}{\sinh(ri+s) \sinh(r(i+1)+s)} &= \frac{1}{e^{r+s} \sinh(r+s)}, \\
\sum_{i=1}^{\infty} \frac{\sinh r}{\cosh(ri+s) \cosh(r(i+1)+s)} &= \frac{1}{e^{r+s} \cosh(r+s)}, \\
\sum_{i=1}^{\infty} \frac{(\pm 1)^i \sinh 2r}{\sinh(ri+s) \sinh(r(i+2)+s)} &= \frac{1}{e^{2r+s} \sinh(2r+s)} \pm \frac{1}{e^{r+s} \sinh(r+s)}, \\
\sum_{i=1}^{\infty} \frac{(\pm 1)^i \sinh 2r}{\cosh(ri+s) \cosh(r(i+2)+s)} &= \frac{1}{e^{2r+s} \cosh(2r+s)} \pm \frac{1}{e^{r+s} \cosh(r+s)}.
\end{aligned}$$

3.2 When $f(i) = 2^i r$ with integer $r > 0$

Using Theorems 1 to 5, we have the next results.

Corollary 13. *Let r, s, M, and N be integers with r, M, and $N > 0$. Then*

$$\sum_{i=1}^{N} \frac{q^{2^i r} u_{2^{i+M}r - 2^i r}}{w_{2^i r} w_{2^{i+M} r}} = \sum_{i=1}^{M} \frac{q^{2^i r} u_{2^{i+N}r - 2^i r}}{w_{2^i r} w_{2^{i+N} r}},$$
$$\sum_{i=1}^{2N} \frac{(\pm 1)^i q^{2^i r} u_{2^{i+2M}r - 2^i r}}{w_{2^i r} w_{2^{i+2M} r}} = \sum_{i=1}^{2M} \frac{(\pm 1)^i q^{2^i r} u_{2^{i+2N}r - 2^i r}}{w_{2^i r} w_{2^{i+2N} r}}.$$

Corollary 14. *Let $r > 0$ be an integer. Then*

$$\sum_{i=1}^{\infty} \frac{q^{2^i r} u_{2^i r}}{w_{2^i r} w_{2^{i+1} r}} = \frac{\beta^{2r}}{A w_{2r}},$$
$$\sum_{i=1}^{\infty} \frac{(\pm 1)^i q^{2^i r} u_{3 \cdot 2^i r}}{w_{2^i r} w_{2^{i+2} r}} = \frac{\beta^{4r}}{A w_{4r}} \pm \frac{\beta^{2r}}{A w_{2r}}.$$

Example 15. To apply our ideas to the case of Fibonacci and Lucas numbers, we set $p = 1$ and $q = -1$, then $\alpha = \varphi$, $\beta = 1 - \varphi$, and $D = 5$. Corollary 14 gives

$$\sum_{i=1}^{\infty} \frac{1}{F_{2^{i+1} r}} = \frac{(1-\varphi)^{2r}}{F_{2r}},$$
$$\sum_{i=1}^{\infty} \frac{F_{2^i r}}{L_{2^i r} L_{2^{i+1} r}} = \frac{(1-\varphi)^{2r}}{\sqrt{5} L_{2r}},$$
$$\sum_{i=1}^{\infty} \frac{(\pm 1)^i F_{3 \cdot 2^i r}}{F_{2^i r} F_{2^{i+2} r}} = \frac{(1-\varphi)^{4r}}{F_{4r}} \pm \frac{(1-\varphi)^{2r}}{F_{2r}},$$
$$\sum_{i=1}^{\infty} \frac{(\pm 1)^i F_{3 \cdot 2^i r}}{L_{2^i r} L_{2^{i+2} r}} = \frac{(1-\varphi)^{4r}}{\sqrt{5} L_{4r}} \pm \frac{(1-\varphi)^{2r}}{\sqrt{5} L_{2r}}.$$

