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Set Partitions and Other Bell Number Enumerated Objects

Fufa Beyene
Department of Mathematics
Addis Ababa University
1176 Addis Ababa
Ethiopia
fufa.beyene@aau.edu.et

Jörgen Backelin
Department of Mathematics
Stockholm University
SE-106 91 Stockholm
Sweden
joeb@math.su.se

Roberto Mantaci
IRIF, Université de Paris
8 Place Aurélie Nemours
F-75013 Paris
France
mantaci@irif.fr

Samuel A. Fufa
Department of Mathematics
Addis Ababa University
1176 Addis Ababa
Ethiopia
samuel.asefa@aau.edu.et

Abstract

In this paper, we study classes of subexcedant functions enumerated by the Bell numbers and present bijections on set partitions. We present a set of permutations whose transposition arrays are the restricted growth functions, thus defining Bell permutations of the second kind. We describe a bijection between Bell permutations of the first (introduced by Poneti and Vajnovzski) and the second kinds. We present two other Bell number enumerated classes of subexcedant functions. Further, we give bijections on set partitions, in particular, an involution that interchanges the sets of merging blocks and successions. We use the bijections to enumerate the distribution of these statistics over set partitions and give some structural and enumeration results.

1 Introduction

Let n be a fixed positive integer and let $[n] := \{1, 2, \ldots, n\}$. A set partition of $[n]$ is a collection of pairwise disjoint non-empty subsets of $[n]$ such that their union forms the whole set $[n]$.

For any set S, the function $\sigma : [n] \longrightarrow S$ corresponds to the word $\sigma(1)\sigma(2)\cdots\sigma(n)$. In particular, a permutation is a word with distinct symbols.

Permutations and set partitions are among the classical objects in enumerative combinatorics. A fundamental reason for this truth is the wide variety of ways to represent a permutation and a set partition combinatorially. A second reason for their richness is the wide variety of interesting statistics. The Eulerian statistic is among the most classical statistics on permutations. The permutation statistics: descents, weak excedances, anti-excedances, and right-to-left minima (maxima) are examples of the Eulerian statistics. On the other hand, we recall the two most basic enumerations for set partitions: the total number of set partitions over $[n]$ and the number of set partitions over $[n]$ having k blocks are the Bell number, $B(n)$ and the Stirling number of the second kind, $S(n,k)$, respectively [6, 13, 19, 21].

Both permutations and set partitions can be coded by subexcedant functions, i.e., functions $f : [n] \mapsto [n]$ such that $1 \leq f(i) \leq i, \forall i \in [n]$ (in some contexts it is rather required that $0 \leq f(i) \leq i-1$).

Some permutation codes with subexcedant functions are very well known (Lehmer code or inversion table, Denert code, and so on) [8, 9, 11, 15]. On the other hand, a way to represent set partitions with subexcedant functions is given by Mansour's definition of canonical form for a set partition P in the standard form, the elements in each block are arranged increasingly, and the blocks are arranged in increasing order of their minima [13]. In the canonical form, any integer $i \in [n]$ is coded with the index of the block of P where it belongs, where P is in its standard form. The canonical forms of set partitions are restricted growth functions (RGF).

Several properties of permutations can easily be read from their corresponding codes, which allows one to prove some results elegantly by reasoning on the codes rather than the coded objects. See, for instance, the article of Baril and Vajnovszki [3] and the article of Foata and Zeilberger [9].

Mantaci and Rakotondrajao [12] studied the bijection ϕ associating a subexcedant function f with the permutation $\sigma = \phi(f) = (n, f_n)(n-1, f_{n-1})\cdots(1, f_1)$, where $f_i = f(i)$, $\forall i \in [n]$ and related the image values of f to the anti-excedances of σ. Later, Baril [1] independently studied a variation of the bijection ϕ, here denoted by χ, given by simply inverting the order of the product of transpositions in the definition of ϕ, and he called the subexcedant function associated with a permutation via this bijection the *transposition array*. Baril [2] also studied, in particular, the positions of weak excedances in a permutation using the corresponding subexcedant function.

Mansour and Munagi [14] studied set partitions according to the number of circular successions, i.e., the number of consecutive element pairs inside a block, assuming the elements arranged around a circle. Mansour and Shattuck [16] studied parity successions, pairs of adjacent elements a and b within some block such that $a \equiv b \pmod 2$, in set partitions. Callan [7] proved that the number of singletons in all set partitions is equal to the number of circular successions by giving a bijection in terms of an algorithm that interchanges singletons and circular successions. Callan also proved that his bijection is an involution on set partitions and that it preserves the non-crossing partitions.

In this paper we study families of subexcedant functions enumerated by the Bell numbers. We present bijections between these classes and the set partitions. We enumerate these classes according to the distribution of some statistics. We also give an involution on set partitions and present some structural and enumeration results.

In Section 3, we study a class of permutations whose transposition arrays are the restricted growth functions. We will call this a class of Bell permutations of the second kind. We prove that the statistic of weak excedances is the Stirling number of the second kind. We enumerate such permutations according to the number of cycles and consider the joint distribution of these statistics. In Subsection 3.1, we describe a bijection between Bell permutations of the second kind and another Bell-counted class of permutations introduced by Poneti and Vajnovszki [18].

Section 4 presents two more families of subexcedant functions enumerated by the Bell numbers and bijections between these classes and set partitions.

Finally, in Section 5, we present bijections on set partitions. In particular, we give an involution that interchanges merging blocks and successions of a set partition. We use the bijections to present some structural and enumeration results. Also, we give the generating function for the joint distribution of these statistics.

2 Notation and preliminaries

2.1 Permutations

Recall that a permutation over $[n]$ is a bijection $\sigma : [n] \mapsto [n]$. Let $\mathfrak{S}_n$ denote the set of all permutations over $[n]$. We write a permutation $\sigma \in \mathfrak{S}_n$ as a word $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ (whence the $\sigma(i)$ also are called *letters*), or in cycle notation as a product of disjoint cycles, where as usual a *cycle* in σ can be written as $(j, \sigma(j), \sigma^2(j), \ldots, \sigma^{t-1}(j))$, where t, the *length* of the cycle, is the smallest positive integer such that $\sigma^t(j) = j$. Cycles of length one are *fixed points*. The cycle notation is $\sigma = C_1C_2\cdots C_k$, where the C_i's are disjoint cycles and the minima of the cycles form an increasing sequence. We let $\mathrm{cyc}(\sigma)$ denote the number of cycles of σ. A *transposition* is a permutation that swaps two integers and fixes all the others.

We say that a permutation σ over $[n]$ has an *excedance (resp., weak excedance, anti-excedance)* in a position i if $\sigma(i) > i$ (resp., $\sigma(i) \geq i, \sigma(i) \leq i$), where $i \in [n]$. We use the notation

$$\begin{aligned}\operatorname{Exc}(\sigma) &:= \{i : 1 \leq i \leq n, \sigma(i) > i\},\\ \operatorname{Wex}(\sigma) &:= \{i : 1 \leq i \leq n, \sigma(i) \geq i\}, \text{ and}\\ \operatorname{Ax}(\sigma) &:= \{i : 1 \leq i \leq n, \sigma(i) \leq i\}.\end{aligned}$$

We also use the notation $\operatorname{exc}(\sigma) := |\operatorname{Exc}(\sigma)|$, $\operatorname{wex}(\sigma) := |\operatorname{Wex}(\sigma)|$, and $\operatorname{ax}(\sigma) := |\operatorname{Ax}(\sigma)|$.

The set of *excedance letters*, *weak excedance letters*, and *anti-excedance letters* of σ are defined as

$$\begin{aligned}\operatorname{ExcL}(\sigma) &:= \{\sigma(i) : i \in \operatorname{Exc}(\sigma)\},\\ \operatorname{WexL}(\sigma) &:= \{\sigma(i) : i \in \operatorname{Wex}(\sigma)\}, \text{ and}\\ \operatorname{AxL}(\sigma) &:= \{\sigma(i) : i \in \operatorname{Ax}(\sigma)\}.\end{aligned}$$

2.2 Subexcedant functions

A subexcedant function is a function $f : [n] \mapsto [n]$ such that $1 \leq f(i) \leq i, \forall i \in [n]$.

We let $\operatorname{SF}(n)$ denote the set of all subexcedant functions over $[n]$. For $f = f_1 f_2 \cdots f_n \in \operatorname{SF}(n)$, we let $\operatorname{Im}(f) := f([n])$, the image set of f, and $\operatorname{im}(f) := |\operatorname{Im}(f)|$. We say that f has a *leftmost (rightmost) occurrence* in a position i if $f_i \notin \{f_1, \ldots, f_{i-1}\}$, i.e., $i = \min(f^{-1}(f_i))$ (or $f_i \notin \{f_{i+1}, \ldots, f_n\}$, i.e., $i = \max(f^{-1}(f_i))$, respectively), where $i \in [n]$. If i is a leftmost (rightmost) occurrence in f, then we say that f_i is a leftmost (or rightmost) letter. The set of fixed points of f is given by

$$\operatorname{Fx}(f) := \{i : 1 \leq i \leq n, f_i = i\}.$$

We let $\operatorname{fx}(f) := |\operatorname{Fx}(f)|$.

2.3 Set partitions

A set partition P of $[n]$ is a collection $B_1, \ldots, B_k$ of nonempty disjoint subsets of $[n]$ such that $\bigcup_{i=1}^{k} B_i = [n]$. The subsets B_i will be referred to as *blocks*. The *block representation* $P = B_1|B_2|\cdots|B_k$ of a set partition P is said to be *standard* if the blocks $B_1, \ldots, B_k$ are sorted in such a way that $\min(B_1) < \min(B_2) < \cdots < \min(B_k)$.

We consider set partitions only in their standard representation.

We let $\operatorname{SP}(n)$ denote the set of all set partitions over $[n]$. We also let $\operatorname{bl}(P)$ denote the number of blocks of a set partition P and $\operatorname{SP}(n,k) := \{P \in \operatorname{SP}(n) : \operatorname{bl}(P) = k\}$.

Recall that $|\operatorname{SP}(n,k)| = S(n,k)$, where $S(n,k)$ is the Stirling number of the second kind.

For $2 \leq i \leq k$, we say that block B_i is *merging* if $\max(B_{i-1}) < \min(B_i)$. A set partition without merging blocks is called *merging-free*.

If the integers $(a-1, a)$ are in the same block of P, then a is said to be a *succession* of P. In literature, "succession" is the first element of the pair, but we prefer to use it for the second element.

We let $\mathrm{Mb}(P)$, $\mathrm{Suc}(P)$, and $\mathrm{Nmb}(P)$ denote the set of the minimum elements of merging blocks of P, the set of successions of P, and the set of the minima of non-merging blocks of P, respectively. We use the notation $\mathrm{mb}(P) := |\,\mathrm{Mb}(P)|$, $\mathrm{suc}(P) := |\,\mathrm{Suc}(P)|$, and $\mathrm{nmb}(P) := |\,\mathrm{Nmb}(P)|$.

Remark 1. For any $P \in \mathrm{SP}(n)$, every element i of $[n]$ is necessarily in one of the first i blocks of P.

