Journal of Integer Sequences, Vol. 26 (2023),
Article 23.1.1

Arithmetic Progressions Among Powerful Numbers

Tsz Ho Chan
Mathematics Department
Kennesaw State University
Marietta, GA 30060
USA
tchan4@kennesaw.edu

Abstract

In this paper, we study k-term arithmetic progressions $N, N+d, \ldots, N+(k-1)d$ of powerful numbers. Unconditionally, we exhibit infinitely many 3-term arithmetic progressions of powerful numbers with $d \leq 5N^{1/2}$. Assuming the *abc*-conjecture, we obtain a nearly tight lower bound on the common difference. We also prove some partial results when $k \geq 4$ and pose some open questions.

1 Introduction and main results

For any integer $k \geq 1$, a non-trivial k-term arithmetic progression (abbreviated as k-AP) is a sequence of the form

$$N, N+d, N+2d, \ldots, N+(k-1)d$$

with initial term N and common difference $d > 0$. Any single number or two different numbers can be considered as a 1-AP or 2-AP respectively. So, from now on, we will assume $k \geq 3$. It is well-known that there are infinitely many 3-APs among perfect squares (e.g., $1, 25, 49$) but there is no 4-AP of perfect squares (first discovered by Fermat). One may ask if there exist k-APs among other interesting arithmetic or polynomial sequences. For instance, Green and Tao [5] recently proved that there are arbitrarily long arithmetic progressions among the prime numbers. In this paper, we are interested in studying arithmetic progressions of powerful numbers A001694 which are square-like.

Definition 1. A number n is *powerful* if $p^2 \mid n$ whenever $p \mid n$ (i.e., its prime factorization $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ satisfies $a_i \geq 2$ for all $1 \leq i \leq r$.)

For example, $72 = 2^3 \cdot 3^2$ is powerful but $24 = 2^3 \cdot 3$ is not. Another common name for powerful number is *squarefull* number. A closely related concept is *squarefree* number.

Definition 2. A number n is *squarefree* if $p^2 \nmid n$ for all prime $p \mid n$ (i.e., its prime factorization $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ satisfies $a_i = 1$ for all $1 \leq i \leq r$.).

For example, $30 = 2 \cdot 3 \cdot 5$ is squarefree but $24 = 2^3 \cdot 3$ is not. By unique prime factorization, one can show that any positive integer and any powerful number can be factored uniquely as $n = a^2 b$ and $n = a^2 b^3$ respectively for some integer $a \geq 1$ and squarefree number $b \geq 1$. Unlike perfect squares, there are arbitrarily long arithmetic progressions among powerful numbers.

Theorem 3. *For any integer $k \geq 3$, there is a k-term arithmetic progression of powerful numbers.*

The above theorem is not new (see remark 2 in [3] for example) but we include a proof for completeness. For $k = 3$, there is a folklore conjecture concerning 3-AP of powerful numbers which seems to have been first posed by Erdős [2].

Conjecture 4. There are no three consecutive powerful numbers.

Later, Mollin and Walsh [6] and Granville [4] reiterated the same conjecture and provided some evidence and interesting consequences. Currently, we are far from being able to prove Conjecture 4. However, Conjecture 4 follows from the famous *abc*-conjecture (see the first half of the proof of Theorem 1.6 with $d = 1$ and $y = 3$ in [1] for example). Let

$$\kappa(m) := \prod_{p \mid m} p$$

denote the *squarefree kernel* or *radical* of an integer m.

Conjecture 5 (*abc*-conjecture)**.** For any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that, for any integers a, b, c with $a + b = c$ and $\gcd(a, b) = 1$, the bound

$$\max\{|a|, |b|, |c|\} \leq C_\epsilon \cdot \kappa(abc)^{1+\epsilon}$$

holds.

In other words, there is no 3-AP of powerful numbers with common difference $d = 1$ under the *abc*-conjecture. Recently, the author [1] studied powerful numbers in short intervals, and it can be deduced that, for any $\epsilon > 0$, there is no 3-AP of powerful numbers with $d \leq N^{1/4-\epsilon}$ for sufficiently large N under the *abc*-conjecture. On the other hand, for any integer $m \geq 1$, the following three expressions

$$(2m^2 - 1)^2, (2m^2 + 2m + 1)^2, (2m^2 + 4m + 1)^2 \tag{1}$$

form a 3-AP of perfect squares with common difference $d = 8m^3 + 12m^2 + 4m$. Hence, there are infinitely many 3-APs of powerful numbers with $d \le 6N^{3/4}$. Thus, we are led to the following natural question.