Example 16. Set $p = e + 1/e$, $q = 1$, $A = 1$, and $B = \pm 1$, then $\alpha = e$ and $\beta = 1/e$. Corollary 14 gives

$$\sum_{i=1}^{\infty} \frac{1}{\sinh 2^{i+1} r} = \frac{1}{e^{2r} \sinh 2r},$$
$$\sum_{i=1}^{\infty} \frac{\tanh 2^i r}{\cosh 2^{i+1} r} = \frac{1}{e^{2r} \cosh 2r},$$
$$\sum_{i=1}^{\infty} \frac{(\pm 1)^i \sinh(3 \cdot 2^i r)}{\sinh 2^i r \sinh 2^{i+2} r} = \frac{1}{e^{4r} \sinh 4r} \pm \frac{1}{e^{2r} \sinh 2r},$$

$$\sum_{i=1}^{\infty} \frac{(\pm 1)^i \sinh(3\cdot 2^i r)}{\cosh 2^i r \cosh 2^{i+2} r} = \frac{1}{e^{4r}\cosh 4r} \pm \frac{1}{e^{2r}\cosh 2r}.$$

Remark 17. The first sum was also discovered by Gould [13, Eq. 24].

3.3 When $f(i) = 3^i r$ with integer $r > 0$

The application of Theorems 1 to 5 yields the next two corollaries.

Corollary 18. *Let r, s, M, and N be integers with r, M, and $N > 0$. Then*

$$\sum_{i=1}^{N} \frac{q^{3^i r} u_{3^{i+M} r - 3^i r}}{w_{3^i r} w_{3^{i+M} r}} = \sum_{i=1}^{M} \frac{q^{3^i r} u_{3^{i+N} r - 3^i r}}{w_{3^i r} w_{3^{i+N} r}},$$

$$\sum_{i=1}^{2N} \frac{(\pm 1)^i q^{3^i r} u_{3^{i+2M} r - 3^i r}}{w_{3^i r} w_{3^{i+2M} r}} = \sum_{i=1}^{2M} \frac{(\pm 1)^i q^{3^i r} u_{3^{i+2N} r - 3^i r}}{w_{3^i r} w_{3^{i+2N} r}}.$$

Corollary 19. *Let $r > 0$ be an integer. Then*

$$\sum_{i=1}^{\infty} \frac{q^{3^i r} u_{2\cdot 3^i r}}{w_{3^i r} w_{3^{i+1} r}} = \frac{\beta^{3r}}{A w_{3r}},$$

$$\sum_{i=1}^{\infty} \frac{(\pm 1)^i q^{3^i r} u_{8\cdot 3^i r}}{w_{3^i r} w_{3^{i+2} r}} = \frac{\beta^{9r}}{A w_{9r}} \pm \frac{\beta^{3r}}{A w_{3r}}.$$

Example 20. To apply our ideas to the case of Fibonacci and Lucas numbers, we set $p = 1$ and $q = -1$, then $\alpha = \varphi$, $\beta = 1 - \varphi$, and $D = 5$. Corollary 19 gives

$$\sum_{i=1}^{\infty} \frac{(-1)^r L_{3^i r}}{F_{3^{i+1} r}} = \frac{(1-\varphi)^{3r}}{F_{3r}},$$

$$\sum_{i=1}^{\infty} \frac{(-1)^r F_{3^i r}}{L_{3^{i+1} r}} = \frac{(1-\varphi)^{3r}}{\sqrt{5} L_{3r}},$$

$$\sum_{i=1}^{\infty} \frac{(\pm 1)^i (-1)^r F_{8\cdot 3^i r}}{F_{3^i r} F_{3^{i+2} r}} = \frac{(1-\varphi)^{9r}}{\sqrt{5} F_{9r}} \pm \frac{(1-\varphi)^{3r}}{\sqrt{5} F_{3r}},$$

$$\sum_{i=1}^{\infty} \frac{(\pm 1)^i (-1)^r F_{8\cdot 3^i r}}{L_{3^i r} L_{3^{i+2} r}} = \frac{(1-\varphi)^{9r}}{\sqrt{5} L_{9r}} \pm \frac{(1-\varphi)^{3r}}{\sqrt{5} L_{3r}}.$$

Remark 21. The first two sums appeared in the paper by Bruckman and Good [7, Eq. (11)]. Shar [28, p. 10] also discovered a similar series.