The *canonical form* of a set partition $P = B_1|B_2|\cdots|B_k$ is an n-tuple $f = f_1f_2\cdots f_n$ indicating for each integer j the index of the block in which it occurs, i.e., $B_j = f^{-1}(j), \forall j \in [k]$. For instance, the canonical form of $P = 1\ 5\ 7|2\ 4|3\ 8|6 \in \mathrm{SP}(8)$ is $f = 12321413$.

Remark 2. The block B_i contains its index i if and only if $i \in \mathrm{Fx}(f)$.

Note that the canonical form of a set partition is a subexcedant function, but not all subexcedant functions are canonical forms of set partitions.

A *restricted growth function* (RGF) over $[n]$ is a function $f : [n] \mapsto [n]$, where $f = f_1\cdots f_n$ such that $f_1 = 1$ and $f_i \le 1 + \max\{f_1, \ldots, f_{i-1}\}$ for $2 \le i \le n$, or equivalently, such that the set $\{f_1, f_2, \ldots, f_i\}$ is an integer interval $\forall i \in [n]$. The canonical forms of set partitions are exactly the restricted growth functions (RGF) [15, p. 2]. We let $\mathrm{RGF}(n)$ denote the set of all restricted growth functions over $[n]$.

3 Bell permutations of the second kind

In this section, we study the class of permutations associated with RGFs under χ, the bijection given by Baril [1]. Since this is a Bell enumerated set, we will call these objects "Bell permutations of the second kind" (Poneti and Vajnovszki [18] already introduced another Bell enumerated family of permutations that they called "Bell permutations").

The bijection χ is given by $\chi : \mathrm{SF}(n) \mapsto \mathfrak{S}_n$, where the permutation $\sigma = \chi(f)$ is defined by the product of transpositions:

$$\sigma = (1, f_1)(2, f_2)\cdots(n, f_n),$$

where the product is taken from right to left. The subexcedant function $f = \chi^{-1}(\sigma)$ is called the transposition array of σ. It is shown in [2] that $\mathrm{Im}(f) = \mathrm{Wex}(\sigma)$. For instance, take $f = 121132342 \in \mathrm{SF}(9)$. Then

$$\begin{aligned}\sigma = \chi(f) &= (1,1)(2,2)(3,1)(4,1)(5,3)(6,2)(7,3)(8,4)(9,2)\\ &= 497812536,\end{aligned}$$

and $\mathrm{Im}(f) = \{1, 2, 3, 4\} = \mathrm{Wex}(\sigma)$.

Remark 3. [2] The rightmost occurrences of f are the weak excedance letters of $\chi(f)$.

Remark 4. Let $f = f_1f_2\cdots f_n$ and $\sigma = \chi(f)$. We have $i \in \mathrm{Fx}(f)$ if and only if i is the minimum element of some cycle of σ.

Beyene and Mantaci [5] essentially proved the following:

Lemma 5. *Let $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n) \in \mathfrak{S}_n$. If $f = \chi^{-1}(\sigma) = f_1 f_2 \cdots f_n$, then $f(i) = \sigma^{-t}(i) \leq i$, where $t \geq 1$ is chosen as small as possible.* □

The following proposition presents an alternative algorithm to compute σ from f as a product of disjoint cycles.

Proposition 6. *If $f \in \mathrm{SF}(n)$, then $\sigma = \chi(f)$ can be constructed as follows. For $i = 1, 2, \ldots, n$:*

- *if $f_i = i$, then add a new singleton cycle: (i),*
- *if $f_i < i$, then insert i after f_i in its cycle.*

□

Example 7. Take $f = 1132532 \in \mathrm{SF}(7)$. Then $\sigma = \chi(f)$ can be obtained as follows:

$$\begin{aligned}
&(1)\\
&(1,2)\\
&(1,2)(3)\\
&(1,2,4)(3)\\
&(1,2,4)(3)(5)\\
&(1,2,4)(3,6)(5)\\
&(1,2,7,4)(3,6)(5) = \sigma.
\end{aligned}$$

The following lemma can easily be deduced from the above proposition and the definition of χ.

Lemma 8. *Let $f \in \mathrm{SF}(n)$ and $\sigma = \chi(f) = C_1 C_2 \cdots C_\ell$. If $\emptyset \neq S \subseteq [n]$, then the following statements are equivalent.*

1. *f has the property*
$$\begin{cases} f_i = \min(S), & \text{if } i \in S;\\ f_i \notin S, & \text{otherwise.} \end{cases}$$
2. *The elements of S form some cycle C_i in σ, and the cycle can be written with its elements forming a decreasing sequence.*
3. *S is the underlying set of some cycle C_i with just one weak excedance.* □

Consider the bijection $\tau : \mathrm{SP}(n) \mapsto \mathrm{RGF}(n)$ given by $\tau(P) = f$, where f is the canonical form of P.

Definition 9. A Bell permutation of the second kind over $[n]$ is a permutation σ obtained from $f \in \mathrm{RGF}(n)$ by applying χ to f, i.e., $\sigma = \chi(f)$.

Let $\mathrm{BP}_2(n) := \chi(\mathrm{RGF}(n))$, the set of all Bell permutations of the second kind over $[n]$, and $\mathrm{BP}_2(n,k) := \{\sigma \in \mathrm{BP}_2(n) : \mathrm{wex}(\sigma) = k\}$.

The restriction of χ to $\mathrm{RGF}(n)$ is a bijection between $\mathrm{RGF}(n)$ and $\mathrm{BP}_2(n)$. Therefore, $\mathrm{BP}_2(n)$ is a Bell number enumerated set, i.e., $|\mathrm{BP}_2(n)| = B(n)$, the nth Bell number.

Since the composition of bijections is a bijection, the map $\lambda = \chi \circ \tau$ is a bijection between $\mathrm{SP}(n)$ and $\mathrm{BP}_2(n)$.

Proposition 10. *Let $P = B_1|B_2|\cdots|B_k$ be a set partition, σ the permutation $\lambda(P)$, and $C_1C_2\cdots C_\ell$ the cycle decomposition in σ. Then*

1. *σ has k weak excedances,*
2. *the set of the weak excedances of σ is exactly the interval $[k] = \{1, 2, \ldots, k\}$, and*
3. *the set of the minimal elements of the cycles of σ is exactly the interval $[\ell]$.*

Proof. The first two items directly follow from Remark 3 and the fact that the number of blocks of P and the cardinality of the image set of its canonical form are equal.

Item 3: by Remark 4, any integer $i \in [n]$ is fixed in f if and only if $i = \min(C_j)$ for some j. We show that if p is the maximum fixed point in f, then any $q < p$ also is fixed. Suppose that there exist a non-fixed point smaller than p. Let t be the maximal of such non-fixed points, i.e., the elements of the interval $[t+1, p]$ are all fixed. So $f_t < t$ and $t \notin \{f_1, f_2, \ldots, f_{t+1} = t+1\}$. Thus $f \notin \mathrm{RGF}(n)$, and this is a contradiction. Therefore, the set of fixed points of f is $[p]$ and hence $p = \ell$. □

The above proposition implies that the distribution of the number of weak excedances on $\mathrm{BP}_2(n)$ is the same as the distribution of the number of blocks on $\mathrm{SP}(n)$ and also that the statistic of the number of cycles on $\mathrm{BP}_2(n)$ has the same distribution as the number of fixed points on $\mathrm{RGF}(n)$. Thus, we have the following.

Corollary 11.

1. *$|\mathrm{BP}_2(n,k)| = S(n,k)$, the Stirling number of the second kind.*
2. *The number of set partitions having ℓ blocks containing their index element is the same as the number of Bell permutations of the second kind having ℓ cycles.*

We consider the following lemma.

Lemma 12. *Let $f' \in \mathrm{SF}(n-1)$, and let $f \in \mathrm{SF}(n)$ be obtained by concatenating some $j \in [n]$ at the end of f'. Let $\sigma' = \chi(f')$ and $\sigma = \chi(f)$. If $j \neq n$, then σ is obtained from σ' by replacing the integer $\sigma'(j)$ with n in σ' and appending $\sigma'(j)$ at the end. If $j = n$, then σ is obtained by simply appending n at the end of σ'.* □

The following lemma gives a recursive procedure to check if a permutation is a Bell Permutation of the second kind.

Lemma 13. *A permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n) \in \mathfrak{S}_n$ whose set of weak excedances is an integer interval $[k]$ is in $\mathrm{BP}_2(n)$ if and only if the permutation $\sigma' \in \mathfrak{S}_{n-1}$ obtained from σ by replacing the integer n with $\sigma(n)$ in $\sigma|_{[n-1]}$ is in $\mathrm{BP}_2(n-1)$.*

Proof. According to Lemma 12, for all permutations σ, if $f = f_1 \cdots f_n = \chi^{-1}(\sigma)$ and $\sigma' \in \mathfrak{S}_{n-1}$ is the permutation obtained from σ by replacing the integer n by $\sigma(n)$, then the transposition array associated with σ' is $f' = f_1 f_2 \cdots f_{n-1}$. Under the hypothesis that $\mathrm{Wex}(\sigma) = \mathrm{Im}(f)$ is an integer interval $[k]$, the following two conditions are trivially equivalent:

1. for $1 \le i \le n$, the set $\{f_1, f_2, \ldots, f_i\}$ is an integer interval with minimum value 1.

2. for $1 \le i \le n-1$, the set $\{f_1, f_2, \ldots, f_i\}$ is an integer interval with minimum value 1.

That is, σ is Bell if and only if σ' is Bell. □

For instance, let $\sigma = 7245613$. We have $\mathrm{Wex}(\sigma) = [5]$, so σ may be a Bell permutation of the second kind. We apply Lemma 13: **72456**13 $\to$ **32456**1 $\to$ **3245**1 $\to$ **324**1 $\to$ **32**1. Since $321 \in \mathrm{BP}_2(3)$, we can conclude that σ and those permutations obtained in the process are Bell permutations of the second kind. But $32541 \notin \mathrm{BP}_2(5)$, because **3254**1 $\to$ **32**14 and $3214 \notin \mathrm{BP}_2(4)$.

We give new proof of the fact that $|\mathrm{BP}_2(n,k)| = S(n,k)$ by showing that the numbers $|\mathrm{BP}_2(n,k)|$ satisfy the recurrence relation of the Stirling number of the second kind.

