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On Some Binomial Coefficient Identities with Applications

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Abstract

In this paper, we present an alternative proof for a generalization of a binomial identity by Simons, initially proven by Munarini. Unlike previous proofs, our approach uses the Taylor theorem and the Wilf-Zeilberger algorithm. We also offer a new binomial sum identity and generalize a combinatorial identity by Alzer and Kouba. Additionally, we derive several harmonic number identities, including the proof for H_n and a formula involving the square of binomial coefficients and harmonic numbers.

1 Introduction

In 2001 Simons [13] discovered a curious binomial identity which can be equivalently written as

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}x^k=\sum_{k=0}^{n}(-1)^{n+k}\binom{n}{k}\binom{n+k}{k}(x+1)^k. \tag{1}$$

This formula has attracted the attention of several researchers, and many alternative proofs of it have appeared in the literature. These authors used a variety of techniques in their proofs. For instance, Batır and Atpınar [4] extended (1) using the Wilf-Zeilberger algorithm.

Chapman [5], using a generating function method, gave an elegant and short proof of (1). Prodinger [11] used Cauchy integral formula to demonstrate a novel and short proof of this identity. Wang and Sun [16] presented a concise proof of it utilizing a linear transformation in the space of polynomials. Gould [6] published an interesting generalization of the Simons identity. Again, Gould [7] used properties of Legendre polynomials to prove curious special function identities. Shattuck [12] presented combinatorial proofs of some Simons-type binomial coefficient identities. Hirschhorn [8] made short comments on Simons' curious identity. Also, see some recent studies by Wei et al. [17] and Xu and Cen [15]. Munarini [9], using the Cauchy integral formula, offered the following nice generalization of (1):

$$\sum_{k=0}^{n}\binom{\alpha}{n-k}\binom{\beta+k}{k}x^k y^{n-k} = \sum_{k=0}^{n}(-1)^{n+k}\binom{\beta-\alpha+n}{n-k}\binom{\beta+k}{k}(x+y)^k y^{n-k}. \tag{2}$$

Note that (2) reduces to (1) when we put $\alpha=\beta=n$ and $y=1$. We can write as follows after replacing x by $\frac{x}{y}$.

$$\sum_{k=0}^{n}\binom{\alpha}{n-k}\binom{\beta+k}{k}x^k = \sum_{k=0}^{n}(-1)^{n+k}\binom{\beta-\alpha+n}{n-k}\binom{\beta+k}{k}(x+1)^k, \tag{3}$$

where n is a non-negative integer and α and β are complex numbers, which are not negative integers. We present a different proof of identity (3) due to Munarini, which generalizes a curious binomial identity of Simons.

Our first aim in the paper is to provide a different proof of (3) using the Taylor theorem and Wilf-Zeilberger algorithm; see [10]. Recently, Alzer and Kouba [1, Eq. (4.1)] proved the following identity

$$4^n\binom{\lambda}{n} = \binom{2\lambda}{n}\sum_{k=0}^{n}\binom{n}{k}\frac{\binom{n-\lambda-1/2}{k}}{\binom{k-\lambda-1/2}{k}}, \tag{4}$$

which is valid for all integers $n\geq 0$ and complex number λ, and they used it to derive several harmonic number sum identities. Secondly, we generalize identity (4) and apply it to deduce other binomial sum identities involving harmonic numbers. Furthermore, we prove a new binomial sum identity stated in Theorem 3. Lastly, applications of our results lead to many interesting and new combinatorial identities involving harmonic numbers. As examples, we prove that

$$H_n = \frac{1}{2}\sum_{k=1}^{n}(-1)^{n+k}\binom{n}{k}\binom{n+k}{k}H_k$$

and

$$\sum_{k=0}^{n}\binom{n}{k}^2 H_k H_{n-k} = \binom{2n}{n}\left((H_{2n}-2H_n)^2 + H_n^{(2)} - H_{2n}^{(2)}\right).$$

We also give a new representation for the Legendre polynomials $P_n(x)$. Many examples of these kinds obtained by taking particular values of the parameters involved are collected in Section 3. To ensure accuracy, all formulas in this paper were verified numerically by *Mathematica*. We think the approach we used in the theorem proofs can be used to prove other combinatorial equalities.