Question 6. We say that $0 < \theta < 1$ is an *admissible exponent* if there exists $C_\theta > 0$ such that there are infinitely many 3-APs of powerful numbers $N, N+d, N+2d$ with common difference $d \le C_\theta N^\theta$. Find the infimum, θ_3, among all such admissible exponents.

The above discussion yields $\frac{1}{4} \le \theta_3 \le \frac{3}{4}$. We shall prove the following optimal result.

Theorem 7. *Assuming the abc-conjecture, we have* $\theta_3 = \frac{1}{2}$.

Analogously, one can define θ_k for k-AP of powerful numbers when $k \ge 4$. We have the following partial results.

Theorem 8. *Assuming the abc-conjecture, we have*

$$\frac{1}{2} \le \theta_4 \le \frac{4}{5}, \frac{1}{2} \le \theta_5 \le \frac{9}{10}, \text{ and } \frac{1}{2} \le \theta_k \le 1 - \frac{1}{10 \cdot 3^{k-5}}$$

for $k \ge 5$.

Note that the upper bounds in Theorems 7 and 8 hold unconditionally and it is their lower bounds that require the *abc*-conjecture.

It would be interesting to see if one can prove $\theta_4 > 1/2$ under the *abc*-conjecture. Another future direction would be narrowing the above ranges for θ_k when $k \ge 4$. One can also ask if there exist infinitely many 3-APs of powerful numbers with common difference $d = o(\sqrt{N})$.

1.1 Notation

Throughout the paper, the letters N, k, m, n, a, b, c and d stand for positive integers while the letters p, p_{ij} and $q_{ij'}$ stand for prime numbers. The symbol $a \mid b$ means that a divides b, the symbol $a \nmid b$ means that a does not divide b, and the symbol $p^n || a$ means that $p^n \mid a$ but $p^{n+1} \nmid a$. The function $\nu_p(a)$ stands for the p-adic valuation of a (i.e., $\nu_p(a) = n$ when $p^n || a$). The symbol $f(x) \ll g(x)$ is equivalent to $|f(x)| \le C g(x)$ for some constant $C > 0$. The symbol $f(x) \ll_\lambda g(x)$ means that the implicit constant may depend on λ. Finally, the symbol $f(x) = o(g(x))$ means that $\lim_{x \to \infty} f(x)/g(x) = 0$.

2 Proof of Theorem 3: long AP of powerful numbers

Proof. We apply induction on k. The base case $k = 3$ follows from (1) on 3-AP among perfect squares. Suppose, for some $k \ge 3$, there is a k-AP among powerful numbers, say

$$a_1^2 b_1^3 < a_2^2 b_2^3 < \cdots < a_k^2 b_k^3 \quad \text{with common difference } d \ge 1.$$

Consider the number $a_k^2 b_k^3 + d = a^2 b$ for some integer a and squarefree number b. Then

$$a_1^2 b_1^3 b^2 < a_2^2 b_2^3 b^2 < \cdots < a_k^2 b_k^3 b^2 < a^2 b^3$$

is a $(k+1)$-AP of powerful numbers with common difference db^2. This finishes the induction proof. $\square$

3 Proof of Theorem 7: 3-AP upper bound

Proof. For the upper bound $\theta_3 \leq 1/2$, we first consider the following 3-AP:

$$x^2 - 2x - 1, x^2, x^2 + 2x + 1.$$

The last two terms are perfect squares. We want the first term to contain a large square factor. The Pell equation

$$X^2 - 2Y^2 = -1$$

has infinitely many integer solutions given by

$$X_m + \sqrt{2} Y_m = (1 + \sqrt{2})^{2m+1} \text{ with integer } m \geq 1.$$

By setting $n = X$ and $\frac{x-1}{2} = Y$, the generalized Pell equation

$$(x-1)^2 - 2n^2 = 2 \tag{2}$$

has infinitely many integer solutions given by

$$x - 1 = 2Y_m \text{ and } n = X_m.$$

Then, equation (2) gives us infinitely many integers x such that $x^2 - 2x - 1 = 2n^2$ for some integer n. Therefore, we have infinitely many 3-APs of powerful numbers, namely

$$N = 2^2(x^2 - 2x - 1) = 2^3 n^2, N + d = 2^2 x^2 = (2x)^2, N + 2d = 2^2(x^2 + 2x + 1) = (2(x+1))^2$$

with common difference

$$d = 2^2(2x + 1) = 8x + 4 \leq 5N^{1/2}.$$

Hence, we have $\theta_3 \leq 1/2$. $\square$

4 Proof of Theorem 7: 3-AP lower bound

First, we need a simple observation.