Example 22. Set $p = e + 1/e$, $q = 1$, $A = 1$, and $B = \pm 1$, then $\alpha = e$ and $\beta = 1/e$. Corollary 19 gives

$$\sum_{i=1}^{\infty} \frac{\cosh 3^i r}{\sinh 3^{i+1} r} = \frac{1}{2e^{3r} \sinh 3r},$$

$$\sum_{i=1}^{\infty} \frac{\sinh 3^i r}{\cosh 3^{i+1} r} = \frac{1}{2e^{3r} \cosh 3r},$$

$$\sum_{i=1}^{\infty} \frac{(\pm 1)^i \sinh(8 \cdot 3^i r)}{\sinh 3^i r \sinh 3^{i+2} r} = \frac{1}{e^{9r} \sinh 9r} \pm \frac{1}{e^{3r} \sinh 3r},$$

$$\sum_{i=1}^{\infty} \frac{(\pm 1)^i \sinh(8 \cdot 3^i r)}{\cosh 3^i r \cosh 3^{i+2} r} = \frac{1}{e^{9r} \cosh 9r} \pm \frac{1}{e^{3r} \cosh 3r}.$$

4 More reciprocal series of Lucas sequences

In this section, we will only deal with the Lucas sequences, (u_n) and (v_n), instead of the generalized Lucas sequences, (w_n). Given any sequences (x_n) and (y_n), one checks that

$$\frac{\pm 4(x_{i+j} y_i - x_i y_{i+j})(x_i x_{i+j} - y_i y_{i+j})}{(x_i \mp y_i)^2 (x_{i+j} \mp y_{i+j})^2} = \frac{(x_i \pm y_i)^2}{(x_i \mp y_i)^2} - \frac{(x_{i+j} \pm y_{i+j})^2}{(x_{i+j} \mp y_{i+j})^2}.$$

Using the same procedure as in §2, we deduce the next two theorems.

Theorem 23. *For an integer-valued function f and positive integers N and M, we have*

$$\sum_{i=1}^{N} \frac{q^{f(i)} u_{f(i+M)-f(i)} u_{f(i+M)+f(i)}}{u_{f(i)}^2 u_{f(i+M)}^2} = \sum_{i=1}^{M} \frac{q^{f(i)} u_{f(i+N)-f(i)} u_{f(i+N)+f(i)}}{u_{f(i)}^2 u_{f(i+N)}^2},$$

$$\sum_{i=1}^{N} \frac{q^{f(i)} u_{f(i+M)-f(i)} u_{f(i+M)+f(i)}}{v_{f(i)}^2 v_{f(i+M)}^2} = \sum_{i=1}^{M} \frac{q^{f(i)} u_{f(i+N)-f(i)} u_{f(i+N)+f(i)}}{v_{f(i)}^2 v_{f(i+N)}^2}.$$

Theorem 24. *For an integer-valued function f and positive integers N and M, we have*

$$\sum_{i=1}^{2N} \frac{(\pm 1)^i q^{f(i)} u_{f(i+2M)-f(i)} u_{f(i+2M)+f(i)}}{u_{f(i)}^2 u_{f(i+2M)}^2} = \sum_{i=1}^{2M} \frac{(\pm 1)^i q^{f(i)} u_{f(i+2N)-f(i)} u_{f(i+2N)+f(i)}}{u_{f(i)}^2 u_{f(i+2N)}^2},$$

$$\sum_{i=1}^{2N} \frac{(\pm 1)^i q^{f(i)} u_{f(i+2M)-f(i)} u_{f(i+2M)+f(i)}}{v_{f(i)}^2 v_{f(i+2M)}^2} = \sum_{i=1}^{2M} \frac{(\pm 1)^i q^{f(i)} u_{f(i+2N)-f(i)} u_{f(i+2N)+f(i)}}{v_{f(i)}^2 v_{f(i+2N)}^2}.$$

In Theorems 25 and 26, we assume further that $f(i) \to \infty$ as $i \to \infty$. When $M = 1$, using the fact that $v_N^2/u_N^2 \to D$ as $N \to \infty$, Theorems 23 and 24 imply the following result:

Theorem 25. *Suppose that $f(i) \to \infty$ as $i \to \infty$. Then*

$$\sum_{i=1}^{\infty} \frac{4q^{f(i)} u_{f(i+1)-f(i)} u_{f(i+1)+f(i)}}{u_{f(i)}^2 u_{f(i+1)}^2} = \frac{v_{f(1)}^2}{u_{f(1)}^2} - D,$$
$$\sum_{i=1}^{\infty} \frac{4q^{f(i)} u_{f(i+1)-f(i)} u_{f(i+1)+f(i)}}{v_{f(i)}^2 v_{f(i+1)}^2} = \frac{1}{D} - \frac{u_{f(1)}^2}{v_{f(1)}^2}.$$

Theorem 26. *Suppose that $f(i) \to \infty$ as $i \to \infty$. Then*

$$\sum_{i=1}^{\infty} \frac{(\pm 1)^i 4q^{f(i)} u_{f(i+2)-f(i)} u_{f(i+2)+f(i)}}{u_{f(i)}^2 u_{f(i+2)}^2} = \frac{v_{f(2)}^2}{u_{f(2)}^2} \pm \frac{v_{f(1)}^2}{u_{f(1)}^2} - D(1 \pm 1),$$
$$\sum_{i=1}^{\infty} \frac{(\pm 1)^i 4q^{f(i)} u_{f(i+2)-f(i)} u_{f(i+2)+f(i)}}{v_{f(i)}^2 v_{f(i+2)}^2} = -\frac{u_{f(2)}^2}{v_{f(2)}^2} \mp \frac{u_{f(1)}^2}{v_{f(1)}^2} + \frac{1}{D}(1 \pm 1).$$

5 Applications, Part II

5.1 When $f(i) = ri + s$ with integers r, s and $r > 0$

The next results are direct consequences of Theorems 23 to 26.

Corollary 27. *Let r, s, M, and N be integers with r, M, and $N > 0$. Then*

$$u_{rM} \sum_{i=1}^{N} \frac{q^{ri+s} u_{r(2i+M)+2s}}{u_{ri+s}^2 u_{r(i+M)+s}^2} = u_{rN} \sum_{i=1}^{M} \frac{q^{ri+s} u_{r(2i+N)+2s}}{u_{ri+s}^2 u_{r(i+N)+s}^2},$$
$$u_{rM} \sum_{i=1}^{N} \frac{q^{ri+s} u_{r(2i+M)+2s}}{v_{ri+s}^2 v_{r(i+M)+s}^2} = u_{rN} \sum_{i=1}^{M} \frac{q^{ri+s} u_{r(2i+N)+2s}}{v_{ri+s}^2 v_{r(i+N)+s}^2},$$
$$u_{2rM} \sum_{i=1}^{2N} \frac{(\pm 1)^i q^{ri+s} u_{2r(i+M)+2s}}{u_{ri+s}^2 u_{r(i+2M)+s}^2} = u_{2rN} \sum_{i=1}^{2M} \frac{(\pm 1)^i q^{ri+s} u_{2r(i+N)+2s}}{u_{ri+s}^2 u_{r(i+2N)+s}^2},$$
$$u_{2rM} \sum_{i=1}^{2N} \frac{(\pm 1)^i q^{ri+s} u_{2r(i+M)+2s}}{v_{ri+s}^2 v_{r(i+2M)+s}^2} = u_{2rN} \sum_{i=1}^{2M} \frac{(\pm 1)^i q^{ri+s} u_{2r(i+N)+2s}}{v_{ri+s}^2 v_{r(i+2N)+s}^2}.$$

Corollary 28. *Let r and s be integers with $r > 0$. Then*

$$\sum_{i=1}^{\infty} \frac{4q^{ri+s} u_r u_{r(2i+1)+2s}}{u_{ri+s}^2 u_{r(i+1)+s}^2} = \frac{v_{r+s}^2}{u_{r+s}^2} - D,$$
$$\sum_{i=1}^{\infty} \frac{4q^{ri+s} u_r u_{r(2i+1)+2s}}{v_{ri+s}^2 v_{r(i+1)+s}^2} = \frac{1}{D} - \frac{u_{r+s}^2}{v_{r+s}^2},$$

$$\sum_{i=1}^{\infty}\frac{(\pm 1)^i 4q^{ri+s}u_{2r}u_{2r(i+1)+2s}}{u_{ri+s}^2 u_{r(i+2)+s}^2} = \frac{v_{2r+s}^2}{u_{2r+s}^2} \pm \frac{v_{r+s}^2}{u_{r+s}^2} - D(1\pm 1),$$
$$\sum_{i=1}^{\infty}\frac{(\pm 1)^i 4q^{ri+s}u_{2r}u_{2r(i+1)+2s}}{v_{ri+s}^2 v_{r(i+2)+s}^2} = -\frac{u_{2r+s}^2}{v_{2r+s}^2} \mp \frac{u_{r+s}^2}{v_{r+s}^2} + \frac{1}{D}(1\pm 1).$$