Now we need to recall some basic tools to prove our examples. The classical gamma function $\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}du$ and the digamma function ψ, logarithmic derivative of the Γ-function satisfy the following relations for all integer $n \geq 0$

$$\Gamma\big(n+\frac{1}{2}\big) = \frac{(2n)!\Gamma(1/2)}{2^{2n}n!}, \tag{5}$$

and for all complex number x, which is not an integer

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}, \tag{6}$$

and

$$\psi(n+1/2) - \psi(1/2) = 2H_{2n} - H_n, \tag{7}$$

where $\gamma = 0.57721...$ is the Euler-Mascheroni constant and H_n is the n th harmonic number; see, for example, [14].

$$\psi(1-x) - \psi(x) = \pi\cot(\pi x) \quad \text{and} \quad \psi'\big(x\big) - \psi'\big(1-x\big) = \pi^2\csc^2(\pi x); \tag{8}$$

see, for example, [14]. A generalized binomial coefficient $\binom{s}{t}$ $(s, t \in \mathbb{C})$ is defined, in terms of the classical gamma function, by

$$\binom{s}{t} = \frac{\Gamma(s+1)}{\Gamma(t+1)\Gamma(s-t+1)}, \quad (s, t \in \mathbb{C}). \tag{9}$$

2 Main results

In the first theorem in this section, we give a different proof of (3) using the Taylor theorem and WZ theory.

Theorem 1. *For all $\alpha, \beta \in \mathbb{C}$, which are not negative integers and non-negative integer n we have*

$$\sum_{k=0}^{n}\binom{\alpha}{n-k}\binom{\beta+k}{k}x^k = \sum_{j=0}^{n}(-1)^{n+j}\binom{\beta-\alpha+n}{n-j}\binom{\beta+j}{j}(x+1)^j. \tag{10}$$

Proof. We define

$$f(x) = \sum_{k=0}^{n}\binom{\alpha}{n-k}\binom{\beta+k}{k}x^k. \tag{11}$$

Since f is a polynomial of degree n in x its Taylor polynomial at $x = -1$ is

$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(-1)}{j!}(x+1)^j. \tag{12}$$

Differentiating f k times, it follows that

$$\frac{f^{(j)}(x)}{j!} = \sum_{k=0}^{n} \binom{\alpha}{n-k}\binom{\beta+k}{k}\binom{k}{j} x^{k-j}.$$

We set $x = -1$ and get

$$\frac{f^{(j)}(-1)}{j!} = \sum_{k=0}^{n} \binom{\alpha}{n-k}\binom{\beta+k}{k}\binom{k}{j} (-1)^{k-j}.$$

Applying this to (12) yields

$$f(x) = \sum_{j=0}^{n} (\sum_{k=0}^{n} \binom{\alpha}{n-k}\binom{\beta+k}{k}\binom{k}{j} (-1)^{k-j})(x+1)^j.$$

Comparing the coefficients of $(x+1)^j$ in this identity and the right-hand side of (10) we conclude that (10) holds if and only if for all $\alpha, \beta \in \mathbb{C}$, which are not negative integers, and all $j = 0, 1, 2, ..., n$ we have

$$\sum_{k=0}^{n} (-1)^{k+j} \binom{\beta+k}{k}\binom{k}{j}\binom{\alpha}{n-k} = (-1)^{n+j}\binom{\beta+j}{j}\binom{\beta-\alpha+n}{n-j}. \tag{13}$$

In order to show that (13) is valid we define

$$F(n,k) = \frac{(-1)^{k+n}\binom{\beta+k}{k}\binom{k}{j}\binom{\alpha}{n-k}}{\binom{\beta+j}{j}\binom{-\alpha+\beta+n}{n-j}}. \tag{14}$$