Lemma 9. *Suppose a and b are positive integers and $p^\delta \mid a^2 b^3$ for some prime p and integer $\delta \geq 1$. Then $\nu_p(ab) \geq \delta/3$.*

Proof. From the definitions of divisibility and p-adic valuation, we have $\delta \le 2\nu_p(a) + 3\nu_p(b)$. Dividing everything by 3, we have $\delta/3 \le 2\nu_p(a)/3 + \nu_p(b) \le \nu_p(a) + \nu_p(b) = \nu_p(ab)$. $\square$

Proof of Theorem 7. Consider any 3-AP of powerful numbers $N, N+d, N+2d$ with

$$N = a_1^2 b_1^3, N+d = a_2^2 b_2^3, \text{ and } N+2d = a_3^2 b_3^3$$

for some integers a_1, a_2, a_3 and squarefree numbers b_1, b_2, b_3. If some prime p divides b_1, b_2 and b_3, then we can consider the reduced 3-AP of powerful numbers

$$\frac{N}{p^3}, \frac{N}{p^3} + \frac{d}{p^3}, \frac{N}{p^3} + \frac{2d}{p^3}.$$

If one could prove a lower bound $d/p^3 \ge C_\theta (N/p^3)^\theta$ with some $0 < \theta < 1$ and $C_\theta > 0$ for the reduced 3-AP, one would also have $d \ge C_\theta N^\theta$ for the original 3-AP. Hence, we may assume $\gcd(b_1, b_2, b_3) = 1$.

Since $(N+d)^2 = N(N+2d) + d^2$, we have $a_2^4 b_2^6 = a_1^2 b_1^3 a_3^2 b_3^3 + d^2$. Let $D^2 = \gcd(a_2^4 b_2^6, d^2)$ which also equals to $\gcd(a_2^4 b_2^6, a_1^2 b_1^3 a_3^2 b_3^3)$ and $\gcd(a_1^2 b_1^3 a_3^2 b_3^3, d^2)$. Note that as $D \mid a_2^2 b_2^3$ and $D \mid d$, we also have $D \mid a_1^2 b_1^3$ and $D \mid a_3^2 b_3^3$. Dividing everything by D^2, we have the equation

$$\Big(\frac{a_2^2 b_2^3}{D}\Big)^2 = \Big(\frac{a_1^2 b_1^3}{D} \frac{a_3^2 b_3^3}{D}\Big) + \Big(\frac{d}{D}\Big)^2$$

where the three terms are pairwise relatively prime. By the *abc*-conjecture, we have

$$\frac{N^2}{D^2} \le \Big(\frac{a_2^2 b_2^3}{D}\Big)^2 \le C_\epsilon \Big(\kappa\Big(\frac{a_1^2 b_1^3}{D} \frac{a_2^2 b_2^3}{D} \frac{a_3^2 b_3^3}{D}\Big) \kappa\Big(\frac{d}{D}\Big)\Big)^{1+\epsilon} \tag{3}$$

as $\kappa(mn) \le \kappa(m)\kappa(n)$. If one simply bounds the right-hand side of (3) by

$$\le C_\epsilon \Big(a_1 b_1 a_2 b_2 a_3 b_3 \frac{d}{D}\Big)^{1+\epsilon} \ll C_\epsilon \Big(\frac{N^{3/2} d}{D}\Big)^{1+\epsilon},$$

solves for d, and applies $D \le d$ as in [1, Theorem1.6], one would obtain $N^{1/4-\epsilon} \ll_\epsilon d$ and the lower bound $\theta_3 \ge 1/4$ only. So, in order to prove Theorem 7, we need a finer analysis. We claim that

$$\kappa\Big(\frac{a_1^2 b_1^3}{D} \frac{a_2^2 b_2^3}{D} \frac{a_3^2 b_3^3}{D}\Big) \le \frac{a_1 b_1 a_2 b_2 a_3 b_3}{D} \tag{4}$$

