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Some Determinantal Representations of Eulerian Polynomials and Their q-Analogues

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Abstract

The author recently q-extended the approach of Qi, Wang, and Guo to derangement numbers, and successfully applied this extended approach to obtain expressions for certain types B and D of derangement polynomials as determinants of order $n+1$. In this paper, we demonstrate that the q-extended approach is applicable to other generating functions, including the Eulerian polynomials $A_n(t), B_n(t)$, and $D_n(t)$ for types A, B, and D, as well as the types A and B "inv" analogues of q-Eulerian polynomials $A_n^{\mathrm{inv}}(t,q)$ and $B_n^{\mathrm{inv}}(t,q)$. We also present a new recurrence relation for $B_n^{\mathrm{inv}}(t,q)$.

1 Introduction

Certain mathematical quantities are expressible as determinants. See [12] for useful tools for evaluating determinants, examples of evaluated determinants in context, and further references. As far as the present work is concerned, a notable example is the nth classical derangement number d_n. Qi, Wang, and Guo [13], by exploiting Lemma 1 below, were able to express d_n as a tridiagonal determinant of order $n+1$. The involved generating function of d_n is of the exponential type. See Section 2 for undefined terms.

An inspection of the approach of Qi et al. reveals that it is q-generalizable. The author [10], by coming up with a q-version of Lemma 1, obtained expressions for types B and D of derangement polynomials $d_n^B(q) := \sum_{\sigma\in\mathcal{D}_n^B} q^{\mathrm{fmaj}(\sigma)}$ and $d_n^D(q) := \sum_{\sigma\in\mathcal{D}_n^D} q^{\mathrm{maj}(\sigma)}$, studied previously in [6, 8], as determinants of order $n+1$, whose generating functions are of the factorial type.

The above mentioned derangement polynomials enumerate signed and even-signed derangements according to a certain statistic. By extrapolating this idea, we can, in general, consider generating functions that enumerate other combinatorial objects according to tuples of statistics. For instance, one can consider the classical Eulerian polynomial $A_n(x)$, which is a univariate generating function of n-permutations by their descent numbers, or q-Eulerian polynomials that are bivariate generating function of permutations by their descent numbers and some other statistics. Examples of the latter include the types A and B "inv" analogues of q-Eulerian polynomials, namely, $A_n^{\mathrm{inv}}(t,q) := \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{des}(\sigma)+1} q^{\mathrm{inv}(\sigma)}$ and $B_n^{\mathrm{inv}}(t,q) := \sum_{\sigma\in B_n} t^{\mathrm{des}_B(\sigma)} q^{\mathrm{inv}_B(\sigma)}$.

The successes in [13, 9, 10] suggest that the original and extended approach of Qi et al. readily apply to generating functions of the exponential or of factorial type and whose closed form expressions are quotients, yielding determinantal expressions of them.

The purpose of this work is to show that the q-extended approach of Qi et al. applies to finding determinantal expressions of k-variate generating functions, with $k = 1, 2$.

The organization of this work is as follows. In Section 2, we gather notation and undefined terms that are needed in the rest of this paper. In Section 3, we compute determinantal expressions of Eulerian polynomials of types A, B, and D. In Section 4, we compute determinantal expressions of the q-Eulerian polynomials $A_n^{\mathrm{inv}}(t,q)$ and $B_n^{\mathrm{inv}}(t,q)$. In Section 5, we explore the relationship between $A_n^{\mathrm{inv}}(t,q)$ and $B_n^{\mathrm{inv}}(t,q)$ and obtain a recurrence relation for $B_n^{\mathrm{inv}}(t,q)$. In Section 6, we end this work by raising two questions.

2 Notation and preliminaries

Let $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the sets of non-negative integers, rational numbers, and real numbers, respectively. Let $a, b \in \mathbb{N}$. Define the interval of integers $[a, b]$ by $\{a, a+1, \dots, b\}$. In case $b < a$, we have $[a, b] = \varnothing$, the empty set. Let $\#S$ denote the cardinality of a finite set S.

Let $n \geq 1$. A square matrix $A = (a_{ij})_{1\leq i,j\leq n}$ of order n is lower Hessenberg if all entries above the superdiagonal are zeros.

Let $\mathfrak{S}_n$ denote the symmetric group of degree n, consisting of all permutations of $1, 2, \dots, n$. An element σ of $\mathfrak{S}_n$ is represented by the associated word $\sigma_1\sigma_2\cdots\sigma_n$, where $\sigma_i = \sigma(i)$ for $i = 1, 2, \dots, n$. An n-permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ is said to be a *derangement* if $\sigma_i \neq i$ for all $i = 1, 2, \dots, n$. Let $\mathcal{D}_n$ denote the set of n-derangements. Define the major index, the descent number, and the inversion number of $\sigma \in \mathfrak{S}_n$ by

$$\begin{aligned}
\mathrm{maj}(\sigma) &:= \sum_{i=1}^{n-1} i\chi(\sigma_i > \sigma_{i+1}),\\
\mathrm{des}(\sigma) &:= \sum_{i=1}^{n-1} \chi(\sigma_i > \sigma_{i+1}),\\
\mathrm{inv}(\sigma) &:= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \chi(\sigma_i > \sigma_j),
\end{aligned}$$

respectively, where $\chi(P) = 1$ or 0 depending on whether the statement P is true or not. This notion of major index applies to any permutations, signed or not. The nth classical Eulerian polynomial is defined by $A_n(x) := \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{des}(\sigma)+1}$. It is well known that the exponential generating function of $(A_n(x))_{n\geqslant 0}$ has the following expression:

$$A(x,z) := \sum_{n\geq 0} A_n(x)\frac{z^n}{n!} = \frac{1-x}{1-xe^{z(1-x)}}, \tag{1}$$

where $A_0(x) := 1$ by convention.