Example 29. To apply our ideas to the case of Fibonacci and Lucas numbers, we set $p = 1$ and $q = -1$, then $\alpha = \varphi$, $\beta = 1 - \varphi$, and $D = 5$. Corollary 28 gives

$$\sum_{i=1}^{\infty}\frac{(-1)^{ri+s}F_{r(2i+1)+2s}}{F_{ri+s}^2 F_{r(i+1)+s}^2} = \frac{L_{r+s}^2}{4F_r F_{r+s}^2} - \frac{5}{4F_r},$$
$$\sum_{i=1}^{\infty}\frac{(-1)^{ri+s}F_{r(2i+1)+2s}}{L_{ri+s}^2 L_{r(i+1)+s}^2} = \frac{1}{20F_r} - \frac{F_{r+s}^2}{4F_r L_{r+s}^2},$$
$$\sum_{i=1}^{\infty}\frac{(\pm 1)^i(-1)^{ri+s}F_{2r(i+1)+2s}}{F_{ri+s}^2 F_{r(i+2)+s}^2} = \frac{L_{2r+s}^2}{4F_r F_{2r+s}^2} \pm \frac{L_{r+s}^2}{4F_r F_{r+s}^2} - \frac{5}{4F_r}(1\pm 1),$$
$$\sum_{i=1}^{\infty}\frac{(\pm 1)^i(-1)^{ri+s}F_{2r(i+1)+2s}}{L_{ri+s}^2 L_{r(i+2)+s}^2} = -\frac{F_{2r+s}^2}{4F_r L_{2r+s}^2} \mp \frac{F_{r+s}^2}{4F_r L_{r+s}^2} + \frac{1}{20F_r}(1\pm 1).$$

Remark 30. The first two sums were a problem in Fibonacci Quarterly, proposed and solved by Gauthier [10, 11]. However, Gauthier made an error in the sign of the first sum. We have also found several papers discussing similar results, but to our knowledge, none of them provided the general form as in Corollary 28. The list of sources (not exhaustive) includes [1, 2, 5, 6, 14, 19, 25].

5.2 When $f(i) = 2^i r$ with integer $r > 0$

Applying Theorems 23 to 26, we get the next two corollaries.

Corollary 31. *Let r, s, M, and N be integers with r, M, and $N > 0$. Then*

$$\sum_{i=1}^{N}\frac{q^{2^i r}u_{2^{i+M}r-2^i r}u_{2^{i+M}r+2^i r}}{u_{2^i r}^2 u_{2^{i+M}r}^2} = \sum_{i=1}^{M}\frac{q^{2^i r}u_{2^{i+N}r-2^i r}u_{2^{i+N}r+2^i r}}{u_{2^i r}^2 u_{2^{i+N}r}^2},$$
$$\sum_{i=1}^{N}\frac{q^{2^i r}u_{2^{i+M}r-2^i r}u_{2^{i+M}r+2^i r}}{v_{2^i r}^2 v_{2^{i+M}r}^2} = \sum_{i=1}^{M}\frac{q^{2^i r}u_{2^{i+N}r-2^i r}u_{2^{i+N}r+2^i r}}{v_{2^i r}^2 v_{2^{i+N}r}^2},$$
$$\sum_{i=1}^{2N}\frac{(\pm 1)^i q^{2^i r}u_{2^{i+2M}r-2^i r}u_{2^{i+2M}r+2^i r}}{u_{2^i r}^2 u_{2^{i+2M}r}^2} = \sum_{i=1}^{2M}\frac{(\pm 1)^i q^{2^i r}u_{2^{i+2N}r-2^i r}u_{2^{i+2N}r+2^i r}}{u_{2^i r}^2 u_{2^{i+2N}r}^2},$$
$$\sum_{i=1}^{2N}\frac{(\pm 1)^i q^{2^i r}u_{2^{i+2M}r-2^i r}u_{2^{i+2M}r+2^i r}}{v_{2^i r}^2 v_{2^{i+2M}r}^2} = \sum_{i=1}^{2M}\frac{(\pm 1)^i q^{2^i r}u_{2^{i+2N}r-2^i r}u_{2^{i+2N}r+2^i r}}{v_{2^i r}^2 v_{2^{i+2N}r}^2}.$$