which would follow from

$$\nu_p\Big(\kappa\Big(\frac{a_1^2 b_1^3}{D} \frac{a_2^2 b_2^3}{D} \frac{a_3^2 b_3^3}{D}\Big)\Big) \le \nu_p(a_1 b_1 a_2 b_2 a_3 b_3) - \nu_p(D) \tag{5}$$

for any prime p. Firstly, if a prime p does not divide $a_1 b_1 a_2 b_2 a_3 b_3$, then (5) is true as both sides are 0. Secondly, if a prime $p \mid a_1 b_1 a_2 b_2 a_3 b_3$ but $p \nmid D$, then left hand side of (5) is exactly 1 while the right-hand side of (5) is $\geq 1-0$. So, inequality (5) is true for such primes. Thus, it remains to consider those primes p which divide both $a_1b_1a_2b_2a_3b_3$ and D. Notice that the left-hand side of (5) is at most 1 for such primes. Suppose we have the following prime factorizations

$$\begin{aligned} b_1 &= p_{11}\cdots p_{1r_1}, a_1 = p_{11}^{\alpha_{11}}\cdots p_{1r_1}^{\alpha_{1r_1}}\cdot q_{11}^{\beta_{11}}\cdots q_{1s_1}^{\beta_{1s_1}}\\ b_2 &= p_{21}\cdots p_{2r_2}, a_2 = p_{21}^{\alpha_{21}}\cdots p_{2r_2}^{\alpha_{2r_2}}\cdot q_{21}^{\beta_{21}}\cdots q_{2s_2}^{\beta_{2s_2}}\\ b_3 &= p_{31}\cdots p_{3r_3}, a_3 = p_{31}^{\alpha_{31}}\cdots p_{3r_3}^{\alpha_{3r_3}}\cdot q_{31}^{\beta_{31}}\cdots q_{3s_3}^{\beta_{3s_3}} \end{aligned}$$

for some integers $r_1, r_2, r_3, s_1, s_2, s_3 \geq 0$, $\alpha_{ij} \geq 0$, $\beta_{ij'} \geq 1$ and primes $p_{ij}, q_{ij'}$ with $q_{ij'} \neq p_{ij}$. Now, consider a fixed prime $p \mid D$ with $\delta := \nu_p(D)$. Note that p does not divide all of the b_1, b_2, b_3 as $\gcd(b_1, b_2, b_3) = 1$.

Case 1: Suppose p does not divide any of the b_1, b_2, b_3. Since $D \mid a_i^2b_i^3$ for $i = 1, 2, 3$, we must have $p \mid a_1, a_2, a_3$. Say $p = q_{1j_1} = q_{2j_2} = q_{3j_3}$ for some $1 \leq j_m \leq s_m$ for $m = 1, 2, 3$. As $p^{2\delta} || D^2 = \gcd(a_2^4b_2^6, a_1^2b_1^3a_3^2b_3^3)$ and $\gcd(p, b_1b_2b_3) = 1$, we have $2\delta = \min(4\beta_{2,j_2}, 2(\beta_{1,j_1} + \beta_{3j_3})) \geq 4$. Hence, we have $2 \leq \delta = \min(2\beta_{2j_2}, \beta_{1j_1} + \beta_{3j_3})$ and

$$\nu_p(a_1b_1a_2b_2a_3b_3) - \nu_p(D) \geq \beta_{1j_1} + \beta_{2j_2} + \beta_{3j_3} - (\beta_{1j_1} + \beta_{3j_3}) = \beta_{2j_2} \geq 1.$$

Case 2: Suppose p divides exactly one of the b_1, b_2, b_3.

Subcase 1: Suppose δ is even. Without loss of generality, say $p \mid b_1$, $p \nmid b_2$, $p \nmid b_3$ as the other cases are similar. Then, we have $p \mid a_2, a_3$ and $p = q_{2j_2} = q_{3j_3}$ for some $1 \leq j_2 \leq s_3$ and $1 \leq j_3 \leq s_3$. As $p^\delta \mid a_2^2b_2^3, a_3^2b_3^3$, we have $\delta/2 \leq \beta_{2j_2}$ and $\delta/2 \leq \beta_{3j_3}$. By Lemma 9, we have

$$\begin{aligned} \nu_p(a_1b_1a_2b_2a_3b_3) - \nu_p(D) &\geq \max(\delta/3, 1) + \beta_{2j_2} + \beta_{3j_3} - \delta\\ &\geq \begin{cases} 1 + \delta/2 + \delta/2 - \delta \geq 1, & \text{when } \delta = 2;\\ \delta/3 + \delta/2 + \delta/2 - \delta > 1, & \text{when } \delta \geq 4. \end{cases} \end{aligned}$$