Let B_n denote the nth hyperoctahedral group, consisting of all signed permutations of $1, 2, \dots, n$, represented by the associated words. The above definition of derangements also applies to signed permutations. Let $\mathcal{D}_n^B$ denote the set of signed n-derangements. Three needed statistics on B_n are the flag major index, the type B descent number, and the type B inversion number, defined respectively by

$$\begin{aligned}
\mathrm{fmaj}(\sigma) &:= 2\,\mathrm{maj}(\sigma) + N(\sigma),\\
\mathrm{des}_B(\sigma) &:= \sum_{i=0}^{n-1} \chi(\sigma_i > \sigma_{i+1}),\\
\mathrm{inv}_B(\sigma) &:= \mathrm{inv}(\sigma) + \sum_{i=1}^{n} |\sigma_i|\chi(\sigma_i < 0),
\end{aligned}$$

where $N(\sigma) := \#\{i\colon \sigma_i < 0\}$ is the number of negative letters of σ, and $\sigma_0 := 0$. The above definition of flag major index is due to Adin-Brenti-Roichman [1], and those of des_B and inv_B are due to Brenti [5], who had also shown that $\mathrm{inv}_B \equiv l_B$, where l_B is the type B length function.

The nth type B Eulerian polynomial is defined by $B_n(x) := \sum_{\sigma\in B_n} x^{\mathrm{des}_B(\sigma)}$. It is known that the exponential generating function of $(B_n(x))_{n\geqslant 0}$ has the following expression:

$$B(x,z) := \sum_{n\geq 0} B_n(x)\frac{z^n}{n!} = \frac{(1-x)e^{z(1-x)}}{1-xe^{2z(1-x)}}, \tag{2}$$

where $B_0(x) := 1$ by convention.

Let D_n denote the nth even-signed permutation group, consisting even-signed permutations of $1, 2, \dots, n$ of length n. An element σ of D_n is a signed n-permutation $\sigma_1\sigma_2\cdots\sigma_n$ with $N(\sigma)$ even. Being a subgroup of B_n, the definition of derangements applies to D_n. Let $\mathcal{D}_n^D$ denote the set of even-signed n-derangements. The type D descent number of $\sigma \in D_n$ is defined by

$$\mathrm{des}_D(\sigma) := \sum_{i=0}^{n-1} \chi(\sigma_i > \sigma_{i+1}),$$

where $\sigma_0 := -\sigma_2$. The nth type D Eulerian polynomial is defined by $D_n(x) := \sum_{\sigma\in D_n} x^{\mathrm{des}_D(\sigma)}$. It is known that the exponential generating function of $(D_n(x))_{n\geqslant 0}$ has the following expression:

$$D(x,z) := \sum_{n\geq 0} D_n(x)\frac{z^n}{n!} = \frac{(1-x)\big(e^{z(1-x)} - xze^{2z(1-x)}\big)}{1-xe^{2z(1-x)}}, \tag{3}$$

where $D_0(x) := 1$ by convention.

Those two classes of derangement polynomials mentioned in the Introduction are

$$d_n^B(q) := \sum_{\sigma\in\mathcal{D}_n^B} q^{\mathrm{fmaj}(\sigma)}, \quad \text{and} \quad d_n^D(q) := \sum_{\sigma\in\mathcal{D}_n^D} q^{\mathrm{maj}(\sigma)},$$

studied previously by the author [6, 8].

Let x and q be commuting indeterminates. Define

$$(x;q)_n := \begin{cases} (1-x)(1-xq)\cdots(1-xq^{n-1}) & \text{if } n\geq 1; \\ 1 & \text{otherwise.} \end{cases}$$

Q-integers, q-factorials and q-binomial coefficients are defined by

$$\begin{aligned} &[n]_q := 1+q+\cdots+q^{n-1}, \\ &[n]_q! := [1]_q[2]_q\cdots[n]_q, \quad \text{and} \\ &\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad 0\leq k\leq n, \end{aligned}$$

respectively.

The original approach of Qi et al. is based on Lemma 1, which is stated in [4, p. 40] as an exercise. A solution of this Bourbaki exercise can be found in [9].