Proposition 14. *For all positive integers $n, k, n \ge 1, 1 \le k \le n$, the number $|\mathrm{BP}_2(n,k)|$ satisfies the recurrence relation*

$$|\mathrm{BP}_2(n,k)| = k|\mathrm{BP}_2(n-1,k)| + |\mathrm{BP}_2(n-1,k-1)|, \ |\mathrm{BP}_2(0,0)| = 1. \tag{1}$$

Proof. We use Lemma 13 to prove the assertion. Any Bell permutation of the second kind $\sigma \in \mathrm{BP}_2(n,k)$ can be uniquely obtained either from a permutation $\sigma' \in \mathrm{BP}_2(n-1,k)$ and an integer $i \in [k]$ or from a permutation $\sigma' \in \mathrm{BP}_2(n-1,k-1)$. More precisely: if $\sigma' \in \mathrm{BP}_2(n-1,k)$ and $i \in [k]$, then σ is obtained from σ' by replacing $\sigma'(i)$ with n and then appending $\sigma'(i)$ at the end, i.e., $\sigma = \sigma'(i,n)$. In this case, $\sigma \in \mathrm{BP}_2(n,k)$, and there are $|\mathrm{BP}_2(n-1,k)|$ possible choices for σ' and k possibilities for i. Hence this contributes $k|\mathrm{BP}_2(n-1,k)|$ to $|\mathrm{BP}_2(n,k)|$. If $\sigma' \in \mathrm{BP}_2(n-1,k-1)$, then σ is obtained from σ' by replacing $\sigma'(k)$ by n and then appending $\sigma'(k)$ at the end, i.e., $\sigma = \sigma'(k,n)$. In this case, $\sigma \in \mathrm{BP}_2(n,k)$, and σ has $|\mathrm{BP}_2(n-1,k-1)|$ possibilities. By combining the two cases, we have (1). □

Let $P \in \mathrm{SP}(n,k)$ and $\mathrm{Mx}(P) = \{\max(B_i) : 1 \le i \le k\}$. By the above proposition, Remark 3, and the fact that the maximum elements of the blocks of P are the rightmost occurrences in $\tau(P)$ we have the following corollaries.

Corollary 15. *We have $\mathrm{Mx}(P) = \mathrm{WexL}(\sigma)$, where $\sigma = \lambda(P)$.*

Corollary 16. *The bistatistics* $(\mathrm{bl}, \mathrm{fx})$ *on the set partitions has the same distribution as* $(\mathrm{wex}, \mathrm{cyc})$ *on the Bell permutations of the second kind.*

Remark 17. The cardinality of the set $\mathrm{BP}_2(n, n-1)$ is equal to the number $S(n, n-1)$ of set partitions over $[n]$ having $n-1$ blocks, which, as is well known, is equal to $\binom{n}{2}$.

OEIS entry number A259691 presents the sequence of the numbers $T(n-1, \ell)$, counting set partitions over $[n]$ where exactly ℓ blocks contain their index element. These numbers satisfy the relation:

$$T(n-1, \ell) = \sum_{i=0}^{n-\ell} \binom{n-\ell}{i} \ell^{i+1} B(n-\ell-i), \tag{2}$$

where $T(n-1, n) = 1$. Thus, by Corollary 11, the number of Bell permutations of the second kind over $[n]$ having exactly ℓ cycles is also equal to $T(n-1, \ell)$.

We now refine (2) by adding a parameter counting the number of weak excedances of the permutation. Consider a permutation $\sigma' \in \mathrm{BP}_2(n-1, k)$, an integer $i \in [k+1]$ representing a weak excedance, and the permutation $\sigma = \sigma'(i, n)$, the product of σ' and the transposition (i, n). Then, by Proposition 6, the numbers of cycles of σ and σ' are equal, except when $i = k+1 = n$ (i.e., both σ' and σ are the identity permutations) and $i = k+1$, in which case $\mathrm{cyc}(\sigma) = \mathrm{cyc}(\sigma') + 1$. Thus, the number $T(n-1, k, \ell)$ of Bell permutations of the second kind over $[n]$ having exactly k weak excedances and ℓ cycles satisfies:

$$T(n-1, k, \ell) = \begin{cases} \delta_{k,n}, & \text{if } \ell = n; \\ \sum\limits_{i=0}^{n-\ell} \binom{n-\ell}{i} \ell^{i+1} S(n-\ell-i, k-\ell), & \text{otherwise.} \end{cases} \tag{3}$$

Here, $\delta_{*,*}$ is the Kronecker delta function. Therefore, we have the following.

Proposition 18. *For* $n \geq 1$, *we have*

$$\sum_{\sigma \in \mathrm{BP}_2(n)} x^{\mathrm{wex}(\sigma)} y^{\mathrm{cyc}(\sigma)} = \sum_{k=1}^{n} \sum_{\ell=1}^{k} T(n-1, k, \ell) x^k y^\ell. \tag{4}$$

□

Corollary 19. *The number of Bell permutations of the second kind over* $[n]$ *having exactly* 1 *cycle equals* $B(n-1), n \geq 1$.

Proof. Let $\ell = 1$ in (3) and take the sum over all $1 \leq k \leq n$. □

3.1 A bijection between Bell permutations of the first and the second kinds

In this subsection, we present a bijection between the set $\mathrm{BP}_1(n)$ of Bell permutations introduced by Poneti and Vajnovszki [18] (which we will call *Bell permutations of the first kind*) and the set $\mathrm{BP}_2(n)$ of Bell permutations of the second kind.

First, we recall the definition of Bell permutations of the first kind. Let $P = B_1|B_2|\cdots|B_k$ be a set partition over $[n]$ in its standard representation and let $\mu : \mathrm{SP}(n) \mapsto \mathrm{BP}_1(n)$, where the permutation $\mu(P)$ is constructed as follows:

- reorder all integers in each block B_i in decreasing order;
- transform each of these blocks into a cycle.

For instance, if $P = 1\ 2\ 7\ 9|3\ 5\ 6|4\ 8$, then $\mu(P) = (9,7,2,1)(6,5,3)(8,4)$.

By Lemma 8, if $\mu(P) = \sigma \in \mathrm{BP}_1(n)$ and $f = \chi^{-1}(\sigma)$ is its transposition array, then $\forall i \in [n]$,

$$f_i = \text{ minimum of the block of } P \text{ containing } i.$$

Recall also that if $\sigma \in \mathrm{BP}_2(n)$ and $f = \chi^{-1}(\sigma) = \tau(P)$ is its transposition array, then $\forall i \in [n]$,

$$f_i = \text{ index of the block of } P \text{ containing } i.$$

Thus, we have the bijection $\beta := \lambda \circ \mu^{-1} : \mathrm{BP}_1(n) \mapsto \mathrm{BP}_2(n)$. As we shall see, β can be described as follows concretely.

Proposition 20. *Let $\sigma = C_1C_2\cdots C_k \in \mathrm{BP}_1(n)$, written in cycle notation, where each cycle is decreasing. Let σ' be constructed from σ according to the rule: for $i = k, k-1, \ldots, 2$, if the integer i is not in the ith cycle, then insert the sequence of elements of the ith cycle after i in the cycle containing i. Then $\sigma' = \beta(\sigma)$.*

Proof. Let $f = \chi^{-1}(\sigma)$, ν be the transformation that normalizes f via the order-preserving bijection of $\mathrm{Im}(f)$ into $[\mathrm{im}(f)]$, and $f' = \nu(f)$. Let $\gamma = \chi \circ \mu^{-1}$. Observe that $\nu = \gamma \circ \tau$. By Lemma 5 and inspection, we have $\chi^{-1}(\sigma') = f' = \nu(f) = \tau \circ \mu^{-1}(\sigma)$. Indeed, if $f^{(i)}$ is the transposition array associated with the permutation obtained after the ith step of the procedure, then $\forall j \in C_i$, one has $f^{(i)}(j) = i$ and the image of such integers j does not change in the following steps. In other words, the following diagram is commutative.

$$\begin{tikzcd}
 & & P \arrow[dr, "\tau"] & & \\
 & f \arrow[ur, "\gamma"] \arrow[rr, "\nu"'] & & f' \arrow[dr, "\chi"] & \\
\sigma \arrow[ur, "\chi^{-1}"] \arrow[rrrr, "\beta"'] & & & & \sigma'
\end{tikzcd}$$

So we have $\sigma' = \chi(f') = \chi \circ \tau \circ \mu^{-1}(\sigma) = \lambda \circ \mu^{-1}(\sigma) = \beta(\sigma)$ indeed. □

For instance, let $\sigma = (9,7,2,1)(6,5,3)(8,4)$. Then σ' is obtained as follows:

$$\sigma = (9,7,2,1)(6,5,3)(8,4) \longrightarrow (9,7,2,1)(6,5,3,8,4) \longrightarrow (9,7,2,6,5,3,8,4,1) = \sigma'.$$

We can also describe directly $\vartheta := \beta^{-1}$ as follows. Take $\sigma' \in \mathrm{BP}_2(n)$ and let $C_1 C_2 \cdots C_l$ be its cycle decomposition. Assume that $\mathrm{Wex}(\sigma) = [k]$. For $i = 2, \ldots, k$, if i is not the minimum of its cycle C_j, then form a new cycle by taking out of C_j the longest sequence of integers greater than i starting immediately after i, and modify the cycles. The resulting permutation is $\sigma = \vartheta(\sigma')$. For instance, let $\sigma = 468912357 = (\mathbf{1},\mathbf{4},9,7,\mathbf{3},\mathbf{5},8)(\mathbf{2},6)$ in cycle notation and with the weak excedances in bold. Then σ is obtained as follows:

$$\begin{aligned}\sigma' = (1,4,9,7,3,5,8)(\mathbf{2},6) &\longrightarrow (1,4,9,7,\mathbf{3})(2,6)(5,8) \longrightarrow (1,\mathbf{4},3)(2,6)(5,8)(9,7) \longrightarrow \\ &(1,4,3)(2,6)(\mathbf{5})(9,7)(8) = \sigma.\end{aligned}$$

Remark 21. Under the bijection $\beta : \sigma \mapsto \sigma'$, the number of cycles of σ is equal to the number of weak excedances of σ'.

The OEIS entry number A026898 enumerates the number of set partitions over $[n+1]$ whose minima form an interval of positive integers starting with 1. By Corollary 11 and Proposition 14, these set partitions correspond to Bell permutations of the second kind over $[n+1]$ having an equal number of weak excedances and the number of cycles. Also, notice that $\mathrm{BP}_1(n) \cap \mathrm{BP}_2(n) = \{\sigma \in \mathrm{BP}_2(n) : \mathrm{wex}(\sigma) = \mathrm{cyc}(\sigma) = \ell\}$. Thus and by (3), we have the following

Corollary 22. *For $n \geq 1$,*

$$|\,\mathrm{BP}_1(n) \cap \mathrm{BP}_2(n)| = 1 + \sum_{\ell=1}^{n} \ell^{n-\ell+1}. \tag{5}$$

□

4 Other classes of Bell enumerated subexcedant functions

In this section, we present two families of subexcedant functions also counted by the Bell numbers.

Let $f = f_1 f_2 \cdots f_n \in \mathrm{SF}(n)$. Recall that i is a leftmost occurrence in f if $f_i \notin \{f_1, \ldots, f_{i-1}\}$, where $i \in [n]$. The integer 1 is a leftmost occurrence. We say that $i > 1$ is a *repetition* in f if it is not a leftmost occurrence.

A subexcedant function f is said to avoid a pattern 212 (or 121) if there do not exist some indices $a < b < c$ such that $f_a = f_c > f_b$ (or $f_a = f_c < f_b$, respectively) [10].

The first family we consider is the set $\mathrm{SF}_1(n)$ of subexcedant functions over $[n]$ such that for $j \in \mathrm{Im}(f)$, the set of all $f^{-1}(j)$ forms an integer interval. For instance, $1133222 \in \mathrm{SF}_1(7)$. The following remark characterizes the set $\mathrm{SF}_1(n)$ in terms of pattern avoidance.

Remark 23. A subexcedant function $f \in \mathrm{SF}_1(n)$ if and only if f is 212 and 121-avoiding.