Subcase 2: Suppose δ is odd. If $p \mid b_1$, $p \nmid b_2$, $p \nmid b_3$, then $p \mid b_1, a_2, a_3$. We have $\nu_p(a_2^4b_2^6) \geq 4$ and $\nu_p(a_1^2b_1^3a_3^2b_3^3) \geq 5$. Hence, we get $\delta \geq 3$ as $p^{2\delta}||D^2 = \gcd(a_2^4b_2^6, a_1^2b_1^3a_3^2b_3^3)$ and δ is odd. If $p \mid b_2$, $p \nmid p_1$, $p \nmid p_3$, then $p \mid b_2, a_1, a_3$. We have $\nu_p(a_2^4b_2^6) \geq 6$ and $\nu_p(a_1^2b_1^3a_3^2b_3^3) \geq 4$. Hence, we get $\delta \geq 3$ by similar reasoning. If $p \mid b_3$, $p \nmid b_1$, $p \nmid b_2$, we also get $\delta \geq 3$ as it is similar to $p \mid b_1$, $p \nmid b_2$, $p \nmid b_3$. Therefore, we obtain $\delta \geq 3$ in all circumstances.

Suppose $p \nmid b_i, b_{i'}$ for some $1 \leq i < i' \leq 3$. Then $p \mid a_i, a_{i'}$. Say $p = q_{ij_i} = q_{i'j_{i'}}$ for some $1 \leq j_i \leq s_i$ and $1 \leq j_{i'} \leq s_{i'}$. Thus, we have $3 \leq \delta \leq 2\beta_{ij_i} - 1$ and $3 \leq \delta \leq 2\beta_{i'j_{i'}} - 1$ as δ is odd. Hence, we have $\beta_{ij_i}, \beta_{i'j_{i'}} \geq \delta/2 + 1/2$ and

$$\nu_p(a_1b_1a_2b_2a_3b_3) - \nu_p(D) \geq 1 + \beta_{ij_i} + \beta_{i'j_{i'}} - \delta > 1.$$

Case 3: Suppose p divides exactly two of the b_1, b_2, b_3.

Subcase 1: Suppose δ is even. Without loss of generality, say $p \mid b_1$, $p \mid b_2$, $p \nmid b_3$ as the other cases are similar. Then $p \mid a_3$ and $p = p_{1j_1} = p_{2j_2} = q_{3j_3}$ for some $1 \leq j_1 \leq r_1$, $1 \leq j_2 \leq r_2$ and $1 \leq j_3 \leq s_3$. As $p^\delta \mid a_3^2 b_3^3$, we have $\delta/2 \leq \beta_{3j_3}$. Also, as $p^\delta \mid a_1^2 b_1^3, a_2^2 b_2^3$, we have $\delta \leq 2\alpha_{1j_1} + 2$ and $\delta \leq 2\alpha_{2j_2} + 2$ since δ is even. Hence, we have

$$\nu_p(a_1 b_1 a_2 b_2 a_3 b_3) - \nu_p(D) \geq (\alpha_{1j_1} + 1) + (\alpha_{2j_2} + 1) + \beta_{3j_3} - \delta \geq \beta_{3j_3} \geq 1.$$

Subcase 2: Suppose δ is odd. If $p \mid b_1$, $p \mid b_2$, $p \nmid b_3$, then $p \mid b_1, b_2, a_3$. We have $\nu_p(a_2^4 b_2^6) \geq 6$ and $\nu_p(a_1^2 b_1^3 a_3^2 b_3^3) \geq 5$. Hence, we get $\delta \geq 3$ as $p^{2\delta} || D^2 = \gcd(a_2^4 b_2^6, a_1^2 b_1^3 a_3^2 b_3^3)$ and δ is odd. If $p \mid b_1$, $p \nmid b_2$, $p \mid b_3$, then $p \mid b_1, a_2, b_3$. We have $\nu_p(a_2^4 b_2^6) \geq 4$ and $\nu_p(a_1^2 b_1^3 a_3^2 b_3^3) \geq 6$. Hence, we get $\delta \geq 3$ by similar reasoning. If $p \nmid b_1$, $p \mid b_2$, $p \mid b_3$, we also get $\delta \geq 3$ as it is similar to the case $p \mid b_1$, $p \mid b_2$, $p \nmid b_3$. Therefore, we obtain $\delta \geq 3$ in all circumstances.