Lemma 1. *Let $u=u(x)$ and $v=v(x)$ be two real functions that are n times differentiable on an interval $I\subset\mathbb{R}$. Then at every point where $v(x)\neq 0$, we have*

$$\frac{d^n}{dx^n}\left(\frac{u}{v}\right) = \frac{(-1)^n}{v^{n+1}} \begin{vmatrix} u & v & 0 & \cdots & 0 & 0 \\ u' & v' & v & \cdots & 0 & 0 \\ u'' & v'' & 2v' & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ u^{(n-1)} & v^{(n-1)} & \binom{n-1}{1}v^{(n-2)} & \cdots & \binom{n-1}{n-2}v' & v \\ u^{(n)} & v^{(n)} & \binom{n}{1}v^{(n-1)} & \cdots & \binom{n}{n-2}v'' & \binom{n}{n-1}v' \end{vmatrix}.$$

Let $\{f_n\}$ be a sequence of numbers or k-variate formal power series independent of x, where $k\geq 1$. If the exponential generating function of $\{f_n\}$,

$$f(x) = \sum_{n\geq 0} f_n\frac{x^n}{n!},$$

can be written as a quotient $u(x)/v(x)$, then with Lemma 1, the approach of Qi et al. allows us to express

$$f_n = \frac{\partial^n}{\partial x^n} f(x)\bigg|_{x=0}$$

as a determinant of order $n+1$.

The author [10] q-extended Lemma 1 to become Lemma 2, where for formal power series $f(t;q) \in \mathbb{Q}[[t,q]]$ in t and q with rational coefficients, its q-derivative with respect to t is defined as

$$D_q f(t;q) := \frac{f(tq;q) - f(t;q)}{(q-1)t}. \tag{4}$$

It is easy to see that $D_q t^n = [n]_q t^{n-1}$. See [2, §2.2] for details of q-calculus.

Lemma 2. *Let $n \geq 1$ and let $u = u(t;q), v = v(t;q) \in \mathbb{Q}[[t,q]]$ be such that their nth q-derivatives $D_q^n u(t;q)$ and $D_q^n v(t;q)$ exist. Then at every point where $v(t;q) \neq 0$, we have*

$$D_q^n\left(\frac{u}{v}\right)(t;q) = \frac{(-1)^n}{v^{n+1}} \begin{vmatrix} u & v & 0 & \cdots & 0 & 0 \\ D_q u & D_q v & v & \cdots & 0 & 0 \\ D_q^2 u & D_q^2 v & \begin{bmatrix}2\\1\end{bmatrix}_q D_q v & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ D_q^{n-1} u & D_q^{n-1} v & \begin{bmatrix}n-1\\1\end{bmatrix}_q D_q^{n-2} v & \cdots & \begin{bmatrix}n-1\\n-2\end{bmatrix}_q D_q v & v \\ D_q^n u & D_q^n v & \begin{bmatrix}n\\1\end{bmatrix}_q D_q^{n-1} v & \cdots & \begin{bmatrix}n\\n-2\end{bmatrix}_q D_q^2 v & \begin{bmatrix}n\\n-1\end{bmatrix}_q D_q v \end{vmatrix},$$

where all the formal power series on the right side are evaluated at $(tq^{-n};q)$.

To simplify the expressions, we write in the sequel $D_q^n u(tq^{-n};q)$ to mean the result of substituting the arguments of the nth q-derivative of $u(t;q)$ by $(tq^{-n};q)$, etc.

Let $(f_n)_{n\geqslant 0}$ be a sequence of formal power series dependent on q and on some other indeterminates but independent of x. If the factorial generating function of $(f_n)_{n\geqslant 0}$,

$$f(x) = \sum_{n\geq 0} f_n \frac{x^n}{[n]_q!},$$

can be written as a quotient $u(x)/v(x)$, then with Lemma 2, the q-extended approach of Qi et al. allows us to express

$$f_n = D_q^n f(x)\big|_{x=0}$$

as a determinant of order $n+1$, where $D_q^n f(x)$ on the right denotes the nth q-derivative of $f(x)$ with respect to x.

3 Determinantal expressions of Eulerian polynomials

The goal of this section is to present determinantal expressions of the Eulerian polynomials $A_n(x), B_n(x)$, and $D_n(x)$.

Theorem 3. *Let $n \geq 1$. The Eulerian polynomial $A_n(x)$ is expressible as the following lower Hessenberg determinant of order $n+1$:*

$$A_n(x) = (-1)^n \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -x & 1 & \cdots & 0 & 0 \\ 0 & -x(1-x) & -2x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -x(1-x)^{n-2} & -\binom{n-1}{1}x(1-x)^{n-3} & \cdots & -\binom{n-1}{n-2}x & 1 \\ 0 & -x(1-x)^{n-1} & -\binom{n}{1}x(1-x)^{n-2} & \cdots & -\binom{n}{n-2}x(1-x) & -\binom{n}{n-1}x \end{vmatrix}.$$