We let $\mathrm{SF}_1(n,k) := \{f \in \mathrm{SF}_1(n) : \mathrm{im}(f) = k\}$. Define the map $\omega : \mathrm{SF}_1(n,k) \mapsto \mathrm{SP}(n, n+1-k)$ by $\omega(f) = P$, where P is the set partition obtained from f as follows: initialize the first block with $f_1 = 1$ as a minimum, the remaining $n-k$ blocks with the repetitions as minima, and finally insert a leftmost occurrence $i > 1$ in the jth block, where $j = |[f_i] \backslash \{f_1, f_2, \ldots, f_{i-1}\}|$.

Example 24. Consider $f = 111334268 \in \mathrm{SF}_1(9,6)$. The set of repetitions of f is $\{2,3,5\}$. So we have 4 blocks initialized as $1\cdots|2\cdots|3\cdots|5\cdots$. Since $|[3]\backslash\{1\}| = 2$, the leftmost occurrence 4 is inserted in the 2-nd block. Since $|[4]\backslash\{1,3\}| = 2$, we insert 6 in the 2-nd block, and so on. Thus we obtain the set partition $\omega(f) = P = 1\ 7|2\ 4\ 6\ 8|3\ 9|5$. Observe that $P \in \mathrm{SP}(9,4)$.

Conversely, assume that the values $f_1 = 1, f_2, \ldots, f_{i-1}$ have already been computed. If i is in the jth block of P and $i > \min(B_j)$, then let f_i be the jth smallest element of the set $[n]\backslash\{f_1, f_2, \ldots, f_{i-1}\}$. If $i = \min(B_j), j > 1$, then let $f_i = f_{i-1}$. It is easy to see that $f = \omega^{-1}(P)$.

Proposition 25. *The map ω is a bijection.* □

Corollary 26. *For $n \geq 1$, we have*

$$|\,\mathrm{SF}_1(n,k)| = S(n, n+1-k),$$

where $S(n,k)$ is the Stirling number of the second kind. □

The second family of subexcedant functions we consider is as follows.

For $f \in \mathrm{SF}(n)$, we define $\mathrm{RmL}(f)$ to be the *subword* of the rightmost letters of f in the order they appear in f, i.e., if $f = f_1 f_2 \cdots f_n$, then $\mathrm{RmL}(f)$ is the subword of f composed of all f_i's such that i is a rightmost occurrence of f. For instance, if $f = 121135623$, then $\mathrm{RmL}(f) = 15623$. Note that $\mathrm{RmL}(f) = \mathrm{Im}(f)$ as a set. Recall that the rightmost letters of f corresponding to the weak excedances of the corresponding permutation $\sigma = \chi(f)$. Thus the subword $\mathrm{RmL}(f)$ increases if and only if the subword of weak excedance letters of σ increases. For the function $f = 121135623$, the corresponding permutation is $\sigma = 489367125$. The subword of its weak excedance letters is 48967 and is not increasing.

We let $\mathrm{SF}_2(n)$ denote the set of subexcedant functions over $[n]$ whose subword of the rightmost letters is increasing. Also, let $\mathrm{SF}_2(n,k) := \{f \in \mathrm{SF}_2(n) : \mathrm{im}(f) = k\}$.

Theorem 27. *The number of permutations in $\mathfrak{S}_n$ having increasing subword of weak excedance letters is the nth Bell number $B(n)$.*

Proof. We give two proofs via the transposition arrays of such permutations. We first prove directly that the cardinality of the set $\mathrm{SF}_2(n,k)$ is equal to $S(n,k)$, which satisfies the relation in (1), and then provide another proof by presenting a bijection between $\mathrm{SF}_2(n)$ and $\mathrm{RGF}(n)$.

Suppose the subword of weak excedance letters of a permutation σ is increasing. Let f be the transposition array of σ, i.e., $f = \chi^{-1}(\sigma)$ with $\mathrm{RmL}(f) = f_{i_1} f_{i_2} \dots f_{i_k}$. Then we have $f_{i_1} < f_{i_2} < \cdots < f_{i_k}$ and $i_1 < i_2 < \cdots < i_k$. Therefore, $f \in \mathrm{SF}_2(n, k)$. We can obtain each such subexcedant function in either of the following ways. Consider a subexcedant function $f \in \mathrm{SF}_2(n-1, k)$. Let a be an element of $\mathrm{Im}(f)$, and let f' be obtained from f by inserting the value a in position a. Then $f' \in \mathrm{SF}_2(n, k)$ and $\mathrm{RmL}(f') = \mathrm{RmL}(f)$. Since there are k possible choices for a, this contributes $k|\mathrm{SF}_2(n-1, k)|$ to the number $|\mathrm{SF}_2(n, k)|$. Consider a subexcedant function $f \in \mathrm{SF}_2(n-1, k-1)$ with $\mathrm{RmL}(f) = f_{i_1} < f_{i_2} < \cdots < f_{i_{k-1}}$, where $i_1 < i_2 < \cdots < i_{k-1}$. Let f' be obtained from f by appending n at its end. Then $f' \in \mathrm{SF}_2(n, k)$ and $\mathrm{RmL}(f') = \langle f_{i_1} < f_{i_2} < \cdots < f_{i_{k-1}} < n \rangle$, where $i_1 < i_2 < \cdots < i_{k-1} < n$. This operation contributes $|\mathrm{SF}_2(n-1, k-1)|$ to the number $|\mathrm{SF}_2(n, k)|$. Hence, by combining the cases, we have the proof.

Alternatively, we present a bijection between the sets $\mathrm{SF}_2(n)$ and $\mathrm{RGF}(n)$. Let $f \in \mathrm{SF}_2(n)$ and f' be the function obtained from f as follows. For $i = n, n-1, \dots, 2, 1$: let $g^{(n)} = f$ and $g^{(i)}$ be the function obtained from $g^{(i+1)}$ by deleting the largest fixed point. Note that $g^{(i)}$ is a subexcedant function over $[i]$. Now let j be the largest fixed point in the function $g^{(i)}$, set $f'_i = j'$, where j' is the normalized value of j under the map ν given in Proposition 20. We note that f' is a restricted growth function, and that $\mathrm{im}(f) = \mathrm{im}(f')$.

Conversely, let $f' \in \mathrm{RGF}(n)$. We obtain f uniquely from f' as follows. Suppose the function $g^{(i-1)}$ has already been computed. This function is a subexcedant function over $[i-1]$. Then at the ith step: if $j = f'_i \leq \mathrm{im}(g^{(i-1)})$, and a is the jth smallest element in $\mathrm{Im}(g^{(i-1)})$, then insert a also as a value in the function $g^{(i-1)}$ in position a; otherwise, let $g^{(i)}_i = i$. It can then be seen that $f = g^{(n)} \in \mathrm{SF}_2(n)$. Therefore, $f \mapsto f'$ is a bijection. $\square$

Example 28. Take $f = 11131338 \in \mathrm{SF}_2(8)$. Then $\mathrm{im}(f) = 3$ and the corresponding RGF $f' = f'_1 f'_2 \cdots f'_8$ is obtained as follows.

$$\begin{array}{ll}
g^{(8)} = 11131338 & f'_8 = 3 \\
g^{(7)} = 1113133 & f'_7 = 1 \\
g^{(6)} = 113133 & f'_6 = 2 \\
g^{(5)} = 11133 & f'_5 = 1 \\
g^{(4)} = 1133 & f'_4 = 2 \\
g^{(3)} = 113 & f'_3 = 2 \\
g^{(2)} = 11 & f'_2 = 1 \\
g^{(1)} = 1 & f'_1 = 1
\end{array}$$

Therefore, $f' = 11221213 \in \mathrm{RGF}(8)$.

5 Bijections on set partitions

In this section, we present some bijections on set partitions. In particular, we give an involution that interchanges the number of merging blocks (that we define below) and the number of successions. We use these bijections to study the generating function for the distribution of these statistics, and to deduce some structural results for set partitions.

For $n \geq 1$, we shall describe a partition of $\mathrm{SP}(n)$ into equivalence classes. The set partitions within each class are closely related. Each class will contain only one merging-free partition. Since there are $B(n-1)$ merging-free partitions, the same is true for the number of classes. The size of each class is a power of two.

Recall that a set partition $P = B_1|B_2|\cdots|B_k \in \mathrm{SP}(n,k)$ in standard form satisfies the condition $\min(B_i) < \min(B_{i+1}), 1 \leq i < k$.

5.1 Merging and successions equivalence

In this subsection, we discuss transforming a merging block of a set partition into succession and vice versa.

Let $\mathcal{T}_n^a := \{P \in \mathrm{SP}(n) : a \in \mathrm{Mb}(P)\}$ and $\mathcal{R}_n^a := \{P \in \mathrm{SP}(n) : a \in \mathrm{Suc}(P)\}$. We always assume that $a \in [2,n]$. Further for any $A \subseteq [2,n]$, let $\mathcal{T}_n^A := \{P \in \mathrm{SP}(n) : \mathrm{Mb}(P) = A\}$ and $\mathcal{R}_n^A := \{P \in \mathrm{SP}(n) : \mathrm{Suc}(P) = A\}$. It can easily be seen that $\mathcal{T}_n^a = \bigcup\limits_{\substack{A \\ a \in A}} \mathcal{T}_n^A$, and similarly

$\mathcal{R}_n^a = \bigcup\limits_{\substack{A \\ a \in A}} \mathcal{R}_n^A$.

Remark 29. We recall [4, Proposition 1.1] that the number $|\mathcal{T}_n^{\emptyset}|$ of merging-free partitions over $[n]$ equals the Bell number $B(n-1), n \geq 1$. Likewise, the sequence of the number of set partitions over $[n]$ having m successions is presented in the OEIS entry number A056857 and Munagi [17].

We define the operation $\mathrm{Swap}_a^{(i,j)}$ on a set partition $P = B_1|B_2|\cdots|B_k$, where i and j are two integers in $[k]$ and $a \in [n]$. If $i = j$ or $a \notin B_i \cup B_j$, we let $\mathrm{Swap}_a^{(i,j)}(P) = P$. Else, we let I_a be the maximal integer interval in $B_i \cup B_j$ that starts with a, and we move the elements of I_a lying in B_i to B_j and vice versa.

For instance, let $P = 1\ 3\ 4\ 6\ 8|2\ 5\ 9|7$. Then $\mathrm{Swap}_3^{(1,2)}(P) = 1\ 5\ 8|2\ 3\ 4\ 6\ 9|7$, $\mathrm{Swap}_7^{(1,3)}(P) = 1\ 3\ 4\ 6\ 7|2\ 5\ 9|8$, $\mathrm{Swap}_3^{(1,1)}(P) = \mathrm{Swap}_3^{(2,3)}(P) = P$, and $\mathrm{Swap}_7^{(2,3)}(P) = 1\ 3\ 4\ 6\ 8|2\ 5\ 7\ 9|$, with the last new block empty. (Here, strictly speaking, $\mathrm{Swap}_7^{(2,3)}(P)$ is not a set partition. However, in our applications of Swap, such an empty block never appears.)

We now define the following maps.