Suppose $p \nmid b_i$ for some $1 \leq i \leq 3$. Then $p \mid a_i$ and $p = q_{ij_i}$ for some $1 \leq j_i \leq s_i$. Thus, we have $3 \leq \delta \leq 2\beta_{ij_i} - 1$ as δ is odd, and $\beta_{ij_i} \geq \delta/2 + 1/2$. By Lemma 9, we have

$$\nu_p(a_1 b_1 a_2 b_2 a_3 b_3) - \nu_p(D) \geq \frac{\delta}{3} + \frac{\delta}{3} + \beta_{ij_i} - \delta \geq \frac{1}{2} + \frac{\delta}{6} \geq \frac{1}{2} + \frac{3}{6} = 1.$$

Consequently, the right-hand side of (5) is at least 1 in all of the above cases. As a result, we have inequalities (5) and (4). Putting (4) into (3), we obtain

$$\frac{N^2}{D^2} \leq C_\epsilon \Big(\frac{a_1 b_1 a_2 b_2 a_3 b_3 d}{D^2} \Big)^{1+\epsilon} \ll C_\epsilon \Big(\frac{N^{3/2} d}{D^2} \Big)^{1+\epsilon}$$

as $a_1^2 b_1^3, a_2^2 b_2^3, a_3^2 b_3^2 \ll N$. Together with $D \geq 1$, we have $N^{1/2 - 2\epsilon} \ll_\epsilon d$ and $\theta_3 \geq 1/2$ as ϵ can be arbitrarily small. $\square$

5 Proof of Theorem 8

Proof. For $k \geq 4$, the lower bound $\theta_k \geq 1/2$ follows from the observation that $\theta_k \geq \theta_3$ and $\theta_3 \geq 1/2$ from Theorem 7 under the *abc*-conjecture.

For the upper bound $\theta_4 \leq 4/5$, we construct 4-APs of powerful numbers as follows. With positive integer a, the following four expressions

$$(x-a)^3(x+a)^2, (x-a)^2 x (x+a)^2, (x-a)^2(x+a)^3, (x-a)^2(x+a)^2(x+2a) \qquad (6)$$

form a 4-AP with common difference $d = a(x-a)^2(x+a)^2$. Note that the first and third terms give powerful numbers for any integer x. If x and $x + 2a$ are powerful, then all four polynomials would result in powerful numbers. We can pick $a = 2$. Note that the Pell equation

$$X^2 - 2Y^2 = 1 \text{ or } 2X^2 - 4Y^2 = 2 \text{ or } 4X^2 = 8Y^2 + 4$$

has solutions

$$X_m + \sqrt{2}Y_m = (3+2\sqrt{2})^m \text{ for positive integer } m.$$

By choosing $x = 8Y_m^2$ and $x+2a = x+4 = 4X_m^2$, we turn (6) into our desired 4-AP of powerful numbers. Observe that the common difference

$$d = 2(x-2)^2(x+2)^2 \le 3((x-2)^3(x+2)^2)^{4/5} = 3N^{4/5}$$

for large enough m (and hence N). We have $\theta_4 \le 4/5$.

For the upper bound $\theta_5 \le 9/10$, one can build upon our 3-AP and 4-AP constructions. With positive integer a, the following five expressions

$$(y-2a)(y-a)^2(y+a)^2, (y-a)^3(y+a)^2, (y-a)^2y(y+a)^2, (y-a)^2(y+a)^3, (y-a)^2(y+a)^2(y+2a) \tag{7}$$

form a 5-AP with common difference $d = a(y-a)^2(y+a)^2$. Note that the second and fourth terms give powerful numbers for any integer y. If $y-2a$, y and $y+2a$ are powerful, then all five terms would be powerful. From our 3-AP construction, we can find infinitely many 3-APs of powerful numbers

$$y-2a = 2^2(x^2-2x-1) = 2^3n^2, y = 2^2x^2 = (2x)^2, y+2a = 2^2(x^2+2x+1) = (2(x+1))^2$$

with

$$2a = 2^2(2x+1) = 8x+4.$$

With the above choices, we obtain the desired 5-AP of powerful numbers with common difference

$$\begin{aligned} d &= (4x+2)(4x^2-4x-2)^2(4x^2+4x+2)^2 \\ &\le 3\Big((4x^2-8x-4)(4x^2-4x-2)^2(4x^2+4x+2)^2\Big)^{9/10} = 3N^{9/10} \end{aligned}$$

for large enough x (and hence N). Thus, we have $\theta_5 \le 9/10$.