Applying the Wilf-Zeilberger algorithm [10], we find the companion function

$$G(n,k) = \frac{(-1)^{n+k}(j-k)(\alpha+k-n)\binom{\beta+k}{k}\binom{k}{j}\binom{\alpha}{n-k}}{(k-n-1)(\alpha-\beta-n-1)\binom{\beta+j}{j}\binom{-\alpha+\beta+n}{n-j}} \tag{15}$$

and the following difference equation

$$F(n,k) - F(n+1,k) = G(n,k+1) - G(n,k).$$

We sum both sides of this equation from $k = 0$ to $k = n+1$ and get

$$\sum_{k=0}^{n+1} \big(F(n,k) - F(n+1,k)\big) = \sum_{k=0}^{n+1} \big(G(n,k+1) - G(n,k)\big).$$

The right-hand side of this equality is telescoping, and thus

$$\sum_{k=0}^{n+1} F(n,k) - \sum_{k=0}^{n+1} F(n+1,k) = G(n,n+2) - G(n,0).$$

It is obvious from (15) that $G(n,0) = G(n,n+2) = 0$. Note that $\binom{\alpha}{-2} = 0$. This results in

$$\sum_{k=0}^{n+1} F(n,k) - \sum_{k=0}^{n+1} F(n+1,k) = 0$$

or equivalently

$$\sum_{k=0}^{n+1} F(n+1,k) - \sum_{k=0}^{n} F(n,k) - F(n,n+1) = 0.$$

But since $F(n,n+1) = 0$, which follows from (14), this gives

$$\sum_{k=0}^{n+1} F(n+1,k) - \sum_{k=0}^{n} F(n,k) = 0.$$

Replacing n by υ here and then summing the resultant identity from $\upsilon = 0$ to $\upsilon = n-1$, we get

$$\sum_{\upsilon=0}^{n-1} \left(\sum_{k=0}^{\upsilon+1} F(\upsilon+1,k) - \sum_{k=0}^{\upsilon} F(\upsilon,k) \right) = 0.$$

This is also a telescoping sum; thus, we obtain

$$\sum_{k=0}^{n} F(n,k) - F(0,0) = 0.$$

Since $0 \le j \le n$ and $F(0,0) = 1$ from (14), so that

$$\sum_{k=0}^{n} F(n,k) = 1,$$

which is equivalent to (13). $\square$

In the next theorem, we generalize the identity (4).

Theorem 2. *Let s and t be complex numbers, which are not negative integers and n be a non-negative integer. Then we have*

$$\sum_{k=0}^{n} \binom{n}{k} \frac{\binom{s}{k}}{\binom{t+k}{k}} = \frac{\binom{n+s+t}{s}}{\binom{s+t}{s}}. \tag{16}$$

Proof. We prove the result using the Wilf-Zeilberger algorithm. Let

$$A(n,k) = \frac{\binom{n}{k}\binom{s}{k}\binom{s+t}{t}}{\binom{t+k}{k}\binom{n+s+t}{s}}. \tag{17}$$

The Wilf–Zeilberger algorithm gives us

$$B(n,k) = \frac{k(t+k)\binom{n}{k}\binom{s}{k}\binom{t+s}{t}}{(k-n-1)(n+t+s+1)\binom{t+k}{k}\binom{n+s+t}{s}} \tag{18}$$

and the pair (A,B) is a WZ-pair. That is,

$$A(n+1,k) - A(n,k) = B(n,k+1) - B(n,k).$$

We sum both sides of this equation from $k = 0$ to $k = n+1$. This gives

$$\sum_{k=0}^{n+1} \big(A(n+1,k) - A(n,k)\big) = \sum_{k=0}^{n+1} \big(B(n,k+1) - B(n,k)\big).$$