Corollary 32. *Let $r > 0$ be an integer. Then*

$$\sum_{i=1}^{\infty} \frac{4q^{2^i r} u_{2^i r} u_{3\cdot 2^i r}}{u_{2^i r}^2 u_{2^{i+1} r}^2} = \frac{v_{2r}^2}{u_{2r}^2} - D,$$

$$\sum_{i=1}^{\infty} \frac{4q^{2^i r} u_{2^i r} u_{3\cdot 2^i r}}{v_{2^i r}^2 v_{2^{i+1} r}^2} = \frac{1}{D} - \frac{u_{2r}^2}{v_{2r}^2},$$

$$\sum_{i=1}^{\infty} \frac{(\pm 1)^i 4q^{2^i r} u_{3\cdot 2^i r} u_{5\cdot 2^i r}}{u_{2^i r}^2 u_{2^{i+2} r}^2} = \frac{v_{4r}^2}{u_{4r}^2} \pm \frac{v_{2r}^2}{u_{2r}^2} - D(1 \pm 1),$$

$$\sum_{i=1}^{\infty} \frac{(\pm 1)^i 4q^{2^i r} u_{3\cdot 2^i r} u_{5\cdot 2^i r}}{v_{2^i r}^2 v_{2^{i+2} r}^2} = -\frac{u_{4r}^2}{v_{4r}^2} \mp \frac{u_{2r}^2}{v_{2r}^2} + \frac{1}{D}(1 \pm 1).$$

Example 33. To apply our ideas to the case of Fibonacci and Lucas numbers, we set $p = 1$ and $q = -1$, then $\alpha = \varphi$, $\beta = 1 - \varphi$, and $D = 5$. Corollary 32 gives

$$\sum_{i=1}^{\infty} \frac{F_{3\cdot 2^i r}}{F_{2^i r} F_{2^{i+1} r}^2} = \frac{L_{2r}^2}{4F_{2r}^2} - \frac{5}{4},$$

$$\sum_{i=1}^{\infty} \frac{F_{2^i r} F_{3\cdot 2^i r}}{L_{2^i r}^2 L_{2^{i+1} r}^2} = \frac{1}{20} - \frac{F_{2r}^2}{4L_{2r}^2},$$

$$\sum_{i=1}^{\infty} \frac{(\pm 1)^i F_{3\cdot 2^i r} F_{5\cdot 2^i r}}{F_{2^i r}^2 F_{2^{i+2} r}^2} = \frac{L_{4r}^2}{4F_{4r}^2} \pm \frac{L_{2r}^2}{4F_{2r}^2} - \frac{5}{4}(1 \pm 1),$$

$$\sum_{i=1}^{\infty} \frac{(\pm 1)^i F_{3\cdot 2^i r} F_{5\cdot 2^i r}}{L_{2^i r}^2 L_{2^{i+2} r}^2} = -\frac{F_{4r}^2}{4L_{4r}^2} \mp \frac{F_{2r}^2}{4L_{2r}^2} + \frac{1}{20}(1 \pm 1).$$

6 Further discussion

It is possible to rewrite Theorems 3 to 6 in terms of α or even consider the more general identity

$$\frac{(A+B)(x_{i+j} y_i - x_i y_{i+j})}{(Ax_i - By_i)(Ax_{i+j} - By_{i+j})} = \frac{x_i + y_i}{Ax_i - By_i} - \frac{x_{i+j} + y_{i+j}}{Ax_{i+j} - By_{i+j}}.$$

The results should be similar except that the limit does not vanish in this case. We leave it to the readers to work it out.

Furthermore, we believe it is possible to further generalize the identity in §4 to include generalized Lucas sequences as well. This will be be an interesting topic to pursue in future research.

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