Proof. By virtue of the exponential generating function (1) of $(A_n(x))_{n\geqslant 0}$, regarding x as a parameter, we let $u(z;x) = 1-x$ and $v(z;x) = 1 - xe^{z(1-x)}$. Then for $k \geq 1$, $\partial_z^k u(z;x) = 0$ and $\partial_z^k v(z;x) = -x(1-x)^k e^{z(1-x)}$. Applying Lemma 1, and letting $z \to 0$, we have

$$\begin{aligned} A_n(x) &= \frac{\partial^n}{\partial z^n} A(x,z)\bigg|_{z=0} \\ &= \frac{(-1)^n}{(1-x)^{n+1}} \begin{vmatrix} 1-x & 1-x & 0 & \cdots & 0 & 0 \\ 0 & -x(1-x) & 1-x & \cdots & 0 & 0 \\ 0 & -x(1-x)^2 & -2x(1-x) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -x(1-x)^{n-1} & -\binom{n-1}{1}x(1-x)^{n-2} & \cdots & -\binom{n-1}{n-2}x(1-x) & 1-x \\ 0 & -x(1-x)^n & -\binom{n}{1}x(1-x)^{n-1} & \cdots & -\binom{n}{n-2}x(1-x)^2 & -\binom{n}{n-1}x(1-x) \end{vmatrix} \\ &= (-1)^n \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -x & 1 & \cdots & 0 & 0 \\ 0 & -x(1-x) & -2x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -x(1-x)^{n-2} & -\binom{n-1}{1}x(1-x)^{n-3} & \cdots & -\binom{n-1}{n-2}x & 1 \\ 0 & -x(1-x)^{n-1} & -\binom{n}{1}x(1-x)^{n-2} & \cdots & -\binom{n}{n-2}x(1-x) & -\binom{n}{n-1}x \end{vmatrix}, \end{aligned}$$

where the last equality follows from dividing each column by $1-x$. □

Theorem 4. *Let $n \geq 1$. The type B Eulerian polynomial $B_n(x)$ is expressible as the following lower Hessenberg determinant of order $n+1$:*

$$B_n(x) = (-1)^n \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 1-x & -2x & 1 & \cdots & 0 & 0 \\ (1-x)^2 & -2^2x(1-x) & -2^2x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (1-x)^{n-1} & -2^{n-2}x(1-x)^{n-2} & -\binom{n-1}{1}2^{n-3}x(1-x)^{n-3} & \cdots & -\binom{n-1}{n-2}2x & 1 \\ (1-x)^n & -2^n x(1-x)^{n-1} & -\binom{n}{1}2^{n-2}x(1-x)^{n-2} & \cdots & -\binom{n}{n-2}2^2x(1-x) & -\binom{n}{n-1}2x \end{vmatrix}.$$

Proof. By virtue of (2), let $u(z;x) = (1-x)e^{z(1-x)}$ and $v(z;x) = 1 - xe^{2z(1-x)}$. Then for $k \geq 1$, $\partial_z^k u(z;x) = (1-x)^{k+1}e^{z(1-x)}$ and $\partial_z^k v(z;x) = -2^k x(1-x)^k e^{z(1-x)}$. Applying Lemma 1, letting $z \to 0$, and followed by dividing each column by $1-x$, we obtain

$$
\begin{aligned}
B_n(x) &= \frac{\partial^n}{\partial z^n} B(x,z)\Big|_{z=0} \\
&= (-1)^n \begin{vmatrix}
1 & 1 & 0 & \cdots & 0 & 0 \\
1-x & -2x & 1 & \cdots & 0 & 0 \\
(1-x)^2 & -2^2x(1-x) & -2^2x & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(1-x)^{n-1} & -2^{n-2}x(1-x)^{n-2} & -\binom{n-1}{1}2^{n-3}x(1-x)^{n-3} & \cdots & -\binom{n-1}{n-2}2x & 1 \\
(1-x)^n & -2^nx(1-x)^{n-1} & -\binom{n}{1}2^{n-2}x(1-x)^{n-2} & \cdots & -\binom{n}{n-2}2^2x(1-x) & -\binom{n}{n-1}2x
\end{vmatrix}.
\end{aligned}
$$

□

Theorem 5. *Let $n \geq 1$. The type D Eulerian polynomial $D_n(x)$ is expressible as the following lower Hessenberg determinant of order $n+1$:*

$$
\begin{aligned}
&D_n(x) \\
&= (-1)^n \begin{vmatrix}
1 & 1 & 0 & \cdots & 0 & 0 \\
(1-2x) & -2x & 1 & \cdots & 0 & 0 \\
(1-x)(1-5x) & -2^2x(1-x) & -2^2x & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(1-x)^{n-2}(1-((n-1)2^{n-2}+1)x) & -2^{n-1}x(1-x)^{n-2} & -\binom{n-1}{1}2^{n-2}x(1-x)^{n-3} & \cdots & -\binom{n-1}{n-2}2x & 1 \\
(1-x)^{n-1}(1-(n2^{n-1}+1)x) & -2^nx(1-x)^{n-1} & -\binom{n}{1}2^{n-1}x(1-x)^{n-2} & \cdots & -\binom{n}{n-2}2^2x(1-x) & -\binom{n}{n-1}2x
\end{vmatrix}.
\end{aligned}
$$