The right-hand side of this equality is a telescoping sum; thus

$$\sum_{k=0}^{n+1} A(n+1,k) - \sum_{k=0}^{n+1} A(n,k) = B(n,n+2) - B(n,0).$$

It is obvious from (18) that $B(n,0) = B(n,n+2) = 0$. This leads to

$$\sum_{k=0}^{n+1} A(n+1,k) - \sum_{k=0}^{n+1} A(n,k) = 0$$

or equivalently

$$\sum_{k=0}^{n+1} A(n+1,k) - \sum_{k=0}^{n} A(n,k) - A(n,n+1) = 0.$$

But since $A(n,n+1) = 0$, which follows from (17), it follows that

$$\sum_{k=0}^{n+1} A(n+1,k) - \sum_{k=0}^{n} A(n,k) = 0.$$

Replacing n by p here and then summing the resultant identity from $p = 0$ to $p = n-1$ we arrive at

$$\sum_{p=0}^{n-1} \left(\sum_{k=0}^{p+1} A(p+1,k) - \sum_{k=0}^{p} A(p,k) \right) = 0.$$

This is also a telescoping sum; thus, we obtain

$$\sum_{k=0}^{n} A(n,k) - A(0,0) = 0.$$

By (17) we have $A(0,0) = 1$, so that

$$\sum_{k=0}^{n} A(n,k) = 1,$$

which proves Theorem 2. $\square$

Note that (16) reduces to (4) when we substitute $s = n - \lambda - \frac{1}{2}$ and $t = -\lambda - \frac{1}{2}$.

Theorem 3. *Let s and p be complex numbers, which are not negative integers, and n be a non-negative integer. Then we have*

$$\sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \binom{s+k}{k} \binom{k}{p} = \binom{n}{p} \binom{s+p}{n}. \tag{19}$$

Proof. We again employ the WZ method. Let

$$P(n,k) = \frac{(-1)^{n+k} \binom{n}{k} \binom{s+k}{k} \binom{k}{p}}{\binom{n}{p} \binom{s+p}{n}}. \tag{20}$$

Using the Wilf-Zeilberger algorithm, this gives us the companion function

$$Q(n,k) = \frac{(-1)^{n+k} k(k-p) \binom{n}{k} \binom{k}{p} \binom{-1-s}{k}}{(-1+k-n)(n-p-s) \binom{n}{p} \binom{s+p}{n}}, \tag{21}$$

and we obtain that the pair (P, Q) satisfies the following difference equation

$$P(n,k) - P(n+1,k) = Q(n,k+1) - Q(n,k).$$

We sum both sides of this equation from $k = 0$ to $k = n+1$ and get

$$\sum_{k=0}^{n+1} \big(P(n,k) - P(n+1,k) \big) = \sum_{k=0}^{n+1} \big(Q(n,k+1) - Q(n,k) \big).$$

The right-hand side of this equality is a telescoping sum, so that

$$\sum_{k=0}^{n+1} P(n,k) - \sum_{k=0}^{n+1} P(n+1,k) = Q(n,n+2) - Q(n,0).$$

It is clear from (21) that $Q(n,0) = Q(n,n+2) = 0$. This leads to

$$\sum_{k=0}^{n+1} P(n,k) - \sum_{k=0}^{n+1} P(n+1,k) = 0$$

or equivalently

$$\sum_{k=0}^{n+1} P(n+1,k) - \sum_{k=0}^{n} P(n,k) - P(n,n+1) = 0.$$

But since $P(n,n+1) = 0$, which follows from (20), this gives

$$\sum_{k=0}^{n+1} P(n+1,k) - \sum_{k=0}^{n} P(n,k) = 0.$$

Replacing n by p here and then summing the resultant identity from $p = 0$ to $p = n-1$ we obtain

$$\sum_{p=0}^{n-1} \left(\sum_{k=0}^{p+1} P(p+1,k) - \sum_{k=0}^{p} P(p,k) \right) = 0.$$

This is also a telescoping sum; thus

$$\sum_{k=0}^{n} P(n,k) - P(0,0) = 0.$$

By (20) we have $P(0,0) = 1$, so that

$$\sum_{k=0}^{n} P(n,k) = 1,$$

which is equivalent to (19). □

3 Applications

In this section we provide many applications of Theorems 1, 2, and 3 by setting particular values for the parameters α, β, t, s and p. All of the examples provided here, based on our assessment, are new in the literature. If we set $\alpha = n$ in (10) and use

$$\binom{\beta}{n-k}\binom{\beta+k}{k} = \binom{n}{k}\binom{\beta+k}{n},$$

which can be easily verified, we get

$$\sum_{k=0}^{n} \binom{n}{k}\binom{\beta+k}{k} x^k = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k}\binom{\beta+k}{n} (1+x)^k, \tag{22}$$

where β and x are complex numbers with $\beta \neq -1, -2, -3, \ldots$ and $n \geq 0$ be an integer.