1. Consider $P = B_1|B_2|\cdots|B_k \in \mathcal{T}_n^a$. Then $a = \min(B_i)$ and $a-1 \in B_j$ for certain i and j. Note then that $\min(B_j) \leq a-1 < \min(B_i)$, whence $j < i$. Define the map $\mu_a : \mathcal{T}_n^a \mapsto \mathcal{R}_n^a$ by $\mu_a(P) = P'$, where P' is obtained from P as follows. Let P^* be the set partition obtained by merging the blocks B_{i-1} and B_i, and put $P' = \mathrm{Swap}_a^{(i-1,j)}(P^*)$. We note that a becomes a succession of $\mu_a(P)$. Later we will show that $\mathrm{Nmb}(P') = \mathrm{Nmb}(P)$. For instance, let $P = 1\,3\,5\,7\,10|2\,4|6\,8|9$. We have $\mathrm{Mb}(P) = \{6,9\}, \mathrm{Suc}(P) = \emptyset$. If $a = 6$, then $i = 3, j = 1$ and $P^* = 1\ 3\ 5\ 7\ 10|2\ 4\ 6\ 8|9$. Thus,

$P' = \mu_6(P) = 1\ 3\ 5\ 6\ 8\ 10|2\ 4\ 7|9 \in \mathcal{R}_{10}^6$. Note that $\mathrm{Mb}(P') = \{9\}, \mathrm{Suc}(P') = \{6\}$, and that $\mathrm{Nmb}(P') = \mathrm{Nmb}(P) = \{1, 2\}$.

2. Consider $P = B_1|B_2|\cdots|B_k \in \mathcal{R}_n^a$. Then $a-1, a \in B_i$ for some i. Define the map $\rho_a : \mathcal{R}_n^a \mapsto \mathcal{T}_n^a$ by $\rho_a(P) = P'$, where P' is obtained from P as follows. Let j be the smallest positive integer such that the elements $1, 2, \ldots, a-1$ are in the first j blocks of P. Apply $\mathrm{Swap}_a^{(i,j)}$ to P and then split the modified block B_j before a. We note that the succession a becomes the minimum element of a merging block of $\rho_a(P)$. Further, $\mathrm{Nmb}(P') = \mathrm{Nmb}(P)$. For instance, let $P = 1\ 3\ 4\ 6\ 9|2\ 5\ 8|7|10$ with $\mathrm{Mb}(P) = \{10\}, \mathrm{Suc}(P) = \{4\}$, and let $a = 4$. So $i = 1, j = 2$, and $\mathrm{Swap}_4^{(1,2)}(P) = 1\ 3\ 5\ 9|2\ 4\ 6\ 8|7|10$. Hence, we have $\rho_4(P) = P' = 1\ 3\ 5\ 9|2|4\ 6\ 8|7|10 \in \mathcal{T}_{10}^4$. Observe that $\mathrm{Mb}(P') = \{4, 10\}, \mathrm{Suc}(P') = \emptyset$, and that $\mathrm{Nmb}(P') = \mathrm{Nmb}(P) = \{1, 2, 7\}$.

Lemma 30.

1. *If* $a \in \mathrm{Mb}(P)$ *and* $P' = \mu_a(P)$, *then* $\mathrm{Mb}(P') = \mathrm{Mb}(P)\backslash\{a\}$, $\mathrm{Suc}(P') = \mathrm{Suc}(P) \cup \{a\}$, *and* $\mathrm{Nmb}(P') = \mathrm{Nmb}(P)$.

2. *If* $a \in \mathrm{Suc}(P)$ *and* $P' = \rho_a(P)$, *then* $\mathrm{Suc}(P') = \mathrm{Suc}(P)\backslash\{a\}$ *and* $\mathrm{Mb}(P') = \mathrm{Mb}(P) \cup \{a\}$, *and* $\mathrm{Nmb}(P') = \mathrm{Nmb}(P)$.

Proof. We provide only the proof of the first item, since the proof of the second is analogous.

Let $P = B_1|\cdots|B_k \in \mathcal{T}_n^a$, where $a \in B_i,\ a-1 \in B_j$ for some $j < i \le k$. So $\max(B_{i-1}) < \min(B_i)$ since B_i is merging. Let I_a denote the interval of integers moved by μ_a (by this we mean the interval moved by the Swap operation in the procedure of μ_a). Let $P' = \mu_a(P) = B_1'|\cdots|B_{k-1}'$. We consider two cases.

If $j = i-1$, then $B_x' = B_x$ for $x < i-1$, $B_{i-1}' = B_{i-1} \cup B_i$, and $B_x' = B_{x+1}$ for $i \le x < k$. This implies $\max(B_{i-1}') = \max(B_i)$ and $\min(B_{i-1}') = \min(B_{i-1})$. So B_{i-1}' (resp., B_i') is merging if and only if B_{i-1} (resp., B_{i+1}) is merging. Thus, $\mathrm{Mb}(P') = \mathrm{Mb}(P)\backslash\{a\}, \mathrm{Suc}(P') = \mathrm{Suc}(P) \cup \{a\}$.

Now suppose that $j < i-1$. In this case $B_x' = B_x$ for $x < j$ or $j < x < i-1$ and $B_x' = B_{x+1}$ for $i \le x < k$, and $\max(B_j) > \min(B_{j+1})$ since $\max(B_j) \ge a-1$ and $\min(B_{j+1}) < a$. Since the integers of the interval I_a are greater than or equal to a and μ_a swaps these integers between B_j and B_i, $\max(B_j') \ge \max(B_j)$. Observe that $\min(B_{j+1}') = \min(B_{j+1})$. Thus, $\max(B_j') > \min(B_{j+1}')$. Further, we have that $\max(B_{i-1}') \le \max(B_i)$ and $\min(B_i') = \min(B_{i+1})$. Hence $\max(B_{i-1}') > \min(B_i')$ if and only if $\max(B_i) > \min(B_{i+1})$. Therefore, no new merging block is created in this process, and hence $\mathrm{Mb}(P') = \mathrm{Mb}(P)\backslash\{a\}$.

On the other hand, let us show that the process does not create any new succession other than a. If $b-1, b \in I_a, b > a$, then either both of them belong to the same block in P, and thus μ_a moves them together to the other block, or they belong to different blocks and thus μ_a swaps them. Thus $\mathrm{Suc}(P') = \mathrm{Suc}(P) \cup \{a\}$.

Furthermore, observe that neither μ_a nor ρ_a moves the minimum element of a non-merging block. Thus $\mathrm{Nmb}(P)$ is preserved under these maps. $\square$

Lemma 31. *We have $\rho_a \circ \mu_a = id_{\mathcal{T}_n^a}$ and $\mu_a \circ \rho_a = id_{\mathcal{R}_n^a}$. In other terms, μ_a and ρ_a are inverses of each other.*

Proof. Since a succession cannot be the minimum element of a block for any set partition P, we have always $\mathrm{Mb}(P) \cap \mathrm{Suc}(P) = \emptyset$. We first prove that $\rho_a \circ \mu_a = id_{\mathcal{T}_n^a}$. Let $P = B_1|B_2|\cdots|B_k \in \mathcal{T}_n^a$, suppose that $a \in B_i$ and $a-1 \in B_j$ with $j < i \le k$. Since B_i is merging and P is in standard form, $B_{i-1} \subseteq [a-1] \subseteq \cup_{\ell=1}^{i-1} B_\ell$. Let I_a be the maximal integer interval moved by μ_a, so $I_a \subseteq B_{i-1} \cup B_i \cup B_j$. After applying μ_a the integer a becomes a succession in $P' = \mu_a(P) = B_1'|B_2'|\cdots|B_{k-1}'$, i.e., $a-1, a \in B_j'$, and since μ_a merges the blocks B_i and B_{i-1}, the block B_{i-1}' in P' is the rightmost block containing some integer(s) smaller than a. Therefore, when ρ_a is applied to P' it splits precisely this block to create a merging block. Thus, if I_a' is the maximal integer interval moved by ρ_a (i.e., by the $\mathrm{Swap}_a^{(i-1,j)}$), then we have $I_a' = I_a$ because $B_{i-1}' \cup B_j' = B_{i-1} \cup B_i \cup B_j$. Therefore, ρ_a reverses the action of μ_a and $\rho_a \circ \mu_a$ is the identity on $\mathcal{T}_n^a$.

Next, we prove that $\mu_a \circ \rho_a = id_{\mathcal{R}_n^a}$. Suppose that $P = B_1|B_2|\cdots|B_k \in \mathcal{R}_n^a$. If ρ_a breaks a succession $a \in B_i$ for some $i \le k$, and creates a merging block, say B_j' for some j, in $P' = \rho_a(P) = B_1'|B_2'|\cdots|B_{k+1}'$, then $a-1 \in B_j'$ and the interval of integers moved by μ_a is the same as the interval moved by ρ_a. So, the map μ_a reverses the action of ρ_a and hence $\mu_a \circ \rho_a = id_{\mathcal{R}_n^a}$. $\square$

Lemma 32. *For any $a \neq b \in [2, n]$, we have*

1. $\mu_a \circ \mu_b = \mu_b \circ \mu_a$ *on* $\mathcal{T}_n^a \cap \mathcal{T}_n^b$,
2. $\rho_a \circ \rho_b = \rho_b \circ \rho_a$ *on* $\mathcal{R}_n^a \cap \mathcal{R}_n^b$, *and*
3. $\mu_a \circ \rho_b = \rho_b \circ \mu_a$ *on* $\mathcal{T}_n^a \cap \mathcal{R}_n^b$.

Proof. Item 1: suppose that $P = B_1|B_2|\cdots|B_k \in \mathcal{T}_n^a \cap \mathcal{T}_n^b$ and assume, without loss of generality, that $a = \min(B_{i_1}) < b = \min(B_{i_2})$ for some $i_1 < i_2 \le k$. Let $I_a := I_{a,P}$ and $I_b := I_{b,P}$ be the maximal integer intervals moved by μ_a and μ_b in P, respectively. Let $a-1 \in B_j$ for some $j < i_1$. Then $I_a \subseteq B_{i_1-1} \cup B_{i_1} \cup B_j$.

Suppose that $b-1 \notin I_a$. Then μ_a does not move $b-1$. Observe then that $I_{a,P}$ is a subset of the maximal integer interval $I_{a,\mu_b(P)}$ moved by μ_a in $\mu_b(P)$. Let $\alpha \notin I_{a,P}$ be the smallest integer greater than a. If $\alpha \in I_{a,\mu_b(P)}$, then α would be in the jth block of $\mu_b(P)$. Since $b-1 \notin I_{a,P}$ (whence $\alpha \le b-1$) and μ_b has only moved integers greater than $b-1$, we would have $\alpha \in B_j$. Therefore, instead, $I_{a,\mu_b(P)} \subseteq I_{a,P}$, and hence $I_{a,\mu_b(P)} = I_{a,P}$. Similarly, $I_{b,\mu_a(P)} = I_{b,P}$. Therefore, we have $\mu_a \circ \mu_b = \mu_b \circ \mu_a$.