Proof. By virtue of (3), let $u(z;x) = (1-x)\big(e^{z(1-x)} - xze^{2z(1-x)}\big)$ and $v(z;x) = 1 - xe^{2z(1-x)}$. For $k \geq 1$,

$$
\begin{aligned}
\partial_z^k u(z;x) &= (1-x)\big((1-x)^k e^{z(1-x)} - x(2^k(1-x)^k z + \tbinom{k}{1}2^{k-1}(1-x)^{k-1})e^{2z(1-x)}\big), \\
\partial_z^k v(z;x) &= -2^k x(1-x)^k e^{z(1-x)}
\end{aligned}
$$

so that

$$
\partial_z^k u(z;x)\big|_{z=0} = (1-x)^k\big(1-(k2^{k-1}+1)x\big), \quad \text{and} \quad \partial_z^k v(z;x)\big|_{z=0} = -2^k x(1-x)^k.
$$

Applying Lemma 1 with $z \to 0$, we have

$$
\begin{aligned}
D_n(x) &= \frac{\partial^n}{\partial z^n} D(x,z)\Big|_{z=0} \\
&= (-1)^n \begin{vmatrix}
1 & 1 & 0 & \cdots & 0 & 0 \\
(1-2x) & -2x & 1 & \cdots & 0 & 0 \\
(1-x)(1-5x) & -2^2x(1-x) & -2^2x & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(1-x)^{n-2}(1-((n-1)2^{n-2}+1)x) & -2^{n-1}x(1-x)^{n-2} & -\binom{n-1}{1}2^{n-2}x(1-x)^{n-3} & \cdots & -\binom{n-1}{n-2}2x & 1 \\
(1-x)^{n-1}(1-(n2^{n-1}+1)x) & -2^nx(1-x)^{n-1} & -\binom{n}{1}2^{n-1}x(1-x)^{n-2} & \cdots & -\binom{n}{n-2}2^2x(1-x) & -\binom{n}{n-1}2x
\end{vmatrix}.
\end{aligned}
$$

□

4 Determinantal expressions of q-Eulerian polynomials

Bivariate extensions of $A_n(x)$ and $B_n(x)$ are available in the literature, namely, $A_n^{\mathrm{inv}}(t,q) := \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{des}(\sigma)+1}q^{\mathrm{inv}(\sigma)}$, whose factorial generating function was computed by Stanley [15]:

$$
A^{\mathrm{inv}}(x;t,q) := \sum_{n\geq 0} A_n^{\mathrm{inv}}(t,q)\frac{x^n}{[n]_q!} = \frac{1-t}{1-te(x(1-t);q)}, \tag{5}
$$

where $e(x;q) := \sum_{n\geq 0} \frac{x^n}{[n]_q!}$ is a q-exponential, and $B_n^{\mathrm{inv}}(t,q) := \sum_{\sigma\in B_n} t^{\mathrm{des}_B(\sigma)} q^{\mathrm{inv}_B(\sigma)}$.

Désarménien and Foata [11] showed that the following "semi" q-recurrence relation holds:

$$A_n^{\mathrm{inv}}(t,q) = \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q A_k^{\mathrm{inv}}(t,q) t(1-t)^{n-1-k}, \tag{6}$$

"semi" in the sense that the summands on the right involve two factors one of which depends on q and the other does not. The author [7] showed that $A_n^{\mathrm{inv}}(t,q)$ satisfies the following "fully" q-recurrence relation:

$$A_{n+1}^{\mathrm{inv}}(t,q) = (1+tq^n)A_n^{\mathrm{inv}}(t,q) + \sum_{k=1}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q q^k A_{n-k}^{\mathrm{inv}}(t,q) A_k^{\mathrm{inv}}(t,q).$$

As far as coefficient extraction is concerned, $A^{\mathrm{inv}}(x;t,q)$ enjoys the following property:

$$D_q^n A^{\mathrm{inv}}(x;t,q)\big|_{x=0} = A_n^{\mathrm{inv}}(t,q),$$

where the q-derivative is taken with respect to x. The key to the applicability of the approach of Qi et al., the original or the q-extended one, is that the exponential or factorial generating function be expressible as a quotient. That of $(A_n^{\mathrm{inv}}(t,q))_{n\geqslant 0}$, i.e., (5), is definitely a quotient. Applying the q-extended approach of Qi et al. readily yields Theorem 6.

Theorem 6. *Let $n \geq 1$. The q-Eulerian polynomial $A_n^{\mathrm{inv}}(t,q)$ is expressible as the following lower Hessenberg determinant of order $n+1$:*

$$A^{\mathrm{inv}}(t,q) = (-1)^n \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -t & 1 & \cdots & 0 & 0 \\ 0 & -t(1-t) & -\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q t & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -t(1-t)^{n-2} & -\begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q t(1-t)^{n-3} & \cdots & -\begin{bmatrix} n-1 \\ n-2 \end{bmatrix}_q t & 1 \\ 0 & -t(1-t)^{n-1} & \begin{bmatrix} n \\ 1 \end{bmatrix}_q t(1-t)^{n-2} & \cdots & -\begin{bmatrix} n \\ n-2 \end{bmatrix}_q t(1-t) & -\begin{bmatrix} n \\ n-1 \end{bmatrix}_q t \end{vmatrix}.$$