For the general upper bound $\theta_k \le 1 - \frac{1}{10\cdot 3^{k-5}}$, we use induction on $k \ge 5$ similar to Theorem 3. The base case $\theta_5 \le 1 - \frac{1}{10\cdot 3^{5-5}}$ is true from $\theta_5 \le 9/10$. Suppose, for some $k \ge 5$, there are infinitely many k-APs among powerful numbers with $d \le C_k N^{1-\frac{1}{10\cdot 3^{k-5}}}$. Say one such AP is

$$N = a_1^2b_1^3 < a_2^2b_2^3 < \cdots < a_k^2b_k^3 \quad \text{with common difference } 1 \le d \le C_k N^{1-\frac{1}{10\cdot 3^{k-5}}}.$$

Consider the number $a_k^2b_k^3 + d = a^2b$ for some integer a and squarefree number b. Multiply everything by b^2, the following $k+1$ numbers

$$N_1 := Nb^2 = a_1^2b_1^3b^2 < a_2^2b_2^3b^2 < \cdots < a_k^2b_k^3b^2 < a^2b^3$$

form a $(k+1)$-AP of powerful numbers with common difference db^2. Note that

$$b \le a^2 b = N + kd \le (1+kC_k)N.$$

Hence, we have

$$\begin{aligned} db^{\frac{2}{10\cdot 3^{k-4}}} &\le d(1+kC_k)^{\frac{2}{10\cdot 3^{k-4}}} N^{\frac{2}{10\cdot 3^{k-4}}} \\ &\le C_k(1+kC_k)^{\frac{2}{10\cdot 3^{k-4}}} N^{1-\frac{1}{10\cdot 3^{k-5}}+\frac{2}{10\cdot 3^{k-4}}} = C_k(1+kC_k)^{\frac{2}{10\cdot 3^{k-4}}} N^{1-\frac{1}{10\cdot 3^{k-4}}} \end{aligned}$$

and

$$db^2 \le C_k(1+kC_k)^{\frac{2}{10\cdot 3^{k-4}}} (Nb^2)^{1-\frac{1}{10\cdot 3^{k-4}}} = C_{k+1} N_1^{1-\frac{1}{10\cdot 3^{(k+1)-5}}}$$

with $C_{k+1} := C_k(1+kC_k)^{\frac{2}{10\cdot 3^{k-4}}}$. This completes the induction proof. □

6 Acknowledgment

The author would like to thank Prof. Lajos Hajdu for pointing out that Theorem 3 has long been known and is even true for perfect powers only. The author also wants to thank the anonymous referee for some helpful corrections.

References

[1] T. H. Chan, A note on powerful numbers in short intervals, *Bull. Australian Math. Soc.*, 2022. Available at https://doi.org/10.1017/S0004972722000995.

[2] P. Erdős, Consecutive integers, *Eureka* **38** (1975-76), 3–8.

[3] L. Hajdu, Perfect powers in arithmetic progression. A note on the inhomogeneous case, *Acta Arith.* **113** (2004), 343–349.

[4] A. Granville, Powerful numbers and Fermat's last theorem, *C. R. Math. Rep. Acad. Sci. Canada* **8** (1986), 215–218.

[5] B. J. Green and T. C. Tao, The primes contain arbitrarily long arithmetic progressions, *Ann. of Math. (2)* **167** (2008), 481–547.

[6] R. A. Mollin and P. G. Walsh, A note of powerful numbers, quadratic fields and the Pellian, *C. R. Math. Rep. Acad. Sci. Canada* **8** (1986), 109–114.

2020 *Mathematics Subject Classification*: Primary 11N25.
Keywords: powerful number, k-full number, arithmetic progression, abc-conjecture.

(Concerned with sequence A001694.)

Received September 30 2022; revised versions received October 1 2022; October 3 2022; December 13 2022. Published in *Journal of Integer Sequences*, December 14 2022.

Return to Journal of Integer Sequences home page.