We now suppose that $b-1 \in I_a$. Then $b-1$ is either in the block B_j or in the block B_{i_1} of P. In either case, $I_a = [a, b-1] \subseteq B_{i_1-1} \cup B_{i_1} \cup B_j$, thus we have $i_1 + 1 = i_2$, i.e., the block containing a and the block containing b in P are adjacent. First, in addition, assume that $b-1 \in B_j$:

$$P = B_1|\cdots|\underbrace{\cdots a-1 \cdots b-1 \cdots}_{B_j}|\cdots|B_{i_1-1}|\underbrace{a\cdots}_{B_{i_1}}|\underbrace{b\cdots}_{B_{i_1+1}}|\cdots|B_k.$$

Consider the product $\mu_b \circ \mu_a$. If $P' = \mu_a(P) = B'_1|B'_2|\cdots|B'_{k-1}$, then $b-1 \in B'_{i_1-1}$ and $b \in B'_{i_1} = B_{i_1+1}$:

$$P' = B'_1|\cdots|\underbrace{\cdots a-1\,a\cdots}_{B'_j}|\cdots|\underbrace{\cdots b-1}_{B'_{i_1-1}}|\underbrace{b\cdots}_{B'_{i_1}}|\cdots|B'_{k-1}.$$

Now when μ_b is applied to P' it simply merges the block B'_{i_1} to B'_{i_1-1} because $I_b \subseteq B'_{i_1}$. Then we have $P'' = \mu_b(P') = B''_1|B''_2|\cdots|B''_{k-2}$ as follows.

$$P'' = B''_1|\cdots|\underbrace{\cdots a-1\,a\cdots}_{B''_j}|\cdots|\underbrace{\cdots b-1\,b\cdots}_{B''_{i_1-1}}|\cdots|\cdots|B''_{k-2}.$$

Now consider the product $\mu_a \circ \mu_b$. Since $b-1 \in B_j, b \in B_{i_1+1}$, the interval $I_b \subseteq B_{i_1+1} \cup B_j$. Thus μ_b move the elements of I_b and merges the modified block B_{i_1+1} with the block B_{i_1}, i.e., we obtain a set partition $P^* = \mu_b(P)$:

$$P^* = B^*_1|\cdots|\underbrace{\cdots a-1\cdots b-1\,b\cdots}_{B^*_j}|\cdots|B^*_{i_1-1}|\underbrace{a\cdots}_{B^*_{i_1}}|\cdots|B^*_{k-1}.$$

Since μ_a moves $b-1$, in this case when μ_a is applied to $P^* = \mu_b(P)$, the interval $I_{a,P^*} = I_{a,P} \cup I_{b,P}$. So μ_a restores those elements that μ_b moved from B_j to B_{i_1+1} back to the jth block of $\mu_b(P)$ and vice-versa. Therefore, $\mu_a(\mu_b(P)) = \mu_a(P^*) = P''$.

In the subcase where $b-1 \in I_a$ and $b-1 \in B_{i_1}$, the argument is similar. Hence $\mu_a \circ \mu_b = \mu_b \circ \mu_a$ in all cases.

For Items 2 and 3, we use the equality in Item 1, and the fact that μ_a and ρ_a are inverses (Lemma 31). So

$$\begin{aligned}\rho_a \circ \rho_b &= \rho_b \circ \rho_a \circ \mu_a \circ \mu_b \circ \rho_a \circ \rho_b\\ &= \rho_b \circ \rho_a \circ \mu_b \circ \mu_a \circ \rho_a \circ \rho_b\\ &= \rho_b \circ \rho_a,\end{aligned}$$

and $\mu_a \circ \rho_b = \rho_b \circ \mu_b \circ \mu_a \circ \rho_b = \rho_b \circ \mu_a \circ \mu_b \circ \rho_b = \rho_b \circ \mu_a$. □

For any $P \in \mathrm{SP}(n)$, $A = \{a_1, \ldots, a_m\} \subseteq \mathrm{Mb}(P)$, and $B = \{b_1, \ldots, b_s\} \subseteq \mathrm{Suc}(P)$, we define $\psi_{A,B}(P) = P'$, where P' is the set partition obtained from P by applying μ_a for each element a of A and applying ρ_b for each element b of B. Thus,

$$\psi_{A,B} = \mu_{a_1} \cdots \mu_{a_m} \rho_{b_1} \cdots \rho_{b_s}.$$

By the preceding lemmas, there is an equivalence relation in the set $\mathrm{SP}(n)$ defined by two set partitions

$$P \equiv P' \iff (\exists A \subseteq \mathrm{Mb}(P)\ \exists B \subseteq \mathrm{Suc}(P) \text{ such that } \psi_{A,B}(P) = P').$$

Let $\Gamma[P]$ denote the equivalence class containing the set partition P.

Proposition 33. *For any $P \in \mathrm{SP}(n)$, we have*

$$|\Gamma[P]| = 2^{\mathrm{mb}(P)+\mathrm{suc}(P)}. \tag{6}$$

Moreover, for any $P' \in \Gamma[P]$, we have

$$\mathrm{Mb}(P') \cup \mathrm{Suc}(P') = \mathrm{Mb}(P) \cup \mathrm{Suc}(P) \tag{7}$$

and

$$\mathrm{Nmb}(P') = \mathrm{Nmb}(P). \tag{8}$$

Proof. Equation (6) follows directly from the fact that for any $A \subseteq \mathrm{Mb}(P) \cup \mathrm{Suc}(P)$, there exists a unique set partition $P' \in \Gamma[P]$ such that $\mathrm{Mb}(P') = A$. The rest follows from Lemma 30. □

Corollary 34. *The number of set partitions over $[n]$ having exactly one non-merging block is $2^{n-1}, n \geq 1$.*

Proof. The set partition over $[n]$ having exactly one non-merging block and no merging block is the trivial set partition, $12 \cdots n$, with $[2, n]$ as a set of successions. Thus, $\{P \in \mathrm{SP}(n) : \mathrm{nmb}(P) = 1\} = \Gamma[12 \cdots n]$, and it has indeed size 2^{n-1} by (6). □

5.2 Enumeration results

In this subsection, we employ the bijections we have defined to give some results on the distribution of $\mathrm{mb}(P)$ and $\mathrm{suc}(P)$, where P is any set partition over $[n]$.

Lemma 35. *For any $A, A', B, B' \subseteq [2, n], n \geq 2$, such that A and B are disjoint, and A' and B' are disjoint, and $A \cup B = A' \cup B'$, the cardinalities of the sets $\mathcal{T}_n^A \cap \mathcal{R}_n^B$ and $\mathcal{T}_n^{A'} \cap \mathcal{R}_n^{B'}$ are equal.*

Proof. The map $\psi_{A \setminus A', B \setminus B'}$ yields a bijection between these sets. □

We note that for any disjoint subsets A and B of $[2, n]$, the restriction of $\psi_{A,B}$ to the set $\mathcal{T}_n^A \cap \mathcal{R}_n^B$ provides a bijection between this set and $\mathcal{T}_n^B \cap \mathcal{R}_n^A$. (Note that $\psi_{\emptyset,\emptyset}$ restricts to the identity on $\mathcal{T}_n^\emptyset \cap \mathcal{R}_n^\emptyset$). Since the collection of such $\mathcal{T}_n^A \cap \mathcal{R}_n^B$ forms a partition of $\mathrm{SP}(n)$, we can put these restrictions together to obtain an involution ψ. In other words, for any set partition P, we let $\psi(P) = \psi_{\mathrm{Mb}(P),\mathrm{Suc}(P)}(P)$.

Example 36. Let $P = 1\ 4\ 5|2\ 6\ 7\ 9|3|8\ 10$. We have $\mathrm{Mb}(P) = \{8\}$, $\mathrm{Suc}(P) = \{5, 7\}$, and $P' = \psi_{\{8\},\{5,7\}}(P) = \rho_5\rho_7\mu_8(P)$. Then $\mu_8(P) = 1\ 4\ 5|2\ 6\ 7\ 8\ 10|3\ 9$, $\rho_7(\mu_8(P)) = 1\ 4\ 5|2\ 6\ 9|3|7\ 8\ 10$, $\rho_5(\rho_7(\mu_8(P))) = 1\ 4|2\ 6\ 9|3|5|7\ 8\ 10 = P'$. So $\mathrm{Mb}(P') = \{5, 7\}$, $\mathrm{Suc}(P') = \{8\}$, and $\mathrm{nmb}(P') = 3 = \mathrm{nmb}(P)$.

In particular (or by Lemma 35), we have

Theorem 37. *Let $n \geq 1$ and*

$$F_n(q,t,r) = \sum_{P \in \mathrm{SP}(n)} q^{\mathrm{mb}(P)} t^{\mathrm{suc}(P)} r^{\mathrm{nmb}(P)}.$$

Then

$$F_n(q,t,r) = F_n(t,q,r).$$

□

Proposition 38. *For any $A \subseteq [2,n], n \geq 2$, the cardinality of the set $\mathcal{T}_n^A$ is given by*

$$|\mathcal{T}_n^A| = B(n-1-|A|).$$

Proof. Let $P \in \mathcal{T}_n^A$, where $A = \{a_1, \ldots, a_m\}$. If $P' = \mu_{a_1} \cdots \mu_{a_m}(P)$, then $P' \in \mathcal{T}_n^{\emptyset}$. We then delete each $a \in A$ obtaining a set partition P'' on $n - |A|$ letters having no merging blocks, i.e., $P'' \in \mathcal{T}_{n-|A|}^{\emptyset}$. So the map $P \mapsto P''$ is a bijection, whence, indeed, by Remark 29 $|\mathcal{T}_n^A| = |\mathcal{T}_{n-|A|}^{\emptyset}| = B(n-1-|A|)$. □

Let $\mathrm{SP}^*(n)$ denote the set of all set partitions $P \in \mathrm{SP}(n)$ such that the removal of n creates a new merging block.

Proposition 39. *We have*

$$\sum_{P \in \mathrm{SP}^*(n+2)} q^{\mathrm{mb}(P)} t^{\mathrm{suc}(P)} r^{\mathrm{nmb}(P)} = n \sum_{Q \in \mathrm{SP}(n)} q^{\mathrm{mb}(Q)} t^{\mathrm{suc}(Q)} r^{\mathrm{nmb}(Q)+1}.$$

Proof. We prove the assertion by providing a bijection between the sets $[2, n+1] \times \mathrm{SP}(n)$ and $\mathrm{SP}^*(n+2)$. Let $\theta : [2, n+1] \times \mathrm{SP}(n) \mapsto \mathrm{SP}^*(n+2)$ be the map associating (a, P) with the set partition P' obtained from (a, P) as follows. Increase every integer greater than or equal to a in P by 1, and insert a to the block containing $a-1$. Now apply ρ_a to the resulting set partition, and insert $n+2$ in the block preceding the merging block created newly. Note that $P' \in \mathrm{SP}^*(n+2)$ and θ is a bijection such that $\mathrm{mb}(P') = \mathrm{mb}(P), \mathrm{suc}(P') = \mathrm{suc}(P)$, and $\mathrm{nmb}(P') = \mathrm{nmb}(P) + 1$. Therefore, we have the assertion. □

Let $h_k(n,m,s) := |\{P \in \mathrm{SP}(n) : \mathrm{bl}(P) = k, \mathrm{mb}(P) = m, \mathrm{suc}(P) = s\}|$. Then we have $h_1(n, 0, n-1) = h_n(n, n-1, 0) = 1, n \geq 1$, and $h_k(n,m,s) = 0$, where $k > n, m \geq k, s \geq n$, or $n < 0$.