Proof. Let $u(x;t,q) := 1-t$ and $v(x;t,q) := 1-te(x(1-t);q)$. It is clear that $u(0;t,q) = v(0;t,q) = 1-t$. For $k \geq 1$, the kth q-derivatives of $u(x;t,q)$ and $v(x;t,q)$ with respect to x are, respectively, $D_q^k u(x;t,q) = 0$ and

$$D_q^k v(x;t,q) = -t\sum_{n\geq k} \frac{(1-t)^n x^{n-k}}{[n-k]_q!} = -t(1-t)^k \sum_{n\geq 0} \frac{[x(1-t)]^n}{[n]_q!}.$$

Evaluating them at $(xq^{-n};t,q)$ followed by letting $x\to 0$, we have $D_q^k u(xq^{-n};t,q)\big|_{x=0} = 0$

and $D_q^k v(xq^{-n};t,q)\big|_{x=0} = -t(1-t)^k$. Applying Lemma 2 with $x \to 0$, we obtain

$$A^{\mathrm{inv}}(t,q)$$
$$= \frac{(-1)^n}{(1-t)^{n+1}} \begin{vmatrix} 1-t & 1-t & 0 & \cdots & 0 & 0 \\ 0 & -t(1-t) & 1-t & \cdots & 0 & 0 \\ 0 & -t(1-t)^2 & -\begin{bmatrix}2\\1\end{bmatrix}_q t(1-t) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -t(1-t)^{n-1} & -\begin{bmatrix}n-1\\1\end{bmatrix}_q t(1-t)^{n-2} & \cdots & -\begin{bmatrix}n-1\\n-2\end{bmatrix}_q t(1-t) & 1-t \\ 0 & -t(1-t)^n & \begin{bmatrix}n\\1\end{bmatrix}_q t(1-t)^{n-1} & \cdots & -\begin{bmatrix}n\\n-2\end{bmatrix}_q t(1-t)^2 & -\begin{bmatrix}n\\n-1\end{bmatrix}_q t(1-t) \end{vmatrix}$$
$$= (-1)^n \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -t & 1 & \cdots & 0 & 0 \\ 0 & -t(1-t) & -\begin{bmatrix}2\\1\end{bmatrix}_q t & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -t(1-t)^{n-2} & -\begin{bmatrix}n-1\\1\end{bmatrix}_q t(1-t)^{n-3} & \cdots & -\begin{bmatrix}n-1\\n-2\end{bmatrix}_q t & 1 \\ 0 & -t(1-t)^{n-1} & \begin{bmatrix}n\\1\end{bmatrix}_q t(1-t)^{n-2} & \cdots & -\begin{bmatrix}n\\n-2\end{bmatrix}_q t(1-t) & -\begin{bmatrix}n\\n-1\end{bmatrix}_q t \end{vmatrix},$$

where the second equality follows from dividing each column by $1-t$. □

Let us now look at $B_n^{\mathrm{inv}}(t,q)$, whose first three members are as follows:

$$\begin{aligned} B_1^{\mathrm{inv}}(t,q) &= 1+qt, \\ B_2^{\mathrm{inv}}(t,q) &= 1+(2q+2q^2+2q^3)t+q^4t^2, \\ B_3^{\mathrm{inv}}(t,q) &= 1+(3q+4q^2+5q^3+4q^4+4q^5+2q^6+q^7)t \\ &\quad +(q^2+2q^3+4q^4+4q^5+5q^6+4q^7+3q^8)t^2+q^9t^3, \end{aligned}$$

Reiner [14, Corollary 4.7] showed, in our terminology, that

$$\sum_{n\geq 0} \frac{x^n \sum_{\pi\in B_n} q^{\mathrm{inv}_B(\pi)} t^{n-\mathrm{des}_B(\pi)}}{[2]_q[4]_q\cdots[2n]_q} = \left(1-\sum_{n\geq 1}\frac{(t-1)^{n-1}x^n}{[n]_q!}\right)^{-1} \sum_{n\geq 0}\frac{(t-1)^n x^n}{[2]_q[4]_q\cdots[2n]_q}.$$

Note that $[2k]_q = (1+q^k)[k]_q = (1+q)[k]_{q^2}$ and that $(1+q)(1+q^2)\cdots(1+q^n) = (-q;q)_n$. The replacements $x \mapsto xt$ and $t \mapsto t^{-1}$ in the preceding factorial generating function identity then yield

$$\begin{aligned} B^{\mathrm{inv}}(x;t,q) &:= \sum_{n\geq 0} B_n^{\mathrm{inv}}(t,q)\frac{x^n}{(-q;q)_n[n]_q!} \\ &= \left(1-\frac{t}{1-t}\sum_{n\geq 1}\frac{(1-t)^n x^n}{[n]_q!}\right)^{-1}\sum_{n\geq 0}\frac{(1-t)^n x^n}{(-q;q)_n[n]_q!}, \end{aligned} \tag{7}$$

which enjoys the following property:

$$D_q^n B^{\mathrm{inv}}(x;t,q)\big|_{x=0} = \frac{B_n^{\mathrm{inv}}(t,q)}{(-q;q)_n}.$$

The q-extended approach of Qi et al. readily applies to $B^{\mathrm{inv}}(x;t,q)$, yielding Theorem 7.