- We set $x = 0$ and $x = -1$ in (22) and get, respectively

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{\beta + k}{n} = (-1)^n \tag{23}$$

and

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{\beta + k}{k} = (-1)^n \binom{\beta}{n}.$$

- Differentiate both sides of (22) with respect to β and then put $\beta = n$ to obtain

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} H_{n+k} - H_n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} = (-1)^n H_n.$$

Using (23) with $\beta = n$, this gives the following explicit representation for H_n.

$$H_n = \frac{1}{2} \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \binom{n+k}{k} H_{n+k}.$$

- Substituting $\beta = -\frac{1}{2}$ in (22) produces

$$\sum_{k=0}^{n} \binom{n}{k} \binom{-\frac{1}{2} + k}{k} x^k = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} \binom{-\frac{1}{2} + k}{n} (1 + x)^k. \tag{24}$$

We have

$$\binom{-1/2 + k}{k} = \frac{\Gamma(k + 1/2)}{k!\Gamma(1/2)} = \frac{1}{4^k} \binom{2k}{k}$$

and

$$\binom{-1/2 + k}{n} = \frac{\Gamma(k + 1/2)}{n!\Gamma(1/2 + k - n)} = \frac{(-1)^{n+k} \binom{2k}{k} \binom{2n-2k}{n-k}}{4^n \binom{n}{k}},$$

both of which follow from Legendre's duplication and reflection formulas for the classical gamma function Γ given in (5), (6), and (7). Replacing these in (24) it follows that

$$\sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \frac{x^k}{2^{2k}} = \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2k}{k} \binom{2n - 2k}{n - k} (1 + x)^k. \tag{25}$$

- Let $P_n(x)$ be the classical Legendre polynomials. From [6, p. 39] (see also [1]) we have

$$t^n P_n\left(\frac{t^2 + 1}{2t}\right) = \frac{1}{4^n} \sum_{k=0}^{n} \binom{2k}{k} \binom{2n - 2k}{n - k} t^{2k}.$$

Making the substitution $x \to t^2 - 1$ in (25), we have

$$\sum_{k=0}^{n} \binom{n}{k}\binom{2k}{k}\left(\frac{t^2-1}{4}\right)^k = \frac{1}{4^n}\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}t^{2k}.$$

From the last two identities, we deduce the following new representation for Legendre polynomials for $t \neq 0$:

$$P_n\left(\frac{t^2+1}{2t}\right) = \frac{1}{t^n}\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\left(\frac{t^2-1}{4}\right)^k.$$

The well-known binomial inversion formula leads us to

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}P_k\left(\frac{t^2+1}{2t}\right)t^k = \binom{2n}{n}\left(\frac{1-t^2}{4}\right)^n.$$

- Differentiating both sides of (19) with respect to p and then setting $p = 0$, we get

$$\sum_{k=0}^{n}(-1)^{n+k}\binom{n}{k}\binom{s+k}{k}H_k = \binom{s}{n}\big(H_n + \psi(s+1) - \psi(s-n+1)\big). \tag{26}$$

For $s = n$ this leads to the following elegant representation for H_n:

$$H_n = \frac{1}{2}\sum_{k=1}^{n}(-1)^{n+k}\binom{n}{k}\binom{n+k}{k}H_k,$$

where we have used $\psi(n+1) + \gamma = H_n$. Setting $s = \frac{1}{2}$ in (26) we get

$$\sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{2k}{k}\frac{H_k}{4^k} = \frac{1}{2^{2n-1}}\binom{2n}{n}\big(H_n - H_{2n}\big).$$