Proposition 40. *For $n \geq 1$, we have*

$$h_k(n,m,s) = \binom{m+s}{m} h_{k-m}(n, 0, s+m). \tag{9}$$

Proof. We start with any set partition over $[n]$ having $k-m$ blocks and no merging blocks. If the set partition has $m+s$ successions, then by applying the maps ρ, we can create m merging blocks in $\binom{m+s}{m}$ ways. Thus, by the product rule, we have the result. □

We now give some consequences of the above proposition.

Proposition 41. *Given $n > s \geq 1$, we have*

$$h_k(n,0,s) = \binom{n-1}{s} h_k(n-s,0,0). \tag{10}$$

Proof. Let $P^{(0)} = P \in \mathcal{T}^{\emptyset}_{n-s} \cap \mathcal{R}^{\emptyset}_{n-s}$. There are $\binom{n-1}{s}$ possible ways to choose a subset of $[2,n]$ having size s. For any such set $A = \{a_1, \ldots, a_s\}$ with $a_1 < \cdots < a_s$ and for $i = 1, \ldots, s$, let $P^{(i)}$ be the set partition obtained from $P^{(i-1)}$ by increasing each integer greater than or equal to a_i by 1, and inserting a_i to the block containing $a_i - 1$. So $P^{(s)}$ is a set partition over $[n]$ with $\mathrm{Suc}(P^{(s)}) = A$. Hence, by the product rule, we obtain the result. $\square$

By combining (9) and (10), we have the following corollary.

Corollary 42. *For $n \geq 1$, we have*

$$h_k(n,m,s) = \binom{n-1}{m,\, s,\, n-m-s-1} h_{k-m}(n-m-s,0,0). \tag{11}$$

We let $G(x,y,z,w) := \sum_{n,k,m,s \geq 0} h_k(n,m,s) x^n y^k z^m w^s$ and $J(x,y) := \sum_{n,k \geq 0} h_k(n,0,0) x^n y^k$. Then we have

Proposition 43. $G(x,y,z,w) = J(x(1-xyz-xw)^{-1}, y)$.

Proof. By (11), we have indeed

$$\begin{aligned}
G(x,y,z,w) &= \sum_{n,k,m,s \geq 0} \binom{n-1+m+s}{m,\, s,\, n-1} x^{m+s} y^m z^m w^s h_k(n,0,0) x^n y^k \\
&= \sum_{n,k \geq 0} \sum_{m \geq 0} \sum_{s \geq 0} \binom{n-1+m+s}{n-1} \binom{m+s}{m} (xyz)^{m+s} y^m z^m w^s h_k(n,0,0) x^n y^k \\
&= \sum_{n,k \geq 0} \sum_{m+s \geq 0} \binom{n-1+m+s}{n-1} \sum_{m \geq 0} \binom{m+s}{m} (xyz)^{m+s} y^m z^m w^s h_k(n,0,0) x^n y^k \\
&= \sum_{n,k \geq 0} \sum_{m+s \geq 0} \binom{n-1+m+s}{n-1} (xyz+xw)^{m+s} h_k(n,0,0) x^n y^k \\
&= \sum_{n,k \geq 0} \frac{1}{(1-xyz-xw)^n} h_k(n,0,0) x^n y^k \\
&= J\left(\frac{x}{1-xyz-xw}, y\right).
\end{aligned}$$

$\square$

We let $\mathrm{SP}^0(n) := \mathcal{T}_n^{\emptyset} \cap \mathcal{R}_n^{\emptyset}$, the set of set partitions having no merging blocks and no successions. So we have $|\mathrm{SP}^0(n,k)| = h_k(n,0,0)$.

Theorem 44. *The numbers $h_k(n,0,0)$ satisfy the following recurrence relation for all positive integers n, k, $n \geq 2, 1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$:*

$$h_k(n,0,0) = (k-1)h_k(n-1,0,0) + (n-2)h_{k-1}(n-2,0,0), \tag{12}$$

where $h_0(n,0,0) = \delta_{n,0}$, $h_1(1,0,0) = 1$.

Proof. Let n and k be fixed positive integers. Let $\mathrm{SP}^0(n,k) = \mathcal{M} \cup \mathcal{N}$, where $\mathcal{M}$ is the subset of $\mathrm{SP}^0(n,k)$ consisting of those set partitions whose removal of n does not create a merging block, and $\mathcal{N} = \mathrm{SP}^0(n,k) \backslash \mathcal{M}$.

Let $P \in \mathrm{SP}^0(n-1,k)$ and P' be the set partition obtained from P by inserting n into any of its blocks except the block containing $n-1$. Then $P' \in \mathcal{M}$. Since there are $k-1$ possibilities where to insert n, we have the first term of the right-hand side of (12).

On the other hand, consider $a \in [2, n-1]$ and $P \in \mathrm{SP}^0(n-2, k-1)$. Let κ be the map that associates (a, P) with the set partition P' obtained as follows: Increase all integers greater than or equal to a in P by 1, split the rightmost block containing element(s) of the set $[a-1]$ after the rightmost element of $[a-1]$, insert n and a to the left and the right blocks of the split block, respectively. Then let the resulting partition be P^*. If $a+1$ is a succession in P^*, then let $P' = \mathrm{Swap}_{a+1}^{(i,j)}(P^*)$, where (i,j) is the pair of indices of the blocks containing a and $a-1$ in P^*; Otherwise, let $P' = P^*$. It can then be seen that $\kappa : [2, n-1] \times \mathrm{SP}^0(n-2, k-1) \mapsto \mathcal{N}$ is a bijection. Therefore, we have $|\mathcal{N}| = (n-2)h_{k-1}(n-2,0,0)$, the second term of the right-hand side of (12). $\square$

Up to a shift on both n and k, this is the same sequence as OEIS entry number A008299, counting set partitions without singletons. Therefore, there should be a natural bijection between these sets, though so far, we couldn't find one.

We now consider the distribution of the number of successions in a set $\mathcal{T}_n^{\emptyset}$ of merging-free partitions having a fixed number of blocks.

Theorem 45. *The numbers $h_k(n,0,s)$ satisfy the following recurrence relation for all positive integers n, k, s, $1 \leq s \leq n-2k+1, 1 \leq 2k-1 \leq n$:*

$$h_k(n,0,s) = h_k(n-1,0,s-1) + (k-1)h_k(n-1,0,s) + (s+1)h_{k-1}(n-1,0,s+1); \tag{13}$$

and $h_k(n,0,0)$ satisfies (12).

Proof. It is possible to obtain any set partition $P' \in \mathcal{T}_n^{\emptyset}$ recursively either from $P \in \mathcal{T}_{n-1}^{\emptyset}$ by inserting n in any of the existing blocks of P or from any $P^* \in \mathcal{T}_{n-1}^{\{a\}}$, where $a \in [2,n]$, by inserting n in the block preceding a merging block of P^*. In the first case, if n is inserted into the block containing $n-1$, then the number of successions increases by 1, but otherwise, it remains the same; anyhow the number of blocks remains the same. This explains the first two terms of the right-hand side of (13). In the second case, $P := \mu_a(P^*)$ has $\mathrm{Suc}(P) = \mathrm{Suc}(P') \cup \{a\}$ and the number of blocks one less than that of P'. Since $P^* = \rho_a(P)$ and a has $\mathrm{suc}(P)$ possibilities, this yields the third term. $\square$

Proposition 46. *Let* $H_k(x,z) = \sum\limits_{n\geq 2k-1}\sum\limits_{s\geq 0} h_k(n,0,s)z^s x^n$*. Then we have*

$$H_k(x,z) = \frac{x}{1-x(k-1+z)}\frac{\partial}{\partial z}\left(H_{k-1}(x,z)\right), k \geq 2. \tag{14}$$

Proof. We define the polynomial $H(n,k;z) = \sum\limits_{s=0}^{n-1} h_k(n,0,s)z^s$. Then, by (13), we have

$$\begin{aligned}\sum_{s\geq 1} h_k(n,0,s)z^s = \sum_{s\geq 1} h_k(n-1,0,s-1)z^s + \sum_{s\geq 1}(s+1)h_{k-1}(n-1,0,s+1)z^s \\ + \sum_{s\geq 1}(k-1)h_k(n-1,0,s)z^s\end{aligned}$$

and

$$\begin{aligned}H(n,k;z) - h_k(n,0,0) = zH(n-1,k;z) + H_z(n-1,k-1;z) - h_{k-1}(n-1,0,1) \\ + (k-1)(H(n-1,k;z) - h_k(n-1,0,0)).\end{aligned}$$

By applying (12) and (10), we obtain

$$H(n,k;z) = (z+k-1)H(n-1,k;z) + \frac{\partial}{\partial z}H(n-1,k-1;z), n \geq 2k-1. \tag{15}$$

We now let $H_k(x,y) = \sum\limits_{n\geq 2k-1} H(n,k;z)x^n$. Then multiplying (15) by x^n and taking the sum over all $n \geq 2k-1$, we obtain

$$H_k(x,z) = \frac{x}{1-x(k-1+z)}\frac{\partial}{\partial z}(H_{k-1}(x,z)).$$

□

We introduce the following definition.

Definition 47. Let $r \geq 0$ and $v = (v_0, v_1, \ldots, v_r)$ be a vector of non-negative integers such that $\sum\limits_{j=0}^{r} v_j = r$ and for $1 \leq i \leq r$, $s_i(v) > i-2$, where $s_i(v) := \sum\limits_{j=0}^{i-1} v_j$.

For any such vector $v = (v_0, v_1, \ldots, v_r)$, we have $v_r \leq 1$. If $v_r = 1$, then we let $v^{(r)} = (v_0, v_1, \ldots, v_{r-1})$. If $v_r = 0$, then for $0 \leq t \leq r-1$ and $v_t > 0$, let $v^{(t)} = (v'_0, v'_1, \ldots, v'_{r-1})$ be the vector obtained from v by setting $v'_t = v_t - \delta_{t,q}$, and deleting v_r. We also let

$$P_v := \prod_{i=1}^{r}(s_i(v) - i + 2),$$

correspondingly for $P_{v^{(t)}}$.

We give the following lemma that will be used to prove Theorem 49.

Lemma 48. *For $r \geq 0$, we have*

$$P_v = \begin{cases} P_{v^{(r)}}, & \text{if } v_r = 1; \\ \sum_{j=0}^{r-1} (v_j + \delta_{j,0}) P_{v^{(j)}}, & \text{if } v_r = 0. \end{cases}$$

Proof. If $v_r = 1$, then the last factor of $P_v = \prod_{i=1}^{r}(s_i(v) - i + 2)$ is $(v_0 + \cdots + v_{r-1} - r + 2) = r - v_r - r + 2 = 1$. Therefore, $P_v = \prod_{i=1}^{r-1}(s_i(v) - i + 2) = P_{v^{(r)}}$.