Theorem 7. *Let $n \geq 1$. The type B q-Eulerian polynomial $B_n^{\mathrm{inv}}(t,q)$ can be expressed as the following lower Hessenberg determinant of order $n+1$:*

$$B_n^{\mathrm{inv}}(t,q)$$
$$= (-1)^n(-q;q)_n \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ \frac{(1-t)}{(-q;q)_1} & -t & 1 & \cdots & 0 & 0 \\ \frac{(1-t)^2}{(-q;q)_2} & -t(1-t) & -\begin{bmatrix}2\\1\end{bmatrix}_q t & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{(1-t)^{n-1}}{(-q;q)_{n-1}} & -t(1-t)^{n-2} & -\binom{n-1}{1}t(1-t)^{n-3} & \cdots & -\begin{bmatrix}n-1\\n-2\end{bmatrix}_q t & 1 \\ \frac{(1-t)^n}{(-q;q)_n} & -t(1-t)^{n-1} & -\begin{bmatrix}n\\1\end{bmatrix}_q t(1-t)^{n-2} & \cdots & -\begin{bmatrix}n\\n-2\end{bmatrix}_q t(1-t) & -\begin{bmatrix}n\\n-1\end{bmatrix}_q t \end{vmatrix}.$$

Proof. Let

$$u(x;t,q) := \sum_{n\geq 0} \frac{(1-t)^n x^n}{(-q;q)_n [n]_q!} \quad \text{and} \quad v(x;t,q) := 1 - \frac{t}{1-t} \sum_{n\geq 1} \frac{(1-t)^n x^n}{[n]_q!}.$$

It is easy to see that $u(0;t,q) = 1$ and $v(0;t,q) = 1$. For $1 \leq k \leq n$, the kth q-derivatives of $u(x;t,q)$ and $v(x;t,q)$ with respect to x are, respectively,

$$D_q^k u(x;t,q) = \sum_{n\geq k} \frac{(1-t)^n x^{n-k}}{(-q;q)_n [n-k]_q!} = (1-t)^k \sum_{n\geq 0} \frac{(1-t)^n x^n}{(-q;q)_{n+k}[n]_q!},$$
$$D_q^k v(x;t,q) = -\frac{t}{1-t} \sum_{n\geq k} \frac{(1-t)^n x^{n-k}}{[n-k]_q!} = -t(1-t)^{k-1} \sum_{n\geq 0} \frac{(1-t)^n x^n}{[n]_q!}.$$

It follows that

$$D_q^k u(xq^{-n};t,q)\big|_{x=0} = \lim_{x\to 0}(1-t)^k \sum_{n\geq 0} \frac{(1-t)^n (xq^{-n})^n}{(-q;q)_{n+k}[n]_q!} = \frac{(1-t)^k}{(-q;q)_k},$$
$$D_q^k v(xq^{-n};t,q)\big|_{x=0} = -\lim_{x\to 0} t(1-t)^{k-1} \sum_{n\geq 0} \frac{(1-t)^n (xq^{-n})^n}{[n]_q!} = -t(1-t)^{k-1}.$$

Since letting $x \to 0$ after evaluating at $(xq^{-n};t,q)$ makes no difference to letting $x \to 0$

directly, it now follows from Lemma 2 that

$$
\begin{aligned}
B_n^{\mathrm{inv}}(t,q) &= (-q;q)_n D_q^n B^{\mathrm{inv}}(xq^{-n};t,q)\big|_{x=0} \\
&= (-1)^n(-q;q)_n \begin{vmatrix}
1 & 1 & 0 & \cdots & 0 & 0 \\
\frac{(1-t)}{(-q;q)_1} & -t & 1 & \cdots & 0 & 0 \\
\frac{(1-t)^2}{(-q;q)_2} & -t(1-t) & -{2 \brack 1}_q t & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{(1-t)^{n-1}}{(-q;q)_{n-1}} & -t(1-t)^{n-2} & -\binom{n-1}{1} t(1-t)^{n-3} & \cdots & -{n-1 \brack n-2}_q t & 1 \\
\frac{(1-t)^n}{(-q;q)_n} & -t(1-t)^{n-1} & -{n \brack 1}_q t(1-t)^{n-2} & \cdots & -{n \brack n-2}_q t(1-t) & -{n \brack n-1}_q t
\end{vmatrix}.
\end{aligned}
$$

□

It is clear that $B_n^{\mathrm{inv}}(t,1) = B_n(t)$. On the other hand, one can move the multiplicative factor $(-q;q)_n$ of the determinantal expression in Theorem 7 into the determinant by multiplying each entry of the first column by $(-q;q)_n$, the resulting quantities $(-q;q)_n/(-q;q)_k$ are q-analogues of powers of 2. In Theorem 4, powers of 2 are are distributed over the last n columns of the determinant. When $q \to 1$, the determinantal expression in Theorem 7 does not reduce to become that in Theorem 4.

5 A recurrence relation for $B_n^{\mathrm{inv}}(t,q)$

Unlike its type A counterpart, a recurrence relation for $B_n^{\mathrm{inv}}(t,q)$ was, up to now, apparently not in the literature. The goal of this final section is to fill in this gap. Before doing so, it is important to note that $A_n^{\mathrm{inv}}(t,q)$ and $B_n^{\mathrm{inv}}(t,q)$ are closely related with each other.