- Differentiating both sides of (19) with respect to p twice, and then setting $p = 0$, we get

$$\begin{aligned}
&\sum_{k=0}^{n}(-1)^{n+k}\binom{n}{k}\binom{s+k}{k}\big(H_k^2 + H_n^{(2)}\big)\\
&= \binom{s}{n}\Big(2H_n\big(\psi(s+1) - \psi(-n+s+1)\big) + \psi(-n+s+1)^2\\
&- 2\psi(s+1)\psi(-n+s+1) - \psi'(-n+s+1) + H_n^2\\
&+ H_n^{(2)} + \psi(s+1)^2 + \psi'(s+1) - \psi'(s-n+1)\Big)
\end{aligned}$$

where ψ is the digamma function. For $s = n$ this gives

$$H_n^2 = \frac{1}{4}\sum_{k=0}^{n}(-1)^{n+k}\binom{n}{k}\binom{n+k}{k}\big(H_k^2 + H_k^{(2)}\big),$$

where $H_k^{(2)} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2}$.

- Applying the binomial inversion formula to (19) we get

$$\sum_{k=0}^{n}\binom{n}{k}\binom{s+p}{k}\binom{k}{p} = \binom{n}{p}\binom{s+n}{n}.$$

Setting $s = n - \frac{1}{2}$ and $p = \frac{1}{2}$ here we obtain

$$\sum_{k=0}^{n}\frac{4^k\binom{n}{k}^2}{\binom{2k}{k}} = \frac{\binom{4n}{2n}}{\binom{2n}{n}}. \tag{27}$$

Using

$$\binom{2n}{n}\binom{n}{k}^2 = \binom{2k}{k}\binom{2n-2k}{n-k}\binom{2n}{2k}$$

Eq. (27) can be rewritten in the following elegant form

$$\sum_{k=0}^{n}\binom{2n}{2k}\binom{2n-2k}{n-k}4^k = \binom{4n}{2n}.$$

- Setting $p = n + \frac{1}{2}$ in (16) we get

$$\sum_{k=0}^{n}\binom{s+k}{k}\binom{2n-2k}{n-k}4^k = \frac{\binom{2n}{n}\binom{2n+2s+1}{2s+1}}{\binom{n+s}{n}}. \tag{28}$$

If we differentiate both sides of (28) with respect to s and then set $s = 0$ we conclude that

$$\sum_{k=0}^{n}\binom{2k}{k}\frac{H_{n-k}}{4^k} = \frac{2n+1}{4^n}\binom{2n}{n}\big(2H_{2n+1} - H_n - 2\big).$$

- From (16) with $s = n$ and $t = 1$ we obtain

$$\sum_{k=1}^{n}k\binom{n}{k}^2 = \frac{n}{2}\binom{2n}{n}.$$

Applying the operator $\frac{\partial^2}{\partial s \partial t}$ both sides of (16), using

$$\sum_{k=0}^{n} \binom{n}{k}^2 H_k = \binom{2n}{n} \big(2H_n - H_{2n}\big),$$

and finally putting $t = 0$ and $s = n$ in the resultant we get

$$\sum_{k=0}^{n} \binom{n}{k}^2 H_k H_{n-k} = \binom{2n}{n} \big((H_{2n} - 2H_n)^2 + H_n^{(2)} - H_{2n}^{(2)}\big).$$

We differentiate both sides of (19) with respect to t twice and then we set $s = n$ and $t = 0$. This gives

$$\sum_{k=0}^{n} \binom{n}{k}^2 \big((H_k)^2 + H_k^{(2)}\big) = \binom{2n}{n} \big((H_{2n} - 2H_n)^2 + 2H_n^{(2)} - H_{2n}^{(2)}\big).$$

Remark 4. We can derive many other examples using our main results, but we are satisfied with these to keep our paper short. The examples presented here demonstrate the usefulness of our main results. Several other applications of (22) can be found in [4].

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