We now assume that $v_r = 0$. By definition $P_{v^{(t)}} = \prod_{i=1}^{r-1}(s_i(v^{(t)}) - i + 2)$ and

$$v^{(t)} = (v_0, v_1, \ldots, v_t - 1, v_{t+1}, \ldots, v_{r-1}).$$

Then

$$\begin{aligned} P_{v^{(t)}} &= \prod_{i=1}^{r-1}(s_i(v^{(t)}) - i + 2) \\ &= \prod_{i=1}^{t}(s_i(v) - i + 2) \cdot \prod_{i=t+1}^{r-1}(s_i(v) - i + 1) \end{aligned} \tag{16}$$

We first use induction on t to prove that

$$\sum_{j=0}^{t}(v_j + \delta_{j,0}) P_{v^{(j)}} = \prod_{i=1}^{t+1}(s_i(v) - i + 2) \cdot \prod_{i=t+1}^{r-1}(s_i(v) - i + 1). \tag{17}$$

Observe that

$$\sum_{j=0}^{0}(v_j + \delta_{j,0}) P_{v^{(j)}} = \prod_{i=1}^{1}(s_i(v) - i + 2) \cdot \prod_{i=1}^{r-1}(s_i(v) - i + 1),$$

and the assertion is true for $t = 0$. Suppose that $t > 0$, and

$$\sum_{j=0}^{t-1}(v_j + \delta_{j,0}) P_{v^{(j)}} = \prod_{i=1}^{t}(s_i(v) - i + 2) \cdot \prod_{i=t}^{r-1}(s_i(v) - i + 1).$$

Now by the induction assumption and (16), we have

$$\begin{aligned}
\sum_{j=0}^{t}(v_j+\delta_{j,0})P_{v^{(j)}} &= \sum_{j=0}^{t-1}(v_j+\delta_{j,0})P_{v^{(j)}} + v_t P_{v^{(t)}} \\
&= \prod_{i=1}^{t}(s_i(v)-i+2)\cdot \prod_{i=t}^{r-1}(s_i(v)-i+1) \\
&\quad + v_t\left(\prod_{i=1}^{t}(s_i(v)-i+2)\cdot \prod_{i=t+1}^{r-1}(s_i(v)-i+1)\right) \\
&= \prod_{i=1}^{t}(s_i(v)-i+2)(s_t(v)-t+1+v_t)\prod_{i=t+1}^{r-1}(s_i(v)-i+1) \\
&= \prod_{i=1}^{t+1}(s_i(v)-i+2)\prod_{i=t+1}^{r-1}(s_i(v)-i+1).
\end{aligned}$$

and thus (17) is proved. Then (17) for $t=r-1$ and the definition of P_v yields the result of the lemma for which $v_r=0$. $\square$

Theorem 49. *The generating function for $H_k(x,z)$ is given by*

$$H_k(x,z)=\frac{x^{2k-1}}{(1-xz)\Pi_{j=0}^{k-1}(1-x(j+z))}\sum_{v}\frac{\Pi_{i=1}^{k-2}(s_i(v)-i+2)}{\Pi_{j=0}^{k-2}(1-x(j+z))^{v_j}}, k\geq 2, \tag{18}$$

where $v=(v_0,v_1,\ldots,v_{k-2})$ as in Definition 47 (with $r=k-2$) with $H_0(x,z)=1$ and $H_1(x,z)=\frac{x}{1-xz}$.

Proof. For $0\leq j\leq k-2$, let $a_j:=1-x(j+z)$ and $a^v:=a_0^{v_0}\cdots a_{k-2}^{v_{k-2}}$. Then from (14) we have $H_k(x,z)=\frac{x}{a_{k-1}}\frac{\partial}{\partial z}(H_{k-1}(x,z))$, and the right-hand side of (18) is

$$\frac{x^{2k-1}}{a_0^2a_1\cdots a_{k-1}}\sum_{v}\frac{P_v}{a^v}=\frac{x^{2k-1}}{a_0^2a_1\cdots a_{k-1}}\left(\sum_{\substack{v,\\ v_{k-2}=1}}\frac{P_v}{a^v}+\sum_{\substack{v,\\ v_{k-2}=0}}\frac{P_v}{a^v}\right).$$

By applying Lemma 48, we have $\sum\limits_{\substack{v,\\ v_{k-2}=1}}\frac{P_v}{a^v}=\sum\limits_{\substack{v,\\ v_{k-2}=1}}\frac{P_{v^{(k-2)}}}{a^v}$, and

$$\sum_{\substack{v,\\ v_{k-2}=0}}\frac{P_v}{a^v}=\sum_{\substack{v,\\ v_{k-2}=0}}\left(\frac{(v_0+1)P_{v^{(0)}}+v_1P_{v^{(1)}}+\cdots+v_{k-2}P_{v^{(k-2)}}}{a^v}\right)$$

$$= \sum_{\substack{v,\\ v_{k-2}=0}} \left(\frac{2P_{v^{(0)}} + P_{v^{(1)}} + \cdots + P_{v^{(k-2)}}}{a^v} + \frac{(v_0-1)P_{v^{(0)}} + \cdots + (v_{k-2}-1)P_{v^{(k-2)}}}{a^v} \right)$$

$$= \sum_{\substack{v,\\ v_{k-2}=0}} \left(\frac{2}{a_0}\frac{P_{v^{(0)}}}{a^{v^{(0)}}} + \frac{1}{a_1}\frac{P_{v^{(1)}}}{a^{v^{(1)}}} + \cdots + \frac{1}{a_{k-2}}\frac{P_{v^{(k-2)}}}{a^{v^{(k-2)}}} + \left(\frac{v_0-1}{a_0}\frac{P_{v^{(0)}}}{a^{v^{(0)}}} + \cdots + \frac{v_{k-3}-1}{a_{k-3}}\frac{P_{v^{(k-3)}}}{a^{v^{(k-3)}}} \right) \right)$$

Therefore, $\frac{x^{2k-1}}{a_0^2 \Pi_{j=1}^{k-1} a_j} \sum\limits_v \frac{P_v}{a^v}$

$$= \frac{x^{2k-1}}{a_0^2 \Pi_{j=1}^{k-1} a_j} \left(\left(\frac{2}{a_0} + \frac{1}{a_1} + \cdots + \frac{1}{a_{k-2}} \right) \sum_{v^{(t)}} \frac{P_{v^{(t)}}}{a^{v^{(t)}}} + \sum_{v^{(t)}} \frac{P_{v^{(t)}}}{a^{v^{(t)}}} \left(\frac{v_0-1}{a_0} + \cdots + \frac{v_{k-3}-1}{a_{k-3}} \right) \right)$$

$$= \frac{x}{a_{k-1}} \frac{\partial}{\partial z} \left(\frac{x^{2k-3}}{a_0^2 a_1 \cdots a_{k-2}} \sum_{v^{(t)}} \frac{P_{v^{(t)}}}{a^{v^{(t)}}} \right)$$

$$= \frac{x}{a_{k-1}} \frac{\partial}{\partial z} (H_{k-1}(x,z))$$

$$= H_k(x,z),$$

indeed. □

By the fact that $\sum\limits_{n\geq k} S(n,k)x^n = \frac{x^k}{\Pi_{j=0}^k (1-jx)}, k \geq 0$, we have

Corollary 50.

$$H_k(x,0) = x^k \sum_{n\geq k-1} S(n,k-1)x^n \sum_v \frac{\Pi_{i=1}^{k-2}(s_i(v)-i+2)}{\Pi_{j=0}^{k-2}(1-jx)^{v_j}}, k \geq 2,$$

with $H_0(x,0) = 1, H_1(x,0) = x$.

Let us use the notation $h_{n,m,s} := \sum_{k=1}^n h_k(n,m,s)$.

Proposition 51. *For* $n \geq 1$, *we have*

$$\sum_{s=0}^{n-1} 2^s h_{n,0,s} = B(n),$$

where $B(n)$ *is the* n*th Bell number.*

Proof. By Proposition 40, we have $h_{n,m,s-m} = \binom{s}{m} h_{n,0,s}$. Thus,

$$\sum_{m=0}^{s} h_{n,m,s-m} = \sum_{m=0}^{s} \binom{s}{m} h_{n,0,s} = 2^s h_{n,0,s}.$$

Hence, taking the sum over all possible s, we have the result. □

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References

[1] J.-L. Baril, Gray code for permutations with fixed number of cycles, *Discrete Math.* **307** (2007), 1557–1571.

[2] J.-L. Baril, Statistics-preserving bijections between classical and cyclic permutations, *Inform. Process. Lett.* **113** (2013), 17–22.

[3] J.-L. Baril and V. Vajnovszki, A permutation code preserving a double Eulerian bistatistic, *Discrete Appl. Math.* **224** (2018), 9–15.

[4] F. Beyene and R. Mantaci, Merging-free partitions and run-sorted permutations, *J. Integer Sequences* **25** (2022), Article 22.7.6.

[5] F. Beyene and R. Mantaci, Permutations with non-decreasing transposition array and pattern avoidance, 2022, preprint. Available at http://arxiv.org/abs/2111.11527.

[6] M. Bona, *Introduction to Enumerative Combinatorics*, McGraw-Hill, 2007.

[7] D. Callan, On conjugates for set partitions and integer compositions, 2005, preprint. Available at http://arxiv.org/math/0508052v3.

[8] D. Dumont and G. Viennot, A combinatorial interpretation of the Seidel generation of Genocchi numbers, *Discrete Math.* **6** (1980), 77–87.

[9] D. Foata and D. Zeilberger, Denert's permutation statistic is indeed Euler-Mahonian, *Stud. Appl. Math.* **83** (1990), 31–59.

[10] V. Jelinek, T. Mansour, and M. Shattuck, On multiple pattern avoiding set partitions, *Adv. Appl. Math.* **50** (2012), 292–326.

[11] D. H. Lehmer, Teaching combinatorial tricks to a computer, in Richard Bellman and Marshall Hall Jr., Editors, *Combinatorial Analysis*, *Proc. Sympos. Appl. Math.* **10** (1960), 179–193.

[12] R. Mantaci and F. Rakotondrajao, A permutation representation that knows what "Eulerian" means, *Discrete Math. Theor. Comput. Sci.* **4** (2001), 101–108.

[13] T. Mansour. *Combinatorics of Set Partitions*, CRC Press, 2013.

[14] T. Mansour and A. O. Munagi, Set partitions with circular successions, *European J. Combin.* **42** (2014), 207–216.

[15] T. Mansour and R. Rastegar, Fixed points of a random restricted growth sequence, *Discrete Appl. Math.* **302** (2021), 171–177.

[16] T. Mansour and M. Shattuck, Parity successions in set partitions, *Linear Algebra Appl.* **439** (2013), 2642–2650.

[17] A. O. Munagi, Set partitions with successions and separations, *Int. J. Math. Math. Sci.* **3** (2005), 451–463.

[18] M. Poneti and V. Vajnovszki, Generating restricted classes of involutions, Bell and Stirling permutations, *European J. Combin.* **31** (2010), 553–564.

[19] G. Rota, The number of partitions of a set, *Amer. Math. Monthly* **71** (1964), 498–504.

[20] N. J. A. Sloane et al., *The On-Line Encyclopedia of Integer Sequences*, 2022. Available at `https://oeis.org`.

[21] R. P. Stanley, *Enumerative Combinatorics* **1**, Cambridge University Press, 2011.

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