Proposition 8. *For $n \geq 1$, we have*

$$
\begin{aligned}
B_n^{\mathrm{inv}}(t,q) &= \sum_{k=0}^{n} {n \brack k}_q \frac{(-q;q)_n (1-t)^{n-k} A_k^{\mathrm{inv}}(t,q)}{(-q;q)_{n-k}}, && (8) \\
&= \sum_{k=0}^{n} {n \brack k}_{q^2} (-q;q)_k (1-t)^{n-k} A_k^{\mathrm{inv}}(t,q). && (9)
\end{aligned}
$$

Proof. An inspection of the denominator of $B^{\mathrm{inv}}(x;t,q)$ reveals that

$$
1 - \frac{t}{1-t}\sum_{n\geq 1} \frac{(1-t)^n x^n}{[n]_q!} = \frac{1 - te(x(1-t);q)}{1-t}
$$

so that

$$
B^{\mathrm{inv}}(x;t,q) = A^{\mathrm{inv}}(x;t,q) \sum_{n\geq 0} \frac{(1-t)^n x^n}{(-q;q)_n [n]_q!}. \tag{10}
$$

Extracting the coefficients of x^n on both sides, we have

$$\frac{B_n^{\text{inv}}(t,q)}{(-q;q)_n[n]_q!} = \sum_{k=0}^{n} \frac{A_k^{\text{inv}}(t,q)(1-t)^{n-k}}{[k]_q!(-q;q)_{n-k}[n-k]_q!},$$

whence (8). Since $[2k]_q! = (1+q)^k[k]_{q^2}! = (-q;q)_k[k]_q!$, we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(-q;q)_k(-q;q)_{n-k}}{(-q;q)_n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2}. \tag{11}$$

Substituting (11) into (8), equality (9) follows. □

Based on Stanley's factorial generating function (5), Björner and Brenti [3, §7.2] generalized the factorial generating functions by descent numbers and lengths to other Coxeter families of which (10) in the preceding proof is a concrete type B realization. Compare (10) with [3, Exercise 7.2.5(b)].

Theorem 9. *For $n \geq 1$, $B_n^{\text{inv}}(t,q)$ satisfies the following recurrence relation:*

$$B_{n+1}^{\text{inv}}(t,q) = \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{q^2} (-q;q)_k[(-q;q)_{n+1-k}t + (1-t)](1-t)^{n-k}A_k^{\text{inv}}(t,q).$$

Proof. Starting from (9) with $n+1$ in place of n, followed by using (6), we obtain

$$\begin{aligned}
B_{n+1}^{\text{inv}}(t,q) &= (-q;q)_{n+1}A_{n+1}^{\text{inv}}(t,q) + \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{q^2} (-q;q)_k(1-t)^{n+1-k}A_k^{\text{inv}}(t,q) \\
&= (-q;q)_{n+1} \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q t(1-t)^{n-k}A_k^{\text{inv}}(t,q) \\
&\quad + \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{q^2} (-q;q)_k(1-t)^{n+1-k}A_k^{\text{inv}}(t,q) \\
&= (-q;q)_{n+1} \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{q^2} \frac{(-q;q)_k(-q;q)_{n+1-k}}{(-q;q)_{n+1}} t(1-t)^{n-k}A_k^{\text{inv}}(t,q) \\
&\quad + \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{q^2} (-q;q)_k(1-t)^{n+1-k}A_k^{\text{inv}}(t,q) \\
&= \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{q^2} (-q;q)_k[(-q;q)_{n+1-k}t + (1-t)](1-t)^{n-k}A_k^{\text{inv}}(t,q).
\end{aligned}$$

It is not apparent that the sum on the right can be written as a combination of $B_k^{\text{inv}}(t,q)$. □

6 Two questions

In the preceding sections, we obtained expressions of Eulerian polynomials of types A, B, and D, as well as the "inv" analogues of types A and B q-Eulerian polynomials, as lower Hessenberg determinants.

Besides representing Eulerian polynomials, determinants also play important roles in the theory of symmetric functions. For instance, the Jacobi-Trudi identity is a determinantal representation of skew Schur functions in terms of homogeneous symmetric functions [16, §7.16]. In view of the present work, it is natural to ask

Question 10. Can a given determinant be interpreted in terms of symmetric functions?

Another occurrence of determinants is in the classical definition of Schur functions [16]:

$$a_{\lambda+\delta}/a_\delta = s_\lambda(x_1, \ldots, x_n),$$

where λ is a partition of n with at most n parts, $\delta = (n-1, n-2, \ldots, 0)$ and $a_{\lambda+\delta} = \det\big(x_i^{\lambda_i+n-j}\big)_{i,j=1}^n$. Note that neither $a_{\lambda+\delta}$ nor a_δ is symmetric in $x_1, \ldots, x_n$, but $a_{\lambda+\delta}/a_\delta$ is. Extending Question 10, more generally, one can ask

Question 11. Can a given quotient of two determinants be interpreted in terms of symmetric functions?

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